Variation of Focal Length of Customized Lens on Dependable Voltage

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1 Setup

The prototype consists of two liquids, polar and non-polar, placed without air gaps between them. These liquids have refractive indices n_1 and n_2, respectively.

A circular voltage belt is placed across the border of the set-up to apply voltage V, enabling control of the system's focal length by altering the curvature of the liquid interface.

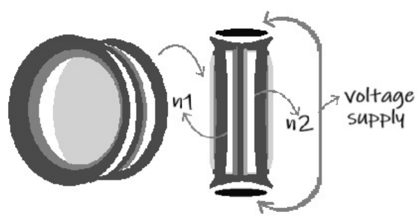

Figure 1: Prototype illustration

1.1 Lens Making Formula

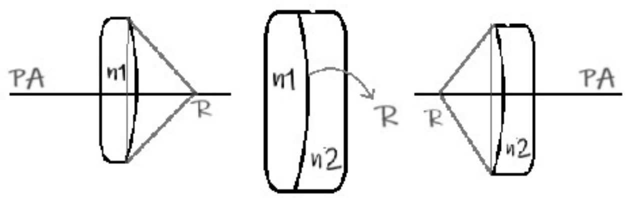

Figure 2: Lens illustration

Since both lenses are plano-convex and plano-concave, the formula for the net focal length is:

$$\frac{1}{f} = \frac{1}{f_1} - \frac{1}{f_2} \tag{1}$$

Let the curvature of the inner surface be R. Then f_1 and f_2 are defined as follows:

For the plano-convex lens, the curvature of the other surface is infinite, making its inverse converge to 0. Therefore:

$$\frac{1}{f_1} = (n_1 - 1)\left(-\frac{1}{R}\right) \tag{2}$$

Similarly, for the plano-concave lens:

$$\frac{1}{f_2} = (n_2 - 1)\left(-\frac{1}{R}\right) \tag{3}$$

1.2 Formula Dependency for Focal Length

Thus, the relation for the net focal length is:

$$\frac{1}{f} = \frac{1}{R}(2 - n_1 - n_2) \tag{4}$$

hence,

$$f = \frac{R}{(2 - n_1 - n_2)} \tag{5}$$

2 Derivation

2.1 Force Equivalent

Equating the forces at the interface, as illustrated in Figure 1:

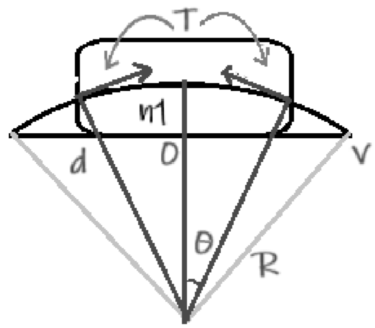

Figure 3: Force equivalence

The vertical component of surface tension is:

$$F_{\text{surface tension}} = 2T\cos\theta \cdot 2\pi d$$

The electrostatic force, due to the applied voltage, is:

$$F_{\text{electrostatic}} = E \cdot q$$

Equating the two:

$$E \cdot q = 2T\cos\theta \cdot 2\pi d$$

Substituting $E = \frac{V}{d}$, where V is the voltage:

$$q = \frac{4\pi T d^2}{V}\cos\theta$$

Differentiating with respect to time t:

$$\frac{dq}{dt} = \frac{4\pi T d^2}{V}\left(-\sin\theta \cdot \frac{d\theta}{dt} - \frac{\cos\theta}{V} \cdot \frac{dV}{dt}\right)$$

Since $\frac{dq}{dt} = I$, and $I = \frac{V}{\text{res}}$, where res is the resistance:

$$\frac{V}{\text{res}} = \frac{4\pi T d^2}{V}\left(-\sin\theta \cdot \frac{d\theta}{dt} - \frac{\cos\theta}{V} \cdot \frac{dV}{dt}\right)$$

Rearranging:

$$V = \text{res} \cdot \frac{4\pi T d^2}{V}\left(-\sin\theta \cdot \frac{d\theta}{dt} - \frac{\cos\theta}{V} \cdot \frac{dV}{dt}\right)$$

2.2 Resistance Calculation

In this section, we calculate the resistance in the radial flow of voltage from the center to the outer layer of the liquid interface, considering the prototype configuration.

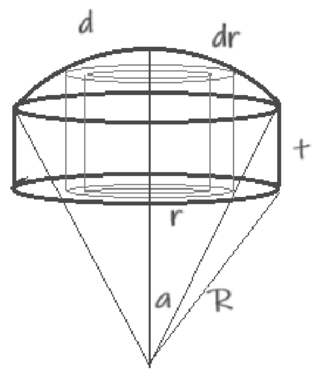

Figure 4: Resistance calculation

The classic formula for resistance is:

$$\text{res} = \rho \cdot \frac{l}{A}, \tag{6}$$

where:

- ρ: resistivity of the medium,
- l: length of the conductive path,
- A: cross-sectional area of the conductive path.

For an infinitesimal section of resistance $d\text{Res}$, we can write:

$$d\text{Res} = \frac{\rho\, dr}{2\pi r \cdot \text{height of section}}. \tag{7}$$

The height of the section is given by:

$$\text{height of section} = th + R \cdot (\cos\alpha - \cos\theta), \tag{8}$$

where:

- th: thickness of the section,
- R: radius of curvature,
- α: angle variable,

- θ: angle at the edge of the section.

We relate the radius r to the angle α using:

$$\sin\alpha = \frac{r}{R} \quad \text{or} \quad r = R \cdot \sin\alpha. \tag{9}$$

Differentiating r with respect to α gives:

$$dr = R \cdot \cos\alpha \, d\alpha. \tag{10}$$

Substituting dr into the expression for $d\text{Res}$, we get:

$$d\text{Res} = \frac{\rho \cdot R \cdot \cos\alpha \, d\alpha}{2\pi R \cdot (th + R \cdot (\cos\alpha - \cos\theta))}. \tag{11}$$

Simplifying further:

$$\frac{d\text{Res}}{d\alpha} = \frac{\rho}{2\pi R} \cdot \frac{\cos\alpha}{th + R \cdot (\cos\alpha - \cos\theta)}. \tag{12}$$

The total resistance of the radial section can be obtained by integrating $d\text{Res}$:

$$\text{Res} = \int_0^\theta \frac{\rho}{2\pi R} \cdot \frac{\cos\alpha}{th + R \cdot (\cos\alpha - \cos\theta)} \, d\alpha. \tag{13}$$

2.3 End formula

previously:

$$\text{Res} = \int_0^\theta \frac{\rho}{2\pi R} \cdot \frac{\cos\alpha}{th + R \cdot (\cos\alpha - \cos\theta)} \, d\alpha. \tag{14}$$

$$\text{Res} = \frac{\rho}{2\pi R^2} \cdot \int_0^\theta 1 - (\frac{th}{R} - \cos\theta) \cdot \frac{1}{th/R + (\cos\alpha - \cos\theta)} \, d\alpha. \tag{15}$$

The general result for the integral is:

$$\int \frac{1}{\cos\theta + m} \, d\theta = \frac{1}{\sqrt{1-m^2}} \cdot \arctan\left(\sqrt{\frac{1-m}{1+m}} \cdot \tan\frac{\theta}{2}\right)$$

Substituting $m = \frac{th}{R} - \cos\theta$, we get:

$$\int \frac{1}{\cos\alpha + \left(\frac{th}{R} - \cos\theta\right)} \, d\alpha = \frac{1}{\sqrt{1 - \left(\frac{th}{R} - \cos\theta\right)^2}} \cdot \arctan\left(\sqrt{\frac{1 - \left(\frac{th}{R} - \cos\theta\right)}{1 + \left(\frac{th}{R} - \cos\theta\right)}} \cdot \tan\frac{\theta}{2}\right)$$

$$\text{Res} = \frac{\rho}{2\pi R^2} \cdot (\theta - (\frac{th}{R} - \cos\theta) \cdot \frac{1}{\sqrt{1 - \left(\frac{th}{R} - \cos\theta\right)^2}} \cdot \arctan\left(\sqrt{\frac{1 - \left(\frac{th}{R} - \cos\theta\right)}{1 + \left(\frac{th}{R} - \cos\theta\right)}} \cdot \tan\frac{\theta}{2}\right)). \tag{16}$$

also

$$V = \text{Res} \cdot \frac{4\pi T d^2}{V} \left(-\sin\theta \cdot \frac{d\theta}{dt} - \frac{\cos\theta}{V} \cdot \frac{dV}{dt} \right)$$

and

$$R = \frac{d}{\sin\theta}$$

in order to reduce the parameters we can assume that the voltage is increasing linearly with time.

3 Energy Conservation

The volumetric energy provided by the change in voltage is equal to the surface energy of the polar liquid.

The electric energy can be expressed as:

$$U_{\text{electric}} = \frac{1}{2}\varepsilon_0 E^2 \cdot \text{liquid volume}$$

The surface energy is given by:

$$U_{\text{surface}} = T \cdot \text{surface area}$$

3.1 Surface Area

To calculate the surface area, we consider a small differential element as shown in the diagram. The total surface area is obtained by integrating over the angle θ:

$$\text{Surface Area} = \int_0^\theta 2\pi R^2 \sin\theta \, d\theta$$

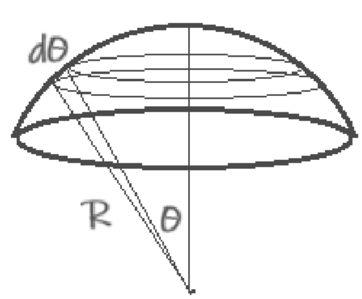

Figure 5: Surface area calculation

Solving the integral:

$$\text{Surface Area} = 2\pi R^2 \int_0^\theta \sin\theta \, d\theta = 2\pi R^2 \left[-\cos\theta\right]_0^\theta$$

Simplify:

$$\text{Surface Area} = 2\pi R^2 (1 - \cos\theta)$$

Here:

- ε_0: Permittivity of free space.
- E: Electric field.
- T: Surface tension of the polar liquid.
- R: Radius of the liquid surface.
- θ: Angle of integration.

3.2 End formula

Equating the energy:

$$\frac{1}{2}\varepsilon_0 E^2 \cdot \text{liquid volume} = T * 2\pi R^2 (1 - \cos\theta)$$

The expressions for electric field (E) and voltage (V) are given as follows:

Electric Field (E)

The electric field is expressed as:

$$E = \sqrt{\frac{8\pi T}{\varepsilon_0 \cdot \text{liquid\_volume}}} \cdot R \cdot \sin\left(\frac{\theta}{2}\right)$$

The voltage is expressed as:

$$V = d \cdot \sqrt{\frac{8\pi T}{\varepsilon_0 \cdot \text{liquid\_volume}}} \cdot R \cdot \sin\left(\frac{\theta}{2}\right)$$

Since $R = \frac{d}{\sin\theta}$, substitute R into V:

$$V = d^2 \cdot \sqrt{\frac{8\pi T}{\varepsilon_0 \cdot \text{liquid\_volume}}} \cdot \frac{\left|\sin\left(\frac{\theta}{2}\right)\right|}{\sin\theta}$$

4 Variation Comparison

4.0.1 energy equivalence

plot θ with respect to v, considering v directly propotional to time (t).

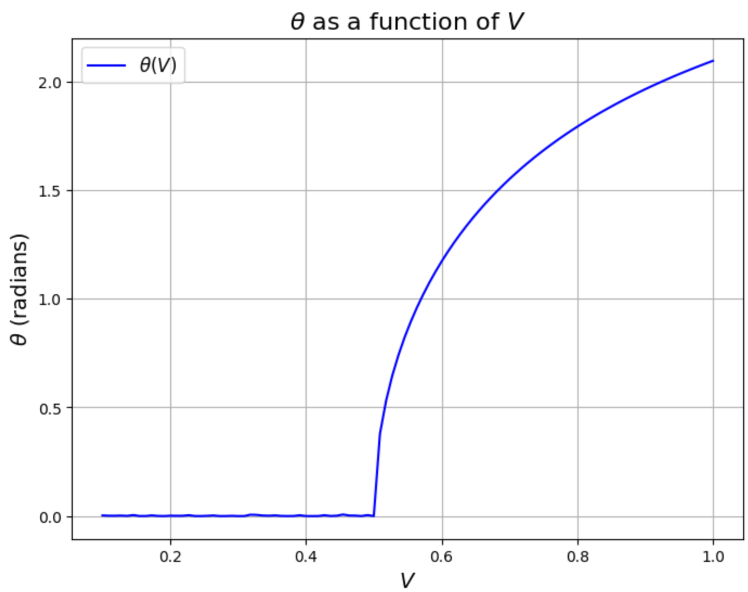

Figure 6: calculation

4.0.2 force equivalence

To plot θ with respect to t, the process involves swapping the axes of the plot, making θ the dependent variable and t the independent variable.

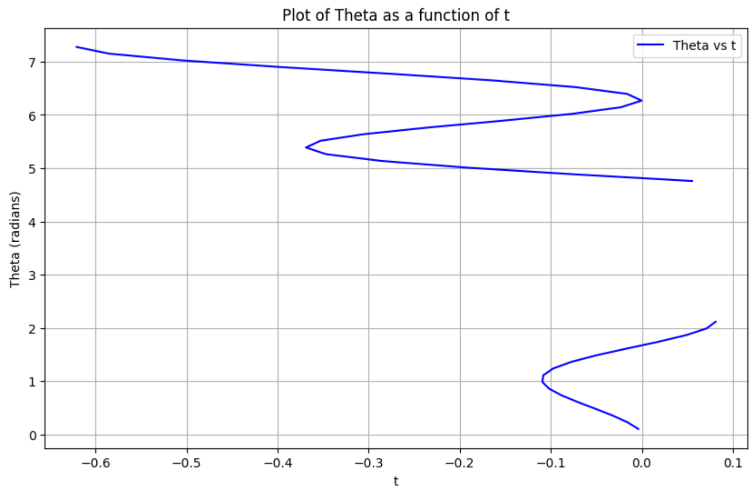

Figure 7: calculation

To ensure that t starts from 0 and progresses to positive values, we will treat t as the independent variable and solve for θ in terms of t. The code needs to reverse the relationship, iteratively finding θ for each t.

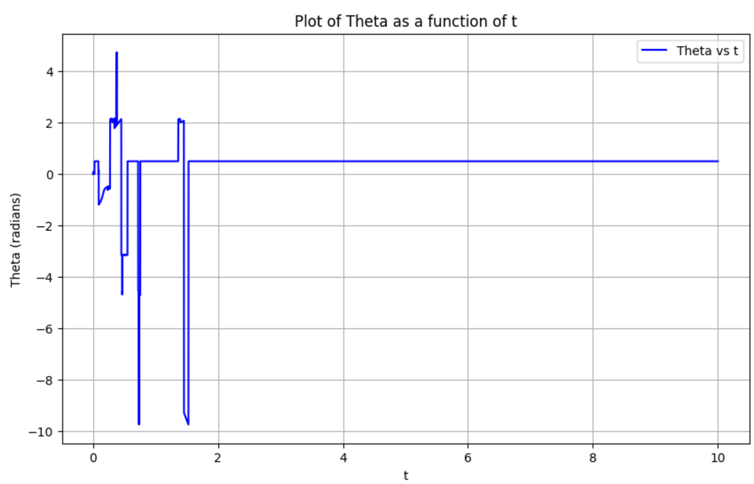

Figure 8: calculation