Markowitz Portfolio Optimization

Group 22

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Problem Overview

- In this project we will use Markowitz model which is a popular Portfolio Optimization technique, to maximize returns while imposing a limit on risk.
- Consider an investor who wishes to allocate capital among N securities at time t=0 and hold them over a single period of time until $t = h$.
- We denote $P_{0,i}$ the (known) price of security i at the beginning of the investment period and $P_{h,i}$ the (random) price of security i at the end of the investment period $t = h$.
- The rate of return of security i over period h is then modeled by the random variable $Ri = P_{h,i}/P_{0,i-1}$, and its expected value is denoted by $\mu_i = E(R_i)$.
- The risk-averse investor seeks to maximize the return of the investment, while trying to keep the investment risk, i. e., the uncertainty of the future security returns R_i on an acceptable low level.

Mathematical Formulation

We make the investment decision at time $t = 0$ by specifying the N-dimensional decision vector x called portfolio, where x_i is the fraction of funds invested into security i. We can then express the random portfolio return as $R_x = \sum_i \boldsymbol{x}_i R_i = \boldsymbol{x}^T R$, where R is the vector of security returns. The optimal $\boldsymbol{x}$ is given based on the following inputs of the portfolio optimization problem:

The expected portfolio return:

$$\mu_{\mathbf{x}} = \mathbb{E}(R_{\mathbf{x}}) = \mathbf{x}^{\mathsf{T}} \mathbb{E}(R) = \mathbf{x}^{\mathsf{T}} \mu.$$

The portfolio variance:

$$\sigma_{\mathbf{x}}^2 = \mathrm{Var}(R_{\mathbf{x}}) = \sum_i \mathrm{Cov}(R_i, R_j) x_i x_j = \mathbf{x}^{\mathsf{T}} \Sigma \mathbf{x}.$$

Here μ is the vector of expected returns, Σ is the covariance matrix of returns, summarizing the risks associated with the securities.
After the above parameters the problem is also referred to as mean–variance optimization (MVO). The choice of variance as the risk measure results that MVO is a quadratic optimization (QO) problem.

Optimization Formulation

There are two methods of formulating the optimisation problem:

Method-1: Explicitly constrain the volatility (variance)

Maximize the expected portfolio return, with the constraint expressing an upper bound on the portfolio risk:

$$\begin{array}{lrcl} \text{maximize} & \boldsymbol{\mu}^{\mathsf{T}}\mathbf{x} & & \\ \text{subject to} & \mathbf{x}^{\mathsf{T}}\boldsymbol{\Sigma}\mathbf{x} & \leq & \gamma^2, \\ & \mathbf{1}^{\mathsf{T}}\mathbf{x} & = & 1. \end{array}$$

Method-2: Using Risk-Aversion coefficient :*δ*

Maximize the utility function of the investor

$$\begin{array}{lrcl} \text{maximize} & \boldsymbol{\mu}^{\mathsf{T}}\mathbf{x} - \frac{\delta}{2}\mathbf{x}^{\mathsf{T}}\boldsymbol{\Sigma}\mathbf{x} & & \\ \text{subject to} & \mathbf{1}^{\mathsf{T}}\mathbf{x} & = & 1. \end{array}$$

In this case we construct the (concave) quadratic utility function $\mu^T \boldsymbol{x} - (\delta/2) \boldsymbol{x}^T \Sigma \boldsymbol{x}$ to represent the risk-averse investor's preferred tradeoff between portfolio return and portfolio risk.

Quadratic Cone for Quadratic Formulation

Assuming that the covariance matrix estimate Σ is positive definite, it is possible to decompose it as $\Sigma = GG^T$, where $G \in R^{N\times k}$. We can do this for example by Cholesky decomposition

Using the decomposition we can write the portfolio variance as $\boldsymbol{x}^T \Sigma \boldsymbol{x} = \boldsymbol{x}^T G G^T \boldsymbol{x} = \| G^T \boldsymbol{x} \|^2{}_2$. This leads to a explicit conic form. We can directly model the squared norm constraint $\| G^T \boldsymbol{x} \|_2{}^2 \leq \gamma^2$ using the rotated quadratic cone as $(\gamma^2, 0.5, G^t \boldsymbol{x}) \in Q_r{}^{k+2}$. This will give us the following conic equivalent:

$$
\begin{array}{lrcl}
\text{maximize} & \boldsymbol{\mu}^\mathsf{T}\mathbf{x} & & \\
\text{subject to} & (\gamma^2, \frac{1}{2}, \mathbf{G}^\mathsf{T}\mathbf{x}) & \in & \mathcal{Q}_\mathrm{r}^{k+2}, \\
 & \mathbf{1}^\mathsf{T}\mathbf{x} & = & 1.
\end{array}
$$

Note: We use the factor γ to avoid short selling

Python Code Overview

1. Define mu (expected returns), sigma (covariance matrix), and gamma (risk tolerance).
2. Use MOSEK API to set up variables and constraints.
3. Define objective function to maximize return.
4. Set risk constraint using a quadratic cone.
5. Solve the model and retrieve results.

```python
with Model('Markowitz_Optimisation') as M:
    x=M.variable("x", num_securities, mf.Domain.greaterThan(0.0))
    M.constraint('budget', mf.Expr.sum(x), mf.Domain.equalsTo(1.0))
    M.objective('obj', mf.ObjectiveSense.Maximize, mf.Expr.dot(mu, x))
    M.constraint('risk', mf.Expr.vstack(gamma, 0.5, mf.Expr.mul(G.T, x)), mf.Domain.inRotatedQCone())
    M.solve()
    expected_return=M.primalObjValue()
    portfolio=x.level()
    print("Optimal Portfolio Allocation:", portfolio)
    print("Expected Portfolio Return:", expected_return)
```

Results and Conclusion

- We used 10 popular securities from NIFTY50 Index
- The time period taken for optimisation is 10 years (10 years of historical data available)
- We calculted daily returns for each security.
- We varied the the risk optimal boundary and obtained an efficient frontier curve
- For $\gamma = 0.02$, we obtain an optimal return of 0.08%