Douady-Earle extensions of Holder continuous and Lipchitz continuous circle homeomorphims

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Abstract

In this article, we prove that Douady-Earle extension $\Phi(f)$ of a Holder continuous circle homeomorphism f is Holder continuous with the same Holder exponent. We also provide some boundary estimates along the radial direction of $\Phi(f)$ when f is Lipchitz.

1 Motivation, Introduction and Preliminaries

Douady-Earle extensions or barycentric extensions of circle homeomorphisms are indispensable tools for studying Teichmueller space of a Riemann surfaces, which are the spaces of possible complex structures on that Riemann surface modulo isotopy fixing the ideal boundary of that Riemann surface [11]. In particular, they are used to show that the Teichmueller spaces are all contractible [6], to obtain alternate characterizations of equivalence in the definitions of asymptotic Teichmueller spaces ([9], [8], [11]). But it is interesting that the definitions of Douady-Earle extensions does not involve any of the above spaces. Following [6], we briefly define, the Douady-Earle or barycentric extension $\Phi(f)$ of a given homeomorphims f of $\mathbb{S}^1$. For a non-atomic probability measure μ on S^1, the vector field V_μ given by:
$V_\mu(w) = \int_{S^1} \frac{t-w}{1-\bar{w}.t} d\mu(t)$
has a unique zero in $\mathbb{D}$ and hence they defined $B(\mu)$ to be the unique zero of the vector field V_μ This zero is called the conformal barycenter of μ. Next, Douady-Earle/Barycentric extension $\Phi(f)$ of f is defined to be : $\Phi(f)(z) = B(f_*\eta_z)$, where η_z is the harmonic measure on S^1 with respect to $z \in \mathbb{D}$ and $f_*\eta_z$ is its pull back by f. Stated more explicitly, $\Phi(f)(z)$ is the unique $w-$zero of the integral $\int_{S^1} \frac{f(t)-w}{1-\bar{w}.f(t)} p(z,t)|dt|$, where $p(z,t) = \frac{1}{2\pi}.\frac{1-|z|^2}{|z-t|^2}$ is the Poisson kernel which appears in the context of complex harmonic extension of a continuous function defined on S^1.

Douady-Earle extension has the following important properties:

(1) Conformal naturality: $\Phi(A \circ f \circ B) = A \circ \Phi(f) \circ B \ \forall A, B \in G = Aut(\mathbb{D})$.
(2) Continuity: let $f_n \to f$ uniformly on $\mathbb{S}^1$, then $\Phi(f_n)$ and all its derivatives converge pointwise to those of $\Phi(f)$.
(3) $\Phi(f)$ is continuous upto the boundary of $\mathbb{D}$, in fact it is a homeomorphism of $\overline{\mathbb{D}}$.

(4) $\Phi(f)|_{\mathbb{D}}$ is real-analytic diffeomorphism of $\mathbb{D}$.
(5) If f is quasisymmetric (respectively, symmetric), then $\Phi(f)$ is quasiconformal(respectively, assymptotically conformal).
(6) If f is the restriction of a biholomorphic automorphism F of $\mathbb{D}$, then $\Phi(f) = F$. In particular $\Phi(Id) = Id$.
(7) $\Phi(f)$ is bi-Lipchitz with respect to the Poincare (=hyperbolic) metric on $\mathbb{D}$.

For all the above properties, consult [6].

Boundary regularity problem of Douady-Earle extensions: Given any operator sending homeomorphisms of $\mathbb{S}^1$ to homeomorphisms of $\overline{\mathbb{D}}$, it is natural to ask the question: if we put further regularity conditions on the first one, whether we can expect the same for the second one? This question has been treated by analysts(see [18]). For complex harmonic extensions, where it is well-known that the complex harmonic extension of a $C^{k,\alpha}$ circle map is also $C^{k,\alpha}, 0 < \alpha < 1$. The corresponding theorem is called **Kellog's theorem**, and a proof of this theorem for $k = 0$ is given in the next section, which closely follows the one given in [18]. In this paper, we prove the corresponding result for Douady-earle extension:

Theorem 1: Let $0 < \alpha < 1$ and f be a $\mathcal{C}^{0,\alpha}$-homeomorphism of the unit circle $\mathbb{S}^1$. then, its Douady-earle extension $\Phi(f)$ is also $\mathcal{C}^{0,\alpha}$-homeomorphism of the unit disk $\mathbb{D}$.

Theorem 2: Let f be a Lipchitz (i.e. $\mathcal{C}^{0,1}$)homeomorphism of $\mathbb{S}^1$. Then $|\Phi(f)(r\zeta) - \Phi(f)(\zeta)| \leq M.(1-r)ln(\frac{1}{1-r})$ as $r \to 1-, \zeta \in \mathbb{S}^1$.

2 $C^{0,\alpha}$-Holder continuity of the harmonic extension of $C^{0,\alpha}$ map on the circle

It is a popular theorem among people working in the areas of partial differential equations and potential theory that the complex harmonic extension of, i.e., the solution to the Dirichlet problem with $C^{k,\alpha}$-boundary data is also $C^{k,\alpha}, 0 < \alpha < 1$. In order to prove the main theorem, we need only the case $k = 0$, which is the following:

Theorem 3(Kellog's theorem). Let $f : \mathbb{S}^1 \to \mathbb{C}$ be $\mathcal{C}^{0,\alpha}(\mathbb{S}^1)$. Then its complex harmonic extension, i.e. Poisson integral $H(f)(z) := \int_{\mathbb{S}^1} f(t)p(z,t)|dt|$ is $\mathcal{C}^{0,\alpha}(\mathbb{D})$.

For a proof of the above, we introduce the notion of *modulus of continuity.* Let the function ϕ be uniformly continuous on a connected set $A \subset \mathbb{C}$. Then its *modulus of continuity* is defined by
$\omega(\delta) = \omega(\delta, \phi, A) = sup\{|\phi(z_1) - \phi(z_2)| : z_1, z_2 \in A, |z_1 - z_2| \leq \delta\}, \geq 0$. This is an increasing continuous function with $\omega(0) = 0$. If A is convex it is easy to see that
(1) $\omega(n\delta) \leq n\omega(\delta), n \in \mathbb{N}$.

If ϕ is analytic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$ then the modulus of continuity in *cud* and $\mathbb{S}^1$ are essentially the same because
(2) $\omega(\delta, \phi, \mathbb{S}^1) \leq \omega(\delta, \phi, \overline{\mathbb{D}}) \leq 3\omega(\delta, \phi, \mathbb{S}^1)$ for $\delta \leq \pi/2$.

The function ϕ is called *Dini-continuous* if:
$\int_0^\pi \frac{\omega(\theta)}{\theta} d\theta < \infty$;
The limit π could be replaced by any positive constant. For *Dini-continuous* ϕ and $0 < \delta < \pi$, we define:
(3) $\omega^*(\delta) = \omega^*(\delta, \phi, A) = \int_0^\delta \frac{\omega(\theta)}{\theta} d\theta + \delta. \int_\delta^\pi \frac{\omega(\theta)}{\theta^2} d\theta$.

Proposition 1. Let ϕ be 2π-periodic and Dini-continuous in $\mathbb{R}$. Then we have the following:

$$a) g(z) := \frac{i}{2\pi}. \int_0^{2\pi} \frac{e^{i\theta} + z}{e^i\theta - z}.\phi(\theta) d\theta, z \in \mathbb{D}$$

has a continuous extension upto $\overline{\mathbb{D}}$.

$$\begin{aligned} b)|g'(z)| &\leq \frac{2}{\pi}.\frac{\omega(1-r)}{1-r} + 2\varphi \int_{1-r}^\pi \frac{\omega(\theta)}{\theta^2} d\theta \\ &\leq 2\pi.\frac{\omega^*(1-r)}{1-r} \forall |z| \leq r \\ c)|g(z_1) - g(z_2)| &\leq 20\omega^*(\delta), \forall |z_1 - z_2| \leq \delta < 1 \end{aligned}$$

.

Note that, the above proposition implies **Kellog's theorem**, since $f : \mathbb{S}^1 \to \mathbb{S}^1$ is 2π-periodic and α-Holder continuous implies that the function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(\theta) = \tilde{f}(\theta) - 2\pi$ is 2π-periodic and α-Holder continuous on $\mathbb{R}$, and hence by $c), g$ is α-Holder continuous on $\mathbb{D}$ [since $\omega^*(\delta) = O(\delta^\alpha)$ for α-Holder continuous maps on $\mathbb{R}$. Hence $H(f) = Re(g)$ is also α-Holder continuous. So the proof of Kellog's theorem boils down to proving the above proposition.

Proof of proposition. It follows from the definition of g that:
(3) $g'(z) = \frac{i}{2\pi} \int_0^{2\pi} \frac{e^{i\theta}}{(e^{i\theta} - z)^2} \phi(\theta) d\theta, z \in \mathbb{D}$.
Since the above integral vanishes for constant function ϕ, substituting $\theta = \nu + \tau$ shows that:

$$g'(re^{i\nu}) = \frac{i}{\pi} \int_{-\pi} \pi \frac{e^i\tau - i\nu}{|e^{i\tau} - r|^2} d\tau$$

. Next, consider the integral over the intervals $[0, 1-r]$ and $[1-r, \pi]$ separately and use the fact that $|e^{i\theta} - r| \geq max\{\theta/\pi, 1-r\}$ to get the inequality b)

The second one follows from (see(2)):
$\frac{1}{3}\omega(\delta) \leq \omega(\delta/3) \leq \int_{\delta/3}^\delta \frac{\omega(\theta)}{\theta} d\theta \leq \int_0^\delta \frac{\omega(\theta)}{\theta} d\theta$. The estimates hold also for $|z| < r$ by maximum principle.

Let $z_1 = r_1.\zeta_1, z_2 = r_2.\zeta_2, \zeta_1, \zeta_2 \in \mathbb{S}^1$ with $|z_1 - z_2| \leq \delta$ and put $r = 1 - \delta$. Integrating the estimate from b) over $[z_1, z_2]$ we obtain:

(4) $|g(z_1) - g(z_2)| \leq 2\pi.\omega^*(\delta)$ for $r_1 \leq r, r_2 \leq r$.

Suppose now that either $r_1 > r$ or $r_2 > r$. from the estimate b) we see that

$$\begin{aligned}
&|g(r\zeta_j) - g(r_j\zeta_j)| \\
&\leq \int_r^{r_j} |g'(\xi.r_j)| d\xi \\
&\leq \frac{2}{\pi}\int_0^{\delta} \frac{\omega(x)}{x} dx + 2\pi. \int_0^{\delta} (\int_x^{\pi} \frac{\omega(\theta)}{\theta^2)} d\theta) dx.
\end{aligned}$$

Exchanging the order of integration we obtain that the last term is

$$= 2\pi. \int_0^{\delta} \frac{\omega(\theta)}{\theta} d\theta + 2\pi.\delta \int_{\delta}^{\pi} \frac{\omega(\theta)}{\theta^2} d\theta$$

so that $|g(r\zeta_j) - g(r_j.\zeta_j)| \leq 7\omega^*(\delta)$ by (3) (definition of ω^*).

Furthermore,
$|g(r\zeta_1) - g(r\zeta_2)| \leq 2\pi\omega^*(\delta)$ by (4). It follows that (c) holds in all cases. This finishes the proof of **Kellog's theorem**.

3 Proof of Theorem 1

Let $H(f)(z) := \int_{\mathbb{S}^1} f(t)p(z,t)|dt|$ denote the complex harmonic extension of the circle homeomorphism f, and let, as usual, $\Phi(f)$ denote the Douady-Earle extension of f. Our way to get the Holder regularity of Douady-earle is to compare the Douady-Earle with the harmonic extension and show that the difference is also Holder continuous. So, we begin with the following lemmas.
Lemma 1.
$|\Phi(f)(z) - H(f)(z)| \leq \int_{\mathbb{S}^1} |f(t) - f(s)| p(z,t)|dt| \forall s \in \mathbb{S}^1, z \in \mathbb{D}$.
Proof of Lemma 1.

We write $\Phi = \Phi(f), H = H(f)$ for short. Note that:

$$\Phi(z) - f(t) - \frac{f(t) - \Phi(z)}{1 - \bar{\Phi(z)}f(t)}.f(t)\bar{\Phi(z)} = \frac{\Phi(z) - f(t)}{1 - \bar{\Phi(z)}f(t)}$$

Multiplying both sides by the Poisson kernel $p(z,t)$ and integarting w.r.t. $|dt|$ gives us:

$$\begin{aligned}
\Phi(z) - H(z) - &\int_{\mathbb{S}^1} \frac{f(t) - \Phi(z)}{1 - \bar{\Phi(z)}f(t)} f(t)\bar{\Phi(z)}p(z,t)|dt| \\
&= -\int_{\mathbb{S}^1} \frac{f(t) - \Phi(z)}{1 - \bar{\Phi(z)}f(t)}.f(t)p(z,t)|dt| = 0
\end{aligned}$$

Therefore,

$$\Phi(f)(z) - H(f)(z) = \int_{\mathbb{S}^1} \frac{|f(t) - \Phi(f)(z)|}{1 - \Phi(\bar{f})(z)f(t)}.(f(t) - f(s)).\Phi(\bar{f})(z).p(z,t)|dt|$$

Taking the aboslute value sign on both sides, we get:

$$|\Phi(f)(z) - H(f)(z)| \leq \int_{\mathbb{S}^1} |f(t) - f(s)|p(z,t)|dt|$$

.

This finsihes the proof of Lemma 1.

Lemma 2. For $0 < \alpha < 1, |z-s|^\alpha \leq \int_{\mathbb{S}^1} |t-s|^\alpha p(z,t)|dt| \leq C.|z-s|^\alpha, C > 0$, the positive constant C is independent of $z \in \mathbb{D}$ and $s \in \mathbb{S}^1$.

Proof of Lemma 2. Call $G_s(z) = |z-s|^\alpha, H_s(z) = \int_{\mathbb{S}^1} |t-s|^\alpha p(z,t)|dt|$. Next, consider the Laplacian of $G_s - H_s$ with respect to z, i.e., for $\Psi(z) = G_s(z) - H_s(z), \Delta\Psi \geq 0$, and that $\Psi = 0$ on $\mathbb{S}^1$. Now maximum principle applied to Ψ immediately implies the left hand side of the inequality.
To prove the right hand side, which is important for our purpose, we first note that:the function on $\mathbb{S}^1$ defined by $t \mapsto |t-1|^\alpha$ is $\mathcal{C}^{0,\alpha}$ on $\mathbb{S}^1$. So by Theorem 3 (Kellog's theorem), its complex harmonic extension is also $\mathcal{C}^{0,\alpha}$ on $\mathbb{D}$.
Therefore, $\int_{\mathbb{S}^1} |t-1|^\alpha p(z,t)|dt| \leq C|z-1|^\alpha$ So, we have

$$\begin{aligned}
\int_{\mathbb{S}^1} |t-s|^\alpha p(z,t)|dt| &= \int_{\mathbb{S}^1} |\frac{t}{s} - 1|^\alpha p(z,t)|dt| \\
&= \int_{\mathbb{S}^1} |\frac{t}{s} - 1|^\alpha p(z,t)|dt| \\
&= \int_{\mathbb{S}^1} |t-1|^\alpha p(\frac{z}{s}, t)|dt| \\
&\leq C.|\frac{z}{s} - 1|^\alpha \\
&= C.|z-s|^\alpha.
\end{aligned}$$

This finishes the proof of lemma 2.

With these lemma, we can finally finish the proof of the Theorem 1.

Proof of Theorem 1. Using lemmas 1 and 2, and Theorem 3 (Kellog's theorem)

$$\begin{aligned}
&|\Phi(f)(z) - \Phi(f)(s)| \\
&\leq |\Phi(f)(z) - H(f)(z)| + |H(f)(z) - H(f)(s)| \\
&\leq C.|z-s|^\alpha + K.|z-s|^\alpha \\
&= (C+K).|z-s|^\alpha
\end{aligned}$$

where C, K are positive constants independent of $z \in \mathbb{D}$ and $s \in \mathbb{S}^1$. This finishes the proof of Theorem 1.

4 Proof of Theorem 2

Again, we denote $\Phi(f)$ by Φ, $H(f)$ by H, and let $\zeta \in \mathbb{S}^1$. Then,

$$
\begin{aligned}
&|\Phi(r\zeta) - \Phi(\zeta)| \\
&\leq |\Phi(r\zeta) - H(r\zeta)| + |H(r\zeta) - H(\zeta)| \\
&\leq \int_{\mathbb{S}^1} |f(t) - f(\zeta)| p(r\zeta, t)|dt| + \int_{\mathbb{S}^1} |f(t) - f(\zeta)| p(r\zeta, t)|dt| \\
&\leq K_1 \int_{\mathbb{S}^1} |t - \zeta| p(r\zeta, t)|dt| + K_2 \int_{\mathbb{S}^1} |t - \zeta| p(r\zeta, t)|dt| \\
&= K. \int_{\mathbb{S}^1} |t - \zeta| p(r\zeta, t)|dt| 4mm(5)
\end{aligned}
$$

The proof of theorem 2 will be completed if we can show that (6)

$$\int_{\mathbb{S}^1} |t - \zeta| p(r\zeta, t)|dt| \leq M.(1 - r) ln(\frac{1}{1 - r})$$

as $r \to 1-$.

Proof of (6):

$$
\begin{aligned}
&\int_{\mathbb{S}^1} |t - \zeta| p(r\zeta, t)|dt| = \int_{\mathbb{S}^1} |t - \zeta| \frac{1 - r^2}{|r\zeta - t|^2} |dt| \\
&= \int_{\mathbb{S}^1} |\zeta - 1|^2 . \frac{1 - r^2}{|r\zeta - t|^2} |dt| \\
&\leq 2(1 - r) \int_{\mathbb{S}^1} \frac{t - \zeta}{|r\zeta - t|^2} |dt| \\
&= 2(1 - r) \int_{\mathbb{S}^1} \frac{t - \zeta}{|r\zeta - t|} . \frac{1}{|r\zeta - t| . |dt|} \\
&\leq 4.(1 - r) \int_{\mathbb{S}^1} \frac{|dt|}{|r\zeta - t|} .
\end{aligned}
$$

Note that, to prove (6), it is enough to assume $\zeta = 1$. So (6) will be proved if we can show that:

$$I(r) := \int_{\mathbb{S}^1} \frac{|dt|}{|r - t|} = O(ln \frac{1}{1 - r})$$

(7)

$$
\begin{aligned}
I(r) &= \int_{-\pi}^{\pi} (1+r^2-2rcos\theta)^{-1/2} d\theta, t = e^{i\theta} \\
&= \int_{-\pi}^{\pi} [(1-r)^2 + 2r(1-cos\theta)]^{-1/2}
\end{aligned}
$$

Choose and fix once and for all $\theta_0 \in (0, \pi/4)$ so that $1 - cos\theta = 2sin^2\frac{\theta}{2} \geq \frac{\theta^2}{4} \forall \theta \in (0, \theta_0)$.

Then we get

$$
\begin{aligned}
&\int_{-\pi}^{\pi} [(1-r)^2 + 2r(1-cos\theta)]^{-1/2} \\
&= \int_{-\theta_0}^{\theta_0} [(1-r)^2 + 2r(1-cos\theta)]^{-1/2} d\theta + \int_{[-\pi,\pi]\setminus[-\theta_0,\theta_0]} (1+r^2-2rcos\theta)^{-1/2} d\theta \\
&= 2\int_{0}^{\theta_0} [(1-r)^2 + 2r(1-cos\theta)]^{-1/2} d\theta + O(\frac{1}{\sqrt{r}})
\end{aligned}
$$

Choosing r sufficiently close to 1, the above becomes:

$$
\begin{aligned}
&\leq 2\int_{0}^{1-r} [(1-r)^2 + 2r.\frac{1}{4}\theta^2]^{-1/2} d\theta + 2\int_{1-r}^{\theta_0} [(1-r)^2 + 2r.\frac{1}{4}\theta^2]^{-1/2} d\theta + O(\frac{1}{\sqrt{r}}) \\
&\leq 2.\int_{0}^{1-r} \frac{1}{1-r} d\theta + 2\int_{1-r}^{\theta_0} \frac{1}{2} r. \frac{d\theta}{\theta} \\
&= 2 + rln\theta_0 + \frac{1}{2} r \ln \frac{1}{1-r} \\
&\leq 2 + \frac{1}{2} ln \frac{1}{1-r} = O(\ln \frac{1}{1-r})
\end{aligned}
$$

as $r \to 1-$.
Hence

$$
I(r) := \int_{\mathbb{S}^1} \frac{|dt|}{|r-t|} = O(ln\frac{1}{1-r}),
$$

which proves (6), and hence theorem 2.

Final remarks and acknowledgements: Motivated by kellog's theorem in $\mathcal{C}^{1,\alpha}$-circle maps, We believe that Douady-Earle extension of a $C^{1,\alpha}$ circle diffeomorphism should be $C^{1,\alpha}$ on $\mathbb{D}$ as well, although we do not have a proof yet. I would like to sincerely thank my thesis advidors Prof. Feng Luo and Prof. Jun Hu for various discussions regarding the above.

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