

# Learning Certified Control Using Contraction Metric

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**Abstract:** In this paper, we solve the problem of finding a certified control policy that drives robots' trajectories from any given initial state and under any bounded disturbance to converge to the desired reference trajectory, with guarantees on the convergence or bounds on the tracking error. Such a controller is crucial in motion planning with safety guarantees. We leverage the advanced theory in Control Contraction Metric and design a learning framework based on neural networks to co-synthesize the metric/certificate and controller. We further provide methods to validate the convergence and bounded error guarantees. We demonstrate the performance of our method using a suite of challenging robotic models, including models with learned dynamics as neural networks. We compare our approach with leading methods using sum-of-squares programming, reinforcement learning, and model predictive control. Results show that our methods indeed can handle a broader class of systems with less tracking error and faster execution speed.

**Keywords:** Learning Tracking Controller, Control Contraction Metric, Convergence Guarantee, Neural Networks

## 1 Introduction

Designing safe motion planning controllers for nonholonomic robotic systems is a critical yet extremely challenging problem. In general, motion planning for robots is notoriously difficult. For example, even planning for a robot (composed of  $n$  polyhedral bodies) in a 3D world that contains a fixed number of polyhedral obstacles is PSPACE-hard [1]. Simultaneously providing formal guarantees on safety and robustness when the robots are facing uncertainties and disturbances makes the problem even more challenging. Sample-based planning techniques [2, 1] can plan safe trajectories by exploring the environment but cannot handle uncertainty and disturbances. A natural thought that has been extensively explored recently is to exploit a separation of concerns that exists in the problem: (A) how to design a reference (also called expected or nominal) trajectory  $x^*(t)$  to drive a robot to its goal safely without any uncertainty or disturbance, and (B) how to design a tracking controller to make sure the actual trajectories  $x(t)$  of the system under disturbances can converge to  $x^*(t)$  with guaranteed bounds for the tracking error between  $x(t)$  and  $x^*(t)$  [3, 4, 5, 6]. Combining controllers from solving (A) and (B) can make sure that the actual behaviors of the robot are all safe.

While the reference trajectories can be found using efficient planning techniques [2, 1, 7] for a very broad class of systems, the synthesis of a guaranteed tracking controller, also called trajectory stabilization [8] is not an automatic process, especially for nonholonomic systems. Control theoretic techniques exist to guide the dual synthesis of control policies and certificates at the same time, where the certificates can make sure certain required properties are provably satisfied when the corresponding controller is run in (closed-loop) composition with the plant. For example, control Lyapunov Function (CLF) [9, 10] ensures the existence of a controller so the controlled system is Lyapunov stable, and Control Barrier Function (CBF) [11, 12, 13] ensures that a controller exists

so the controlled system always stays in certain safety invariant sets defined by the corresponding barrier function. Other methods pre-compute the tracking error through reachability analysis [4], Funnels [14, 5], and Hamilton-Jacobi analysis [3, 15]. Despite the benefit brought by these certificates (e.g. Lyapunov Function, Barrier Function), finding the correct function representation for the certificates is non-trivial. Various methods have been studied to learn the certificate as polynomials [16, 10], support vectors [17], and neural networks (NN) [9, 13, 18, 19, 20]. Unlike reinforcement learning (RL) [21, 22, 23, 24] which focuses on learning a control policy that maximizes an accumulated reward and usually lacks formal safety guarantees, certificate-guided controller learning focuses on learning a sufficient condition for the desired property. In this paper, we follow this idea and learn a tracking controller by learning a convergence certificate simultaneously as guidance.

In this paper, we leverage recent advances in Control Contraction Metric (CCM) [25, 26] theory that extends the contraction theory to controlled systems to prove the existence of a tracking controller so the closed-loop system is contracting. In the CCM theory [25], it has been shown that a valid CCM implies the existence of a contracting tracking controller for any reference trajectories. In our framework, we model both the controller and the metric function as neural networks and optimize them jointly with the cost functions inspired by the CCM theory. Due to the constraints imposed during training, the tracking error of the learned controller is guaranteed to be bounded even with external disturbances.

Synthesis of CCM certificate has been formulated as solving a Sum-of-Squares (SoS) [27], or Linear Matrix Inequality (LMI) [8] problem even when the model dynamics are unknown. However, such SoS-based or LMI-based techniques have to rely on an assumption on the special control input structure of a class of underactuated systems (see Sec. 3.1 for details) and therefore cannot be applied to general robotic systems. Moreover, the above methods only learn the CCM. The controller needs to be found separately by computing geodesics of the CCM, which cannot be solved exactly and has high computational complexity. To use SoS, the system dynamics need to be polynomial equations or can be approximated as polynomial equations. The degree and template of the polynomials in SoS play a crucial role in determining whether a solution exists and need to be tuned for each system. In [28], the authors proposed a synthesis framework using recurrent NNs to model the contraction metric and this framework works on nonlinear systems which are a convex combination of multiple state-dependent coefficients (i.e.  $f(x, t)$  is written as  $A(x, t)x$ ). Again, the controller in [28] is constructed from the learned metric. In contrast, our approach can simultaneously synthesize the controller and the CCM certificate without any additional structural assumptions for the system.

We provide two methods to prove convergence guarantees of the learned controller and CCM (Section 3.2). The first method provides deterministic guarantees by leveraging the Lipschitz continuity of the CCM's condition. We show that for any given reference trajectory, we can prove the validity of the CCM by using enough samples. Even if the CCM's condition does not hold globally for every state in the state space, the resulting trajectories still often converge to the reference trajectories in our experience. This motivates our second approach based on conformal regression to give probabilistic guarantees on the convergence of the tracking error. Both the two methods can provide upper bounds on the tracking error, using which one can explore trajectory planning methods such as sampling-based methods (e.g. RRT [29], PRM [2]), model predictive control (MPC) [7], and satisfiability-based methods [30] to find safe reference trajectories to accomplishing the safe motion planning missions. We compare our approach Certified Control Using Contraction Metric (C3M) with the SoS [27], RL [31], and MPC [7, 32, 33] on several representative robotic systems, including the ones whose dynamics are learned from physical data using neural networks. We show that C3M outperforms other methods by being the only approach that can find converging tracking controllers for all the benchmarks. We provide two metrics for evaluating the tracking performance and show that controllers achieved through C3M have a much smaller tracking error. In addition, controllers found by C3M can be executed in sub-milliseconds at each step and is much faster than methods that only learn the metric (e.g. SoS programming) and online control methods (e.g. MPC).

## 2 Preliminaries and notations

For a symmetric matrix  $A \in \mathbb{R}^{n \times n}$ , the notation  $A \succ 0$  ( $A \prec 0$ ) means  $A$  is positive (negative) definite, and  $\mathbb{R}$  denotes the set of real numbers. Also,  $A \succ B$  means  $A - B \succ 0$ . The set of positive definite  $n \times n$  matrices is denoted by  $S_n^{>0}$ . Non-negative reals are denoted by  $\mathbb{R}^{\geq 0} := [0, +\infty)$ . For a matrix-valued function  $M(x) : \mathbb{R}^n \mapsto \mathbb{R}^{n \times n}$ , we denote the element-wise Lie derivative along a

vector  $v \in \mathbb{R}^n$  by  $\partial_v M := \sum_i v^i \frac{\partial M}{\partial x^i}$ . Unless otherwise stated,  $x^i$  denotes the  $i$ -th element of vector  $x$ . For  $A \in \mathbb{R}^{n \times n}$ , we denote  $A + A^\top$  by  $\hat{A}$ .

**Dynamical system** We consider the control-affine systems with disturbances of the form

$$\dot{x}(t) = f(x(t)) + B(x(t))u(t) + d(t), \quad (1)$$

where  $x(t) \in \mathcal{X} \subseteq \mathbb{R}^n$ ,  $u(t) \in \mathcal{U} \subseteq \mathbb{R}^m$ , and  $d(t) \in \mathcal{D} \subseteq \mathbb{R}^n$  for all  $t \in \mathbb{R}^{\geq 0}$  are states, inputs, and disturbances respectively. Here,  $\mathcal{X}, \mathcal{U}$  and  $\mathcal{D}$  are compact sets that represent the state, input, and disturbance space respectively. We assume that  $f : \mathbb{R}^n \mapsto \mathbb{R}^n$  and  $B : \mathbb{R}^n \mapsto \mathbb{R}^{n \times m}$  are smooth functions, the control input  $u : \mathbb{R}^{\geq 0} \mapsto \mathcal{U}$  is a piece-wise continuous function, and the right-side of Eq. (1) holds at points of discontinuity. Given an input signal  $u : \mathbb{R}^{\geq 0} \mapsto \mathcal{U}$ , a disturbance signal  $d : \mathbb{R}^{\geq 0} \mapsto \mathcal{D}$ , and an initial state  $x_0 \in \mathcal{X}$ , a (state) trajectory of the system is a function  $x : \mathbb{R}^{\geq 0} \mapsto \mathbb{R}^n$  such that  $x(t), u(t), d(t)$  satisfy Eq. (1) and  $x(0) = x_0$ . The goal of this paper is to design a tracking controller  $u(t)$  such that the controlled state trajectory  $x(t)$  can track *any* target trajectory  $x^*(t)$  generated by the reference control  $u^*(t)$  when  $x(0)$  is in a neighborhood of  $x^*(0)$  and there is no disturbance (i.e.  $d(t) = 0$ ). When  $d(t) \neq 0$ , the tracking error can be bounded (Sec. 3.3). Fig. 1 (a) illustrates the reference trajectory and the actual trajectory controlled by the proposed method of a Dubin's vehicle model, which is a classical nonholonomic system.

**Control contraction metric theory** Contraction theory analyzes the incremental stability of a system by considering the evolution of the distance between any pair of arbitrarily close neighboring trajectories. Let us first consider a time-invariant autonomous system of the form  $\dot{x} = f(x)$ . Given a pair of neighboring trajectories, denote the infinitesimal displacement between them by  $\delta_x$ , which is also called a *virtual displacement*. The evolution of  $\delta_x$  is dominated by a linear time varying (LTV) system:  $\dot{\delta}_x = \frac{\partial f}{\partial x}(x)\delta_x$ . Thus, the dynamics of the squared distance  $\delta_x^\top \delta_x$  is given by  $\frac{d}{dt}(\delta_x^\top \delta_x) = 2\delta_x^\top \dot{\delta}_x = 2\delta_x^\top \frac{\partial f}{\partial x} \delta_x$ . If the symmetrical part of the Jacobian  $\frac{\partial f}{\partial x}$  is uniformly negative definite, i.e. there exists a constant  $\lambda > 0$  such that for all  $x$ ,  $\frac{1}{2} \frac{\partial f}{\partial x} \preceq -\lambda \mathbf{I}$ , then  $\delta_x^\top \delta_x$  converges to zero exponentially at rate  $2\lambda$ . Hence, all trajectories of this system will converge to a common trajectory [34]. Such a system is referred to be *contracting*.

The above analysis can be generalized by introducing a *contraction metric*  $M : \mathbb{R}^n \mapsto S_n^{>0}$ , which is a smooth function. Then,  $\delta_x^\top M(x) \delta_x$  can be interpreted as a Riemannian squared length. Since  $M(x) \succ 0$  for all  $x$ , if  $\delta_x^\top M(x) \delta_x$  converges to 0 exponentially, then the system is contracting. The converse is also true. As shown in [34], if a system is contracting, then there exists a contraction metric  $M(x)$  and a constant  $\lambda > 0$  such that  $\frac{d}{dt}(\delta_x^\top M(x) \delta_x) < -\lambda \delta_x^\top M(x) \delta_x$  for all  $x$  and  $\delta_x$ .

Contraction theory can be further extended to control-affine systems. First, let us ignore the disturbances in system (1), then the dynamics of the corresponding virtual system is given by  $\dot{\delta}_x = A(x, u)\delta_x + B(x)\delta_u$ , where  $A(x, u) := \frac{\partial f}{\partial x} + \sum_{i=1}^m u^i \frac{\partial b_i}{\partial x}$ ,  $b_i$  is the  $i$ -th column of  $B$ ,  $u^i$  is the  $i$ -th element of  $u$ , and  $\delta_u$  is the infinitesimal difference of  $u$ . A fundamental theorem in Control Contraction Metric (CCM) theory [25] says if there exists a metric  $M(x)$  such that the following conditions hold for all  $x$  and some  $\lambda > 0$ ,

$$B_\perp^\top \left( -\partial_f W(x) + \frac{\partial f(x)}{\partial x} W(x) + 2\lambda W(x) \right) B_\perp \prec 0, \quad (2)$$

$$B_\perp^\top \left( \partial_{b_j} W(x) - \frac{\partial b_j(x)}{\partial x} W(x) \right) B_\perp = 0, \quad j = 1, \dots, m, \quad (3)$$

where  $B_\perp(x)$  is an annihilator matrix of  $B(x)$  satisfying  $B_\perp^\top B = 0$ , and  $W(x) = M(x)^{-1}$  is the dual metric of  $M(x)$ , then there exists a tracking controller  $k(x, x^*, u^*)$  such that the closed-loop system controlled by  $u = k(x, x^*, u^*)$  is contracting with rate  $\lambda$  under metric  $M(x)$  [25].

### 3 Learning tracking controllers using CCM with formal guarantees

In this paper, we utilize machine learning methods to jointly learn a parametric Contraction Metric and a tracking controller for System (1) with known dynamics. The overall learning system is illustrated in Fig. 1 (b). Different from RL-based methods, the proposed method is fully offline and

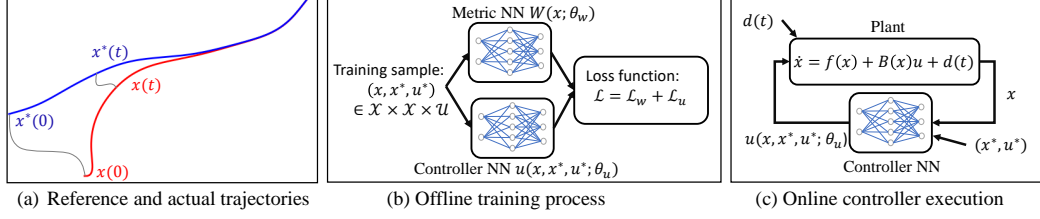


Figure 1: Components of the proposed method. (a) Trajectories of the Dubins car model projected to the positions of the car, whose dynamics are given by:  $\dot{x} = v \cos(\theta)$ ,  $\dot{y} = v \sin(\theta)$ ,  $\dot{v} = a$ ,  $\dot{\theta} = \omega$ , where  $x, y$  are the positions,  $v$  is the velocity,  $\theta$  is the heading direction,  $a$  is the acceleration, and  $\omega$  is the angular velocity. (b) The training process for the CCM and the controller. (c) Illustration on how the learned controller is used.

simulations are not required for training, i.e. the proposed method explores the state space instead of trajectory space. The data set for training is of the form  $\{(x_i, x_i^*, u_i^*) \in \mathcal{X} \times \mathcal{X} \times \mathcal{U}\}_{i=1}^N$ , and samples are drawn independently. The contraction metric and the tracking controller are parameterized using neural networks. In what follows we first ignore the disturbance  $d(t)$  then show in Sec. 3.3 that with  $d(t) \neq 0$  tracking error using the learned controller can still be bounded.

### 3.1 Controller and metric learning

The controller  $u(x, x^*, u^*; \theta_u)$  is a neural network with parameters  $\theta_u$ , and the dual metric  $W(x; \theta_w)$  is also a neural network with parameters  $\theta_w$ . By design, the controller satisfies that  $x = x^* \Rightarrow u(x, x^*, u^*) = u^*$ , and  $W(x; \theta_w)$  is a symmetric matrix satisfying  $W(x; \theta_w) \succ \underline{w}\mathbf{I}$  for all  $x$  and all  $\theta_w$ , where  $\underline{w}$  is a hyper-parameter. Plugging the parametric feedback controller into System (1) with  $d(t) = 0$ , the dynamics of the generalized squared length of the virtual displacement under metric  $M(x; \theta_w) = W(x; \theta_w)^{-1}$  is given by  $\frac{d}{dt}(\delta_x^T M(x; \theta_w) \delta_x) = \delta_x^T (\dot{M} + \overline{M}(A + BK)) \delta_x$ , where  $K = \frac{\partial u}{\partial x}$ , and  $\dot{M} = \sum_{i=1}^n \frac{\partial M}{\partial x^i} \dot{x}^i$ , or using the Lie derivative notation  $\dot{M} = \partial_{f(x)+B(x)u} M$ . Then, a sufficient condition [25] for the closed-loop system being contracting is that for all  $x, x^* \in \mathcal{X}$  and all  $u^* \in \mathcal{U}$ , there exists  $\lambda > 0$  such that

$$\dot{M} + \overline{M}(A + BK) + 2\lambda M \prec 0. \quad (4)$$

Since we have that  $x = x^* \Rightarrow u(x, x^*, u^*) = u^*$ , the reference trajectory  $(x^*(t), u^*(t))$  solves the closed-loop dynamics and thus a valid trajectory of the closed-loop system. If the closed-loop system is contracting, then starting from any initial point, the trajectory will converge to  $x^*(t)$ .

**Proposition 1.** *If condition (4) holds and  $\overline{m} \geq \underline{m} > 0$  satisfying  $\underline{m}\mathbf{I} \prec M(x) \prec \overline{m}\mathbf{I}$  for all  $x$ , then the initial difference between the reference and the actual trajectory is exponentially forgotten, i.e.  $\|x(t) - x^*(t)\|_2 \leq \sqrt{\overline{m}/\underline{m}} e^{-\lambda t} \|x(0) - x^*(0)\|_2$ .*

Denoting the LHS of Eq.(4) by  $C_u(x, x^*, u^*; \theta_w, \theta_u)$  and the uniform distribution over the data space  $\mathcal{S} := \mathcal{X} \times \mathcal{X} \times \mathcal{U}$  by  $\rho(\mathcal{S})$ , then the contraction risk of the system is defined as

$$\mathcal{L}_u(\theta_w, \theta_u) = \mathbb{E}_{(x, x^*, u^*) \sim \rho(\mathcal{S})} L_{PD}(-C_u(x, x^*, u^*; \theta_w, \theta_u)), \quad (5)$$

where  $L_{PD}(\cdot) \geq 0$  is for penalizing non-positive definiteness, and  $L_{PD}(A) = 0$  iff.  $A \succeq 0$ .

**Proposition 2.** *Let  $u(x, x^*, u^*; \theta_u^*)$  and  $W(x; \theta_w^*)$  satisfy inequality (4), then  $(\theta_w^*, \theta_u^*)$  is a global optimizer for  $\mathcal{L}_u(\theta_w, \theta_u)$ .*

The proof of Proposition 2 can be found in the supplementary materials. As shown in Proposition 1, the overshoot of the tracking error is effected by the uniform condition number  $\overline{m}/\underline{m}$  of the metric. Since the smallest eigenvalue of the dual metric is lower bounded by  $\underline{w}$  by design, penalizing large condition numbers is equivalent to penalizing the largest eigenvalues, and thus the following risk function is used

$$\mathcal{L}_c(\theta_w) = \mathbb{E}_{(x, x^*, u^*) \sim \rho(\mathcal{S})} L_{PD}(\overline{w}\mathbf{I} - W(x; \theta_w)), \quad (6)$$

where  $\overline{w}$  is a hyper-parameter. Another loss term can be added to penalize the large control inputs so the inputs are within actuation limit. However, we found in experiments that the magnitude of

the output of the controller is always reasonably small and such a loss term is always 0. Therefore, such a penalty on input is not necessary in our benchmarks.

However, jointly learning a metric and a controller by minimizing  $\mathcal{L}_u$  solely is hard and leads to poor results. Inspired by the CCM theory, we add two auxiliary loss terms for the dual metric to mitigate the difficulty of minimizing  $\mathcal{L}_u$ . As shown in Sec. 2, a valid CCM should satisfy conditions (2) and (3). Intuitively, imposing those two constraints on the dual metric  $W(x; \theta_w)$  provides more guidance for optimization. More discussion can be found in supplementary materials. Denoting the LHS of Eq. (2) and (3) by  $C_1(x; \theta_w)$  and  $\{C_2^j(x; \theta_w)\}_{j=1}^m$  respectively, the following risk functions are used

$$\mathcal{L}_{w1}(\theta_w) = \mathbb{E}_{(x, x^*, u^*) \sim \rho(\mathcal{S})} L_{PD}(-C_1(x; \theta_w)); \mathcal{L}_{w2}(\theta_w) = \sum_{j=1}^m \mathbb{E}_{(x, x^*, u^*) \sim \rho(\mathcal{S})} \|C_2^j(x; \theta_w)\|_F, \quad (7)$$

where  $\|\cdot\|_F$  is the Frobenius norm.

Please note that in some cases, condition (3) can be naturally satisfied for all  $\theta_w$  by designing  $W(\cdot)$  appropriately. As stated in [8], if the matrix  $B(x)$  is sparse and of the form  $B(x) = \begin{bmatrix} 0_{(n-m) \times m} \\ b(x) \end{bmatrix}$ , where  $b(x)$  is an invertible matrix, condition (3) can be naturally satisfied for all  $x$  and all  $\theta_w$  by making the upper-left  $(n-m) \times (n-m)$  block of  $W(x; \theta_w)$  not a function of the last  $(n-m)$  elements of  $x$ . In [8, 27] such sparsity assumptions are necessary for their approach to work. For dynamical models satisfying this assumption, we can make use of this property by designing  $W$  to satisfy such a structure and eliminate  $\mathcal{L}_{w2}$  since it will always be 0. For models not satisfying this assumption, we will show in Sec. 4 the impact of the loss term  $\mathcal{L}_{w2}$ .

In order to train the neural network using sampled data, the following *empirical risk* function is used

$$\mathcal{L}(\theta_u, \theta_w) = \frac{1}{N} \sum_{i=1}^N \left[ L_{PD}(-C_u(x_i, x_i^*, u_i^*; \theta_u, \theta_w)) + L_{PD}(-C_1(x_i; \theta_w)) + \sum_{j=1}^m \|C_2^j(x_i; \theta_w)\|_F + L_{PD}(\bar{w}\mathbf{I} - W(x_i; \theta_w)) \right], \quad (8)$$

where  $\{(x_i, x_i^*, u_i^*)\}_{i=1}^N$  are drawn independently from  $\rho(\mathcal{S})$ , and  $L_{PD}$  is implemented as follows. Given a matrix  $A \in \mathbb{R}^{n \times n}$ , we randomly sample  $K$  points  $\{p_i \in \mathbb{R}^n \mid \|p_i\|_2 = 1\}_{i=1}^K$ . Then, the loss function is calculated as  $L_{PD}(A) = \frac{1}{K} \sum_{i=1}^K \min\{0, -p_i^\top A p_i\}$ .

### 3.2 Theoretical convergence guarantees

As discussed in Sec. 3.1, satisfaction of inequality (4) for all  $(x, x^*, u^*) \in \mathcal{S}$  is a strong guarantee to show the validity of  $M(x)$  as a contraction metric and therefore provide convergence guarantees. SMT solvers are valid tools for verifying the satisfaction of (4) [9]. However, SMT solvers for NN have poor scalability and cannot handle the neural networks used in our experiments. Moreover, we observed in experiments that even if inequality (4) does not hold for all  $(x, x^*, u^*) \in \mathcal{S}$ , the learned controller still has perfect performance. Therefore, we introduce two approaches to provide slightly weaker versions of the guarantees on inequality (4) and convergence, which in practice are easier to validate and still give the needed level of assurance.

**Deterministic guarantees using Lipschitz continuity.** For a Lipschitz continuous function  $f : \mathcal{X} \mapsto \mathbb{R}$  with Lipschitz constant  $L_f$ , discretizing the domain  $\mathcal{X}$  such that the distance between any grid point and its nearest neighbor is less than  $\tau$ , if  $f(x_i) < -L_f \tau$  holds for all grid point  $x_i$ , then  $f(x) < 0$  holds for all  $x \in \mathcal{X}$ . We show in the following proposition that the largest eigenvalue of the LHS of inequality (4) indeed has a Lipschitz constant if both the dynamics and the learned controller and metric have Lipschitz constants.

**Proposition 3.** *Let  $A$ ,  $B$ ,  $K$ , and  $M$  be functions of  $x$ ,  $x^*$ , and  $u^*$ . If  $\dot{M}$ ,  $M$ ,  $A$ ,  $B$ , and  $K$  all have Lipschitz constants  $L_{\dot{M}}$ ,  $L_M$ ,  $L_A$ ,  $L_B$ , and  $L_K$  respectively, and 2-norms of the last four are all bounded by  $S_M$ ,  $S_A$ ,  $S_B$ , and  $S_K$  respectively, then  $\lambda_{\max}(\dot{M} + \overline{M(A + BK)} + 2\lambda M)$  has a Lipschitz constant  $L_{\dot{M}} + 2(S_M L_A + S_A L_M + S_M S_B L_K + S_B S_K L_M + S_M S_K L_B + \lambda L_M)$ .*



The proof of Proposition 3 is provided in the supplementary material. Combining discrete samples and the Lipschitz constant from Proposition 3 can guarantee the strict satisfaction of inequality (4). The Lipschitz constants for the dynamics  $f$ ,  $B$  can be computed directly as the upper bound of their first derivative. The Lipschitz constants for the metric and controller NNs can be computed using various existing tools such as the methods in [35]. In the supplementary material, we also use the Dubin’s vehicle model as an example to illustrate the proposed validation process.

**Probabilistic guarantees.** We observed that even if inequality (4) does not always hold for all points in  $\mathcal{S}$ , the learned controller can still drive all simulated trajectories to converge to reference trajectories. This motivates us to derive probabilistic guarantees on the convergence as an alternative in the trajectory space. The probabilistic guarantee is derived based on a simple result from conformal prediction [36]. Let us consider the process of evaluating a tracking controller using the value of the tracking error  $\|x^*(t) - x(t)\|_2$  or the convergence rate  $\lambda$  where  $\|x^*(t) - x(t)\|_2 \leq Ce^{-\lambda t}\|x^*(0) - x(0)\|_2$  with given  $C \geq 1$ . Given an *evaluation configuration* including an initial error  $e_i = \|x_i^*(0) - x_i(0)\|_2$  and a reference control signal  $u_i^*(t)$ , the quality of the tracking trajectory is evaluated by a scalar metric  $m_i$  which can be either the accumulated tracking error or the convergence rate. The following proposition gives a probabilistic guarantee on the quality metric of the tracking controller.

**Proposition 4.** *Given a set of i.i.d. evaluation configurations, let  $\{m_i\}_{i=1}^n$  be the quality metric for each configuration. Then, for a new i.i.d. evaluation configuration and the corresponding quality metric  $m_{n+1}$ ,  $\Pr(m_{n+1} \geq q_{1-\alpha}) \leq 1 - \alpha$ , where  $q_{1-\alpha}$  is the  $(1 - \alpha)$ -th quantile of  $\{m_i\}_{i=1}^n$ .*

Note that Proposition 4 asserts marginal coverage guarantees, which should be distinguished from conditional guarantees. Such probabilistic guarantees work on the trajectory space and therefore can be applied to any method that finds tracking controllers. In Sec. 4, we evaluate both the average tracking error and the convergence rate  $\lambda$  for various methods, and the latter can produce high-probability reachable sets for safe motion planning algorithms.

### 3.3 Robustness of the learned controller

Even if the system has nonzero bounded disturbance, the learned controller is still robust by bounding the tracking error as follows.

**Theorem 5.** *Given  $M(x)$  and  $u(x, x^*, u^*)$  satisfying inequality (4), since  $M(x) \in \mathcal{S}_n^{>0}$ , there exist  $\bar{m} \geq \underline{m} > 0$  such that  $\underline{m}\mathbf{I} \preceq M(x) \preceq \bar{m}\mathbf{I}$  for all  $x$ . Assume that the disturbance is uniformly bounded as  $\|d(t)\|_2 \leq \epsilon$ . Now considering two trajectories  $x_1(t)$  and  $x_2(t)$  of the unperturbed and the perturbed closed-loop system for the same reference, then the distance between those two trajectories can be bounded as  $\|x_1(t) - x_2(t)\|_2 \leq \frac{R_0}{\sqrt{\underline{m}}}e^{-\lambda t} + \sqrt{\frac{\bar{m}}{\underline{m}}} \cdot \frac{\epsilon}{\lambda}(1 - e^{-\lambda t})$ , where  $R_0 = \int_{x_1(0)}^{x_2(0)} \sqrt{\delta_x^\top M(x) \delta_x}$  is the initial geodesic distance between  $x_1(0)$  and  $x_2(0)$  under metric  $M(x)$ .*

The proof of Theorem 5 is inspired by Theorem 1 in [28] and is provided in the supplementary material. In practice  $\bar{m}$  and  $\underline{m}$  can be computed using samples and the Lipschitz constant of the eigenvalues of  $M(x)$  as discussed in Sec. 3.2.

## 4 Evaluation of performance

We evaluate the above controller-CCM co-synthesis (C3M) approach on 9 representative case studies (5 of them are reported in the supplementary material), including high-dimensional (up to 9 state variables and 3 control variables) models and a system with learned dynamics as neural networks. All our experiments were conducted on a Linux workstation with two Xeon Silver 4110 CPUs, 32 GB RAM, and a Quadro P5000 GPU. Please note that the execution time reported in Tab. 1 were all evaluated on CPU. All code will be publicly available upon acceptance.

**Comparison methods.** We compare the performance of C3M with 3 different leading approaches for synthesizing tracking controllers, including both model-based and model-free methods. To be specific, the methods include (1.) **SoS**: the SoS-based method proposed in [27], using the official implementation<sup>1</sup>. (2.) **MPC**: an open-source implementation of MPC on PyTorch [7], which solves

<sup>1</sup><https://github.com/StanfordASL/RobustMP>

nonlinear implicit MPC problems using differential dynamic programming and first-order approximation of the dynamics [37]. (3.) **RL**: the Proximal Policy Optimization (PPO) RL algorithm [31] implemented in Tianshou<sup>2</sup>. Note that although RL is often used in a model-free setting, the advances in deep neural networks have made RL an outstanding tool for controller learning.

**Studied systems.** We study 4 representative system models adopted from classical benchmarks: (1). **PVTOL** models a planar vertical-takeoff-vertical-landing (PVTOL) system for drones and is adopted from [27, 8]. (2). **Quadrotor** models a physical quadrotor and is adopted from [27]. (3). **Neural lander** models a drone flying close to the ground so that ground effect is prominent and therefore could not be ignored [38]. Note that in this model the ground effect is learned from empirical data of a physical drone using a 4-layer neural network [38]. Due to the neural network function in the dynamics, the SoS-based method as in [27] cannot handle such a model since neural networks are hard to be approximated by polynomials. (4). **SEGWAY** models a real-world Segway robot and is adopted from [13]. Since this model does not satisfy the sparsity assumption for  $B$  in Sec. 3.1, SoS-based method cannot handle it. With this model, we also study the impact of the regularization term  $\mathcal{L}_{w2}$  in Equation (7) for learning a tracking controller.

We report the performance of C3M on 5 other examples in the supplementary material, including a 10D quadrotor model, a cart-pole model, a pendulum model, a two-link planar robot arm system, and the Dubin’s vehicle model in Fig. 1. The detailed dynamics of the above 4 representative system models are also reported in the supplementary material.

**Implementation details.** For all systems studied here, we model the dual metric as  $W(x) = C(x; \theta_w)^\top C(x; \theta_w) + w\mathbf{I}$  where  $C(x; \theta_w) \in \mathbb{R}^{n \times n}$  is a 2-layer neural network, of which the hidden layer contains 128 neurons. We use a relative complex structure for the controller. First, two weight matrices of the controller  $w_1 = w_1(x, x^*; \theta_{u1})$  and  $w_2 = w_2(x, x^*; \theta_{u2})$  are modeled using two 2-layer neural networks with 128 neurons in the hidden layer, where  $\theta_{u1}$  and  $\theta_{u2}$  are the parameters. Then the controller is given by  $u(x, x^*, u^*; \theta_u) = w_2 \cdot \tanh(w_1 \cdot (x - x^*)) + u^*$ , where  $\theta_u = \{\theta_{u1}, \theta_{u2}\}$  and  $\tanh(\cdot)$  is the hyperbolic tangent function. A training set with 130K samples is used. We use Adam [39] as the optimizer and train the neural network for 20 epochs.

The process of generating random reference trajectories  $x^*$  is critical in the evaluation of tracking controllers and also a fundamental part in RL training. We pre-define a group of sinusoidal signals with some fixed frequencies and randomly sampled a group of weights, then the reference control inputs  $u^*(t)$  are calculated as the linear combination of the sinusoidal signals. The initial states of the reference trajectories  $x^*(0)$  are uniformly randomly sampled from a compact set. Using  $x^*(0)$  and  $u^*(t)$ , we can get the reference trajectories  $x^*(t)$  following the system dynamics (1) with  $d(t) = 0$ . For evaluation, the initial errors  $x_e(0)$  are uniformly randomly sampled from a bounded set, and the initial states  $x(0)$  are computed as  $x(0) = x^*(0) + x_e(0)$ . Trajectories of the closed-loop systems are obtained using Euler’s integration method for a bounded time horizon  $t \in [0, T]$ .

To make use of the model-free PPO RL library, we formulate some key concepts as follows. The state of the environment at time  $t$  is the concatenation of  $x(t)$  and  $x^*(t)$ . At the beginning of each episode, the environment randomly samples two initial points as  $x(0)$  and  $x^*(0)$  respectively, and a reference control input  $u^*(t)$  using the aforementioned sampling method. At each transition, the environment takes the action  $u(t)$  from the agent, and returns the next state as  $x(t + \Delta t) = x(t) + \Delta t(f(x(t)) + B(x(t))(u(t) + u^*(t)))$  and  $x^*(t + \Delta t) = x^*(t) + \Delta t(f(x^*(t)) + B(x^*(t))u^*(t))$ , and the reward as  $r_t = 1/(1 + \|w \circ (x(t + \Delta t) - x^*(t + \Delta t))\|_2)$ , where  $w$  are predefined weights for each state. For a fair comparison, RL and our method share the same architecture for the controller.

For the MPC-PyTorch library, we observe that the time horizon for computing the control sequence  $T$ , also called the receding time window, plays a crucial role. With smaller  $T$ , the resulting controller cannot always produce trajectories that converge to the reference trajectory. The larger  $T$  is, the smaller the tracking errors are. However, a large  $T$  can also cause the computational time to increase. As a trade-off, we set  $T$  to be 50 time units in all cases.

**Results and discussion** Results of all 4 methods on 4 representative benchmarks are reported in Tab. 1 and Fig. 2. For each benchmark, a group of evaluation configurations was randomly sampled using the aforementioned sampling method, and then each method was evaluated with these configurations. In Fig. 2 we show the normalized tracking error  $x_e(t) = \|x(t) - x^*(t)\|_2 / \|x(0) - x^*(0)\|_2$  for each model and each available method. In Tab. 1, we evaluate the tracking controllers using two

<sup>2</sup><https://github.com/thu-ml/tianshou>

Model	Dim		Execution time per step (ms)				Tracking error (AUC)				Convergence rate ( $\lambda/C$ )			
	$n$	$m$	C3M	SoS	MPC	RL	C3M	SoS	MPC	RL	C3M	SoS	MPC	RL
PVTOL	6	2	0.41	3.4	1968	0.41	0.659	0.975	0.892	0.735	1.153/3.95	0.864/4.91	0.489/4.95	0.799/3.84
Quadrotor	9	3	0.40	12.6	3535	0.40	0.772	1.103	0.977	1.416	1.078/3.63	0.821/3.30	0.401/2.23	0.187/1.37
Neural lander	6	3	0.36	-	3385	0.36	0.588	-	0.713	0.793	1.724/2.89	-	0.822/1.34	0.606/1.30
Segway	4	1	0.29	-	-	0.29	0.704	-	-	1.408	0.446/3.11	-	-	0.168/8.12

Table 1: Comparison results of C3M vs. other methods.  $n$  and  $m$  are dimensions of the state and input space.

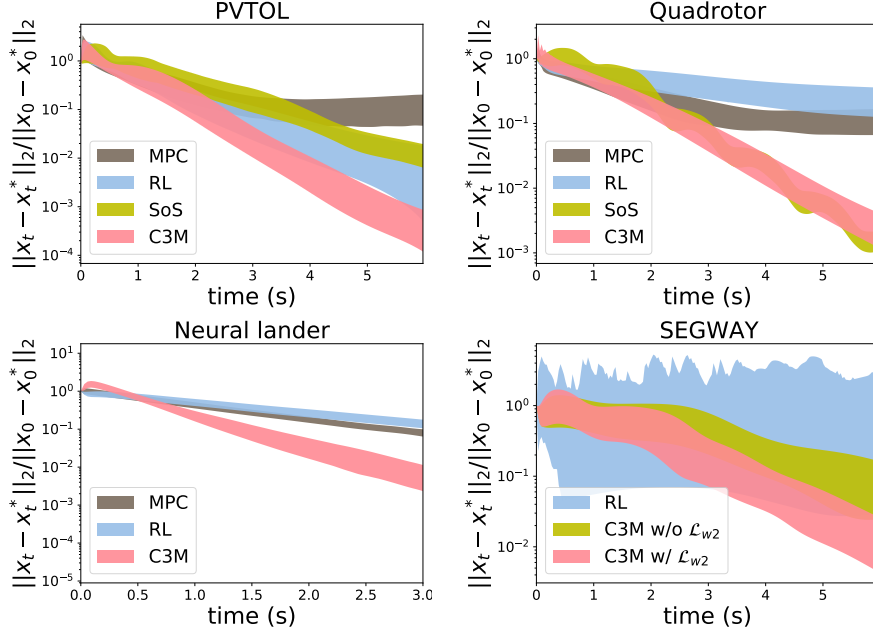


Figure 2: Normalized tracking error on the benchmarks using different approaches. The  $y$  axes are in log-scale. The tubes are tracking errors between mean plus and minus one standard deviation over 100 trajectories.

quality metrics as discussed in Sec. 3.2. (1). Total tracking error: Given the normalized tracking error  $x_e(t)$  for  $t \in [0, T]$ , the total tracking error is just the area under this curve (AUC). (2). Convergence rate: Given a set of tracking error curves, first we do the following for the error curve  $x_e(t)$  that possesses highest overshoot. We search for  $\lambda > 0$  and  $C \geq 1$  such that  $x_e(t) \leq Ce^{-\lambda t}$  for all  $t \in [0, T]$  and the AUC of  $Ce^{-\lambda t}$  is minimized. Then,  $C$  is fixed, and the convergence rate  $\lambda$  is calculated for each curve. After obtaining the metrics for all the error curves, we report the metric corresponds to  $\alpha = 0.05$  using the statistical method in Sec. 3.2. Plots of the actual trajectories can be found in the supplementary materials.

We also have the following observations when conducting the comparison. (1.) On PVTOL and the quadrotor models, both SOS and our method successfully found a tracking controller such that the tracking error decreases rapidly. However, the SOS-based method entails the calculation of geodesics on the fly for control synthesis, while our method learns a neural controller offline, which makes a difference in running time. Also, due to the aforementioned limitations on control matrix sparsity, the SOS-based method cannot be applied to the neural lander and Segway. (2.) The performance of the RL method varies with tasks. On PVTOL and the neural lander models, RL achieved comparable results with model-based methods. However, since the state space of the quadrotor model is larger, RL failed to find a contracting tracking controller within a reasonable time. As for the Segway model, although we have tried our best to tune the hyper-parameters, RL failed to find a reasonable controller. More details can be found in the supplementary materials. (3.) Running time of MPC is not practical for online control tasks. Also, the MPC library used in experiments failed to handle the Segway model due to numerical issues. (4.) The results of our methods with and without penalty term  $\mathcal{L}_{w2}$  on Segway verify the sufficiency of this cost term when the sparsity assumption for  $B$  is not satisfied.



## 5 Conclusion

In this paper, we showed that a certified tracking controller with guaranteed convergence and tracking error bounds can indeed be learned by learning a control contraction metric simultaneously as guidance. Results indicated that such a learned tracking controller can better track the reference or nominal trajectory, and the execution time of such a controller is much smaller than online control methods such as MPC. In future works, we plan to extend our learning framework to systems with unknown dynamics and control non-affine systems.

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## A Proof of Proposition 2

*Proof.* If  $u(x, x^*, u^*; \theta_u^*)$  and  $W(x; \theta_w^*)$  satisfy inequality (4) in the main paper for all  $x, x^*$  and  $u^*$ , then  $L_{PD}(-C_u(x, x^*, u^*; \theta_w, \theta_u)) = 0$  holds for all  $x, x^*$  and  $u^*$ , and thus  $\mathcal{L}_u(\theta_w^*, \theta_u^*) = 0$ . The proposition follows from the fact that  $L_u(\theta_w, \theta_u)$  is non-negative.  $\square$

## B Justification for the CCM regularization terms

In theory, minimizing  $\mathcal{L}_u$  is enough for learning a tracking controller, since inequality (4) implies the contraction of the closed-loop system. However, in the experiments, we found that minimizing  $\mathcal{L}_u$  solely is hard. Here,  $\mathcal{L}_{w1}$  and  $\mathcal{L}_{w2}$  are used as auxiliary loss terms to mitigate the difficulty of minimizing  $\mathcal{L}_u$ .

Furthermore, conditions (2) and (3) are sufficient but not necessary for the existence of contracting controller. In the derivation of conditions (2) and (3), the inequality (inequality (4) in the main paper) which indicates contracting is required to be hold for all unbounded  $u$ , which is not necessary in the sense that  $u$  is usually bounded in a control problem. Thus, even if conditions (2) and (3) do not exactly hold, a contracting tracking controller may still exist.

All the benchmark models used in experiments are under-actuated where the rank of the control matrix  $r(B) < n$ . In cases where  $r(B) = n$  for all  $x$  and thus there does not exist a non-zero annihilator matrix  $B_\perp$  such that  $B_\perp^\top B = 0$ , one can easily verify that any uniformly positive definite metric function  $M(x)$  can be a valid CCM. The control synthesis is also straightforward in this case. One can directly compute the instantaneous control  $u$  given the desired  $\dot{x}$  since  $B(x)$  is invertible. Moreover, as shown in [25], existence of a CCM is guaranteed for feedback linearizable systems. In that case, a valid CCM can be constructed analytically from the variable and feedback transformation. We refer the interested readers to [25] for more details.

## C Proof of Proposition 3 and CCM validation process example

### C.1 Proof of Proposition 3

We denote the operator 2-norm for matrices by  $\|\cdot\|_2$ . The following lemmas are going to be used for the proof of Proposition 3.

**Lemma 6.** *For any two symmetric matrices  $A, B \in \mathbb{R}^{n \times n}$ , the difference of their eigenvalues satisfies:*

$$|\lambda_{\max}(A) - \lambda_{\max}(B)| \leq \|A - B\|_2.$$

Lemma 6 is a well-known result that uses the Courant-Fischer minimax theorem. The detailed proof can be found at [40].

**Lemma 7.** *For any two functions  $A : \mathbb{R}^n \rightarrow \mathbb{R}^{k \times \ell}$  and  $B : \mathbb{R}^n \rightarrow \mathbb{R}^{\ell \times s}$ , if  $A$  and  $B$  both have Lipschitz constants  $L_A$  and  $L_B$  with respect to 2-norm respectively, moreover, if  $\forall x, \|A(x)\|_2 \leq S_A$  and  $\|B(x)\|_2 \leq S_B$ , then for any  $x, y \in \mathbb{R}^n$ ,*

$$\|A(x)B(x) - A(y)B(y)\|_2 \leq (S_A L_B + S_B L_A) \|x - y\|_2.$$

*Proof.*

$$\begin{aligned} \|A(x)B(x) - A(y)B(y)\|_2 &= \|A(x)B(x) - A(x)B(y) + A(x)B(y) - A(y)B(y)\|_2 \\ &\leq L_B \|A(x)\|_2 \|x - y\|_2 + L_A \|B(y)\|_2 \|x - y\|_2 \\ &\leq (S_A L_B + S_B L_A) \|x - y\|_2. \end{aligned} \tag{9}$$

$\square$

The proof of Proposition 3 follows from the above lemmas.

*Proof.* For all  $X$  and  $Y \in \mathcal{X} \times \mathcal{X} \times \mathcal{U}$ , we have

$$\begin{aligned} & \left| \lambda_{\max} \left( \dot{M}(X) + (A(X) + B(X)K(X))^{\top} M(X) + M(X)(A(X) + B(X)K(X)) + 2\lambda M(X) \right) \right. \\ & \quad \left. - \lambda_{\max} \left( \dot{M}(Y) + (A(Y) + B(Y)K(Y))^{\top} M(Y) + M(Y)(A(Y) + B(Y)K(Y)) + 2\lambda M(Y) \right) \right| \\ & \leq \|\dot{M}(X) - \dot{M}(Y)\|_2 + 2\|M(X)A(X) - M(Y)A(Y)\|_2 \\ & \quad + 2\|M(X)B(X)K(X) - M(Y)B(Y)K(Y)\|_2 + 2\lambda\|M(X) - M(Y)\|_2 \\ & \leq (L_{\dot{M}} + 2(S_M L_A + S_A L_M + S_M S_B L_K + S_B S_K L_M + S_M S_K L_B + \lambda L_M)) \|X - Y\|_2. \end{aligned}$$

□

## C.2 CCM validation on Dubin's vehicle

First, the Lipschitz constants and bounds on 2-norms are derived as follows.

**$L_A$  and  $S_A$ :** By definition,  $A = \frac{\partial f}{\partial x} + \sum \frac{\partial b_i}{\partial x} u^i = \begin{bmatrix} 0 & 0 & -v \cos(\theta) & \cos(\theta) \\ 0 & 0 & v \sin(\theta) & \sin(\theta) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ . Considering a

vector-valued function  $\tilde{A} = [-v \cos(\theta), v \sin(\theta), \cos(\theta), \sin(\theta)]$  and its Lipschitz constant  $L_{\tilde{A}}$  w.r.t.

the Frobenius norm, since the Jacobian of  $\tilde{A}$  is given by  $\frac{\partial \tilde{A}}{\partial x} = \begin{bmatrix} -\cos(\theta) & v \sin(\theta) \\ \sin(\theta) & v \cos(\theta) \\ 0 & -\sin(\theta) \\ 0 & \cos(\theta) \end{bmatrix}$ , we have

$L_{\tilde{A}} = \sup \|\frac{\partial \tilde{A}}{\partial x}\|_F = \sqrt{2 + v_h^2}$ , where  $v_h$  is the upper bound of the velocity. Obviously, under the Frobenius norm, the Lipschitz constants of  $A$  and  $\tilde{A}$  coincide. Moreover, for any matrix  $A$ , we have  $\|A\|_2 \leq \|A\|_F$ , and thus  $L_A \leq L_{\tilde{A}} = \sqrt{2 + v_h^2}$ . As for the norm bound,  $S_A = \sup \|A\|_2 \leq \sup \|A\|_F = \sqrt{1 + v_h^2}$ .

**$L_B$  and  $S_B$ :** By definition,  $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ , which is a constant. Thus,  $L_B = 0$ , and  $S_B =$

$\sigma_{\max}(B) = 1$ , where  $\sigma_{\max}$  is the largest singular value.

**$L_M$  and  $S_M$ :** By definition, we have  $W(x) = C(x)^{\top} C(x) + \underline{w} \mathbf{I}$  and  $M(x) = W(x)^{-1}$ , and thus  $S_M = \sup \|M(x)\|_2 \leq \frac{1}{\underline{w}}$ . Moreover,  $\|M(x) - M(y)\|_2 = \|M(x)(W(x) - W(y))M(y)\|_2 \leq \frac{1}{\underline{w}^2} \|W(x) - W(y)\|_2$ . Thus, we have  $L_M \leq \frac{1}{\underline{w}^2} L_W$ . As for  $L_W$ , by Lemma 7,  $L_W \leq 2S_C L_C$ , where  $L_C$  can be evaluated using tools for estimating the Lipschitz constants of NNs. In our experiments, we use the official implementation<sup>3</sup> of [35]. After obtaining  $L_C$ ,  $S_C$  can be upper bounded using samples and the Lipschitz constant  $L_C$ .

**$L_{\dot{M}}$ :** By definition,  $\dot{M} = \sum_{i=1}^n \frac{\partial M}{\partial x_i} \dot{x}_i$ . We randomly sampled 10K points and evaluate the gradient of  $\dot{M}$ , then the maximum of the norm of the gradient is returned as a Lipschitz constant.

**$L_K$  and  $S_K$ :** For simplicity of the analysis, we use a simpler neural controller  $u = k(x, x^*)x_e + u^*$ , where  $k$  is a two-layer neural network and  $x_e := (x - x^*)$ . Then,  $K = \frac{\partial u}{\partial x} = k(x, x^*) + \sum_{i=1}^n x_e^i \frac{\partial k_i}{\partial x}$ , where  $k_i$  is the  $i$ -th column of  $k$  and  $x_e^i$  is the  $i$ -th element of  $x_e$ . Thus,  $S_K \leq S_k + \sum_{i=1}^n \sup |x_e^i| L_{k_i|x}$ , where  $L_{k_i|x}$  is the Lipschitz constant of  $k_i$  w.r.t.  $x$ . Also,  $L_K \leq L_k + \sum_{i=1}^n \sup |x_e^i| L_{\frac{\partial k_i}{\partial x}} + \sqrt{2} \sum_{i=1}^n L_{k_i|x}$ , where we use the facts that  $x_e^i = x_i - x_i^* \implies L_{x_e^i} = \sqrt{2}$ , and  $S_{\frac{\partial k_i}{\partial x}} \leq L_{k_i|x}$ . As for  $L_{\frac{\partial k_i}{\partial x}}$ , we have  $\frac{\partial k_i}{\partial x} = A_2^i \cdot \text{diag}(\text{dtanh}(A_1[x, x^*])) \cdot A_{11}$ , where  $A_1$  is the weights in the first layer,  $A_{11}$  is the weights associated to  $x$  in the first layer,  $A_2^i$  is the weights

<sup>3</sup><https://github.com/arobey1/LipSDP>

associated to the  $i$ -th column of  $k$  in the second layer, and  $d\tanh$  is the derivative of the  $\tanh$  function. Thus,  $L_{\frac{\partial k_i}{\partial x}} \leq \|A_2^i\| \cdot \|A_{11}\| \cdot \|A_1\| \cdot L_{d\tanh}$ .

**Validation process:** We successfully verified the trained model for a state space where  $[-5, -5, -\pi, 1] \leq x, x^* \leq [5, 5, \pi, 2]$ ,  $[-0.1, -0.1, -0.1, -0.1] \leq x - x^* \leq [0.1, 0.1, 0.1, 0.1]$ , and  $[-1, 0] \leq u^* \leq [1, 0]$  with  $\lambda = 0.1$ . The derived Lipschitz constant using Theorem 3 for the trained model was 616.8. After that, we discretized the state space and evaluated the maximum eigenvalues of the LHS of inequality (4) at the grid points, and the results gave a theoretical guarantee on the satisfaction of inequality (4) for all  $x, x^*$ , and  $u^*$ .

## D Proof of Proposition 4

The following lemma from [41] is going to be used.

**Lemma 8.** *Consider a set of i.i.d. random variables  $\{U_1, \dots, U_n\}$ , then the following inequality holds for a new i.i.d. sample  $U_{n+1}$ ,*

$$\Pr(U_{n+1} \geq q_{1-\alpha}) \leq 1 - \alpha,$$

where  $q_{1-\alpha}$  is  $(1 - \alpha)$ -th quantile of  $\{U_i\}_{i=1}^n$ , i.e.

$$q_{1-\alpha} = \begin{cases} U_{(\lceil (1-\alpha)(n+1) \rceil)} & \text{if } \lceil (1-\alpha)(n+1) \rceil \leq n, \\ \infty & \text{otherwise} \end{cases},$$

where  $U_{(1)} \leq \dots \leq U_{(n)}$  is the order statistics of  $\{U_i\}_{i=1}^n$ .

The above lemma follows from an obvious fact that the rank of  $U_{n+1}$  in  $\{U_i\}_{i=1}^{n+1}$  is uniformly distributed in  $\{1, \dots, n+1\}$ .

Since the evaluation configurations are all i.i.d., the corresponding quality metrics  $\{m_i\}_{i=1}^{n+1}$  are also i.i.d. The proposition immediately follows from the above lemma.

## E Proof of Theorem 5

*Proof.* Since  $\forall x, M(x) \in S_n^{>0}$ , there exists  $\Theta(x)$  such that  $M(x) = \Theta(x)^\top \Theta(x)$  for all  $x$ . Now, let us consider the smallest path integral between  $x_1(t)$  and  $x_2(t)$  w.r.t. metric  $M(x)$  and denote it by  $R(t) = \int_{x_1(t)}^{x_2(t)} \sqrt{\delta_x^\top M(x) \delta_x} = \int_{x_1(t)}^{x_2(t)} \|\Theta(x) \delta_x\|_2$ . Then differentiating  $R(t)$  w.r.t.  $t$  yields  $\dot{R} \leq -\lambda R + \|\Theta d\|_2 \leq -\lambda R + \epsilon \sqrt{m}$ . (see [34], pp. 11 (vii)) By comparison lemma (see [42], pp. 102),  $R(t)$  is bounded as  $R(t) \leq R(0)e^{-\lambda t} + \frac{\epsilon}{\lambda} \sqrt{m}(1 - e^{-\lambda t})$ . Since  $\Theta$  is uniformly bounded as  $\Theta(x) \succeq \sqrt{m}\mathbf{I}$ , we have  $R(t) \geq \sqrt{m}\|x_1(t) - x_2(t)\|_2$ , which follows from the fact that the smallest path integral w.r.t. a constant metric is the path integral along the straight line. Combining these two inequalities yields the theorem.  $\square$

## F More details on the experimental results

### F.1 Hyper-parameters

For all the models, we set  $\underline{w} = 0.1$  and  $\overline{w} = 10$ . For PVTOL, Quadrotor, Neural lander, Segway, Quadrotor2, and Cart-pole, we set  $\lambda = 0.5$ . For Dubin's car, we set  $\lambda = 1$ . For TLPRA, we set  $\lambda = 2$ . For Pendulum, we set  $\lambda = 3$ .

For the evaluation of RL, we trained the model for 1K epochs (1K steps per epoch, and thus 1M steps in total). The best model was used for evaluation.

### F.2 Dynamics of the 4 benchmarks discussed in Section 4

As mentioned in the main paper, for each benchmark model, we have to define the state space  $\mathcal{X}$  and the reference control space  $\mathcal{U}$ . Also, for evaluation purpose, the initial state of the reference trajectory is sampled from a set  $\mathcal{X}_0$ , and the initial error is sampled from a set  $\mathcal{X}_{e0}$ . All these sets are defined as hyper-rectangles. For two vectors  $x, y \in \mathbb{R}^n$ ,  $x \leq y$  or  $x \geq y$  means element-wise inequality.

**PVTOL** models a planar vertical-takeoff-vertical-landing (PVTOL) system for drones and is adopted from [27, 8]. The system includes 6 variables  $x := [p_x, p_z, \phi, v_x, v_z, \dot{\phi}]$ , where  $p_x$  and  $p_z$  are the positions,  $v_x$  and  $v_z$  are the corresponding velocities,  $\phi$  is the roll, and  $\dot{\phi}$  is the angular rate. The 2-dimensional control input  $u$  corresponds to the thrusts of the two rotors. The dynamics of the system are as follows:

$$\dot{x} = \begin{bmatrix} v_x \cos(\phi) - v_z \sin(\phi) \\ v_x \sin(\phi) + v_z \cos(\phi) \\ \dot{\phi} \\ v_z \dot{\phi} - g \sin(\phi) \\ -v_x \dot{\phi} - g \cos(\phi) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1/m & 1/m \\ l/J & -l/J \end{bmatrix} u,$$

where  $m = 0.486$ ,  $J = 0.00383$ ,  $g = 9.81$ , and  $l = 0.25$ . For the sets, we set  $\mathcal{X} = \{[-35, -2, -\frac{\pi}{3}, -2, -1, -\frac{\pi}{3}] \leq x \leq [0, 2, \frac{\pi}{3}, 2, 1, \frac{\pi}{3}]\}$ ,  $\mathcal{U} = \{[mg/2 - 1, mg/2 - 1] \leq u \leq [mg/2 + 1, mg/2 + 1]\}$ ,  $\mathcal{X}_0 = \{[0, 0, -0.1, 0.5, 0, 0] \leq x_0 \leq [0, 0, 0.1, 1, 0, 0]\}$ , and  $\mathcal{X}_{e0} = \{[-0.5, -0.5, -0.5, -0.5, -0.5, -0.5] \leq x_{e0} \leq [0.5, 0.5, 0.5, 0.5, 0.5, 0.5]\}$ .

**Quadrotor** is adopted from [27]. The state variables are given by  $x := [p_x, p_y, p_z, v_x, v_y, v_z, f, \phi, \theta, \psi]$ , where  $p_k$  and  $v_k$  for  $k \in \{x, y, z\}$  are positions and velocities, and  $\phi$ ,  $\theta$  and  $\psi$  are roll, pitch, and yaw respectively, and  $f > 0$  is the net (normalized by mass) thrust generated by the four rotors. The control input is considered as  $u := [f, \phi, \theta, \psi]$ . The dynamics of the system are as follows:

$$\dot{x} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ -f \sin(\theta) \\ f \cos(\theta) \sin(\phi) \\ g - f \cos(\theta) \cos(\phi) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} u,$$

where  $g = 9.81$ . For the sets, we set  $\mathcal{X} = \{[-30, -30, -30, -1.5, -1.5, -1.5, 0.5g, -\frac{\pi}{3}, -\frac{\pi}{3}, -\frac{\pi}{3}] \leq x \leq [30, 30, 30, 1.5, 1.5, 1.5, 2g, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}]\}$ ,  $\mathcal{U} = \{[-1, -1, -1, -1] \leq u \leq [1, 1, 1, 1]\}$ ,  $\mathcal{X}_0 = \{[-5, -5, -5, -1, -1, -1, g, 0, 0, 0] \leq x_0 \leq [5, 5, 5, 1, 1, 1, g, 0, 0, 0]\}$ , and  $\mathcal{X}_{e0} = \{[-0.5, -0.5, -0.5, -0.5, -0.5, -0.5, -0.5, -0.5, -0.5, -0.5] \leq x_{e0} \leq [0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5]\}$ .

**Neural lander** models a drone flying close to the ground so that ground effect is prominent and therefore could not be ignored [38]. The state is give by  $x := [p_x, p_y, p_z, v_x, v_y, v_z]$ , where  $p_k$  and  $v_k$  for  $k \in \{x, y, z\}$  are the positions and velocities. The 3-dimensional control input is given by  $u := [a_x, a_y, a_z]$  which are the acceleration in three directions. Note that in this model the ground effect is learned from empirical data of a physical drone using a 4-layer neural network [38]. Due to the neural network function in the dynamics, the SoS-based method as in [27] cannot handle such a model since neural networks are hard to be approximated by polynomials. However, C3M can handle this model since it doesn't impose hard constraints and only requires the ability to evaluate the dynamics and its gradient. The dynamics of the system are as follows:

$$\dot{x} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ Fa_1/m \\ Fa_2/m \\ Fa_3/m - g \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u,$$

where  $m = 1.47$ ,  $g = 9.81$ , and  $Fa_i = Fa_i(z, v_x, v_y, v_z)$  for  $i = 1, 2, 3$ , are neural networks. For the sets, we set  $\mathcal{X} = \{[-5, -5, 0, -1, -1, -1] \leq x \leq [5, 5, 2, 1, 1, 1]\}$ ,  $\mathcal{U} = \{[-1, -1, -3] \leq u \leq [1, 1, 9]\}$ ,  $\mathcal{X}_0 = \{[-3, -3, 0.5, 1, 0, 0] \leq x_0 \leq [3, 3, 1, 1, 0, 0]\}$ , and  $\mathcal{X}_{e0} = \{[-1, -1, -0.4, -1, -1, 0] \leq x_{e0} \leq [1, 1, 1, 1, 1, 0]\}$ .



**SEGWAY** models a real-world Segway robot and is adopted from [13]. The state variables are given by  $x := [p_x, \theta, v_x, \omega]$  where  $p_x$  and  $v_x$  are the position and velocity, and  $\theta$  and  $\omega$  are the angle and angular velocity. The control input is the torque of the motor. The dynamics of the system are as follows:

$$\dot{x} = \begin{bmatrix} v_x \\ \omega \\ \frac{\cos(\theta)(9.8 \sin(\theta) - 1.8u + 11.5v) - 10.9u + 68.4v - 1.2\omega^2 \sin(\theta)}{\cos(\theta) - 24.7} \\ \frac{(9.3u - 58.8v) \cos(\theta) + 38.6u - 243.5v - \sin(\theta)(208.3 + \omega^2 \cos(\theta))}{\cos^2(\theta) - 24.7} \end{bmatrix}.$$

For the sets, we set  $\mathcal{X} = \{[-5, -\frac{\pi}{3}, -1, -\pi] \leq x \leq [5, \frac{\pi}{3}, 1, \pi]\}$ ,  $\mathcal{U} = \{0 \leq u \leq 0\}$ ,  $\mathcal{X}_0 = \{[0, 0, 0, 0] \leq x_0 \leq [0, 0, 0, 0]\}$ , and  $\mathcal{X}_{e0} = \{[-1, -\frac{\pi}{3}, -0.5, -\pi] \leq x_{e0} \leq [1, \frac{\pi}{3}, 0.5, \pi]\}$ .

### F.3 Additional benchmarks

**Cart-pole** is a well-known model for evaluating control algorithms. The state variables are given by  $x := [p_x, \theta, v_x, \omega]$ . The dynamics are given by

$$\dot{x} = \begin{bmatrix} v_x \\ \omega \\ \frac{u + m_p \sin(\theta)(l\omega^2 - g \cos(\theta))}{m_c + m_p \sin^2(\theta)} \\ \frac{u \cos(\theta) + m_p l \omega^2 \cos(\theta) \sin(\theta) - (m_c + m_p)g \sin(\theta)}{l(m_c + m_p \sin^2(\theta))} \end{bmatrix},$$

where  $m_c = 1$ ,  $m_p = 1$ ,  $g = 9.8$ , and  $l = 1$ . For the sets, we set  $\mathcal{X} = \{[-5, -\frac{\pi}{3}, -1, -1] \leq x \leq [5, \frac{\pi}{3}, 1, 1]\}$ ,  $\mathcal{U} = \{0 \leq u \leq 0\}$ ,  $\mathcal{X}_0 = \{[0, 0, 0, 0] \leq x_0 \leq [0, 0, 0, 0]\}$ , and  $\mathcal{X}_{e0} = \{[-0.3, -0.3, -0.3, -0.3] \leq x_{e0} \leq [0.3, 0.3, 0.3, 0.3]\}$ .

**Quadrotor2** is adopted from [3]. The state variables are given by  $x := [p_x, p_y, p_z, v_x, v_y, \theta_x, \theta_y, \omega_x, \omega_y, v_z]$ . The control input is given by  $u := [a_x, a_y, a_z]$ . The dynamics are given by

$$\dot{x} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ g \tan(\theta_x) \\ g \tan(\theta_y) \\ -d_1 \theta_x + \omega_x \\ -d_1 \theta_y + \omega_y \\ -d_0 \theta_x + n_0 a_x \\ -d_0 \theta_y + n_0 a_y \\ k_T a_z - g \end{bmatrix},$$

where  $d_0 = 10$ ,  $d_1 = 8$ ,  $n_0 = 10$ ,  $k_T = 0.91$ , and  $g = 9.81$ . For the sets, we set  $\mathcal{X} = \{[-15, -15, -15, -2, -2, -\frac{\pi}{3}, -\frac{\pi}{3}, -\frac{\pi}{3}, -\frac{\pi}{3}, -2] \leq x \leq [15, 15, 15, 2, 2, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, 2]\}$ ,  $\mathcal{U} = \{[-10, -10, 0] \leq u \leq [10, 10, 1.5g]\}$ ,  $\mathcal{X}_0 = \{[-2, -2, -2, -1, -1, -0.5, -0.5, -0.5, -0.5, -1] \leq x_0 \leq [2, 2, 2, 1, 1, 0.5, 0.5, 0.5, 0.5, 1]\}$ , and  $\mathcal{X}_{e0} = \{[-0.5, -0.5, -0.5, -0.5, -0.5, -0.5, -0.5, -0.5, -0.5, -0.5] \leq x_{e0} \leq [0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5]\}$ .

**Pendulum** is a simple model for evaluating control methods. The state variables are given by  $x := [\theta, \dot{\theta}]$ , where  $\theta$  is the angle, and  $\dot{\theta}$  is the angular velocity. The one-dimensional control input corresponds to the torque of the motor. The dynamics are as follows:

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \frac{mgl \sin(\theta) - 0.1\dot{\theta} + u}{ml^2} \end{bmatrix},$$

where  $g = 9.81$ ,  $m = 0.15$ , and  $l = 0.5$ . As for the sets, we set  $\mathcal{X} = \{[-\frac{\pi}{3}, -\frac{\pi}{3}] \leq x \leq [\frac{\pi}{3}, \frac{\pi}{3}]\}$ ,  $\mathcal{U} = \{-1 \leq u \leq 1\}$ ,  $\mathcal{X}_0 = \{[0, 0] \leq x_0 \leq [0, 0]\}$ , and  $\mathcal{X}_{e0} = \{[-\frac{\pi}{4}, -\frac{\pi}{4}] \leq x_{e0} \leq [\frac{\pi}{4}, \frac{\pi}{4}]\}$ .

**Dubin’s car.** The state of the car is given by  $x := [p_x, p_y, \theta, v]$ , where  $p_x, p_y$  are the positions,  $\theta$  is the heading direction, and  $v$  is the velocity. Its motion is controlled by two inputs: angular velocity  $\omega$  and linear acceleration  $a$ . The car’s dynamics are given by:

$$\dot{x} = \begin{bmatrix} v \cos(\theta) \\ v \sin(\theta) \\ \omega \\ a \end{bmatrix}.$$

For the sets, we set  $\mathcal{X} = \{[-5, -5, -\pi, 1] \leq x \leq [5, 5, \pi, 2]\}$ ,  $\mathcal{U} = \{[-1, 0] \leq u \leq [1, 0]\}$ ,  $\mathcal{X}_0 = \{[-2, -2, -1, 1.5] \leq x_0 \leq [2, 2, 1, 1.5]\}$ , and  $\mathcal{X}_{e0} = \{[-1, -1, -1, -1] \leq x_{e0} \leq [1, 1, 1, 1]\}$ . There is a simpler Dubin’s car model with three state variables  $x := [p_x, p_y, \theta]$  and a two-dimensional control input  $u := [v, \omega]$ . The dynamics are similar to the above ODEs without  $\dot{v} = a$ . However, we notice that in the simpler model  $f(x) \equiv 0$  and  $B(x)$  is not full-rank. Therefore, Eq. (2) in the paper is not feasible and the loss term  $\mathcal{L}_{w1}$  does not make sense. If we train the model only with the loss  $\mathcal{L}_u$  as in Eq. (5), the resulting controller is not performing well.

**TLPRA** models a Two-Link Planar Robot Arm system and is adopted from [43]. The state variables are given by  $x := [q_1, q_2, \dot{q}_1, \dot{q}_2]$ , where  $q$  and  $\dot{q}$  are the rotational angle and angular velocity of the joints respectively. The control input is given by  $u := [\tau_1, \tau_2]$  which are the torques applied on the joints. The dynamics are given by

$$\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} \alpha + \beta + 2\eta \cos(q_2) & \beta + \eta \cos(q_2) \\ \beta + \eta \cos(q_2) & \beta \end{bmatrix}^{-1} \begin{bmatrix} \tau_1 + \eta(2\dot{q}_1\dot{q}_2 + \dot{q}_2^2) \sin(q_2) - \alpha e_1 \cos(q_1) - \eta e_1 \cos(q_1 + q_2) \\ \tau_2 - \eta\dot{q}_1^2 \sin(q_2) - \eta e_1 \cos(q_1 + q_2) \end{bmatrix},$$

where  $\alpha = (m_1 + m_2)a_1^2$ ,  $\beta = m_2a_2^2$ ,  $\eta = m_2a_1a_2$ ,  $e_1 = g/a_1$ ,  $g = 9.8$ ,  $m_1 = 0.8$ ,  $m_2 = 2.3$ ,  $a_1 = 1$ , and  $a_2 = 1$ . For the sets, we set  $\mathcal{X} = \{[-\frac{\pi}{2}, -\frac{\pi}{2}, -\frac{\pi}{3}, -\frac{\pi}{3}] \leq x \leq [\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}]\}$ ,  $\mathcal{U} = \{[0, 0] \leq u \leq [0, 0]\}$ ,  $\mathcal{X}_0 = \{[\frac{\pi}{2}, 0, 0, 0] \leq x_0 \leq [\frac{\pi}{2}, 0, 0, 0]\}$ , and  $\mathcal{X}_{e0} = \{[-0.3, -0.3, 0, 0] \leq x_{e0} \leq [0.3, 0.3, 0, 0]\}$ .

#### F.4 Experimental results on additional benchmarks

Results comparing the performance of C3M and other approaches on the additional 5 models are shown in Fig. 3 and Tab. 2. On some simple models including Pendulum, Dubin’s car, Cart-pole, and TLPRA, both RL and C3M achieve perfect results, and C3M demonstrates lower variance. On the higher-dimensional system, Quadrotor2, C3M outperforms RL with lower tracking error and lower variance. Among these models, the SoS-based method cannot handle Cart-pole since it does not satisfy the sparse assumption for the control matrix  $B$  and TLPRA since the dynamics is too complicated and hard to be approximated with low-degree polynomials. We did not evaluate the SoS-based method on other models since the case-by-case hyper-parameters tuning is too time consuming without domain expertise on the models. For Cart-pole, Pendulum, and TLPRA, the MPC-PyTorch library had numerical errors.

Model	Dim		Execution time per step (ms)			Tracking error (AUC)			Convergence rate ( $\lambda/C$ )		
	$n$	$m$	C3M	MPC	RL	C3M	MPC	RL	C3M	MPC	RL
CAR	4	2	0.46	2300	0.46	0.290	0.509	0.275	1.970/2.69	0.493/1.49	1.836/4.99
Quadrotor2	10	3	0.45	7460	0.45	1.239	1.673	2.081	0.400/3.52	0.451/3.98	0.237/3.78
Cart-pole	4	1	0.45	-	0.45	0.865	-	1.207	1.100/4.85	-	0.509/4.10
Pendulum	2	1	0.37	-	0.37	0.091	-	0.092	5.772/6.82	-	2.675/9.46
TLPRA	4	2	0.42	-	0.42	1.082	-	1.086	2.035/4.91	-	1.883/6.82

Table 2: Comparison results of C3M vs. other methods. Here,  $n$  and  $m$  are dimensions of the state and input space. The MPC library used in experiments failed to handle some models due to numerical issues (we got some **nan** errors).

#### F.5 Sampled trajectories under disturbance

We show some plots of actual trajectories of the controlled systems in Fig. 4. We simulate the controlled Dubin’s car and Quadrotor2 system using different control strategy and with different

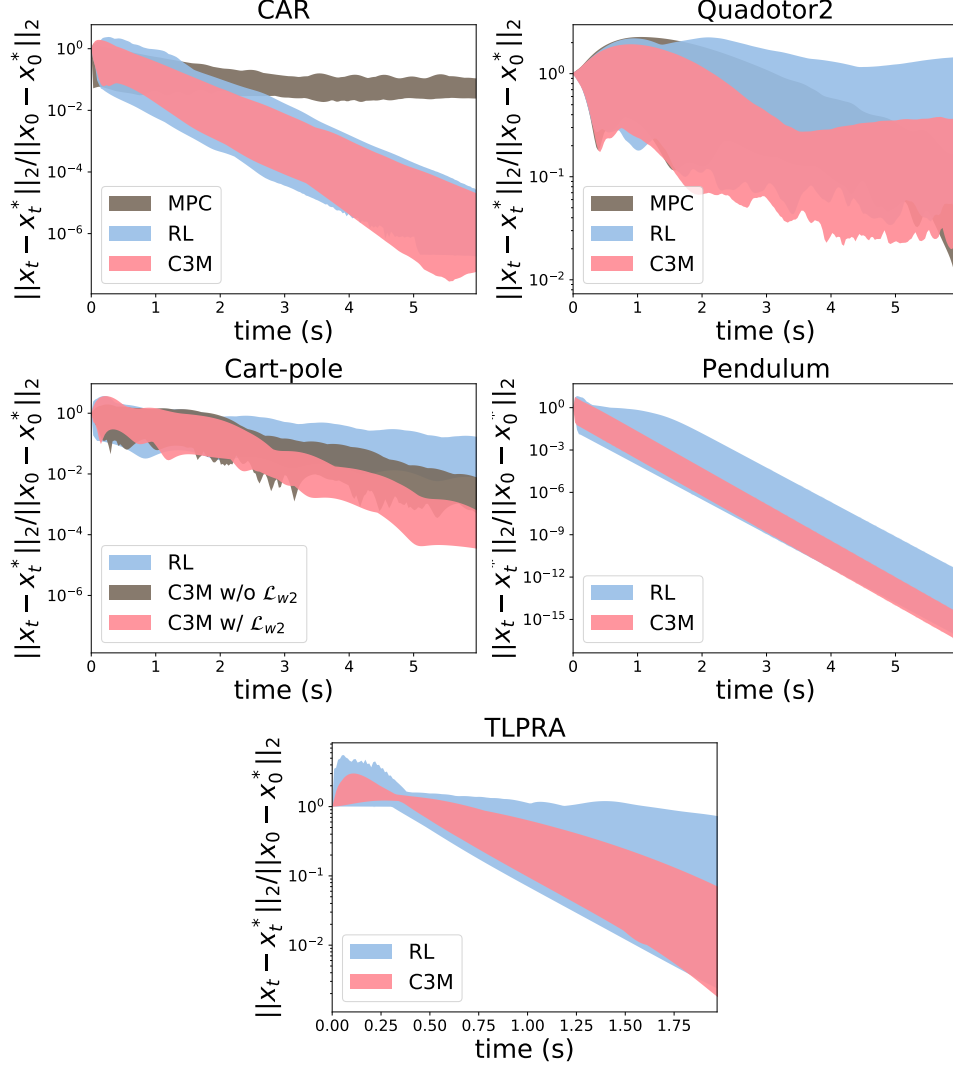


Figure 3: Normalized tracking error on the benchmarks using different approaches. The  $y$  axes are in log-scale. The tubes are tracking errors between mean plus and minus one standard deviation over 100 trajectories.

levels of disturbances, where elements of the disturbance are independently Gaussian distributed, i.e.  $d_i(t) \sim \mathcal{N}(0, \sigma)$  for  $i = 1, 2, \dots, n$ . Since the controlled system is proved to be contracting using C3M, the tracking error is still bounded even under additive random disturbances. We can also see from Fig. 4 that even with disturbance, the trajectories of the closed-loop system using C3M has less tracking error than the system with controllers using RL or MPC.

## G More discussions on related work

In the broad area of safe motion planning, driving the robots to some waypoints or follow some nominal trajectory without hitting obstacles in dynamically changing environments is a very challenging task, especially when robotics systems are nonlinear and nonholonomic. In this section, we provide more details on some related techniques in this area.

Sample-based planning methods including rapidly-exploring random trees (RRT) [29], probabilistic road maps (PRM) [2], fast marching tree (FMT) [44], and many others [45, 46, 47], plan safe open-loop trajectories by exploring the environment using samples. However, it is difficult to use sample-based planning to handle uncertainty and disturbances with safety guarantees. In order to plan safe trajectories in the presence of disturbances, the uncertainty should be handled appropriately. The

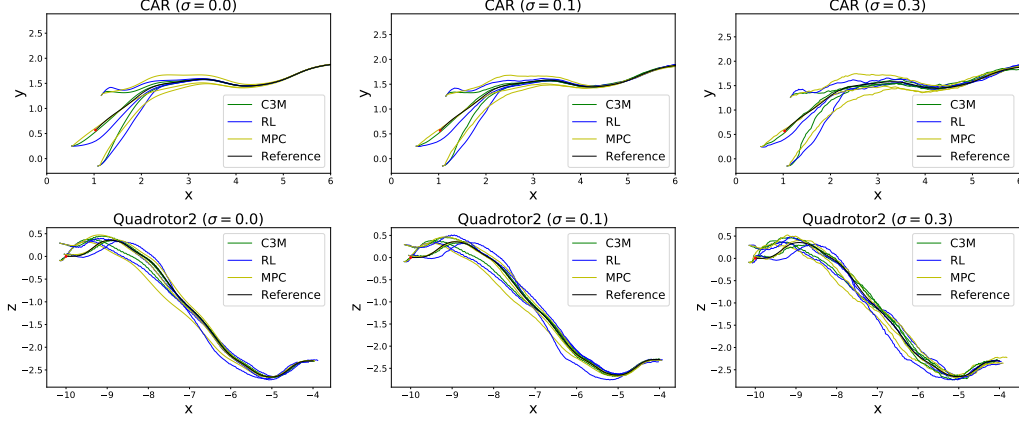


Figure 4: Actual trajectories of the controlled systems, where elements of the disturbance are independently Gaussian distributed, i.e.  $d_i(t) \sim \mathcal{N}(0, \sigma)$  for  $i = 1, 2, \dots, n$ . Red dots indicate the starting points of the reference trajectories.

disturbances are usually assumed to be bounded, and the safety can be guaranteed by dealing with the worst case safely.

One may model safe motion planning problems as differential games, where the disturbances are modeled as an adversarial agent. Backward reachable sets and the corresponding optimal controller can be computed by solving Hamilton-Jacobi PDE using level-set methods [48, 49, 50, 3, 51, 52]. However, this class of methods suffers from poor scalability. Solving Hamilton-Jacobi PDE for high-dimensional models is computationally expensive. Moreover, such high computational complexity limits their capability of solving online planning problems for dynamical environments. In comparison, our learned tracking controller can be executed in sub-millisecond level and therefore supports online planning. Moreover, because of the learned contraction metric, we can pre-compute the tracking error bound, which can be further used to understand how far the nominal trajectories should be from the obstacles.

Barrier functions are another widely-used class of certificates for safety. A barrier function is a function of state whose time-derivative on the zero-level set is uniformly negative, which makes the zero-level set an invariant. In [53, 54], barrier functions are computed for given controllers and any admissible disturbances and are used for verification of the closed-loop system. As an extension to controlled systems, control barrier functions [11, 12, 13] enable synthesis of safety-guaranteed controllers and thus have gained increasing popularity. In [11], the authors jointly leveraged control barrier functions (CBF) and control Lyapunov functions (CLF) as safety and objective certificates respectively for control synthesis in the context of adaptive cruise control and lane keeping. However, finding valid CBF for general models is hard. Recently, the advances of machine learning have been utilized to help search for CBF [13].

The idea of synthesizing a controller and bounding the tracking error meanwhile through pre-computation has gained increasing popularity. It enables safe motion planning for dynamically changing environments. Control Lyapunov functions can be used for this purpose [10]. Learning-based CLF has also been studied in some recent works [20, 17, 19]. However, in order to use CLF for synthesizing controller to track reference or nominal trajectories instead of driving the system to a fixed goal set, the original dynamics has to be converted into error dynamics depending on the reference trajectory, which limits the diversity of available reference trajectories. Similarly, funnel library methods have been studied in [55, 5], where tracking controllers for a fixed set of reference trajectories and the corresponding tracking error are computed offline. In the online planning phase, the safety of the reference trajectories are examined using the funnels corresponding to them, and the safest one is chosen. Again, this class of methods lack diversity of available reference trajectories which is critical to safe motion planning in complicated environments. LQR-Trees [14] builds a tree of LQR controllers backward from the goal set such that the total contraction regions of the nodes cover the whole state space, however, it cannot handle scenarios where the environments are unknown until runtime. Tube Model Predictive Control (TMPC) is another class of related techniques, where a tracking controller is computed such that the actual trajectory remains in a tube

centered at the planned MPC nominal trajectory in the presence of bounded disturbances. TMPC techniques have been studied for linear system in many works [56, 57, 58]. As for non-linear systems, linearization and Lipschitz-continuity-based reachability analysis are used [59, 60, 61]. However, in the latter case, the tube are often too conservative to be used in motion planning with limited freespace.

Contraction analysis [34] is another series of methods for analyzing the incremental stability of systems. Recently, it has been extended to controlled system in [25] and thus enabled tracking control synthesis for arbitrary reference trajectories with guaranteed bound on tracking error. The most challenging part in contraction analysis is the search for a valid contraction metric which entails solving Linear Matrix Inequalities (LMIs). In [62], the authors proposed to solve this feasibility problem with Sum-of-Squares (SoS) programming. In [27], the authors extended the SoS-based method to controlled systems, i.e. search for a *control* contraction metric (CCM) instead of an ordinary one, and proposed a more general method for synthesizing control given a valid CCM. However, in order to apply SoS-based methods, the dynamics of the system has to be represented by polynomials or can be approximated by polynomials. Furthermore, the method proposed in [27] relies on an assumption on the structure which encodes the controllability of the system. Different from ours, the method proposed in [27] searches for a metric using the structural assumption and synthesize the controller using geodesics, which entails heavy computation. Moreover, geodesics cannot be computed exactly, and thus the authors in [27] resorted to the optimization-based method proposed in [63] as an approximation of the geodesics.

Combining machine learning and contraction theory has also been studied in some recent work. In [8], the authors proposed a learning-based method for identifying an unknown system with an additional constraint that such learned dynamics admit a valid CCM. In [28], the authors proposed a control synthesis method for nonlinear systems where the dynamics is written as a convex combination of multiple state-dependent coefficients (i.e.  $f(x, t)$  written as  $A(x, t)x$ ) and made use of RNNs to model a time-varying metric. Again, different from our method, the controller is not jointly learned with the metric. In contrast, our approach can simultaneously synthesize the controller and the CCM certificate without any additional structural assumptions for the system.



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