BOUNDARY DIFFERENTIABILITY OF DOUADY-EARLE EXTENSIONS OF DIFFEOMORPHISMS OF $\mathbb{S}^n$

JUN HU* AND SUSOVAN PAL

ABSTRACT. Let f be a homeomorphism of a unit sphere $\mathbb{S}^n$ of dimension n and $\Phi(f)$ the Douady-Earle extension of f to the closed unit ball $\overline{\mathbb{B}^{n+1}}$ bounded by $\mathbb{S}^n$. We show that if f is C^1-smooth in a neighborhood of a point p on $\mathbb{S}^n$ and $Df(p)$ is not singular, then $\Phi(f)$ is C^1 at p. It follows that when $n = 1$, $\Phi(f)$ is a C^1 diffeomorphism of $\overline{\mathbb{D}}$ if f is a C^1 diffeomorphism of $\mathbb{S}^1$; when $n > 1$, $\Phi(f)$ is a C^1 smooth map from $\overline{\mathbb{B}^{n+1}}$ to itself if f is a C^1 diffeomorphism of $\mathbb{S}^n$. In particular, when $n = 1$, if an orientation-preserving homeomorphism f of the unit circle $\mathbb{S}^1$ has the same local smoothness in a neighborhood of p on $\mathbb{S}^1$, then $\Phi(f)$ has a complex derivative at p, which is equal to $f'(p)$, and $\frac{\partial}{\partial z}\Phi(f)(z)$ and $\frac{\partial}{\partial \bar{z}}\Phi(f)(z)$ converge to $f'(p)$ and 0 respectively as z approaching p.

1. Introduction

Douady-Earle extensions of orientation-preserving homeomorphisms of the unit circle are of particular interest in the study of contractibility and complex structures of Teichmüller spaces and asymptotic Teichmüller spaces. Our understanding of differentiability of the extensions of circle diffeomorphisms at boundary points evolves into a proof that can be generalized to derive a corresponding result for Douady-Earle extensions of diffoemorphisms of a unit sphere of any dimension n. Therefore, we separate the introduction and proof of our main theorem in one-dimensional case from other higher dimensional cases. In this section, we first give a brief account of properties of the extensions of orientation-preserving circle homeomorphisms; then we recall an early result of Earle on a sufficient condition for the existence of angular derivative for the extension of a circle homeomorphism at a boundary point; thirdly we state a differentiability theorem for the extensions of circle diffeomorphisms at boundary points; and finally, we state a corresponding theorem for the extensions of diffeomorphisms of a unit sphere of any dimension n.

Let $\mathbb{S}^1$ be the unit circle on the complex plane and centered at the origin, and let $\mathbb{D}$ (resp. $\overline{\mathbb{D}}$) be the open (resp. closed) unit disk bounded by $\mathbb{S}^1$. In [5], Douady and Earle extended homeomorphisms f of $\mathbb{S}^1$ to homeomorphisms $\Phi(f)$ of $\overline{\mathbb{D}}$ satisfying the following two properties:

(1) The extension of the identity map on $\mathbb{S}^1$ is the identity map on $\overline{\mathbb{D}}$.

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(2) Given any two conformal automorphisms A and B of $\mathbb{D}$,

$$\Phi(A \circ f \circ B) = A \circ \Phi(f) \circ B.$$

It follows that $\Phi(A|_{\mathbb{S}^1}) = A$ for any conformal automorphism A of $\mathbb{D}$. This is the so-called *conformal naturality* of the extension $\Phi(f)$ and such extensions are called *Douady-Earle extensions* or *conformally natural extensions* of circle homeomorphisms. Combined with other properties, these extensions have applications in the study of properties of Teichmüller spaces, such as contractibility of Teichmüller spaces ([5]) and aysmptotically conformal Teichmüller spaces ([7]). As the same as in these applications, the conformal naturality plays a key role in the course of developing the main theorems of this paper. By contrast, the definition of the extension $\Phi(f)$ of f is not much used explicitly in our proofs. For this reason, we recall the definition of the Douady-Earle extension without elaborations. Instead, we put more effort to summarize other properties of $\Phi(f)$.

Given a point $z \in \mathbb{D}$, let η_z be the harmonic measure on $\mathbb{S}^1$ viewed from z. Then $\Phi(f)(z)$ is defined to be the conformal barycenter $B(\mu)$ of the pushforward measure $\mu = f_*\eta_z$ of η_z under f, which is defined as a point $w \in \mathbb{D}$ such that

$$V_\mu(w) = (1 - |w|^2) \int_{\mathbb{S}^1} \frac{\zeta - w}{1 - \bar{w}\zeta} d\mu(\zeta) = 0.$$

The existence and uniqueness of $B(\mu)$ was proved in [5] by using Poincaré-Hopf Index Theorem. An elementary and direct proof was given in [10]. Clearly, for any $z \in \mathbb{S}^1$, $\Phi(f)(z)$ is defined to be $f(z)$.

In [5], Douady and Earle proved that $\Phi(f)$ is a homeomorphism of $\overline{\mathbb{D}}$ for any orientation-preserving homeomorphism f of $\mathbb{S}^1$ and furthermore $\Phi(f)$ is quasiconformal if f admits a quasiconformal extension to $\overline{\mathbb{D}}$. Then Earle, Markovic and Šărić showed in [7] that $\Phi(f)$ is asymptotically conformal if f admits an asymptotically conformal extension to $\overline{\mathbb{D}}$. Using the Beurling-Ahlfors extension ([4], included in [3] and [11]), we know that if f is quasisymmetric (respectively, symmetric), then f admits a quasiconformal (resp. asymptotically conformal) extension to $\overline{\mathbb{D}}$. Therefore, the Douady-Earle extension $\Phi(f)$ of f is quasiconformal (respectively, asymptotically conformal) if f is quasisymmetric (respectively, symmetric). Lately, Hu and Muzician introduced in [8] a method to estimate the maximal dilatation of $\Phi(f)$ in terms of the cross-ratio distortion norm of f, which enables them to conclude the quasiconformality of $\Phi(f)$ directly from the quasisymmetry of f. Furthermore, it was shown in [9] that local quasisymmetry (resp. local symmetry) of f implies local quasiconformality (resp. locally asymptotical conformality) of $\Phi(f)$, and these local implications can be developed first so that global implications follow from them.

The Douady-Earle extension $\Phi(f)$ has another interesting feature; that is, $\Phi(f)$ is real-analytic on $\mathbb{D}$ with non-singular Jacobian at every point on $\mathbb{D}$. Thus, $\Phi(f)$ is a diffeomorphism of $\mathbb{D}$. But one can only expect $\Phi(f)$ to have continuity at any boundary point $p \in \mathbb{S}^1$ since f is just a homeomorphism of $\mathbb{S}^1$. It gives rise to a question: what regularity does $\Phi(f)$ have at a boundary point p if f is differentiable at p or C^1 on a neighborhood of p on $\mathbb{S}^1$? Earle showed in [6] the following result provided that f is differentiable at p and $f'(p) \neq 0$.

Earle's Theorem (Existence of angular derivative)**.** *Let f be an orientation-preserving homeomorphism of $\mathbb{S}^1$. If f is differentiable at a point $p \in \mathbb{S}^1$ and*

$f'(p) \neq 0$, then the difference quotient $\frac{\Phi(f)(z)-\Phi(f)(p)}{z-p}$ converges to $f'(p)$ as a point z of $\mathbb{D}$ approaches p non-tangentially. Furthermore, $\frac{\partial}{\partial z}\Phi(f)(z)$ and $\frac{\partial}{\partial \bar{z}}\Phi(f)(z)$ converge to $f'(p)$ and 0 respectively as z approaches p non-tangentially.

In the previous theorem, by *a non-tangential limit* of an expression involving z as z approaching p, we mean that for any small positive real number ϵ, the expression has a limit as along as $z \in \mathbb{D} \cap \Omega_\epsilon$ and approaches p, where Ω_ϵ is the region in the complex plane bounded by two rays emanating from p and having angle $\pi/2 - \epsilon$ with the radius through p. By definition, the existence of the limits of the three expressions and the pattern of the limiting values in Earle's Theorem entitle the map $\Phi(f)$ to be said to have the *angular derivative* $f'(p)$ at p ([6]).

In [6], Earle also pointed out that there is an analogous result for the extensions of orientation-reversing homeomorphisms of $\mathbb{S}^1$. Since this paper considers extensions of all homeomorphisms of $\mathbb{S}^1$, it is helpful to know the exact statement of the analogue, which is as follows.

Analogue of Earle's Theorem. *Let f be an orientation-reversing homeomorphism of $\mathbb{S}^1$. If f is differentiable at a point $p \in \mathbb{S}^1$ and $f'(p) \neq 0$, then the difference quotient $\frac{\Phi(f)(z)-\Phi(f)(p)}{\bar{z}-\bar{p}}$ converges to $f'(p)$ as a point z of $\mathbb{D}$ approaches p non-tangentially. Furthermore, $\frac{\partial}{\partial z}\Phi(f)(z)$ and $\frac{\partial}{\partial \bar{z}}\Phi(f)(z)$ converge to 0 and $f'(p)$ respectively as z approaches p non-tangentially.*

Earle's Theorem and its analogue invite curiosities to study the limits of those three expressions as z approaches p tangentially or what more can imply the existence of such limits. In this paper, we first show that those expressions do have limits as z approaches p provided that f is C^1 in a neighborhood of p on $\mathbb{S}^1$ and $f'(p) \neq 0$.

Theorem 1 (Differentiability at a boundary point)**.** *Let f be a homeomorphism of $\mathbb{S}^1$ and $p \in \mathbb{S}^1$. Assume that f is C^1 in a neighborhood of p on $\mathbb{S}^1$ and $f'(p) \neq 0$.*

If f is orientation-preserving, then the difference quotient $\frac{\Phi(f)(z)-\Phi(f)(p)}{z-p}$ converge to $f'(p)$ as a point z of $\mathbb{D}$ approaches p, and furthermore $\frac{\partial}{\partial z}\Phi(f)(z)$ and $\frac{\partial}{\partial \bar{z}}\Phi(f)(z)$ converge to $f'(p)$ and 0 respectively as z approaches p.

If f is orientation-reserving, then the difference quotient $\frac{\Phi(f)(z)-\Phi(f)(p)}{\bar{z}-\bar{p}}$ converges to $f'(p)$ as a point z of $\mathbb{D}$ approaches p, and furthermore $\frac{\partial}{\partial z}\Phi(f)(z)$ and $\frac{\partial}{\partial \bar{z}}\Phi(f)(z)$ converge to 0 and $f'(p)$ respectively as z approaches p.

It implies the following theorem.

Theorem 2 (Global C^1 diffoemorphism)**.** *For any C^1 diffoemorphism f of $\mathbb{S}^1$, the Douady-Earle extension $\Phi(f)$ of f is a C^1 diffoemorphism of $\overline{\mathbb{D}}$.*

Here by a C^1 *diffeomorphism* F of $\overline{\mathbb{D}}$ we mean it is a homeomorphism of $\overline{\mathbb{D}}$ and a diffeomorphism of $\mathbb{D}$, and furthermore it has a C^1 diffeomorphic extension on an open neighborhood of $\overline{\mathbb{D}}$ in $\mathbb{R}^2$.

Remark 1. (1) Let f be an orientation-preserving C^1 diffeomorphism of $\mathbb{S}^1$. Then f is symmetric. As pointed out in a previous paragraph, using the Beurling-Ahlfors extension of f and a result in [7], one concludes that $\Phi(f)$ is an asymptotically conformal homeomorphism of $\mathbb{D}$. Note that our Theorem 1 does recover this fact in the case when f is an orientation-preserving C^1 diffoemorphism of $\mathbb{S}^1$.

(2) Growing out of ideas in [2] and unpublished ideas of Milnor, an effective algorithm for finding the images of points under $\Phi(f)$, called the MAY iterator, was formally introduced in [1]. With availability of computer facilities, one may use the MAY iterator to explore properties of Douady-Earle extensions of circle homeomorphisms of different regularities. For example, one may verify Earle's Theorem and Theorem 1 numerically with some examples.

In [6], Earle proved his theorem on the existence of angular derivative by arranging arguments and reasonings with maps defined on $\overline{\mathbb{D}}$ and $\mathbb{S}^1$. After understanding his proof, we asked ourselves how the proof would go if we considered the corresponding maps defined on the upper half plane $\mathbb{H}$ and the extended real line $\hat{\mathbb{R}}$. The click of this idea leads to a very short proof of Earle's Theorem. From there, we developed our Theorem 1. Furthermore, we generalized Earle's Theorem and our Theorems 1 and 2 to the Douady-Earle extensions of homeomorphisms of the unit sphere $\mathbb{S}^n$ of any dimension n.

Let $\mathbb{S}^n$ be the unit sphere in $\mathbb{R}^{n+1}$ and centered at the origin and $\mathbb{B}^{n+1}$ (resp. $\overline{\mathbb{B}^{n+1}}$) the open (resp. closed) unit ball in $\mathbb{R}^{n+1}$ bounded by $\mathbb{S}^n$. Now let f be a homeomorphism of $\mathbb{S}^n$. When $n \geq 2$, the Douady-Earle extension $\Phi(f)$ of f is defined by using conformal barycenters in the exact same way as the case when $n = 1$ (see [5]). The extensions have the conformal naturality; that is,

(1) The extension of the identity map on $\mathbb{S}^n$ is the identity map on $\overline{\mathbb{B}^{n+1}}$;

(2) For any two conformal automorphisms A and B of $\mathbb{B}^{n+1}$,

$$\Phi(A \circ f \circ B) = A \circ \Phi(f) \circ B.$$

But in this case (when $n \geq 2$), it is pointed out in [5] that in general $\Phi(f)$ is not necessarily to be a homeomorphism of $\overline{\mathbb{B}^{n+1}}$, although it is true, in fact, a quasiconformal homeomorphism of $\overline{\mathbb{B}^{n+1}}$, provided that f is quasiconformal with maximal dilatation small enough (due to Tukia).

In the second part of this paper, we show the followings.

Theorem 3 (Existence of conic differentiability in any dimension)**.** *If a homeomorphism f of $\mathbb{S}^n$ is differentiable at a point $\vec{p} \in \mathbb{S}^n$ and $Df(\vec{p})$ is non-singular, then there exists a non-singular linear map L from $\mathbb{R}^{n+1}$ into itself, which preserves a half space of $\mathbb{R}^{n+1}$ and whose determinant has the same sign as $Df(\vec{p})$, such that*

$$\Phi(f)(\vec{x}) - \Phi(f)(\vec{p}) = L(\vec{x} - \vec{p}) + o(||\vec{x} - \vec{p}||)$$

as a point $\vec{x}$ of $\mathbb{B}^{n+1}$ approaches $\vec{p}$ non-tangentially. Furthermore, $D\Phi(f)(\vec{x})$ converges to L as $\vec{x}$ approaches $\vec{p}$ non-tangentially.

In the previous theorem, by *a point $\vec{x} \in \mathbb{B}^{n+1}$ approaching a point $\vec{p} \in \mathbb{S}^n$ non-tangentially* we mean $\vec{x}$ approaches $\vec{p}$ in a region of $\mathbb{R}^{n+1}$ bounded by a regular cone taking the radius passing $\vec{p}$ as its rotation axis and $\vec{p}$ as the tip of the cone. Similar to Earle's terminology on angular derivative, we say that the map $\Phi(f)$ in the theorem has a *conic differentiability* at the boundary point $\vec{p}$.

Theorem 4 (Differentiability at a boundary point in any dimension)**.** *Let f be a homeomorphism of $\mathbb{S}^n$ and $\vec{p} \in \mathbb{S}^n$. If f is C^1 in a neighborhood of $\vec{p}$ on $\mathbb{S}^n$ and $Df(\vec{p})$ is non-singular, then there exists a linear map L from $\mathbb{R}^{n+1}$ into itself, which preserves a half space of $\mathbb{R}^{n+1}$ and whose determinant has the same sign as $Df(\vec{p})$, such that*

$$\Phi(f)(\vec{x}) - \Phi(f)(\vec{p}) = L(\vec{x} - \vec{p}) + o(||\vec{x} - \vec{p}||)$$

as a point $\vec{x}$ of $\mathbb{B}^{n+1}$ approaches $\vec{p}$. Furthermore, $D\Phi(f)(\vec{x})$ converges to L as $\vec{x}$ approaches $\vec{p}$.

Remark 2. (1) We will present two approaches to prove the differentiability of $\Phi(f)$ at the point $\vec{p}$ in the previous theorem, from which two different expressions for the linear map L are obtained (see the proof of this theorem in the last section).

(2) There is a difference between one-dimensional case and higher dimensional cases. When $n = 1$, one can easily find the Douady-Earle extension of the linear approximation $D(f)(\vec{p})$ of the boundary map f at a point $\vec{p}$ (if it is differentiable at that point), which is a conformal or anti-conformal automorphism of a half plane. This feature no longer holds for the extension of $D(f)(\vec{p})$ when $n > 1$. Because of this difference, the two expressions of L in the case of $n = 1$ can be easily observed to be equal, but not in other cases of $n \geq 1$.

Theorem 4 implies:

Theorem 5 (Global C^1 smoothness in any dimension)**.** *For any C^1 diffeomorphism f of $\mathbb{S}^n$, the Douady-Earle extension $\Phi(f)$ of f is a C^1 smooth map from $\overline{\mathbb{B}^{n+1}}$ to itself.*

Here a C^1 *smooth map* F from $\overline{\mathbb{B}^{n+1}}$ to itself can be defined in a similar way as the case when $n = 1$.

The paper is arranged as follows. In the second section, we show Theorems 1 and 2. We also present short proofs of Earle's Theorem and its analogue by arranging arguments with the maps defined on the upper half plane and the extended real line. Then in the third section, we prove Theorems 3, 4 and 5.

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2. Proof of Theorem 1

Because of the conformal naturality, we introduce the following normalization to the map f considered in Theorem 1. Pre-composed and post-composed by rotations around the origin, we may assume that $p = 1$ and $f(1) = 1$. Then followed by post-composition by a translation preserving $\mathbb{S}^1$ and fixing 1, we may assume that f fixes -1 too. Finally, post-composed by a hyperbolic transformation fixing -1 and 1, we may assume that $f'(1) = 1$ (resp. $f'(1) = -1$) if the derivative of f at 1 is positive (resp. negative). In summary, we may assume that $p = 1$, $f(1) = 1$, $f(-1) = -1$, and $f'(1) = 1$ or -1 (corresponding to an orientation preserving or reversing homeomorphism f).

In addition to the conformal naturality, a functional property of Douady-Earle extensions of circle homeomorphisms developed by Douady and Earle in [5] is another important key to develop proofs of our theorems.

Proposition 1 (Douady-Earle [5])**.** *Let $\mathcal{H}(\mathbb{S}^1)$ (respectively, $\mathcal{H}(\overline{\mathbb{D}})$) be the space of homeomorphisms of $\mathbb{S}^1$ (respectively, $\overline{\mathbb{D}}$) equipped with C^0-topology, and $\mathcal{D}iff(\mathbb{D})$ the space of diffeomorphisms of the open unit disk with C^∞-topology. Then the map $\Phi : \mathcal{H}(\mathbb{S}^1) \to \mathcal{D}iff(\mathbb{D}) \cap \mathcal{H}(\overline{\mathbb{D}})$ is continuous.*

As pointed out in the introduction, by arranging arguments and reasonings with corresponding maps defined on the extended real line $\hat{\mathbb{R}}$ and the upper half plane $\mathbb{H}$, we are able to provide quite simple proofs of Theorem 1, and Earle's Theorem and its analogue. In order to do so, we conjugate f and its Douady-Earle extension $\Phi(f)$ by a Möbius transformation between the unit disk $\mathbb{D}$ and the upper half plane $\mathbb{H}$. That is, let $h(z) = -i\frac{z-1}{z+1}$, $\tilde{f} = h \circ f \circ h^{-1}$ and $\Phi(\tilde{f}) = h \circ \Phi(f) \circ h^{-1}$. Then $\tilde{f}$ is a homeomorphism of $\mathbb{R}$, and $\tilde{f}(0) = 0$ and $\tilde{f}'(0) = 1$ or -1. We also call $\Phi(\tilde{f})$ the Douady-Earle extension of $\tilde{f}$. Note also that $h(0) = i$, and we will view i as a "center" of the hyperbolic plane $\mathbb{H}$. Based on these settings, we need the following corollary of the previous proposition.

Corollary 1. *Let $\mathcal{H}(\mathbb{R})$ be the space of homeomorphisms of $\mathbb{R}$ with the topology of uniform convergence of homeomorphisms on compact subsets of $\mathbb{R}$, and let $\mathcal{D}iff(\mathbb{H})$ be the space of diffeomorphisms of $\mathbb{H}$ with the topology of uniform convergence of homeomorphisms and all derivatives on compact subsets of $\mathbb{H}$. Then with respect to these two topologies, $f \mapsto \Phi(f) = h \circ \Phi(h^{-1} \circ f \circ h) \circ h^{-1}$ is continuous.*

The proof of Theorem 1 is reduced to show the following two propositions.

Proposition 2. *Let f be a homeomorphism of $\mathbb{R}$, and assume that f is C^1 in a neighborhood of 0, $f(0) = 0$ and $f'(0) = 1$. Let F denote the Douady-Earle extension $\Phi(f)$ of f. Then as a point z of $\mathbb{H}$ approaching 0,*

$$\lim_{z\to 0} \frac{F(z)}{z} = 1, \quad \lim_{z\to 0} \frac{\partial F(z)}{\partial z} = 1 \quad \textit{and} \quad \lim_{z\to 0} \frac{\partial F(z)}{\partial \bar{z}} = 0.$$

Proof. For brevity of notation, we use $\epsilon(t)$ to denote a quantity that approaches 0 as an involved real variable t goes to 0.

Let b be a positive real number and $m_b(z) = bz$. Given any real number a, let $t_a(z) = z + a$. We first show that

$$F(a + bi) = f(a) + bi + \epsilon(|a| + b)b. \tag{1}$$

where $\epsilon(|a| + b)$ goes to 0 as $|a| + b$ approaches 0. By the conformal naturality,

$$\frac{F(a + bi) - f(a)}{b} = \frac{\Phi(f) \circ t_a \circ m_b(i) - f(a)}{b}$$

$$= m_{1/b} \circ t_{-f(a)} \circ \Phi(f) \circ t_a \circ m_b(i) = \Phi(m_{1/b} \circ t_{-f(a)} \circ f \circ t_a \circ m_b)(i).$$

Let $f_{(a,b)} = m_{1/b} \circ t_{-f(a)} \circ f \circ t_a \circ m_b$. Then

$$\frac{F(a + bi) - f(a)}{b} = \Phi(f_{(a,b)})(i).$$

Now we rewrite the boundary map $f_{(a,b)}$ as

$$f_{(a,b)}(x) = \frac{f(a + bx) - f(a)}{b} = f'(a)x + \epsilon(|bx|)x. \tag{2}$$

Since f is C^1 in a neighborhood of 0 and $f'(0) = 1$, $f_{(a,b)}$ converges to the identity map Id uniformly on every compact subset of $\mathbb{R}$ as a and b go to 0. By Corollary 1, as a and b approach 0, we obtain that $\Phi(f_{(a,b)})(i)$ converges to $\Phi(Id)(i) = i$, and $\frac{\partial}{\partial z}\Phi(f_{(a,b)})(i)$ and $\frac{\partial}{\partial \bar{z}}\Phi(f_{(a,b)})(i)$ converge to $\frac{\partial}{\partial z}\Phi(Id)(i) = 1$ and $\frac{\partial}{\partial \bar{z}}\Phi(Id)(i) = 0$, respectively. It is clear that $\Phi(f_{(a,b)})(i)$ converging to i as a and b approaching 0 implies the estimate (1). Furthermore, applying the chain rule of taking derivatives to the right side of $\Phi(f_{(a,b)})(z) = \frac{F(a+bz)-f(a)}{b}$, we obtain

$$\frac{\partial}{\partial z}\Phi(f_{(a,b)})(z) = \frac{\partial}{\partial z}F(a+bz) \ \text{ and } \ \frac{\partial}{\partial \bar{z}}\Phi(f_{(a,b)})(z) = \frac{\partial}{\partial \bar{z}}F(a+bz).$$

Now by letting $z = i$, we can see that $\frac{\partial}{\partial z}F(a+bi)$ and $\frac{\partial}{\partial \bar{z}}F(a+bi)$ converge to 1 and 0 respectively as the same as $\frac{\partial}{\partial z}\Phi(f_{(a,b)})(i)$ and $\frac{\partial}{\partial \bar{z}}\Phi(f_{(a,b)})(i)$ when a and b go to 0.

It remains to show that $\frac{F(a+bi)}{a+bi}$ converges to 1 as $a+bi$ goes to 0. Using again the assumption of f in a neighborhood of 0, the estimate of $F(a+bi)$ in (1) can be further expressed as

$$F(a+bi) = a + \epsilon(a)a + bi + \epsilon(|a|+b)b = a + bi + \epsilon(a)a + \epsilon(|a|+b)b. \tag{3}$$

Thus

$$\frac{F(a+bi)}{a+bi} = 1 + \frac{\epsilon(a)a + \epsilon(|a|+b)b}{a+bi}.$$

Clearly,

$$|\frac{\epsilon(a)a + \epsilon(|a|+b)b}{a+bi}| = |\frac{\epsilon(a)a + \epsilon(|a|+b)b}{\sqrt{a^2+b^2}}| \leq |\epsilon(a)| + |\epsilon(|a|+b)|,$$

which converges to 0 as $a+bi$ approaches 0. By letting $z = a+bi$, we have shown that $\lim_{z\to 0}\frac{F(z)}{z} = 1$. We complete the proof. ☐

Proposition 3. *Let f be a homeomorphism of $\mathbb{R}$, and assume that f is C^1 in a neighborhood of 0, $f(0) = 0$ and $f'(0) = -1$. Let F denote the Douady-Earle extension $\Phi(f)$ of f. Then as a point z of $\mathbb{H}$ approaching 0,*

$$\lim_{z\to 0}\frac{F(z)}{\bar{z}} = -1, \ \lim_{z\to 0}\frac{\partial F(z)}{\partial \bar{z}} = -1 \ \text{ and } \ \lim_{z\to 0}\frac{\partial F(z)}{\partial z} = 0.$$

Proof. Under the same notation introduced in the proof of the previous proposition, one can see that the expression of the boundary map $f_{(a,b)}$ given in (2) now implies that $f_{(a,b)}$ converges to $-Id$ uniformly on every compact subset of $\mathbb{R}$ as a and b approach 0. One extra work to complete the proof of this proposition is to find the Douady-Earle extension of $-Id$. Let $f = -Id$ and $h(z) = -i\frac{z-1}{z+1}$. Then $g(z) = h^{-1}\circ f\circ h(z) = \frac{1}{z} = \bar{z}$ since $z \in \mathbb{S}^1$. Using the definition of the conformal barycenter, one can easily verify that $\Phi(g)(0) = 0$. For any point $z_0 \in \mathbb{D}$, let $g_{z_0}(z) = \frac{z-z_0}{1-\bar{z}_0 z}$. Then $g_{\bar{z}_0}\circ g\circ g_{z_0}^{-1}(z) = \bar{z}$ for any $z \in \mathbb{S}^1$. Thus

$$\Phi(g_{\bar{z}_0}\circ g\circ g_{z_0}^{-1})(0) = 0.$$

Using the conformal naturality, we have shown that $\Phi(g)(z_0) = \bar{z}_0$ for any $z_0 \in \mathbb{D}$. It follows that $\Phi(f)(z) = h\circ\Phi(g)\circ h^{-1}(z) = -\bar{z}$ for each z in the upper half plane $\mathbb{H}$.

The rest proof of this proposition follows the exact same strategies to complete the proof of the previous proposition. ☐

The proofs of Earle's Theorem and its analogue are reduced to show the following two propositions.

Proposition 4. *Let f be a homeomorphism of $\mathbb{R}$, and assume that $f(0) = 0$ and $f'(0) = 1$. Let F denote the Douady-Earle extension $\Phi(f)$ of f. Then given any real number $M > 0$, if a point $z = a + bi$ of $\mathbb{H}$ approaching 0 with $|a/b| \leq M$, then*

$$\lim_{z\to 0} \frac{F(z)}{z} = 1, \ \lim_{z\to 0} \frac{\partial F(z)}{\partial z} = 1 \ \textit{and} \ \lim_{z\to 0} \frac{\partial F(z)}{\partial \bar{z}} = 0.$$

Proof. Let us use the same notation as introduced in the proof of Proposition 2. Using the expression (2) of $f_{(a,b)}$, we rewrite

$$\begin{aligned} f_{(a,b)}(x) &= \frac{f(a+bx) - f(a)}{b} \\ &= \frac{[a + bx + \epsilon(|a+bx|)(a+bx)] - [a + \epsilon(|a|)a]}{b} \\ &= x + \epsilon(|a+bx|)(a/b + x) - \epsilon(|a|)a/b. \end{aligned}$$

Now we can see that if both a and $b > 0$ approach 0 with $|a/b| \leq M$, then $f_{(a,b)}$ converges to the identity map on every compact subset of $\mathbb{R}$. The rest of the proof is exactly same as presented in the proof of the previous proposition with $z = a+bi$ approaching 0 in the upper half plane arbitrarily replaced by $z = a+bi$ approaching 0 with $|a/b| \leq M$. To save space, we skip it. □

Proposition 5. *Let f be a homeomorphism of $\mathbb{R}$, and assume that $f(0) = 0$ and $f'(0) = -1$. Let F denote the Douady-Earle extension $\Phi(f)$ of f. Then given any real number $M > 0$, if a point $z = a + bi$ of $\mathbb{H}$ approaching 0 with $|a/b| \leq M$, then*

$$\lim_{z\to 0} \frac{F(z)}{\bar{z}} = -1, \ \lim_{z\to 0} \frac{\partial F(z)}{\partial \bar{z}} = -1 \ \textit{and} \ \lim_{z\to 0} \frac{\partial F(z)}{\partial z} = 0.$$

Proof. The proof follows the exact same strategies to show the previous proposition after rewriting the boundary map $f_{(a,b)}$ in this case as follows:

$$\begin{aligned} f_{(a,b)}(x) &= \frac{f(a+bx) - f(a)}{b} \\ &= \frac{[-(a + bx) + \epsilon(|a+bx|)(a+bx)] - [-a + \epsilon(|a|)a]}{b} \\ &= -x + \epsilon(|a+bx|)(a/b + x) - \epsilon(|a|)a/b. \end{aligned}$$

□

Proof of Theorem 2. For each point z on the extended complex plane, let z^* be the mirror image of z with respect to $\mathbb{S}^1$, that is, $z^* = 1/\bar{z}$. Then extend $F = \Phi(f)$ to the exterior of the unit disk by defining $F(z) = (F(z^*))^*$ for each z outside the unit disk $\overline{\mathbb{D}}$. Using Theorem 1, one can see that the extended map F is a diffeomorphism of the extended complex plane. □

3. Proofs of Theorems 3, 4 and 5

Let n be a positive integer, $\mathbb{S}^n$ the unit sphere in $\mathbb{R}^{n+1}$ centered at the origin, and $\mathbb{B}^{n+1}$ (resp. $\overline{\mathbb{B}^{n+1}}$) the open (resp. closed) unit ball in $\mathbb{R}^{n+1}$ bounded by $\mathbb{S}^n$. Assume that f is a homeomorphism of $\mathbb{S}^n$ and $\Phi(f)$ is the Douady-Earle extension of f. Since $\Phi(f)$ is not necessarily a homeomorphism of $\overline{\mathbb{B}^{n+1}}$ when $n \geq 2$, the functional property of $f \mapsto \Phi(f)$ for $n = 1$ needs to be modified to work for extensions in all cases.

Proposition 6 (Douady-Earle [5]). *Let $\mathcal{H}(\mathbb{S}^n)$ be the space of homeomorphisms of $\mathbb{S}^n$ equipped with C^0-topology, $\mathcal{C}(\overline{\mathbb{B}^{n+1}})$ the space of continuous maps from $\overline{\mathbb{B}^{n+1}}$ into itself with C^0-topology, and $\mathcal{C}^\infty(\mathbb{B}^{n+1})$ the space of smooth maps from $\mathbb{B}^{n+1}$ into itself with C^∞-topology. Then the map $\Phi : \mathcal{H}(\mathbb{S}^n) \to \mathcal{C}^\infty(\mathbb{B}^{n+1}) \cap \mathcal{C}(\overline{\mathbb{B}^{n+1}})$ is continuous.*

Let $\vec{e}_k$ be the unit vector of $\mathbb{R}^{n+1}$ with the k^{th} entry equal to 1, where $1 \leq k \leq n+1$. We also introduce a normalization for the homeomorphism f considered in Theorem 3 or 4. That is, pre-composed and post-composed by rotations around the origin, we may assume that $\vec{p} = \vec{e}_1$ and $f(\vec{e}_1) = \vec{e}_1$; then followed by post-composition by a translation fixing $\vec{e}_1$, we may further assume that f fixes $-\vec{e}_1$. Let $\mathbb{H}^{n+1}$ be the upper half space of $\mathbb{R}^{n+1}$ consisting of all points of $\mathbb{R}^{n+1}$ with positive last entries. Let h be a conformal map from $\mathbb{B}^{n+1}$ onto $\mathbb{H}^{n+1}$ such that $h(\vec{0}) = \vec{e}_{n+1}$, $h(\vec{e}_1) = \vec{0}$ and $h(-\vec{e}_1) = \infty$. Let $\mathbb{R}^n$ be the subspace of $\mathbb{R}^{n+1}$ consisting of all points of $\mathbb{R}^{n+1}$ with last entries equal to 0. Assume that $\tilde{f} = h \circ f \circ h^{-1}$ and $\Phi(\tilde{f}) = h \circ \Phi(f) \circ h^{-1}$. Then $\tilde{f}$ is a homeomorphism of $\mathbb{R}^n$, and $\tilde{f}(\vec{0}) = \vec{0}$ and $\Phi(\tilde{f})$ is called *the Douady-Earle extension* of $\tilde{f}$. Similar to the one dimensional case, we need the following corollary.

Corollary 2. *Let $\mathcal{H}(\mathbb{R}^n)$ be the space of homeomorphisms of $\mathbb{R}^n$ with the topology of uniform convergence of homeomorphisms on compact subsets of $\mathbb{R}^n$, and $\mathcal{H}(\mathbb{H}^{n+1})$ the space of smooth maps from $\mathbb{H}^{n+1}$ to $\mathbb{H}^{n+1}$ with the topology of uniform convergence of maps and all derivatives on compact subsets of $\mathbb{H}^{n+1}$. Then with respect to these two topologies, $f \mapsto \Phi(f) = h \circ \Phi(h^{-1} \circ f \circ h) \circ h^{-1}$ is continuous.*

The proof of Theorem 4 is reduced to prove the following proposition.

Proposition 7. *Let f be a homeomorphism of $\mathbb{R}^n$, and assume that f is C^1 in a neighborhood of $\vec{0}$, $f(\vec{0}) = \vec{0}$ and $Df(\vec{0})$ is non-singular. Let F denote the Douady-Earle extension $\Phi(f)$ of f. Then there exists a linear map L from $\mathbb{R}^{n+1}$ into itself preserving the upper half space $\mathbb{H}^{n+1}$ of $\mathbb{R}^{n+1}$ such that*

$$F(\vec{x}) = L(\vec{x}) + o(||\vec{x}||)$$

as a point $\vec{x}$ of $\mathbb{H}^{n+1}$ approaches $\vec{0}$. Furthermore, $DF(\vec{x})$ converges to L as $\vec{x}$ approaches $\vec{0}$.

Different from the proof of Proposition 2, we need the following lemma to prove the continuity of differentiation at $\vec{0}$, from which we also obtained a second expression of the linear map L in Proposition 7.

Lemma 1. *Consider $\mathbb{R}^n$ as the boundary of $\mathbb{H}^{n+1}$ in $\mathbb{R}^{n+1}$. Assume that a map $F : \mathbb{H}^{n+1} \cup \mathbb{R}^n \to \mathbb{H}^{n+1} \cup \mathbb{R}^n$ is continuous and C^1 on $\mathbb{H}^{n+1}$. If $DF(\vec{x})$ converges to L as a point $\vec{x}$ of $\mathbb{H}^{n+1}$ goes to $\vec{0}$, then F has a linear approximation L at $\vec{0}$ from interior; that is, for any $\vec{x} \in \mathbb{H}^{n+1}$,*

$$F(\vec{x}) - F(\vec{0}) = L\vec{x} + o(||\vec{x}||).$$

Proof. Given any point $\vec{x} \in \mathbb{H}^{n+1}$, let $\vec{s}(t) = t\vec{x}$, $t \in [0,1]$, be the line segment between $\vec{0}$ and $\vec{x}$. Then by the fundamental theorem of calculus,

$$F(\vec{x}) - F(\vec{0}) = \int_0^1 \frac{d}{dt} F(\vec{s}(t))dt.$$

Clearly, $\frac{d}{dt}F(\vec{s}(t)) = DF(\vec{s}(t))\vec{x}$, which is close to $L\vec{x}$ for any $t \in [0,1]$ as $||\vec{x}||$ is small. Precisely,

$$DF(\vec{s}(t))\vec{x} = L\vec{x} + \epsilon(||\vec{x}||)||\vec{x}||$$

for any $t \in [0,1]$. Thus

$$F(\vec{x}) - F(\vec{0}) = L\vec{x} + o(||\vec{x}||).$$

□

Proof of Proposition 7. Let b be a positive real number and $m_b(\vec{x}) = b\vec{x}$, where $\vec{x} \in \mathbb{R}^{n+1}$. Given a point $\vec{a} \in \mathbb{R}^n$, let $t_{\vec{a}}(\vec{x}) = \vec{x} + \vec{a}$, where $\vec{a}$ is also viewed as a point of $\mathbb{R}^{n+1}$ with the last entry equal to 0. We first show that

$$F(\vec{a} + b\vec{e}_{n+1}) = f(\vec{a}) + b\Phi(Df(\vec{0}))(\vec{e}_{n+1}) + \epsilon(||\vec{a}|| + b)b, \tag{4}$$

where $f(\vec{a})$ is also viewed as a point of $\mathbb{R}^{n+1}$ with the last entry equal to 0, and $\epsilon(||\vec{a}|| + b)$ means a quantity approaching 0 as $|a| + b$ approaching 0.

By the conformal naturality,

$$\frac{F(\vec{a} + b\vec{e}_{n+1}) - f(\vec{a})}{b} = \Phi(m_{1/b} \circ t_{-f(\vec{a})} \circ f \circ t_{\vec{a}} \circ m_b)(\vec{e}_{n+1}).$$

Let $f_{(\vec{a},b)} = m_{1/b} \circ t_{-f(\vec{a})} \circ f \circ t_{\vec{a}} \circ m_b$. Then

$$\frac{F(\vec{a} + b\vec{e}_{n+1}) - f(\vec{a})}{b} = \Phi(f_{(\vec{a},b)})(\vec{e}_{n+1}).$$

Now we rewrite the boundary map $f_{(\vec{a},b)}$ as

$$f_{(\vec{a},b)}(\vec{x}) = \frac{f(\vec{a} + b\vec{x}) - f(\vec{a})}{b} = Df(\vec{a})\vec{x} + \epsilon(||b\vec{x}|)\vec{x}. \tag{5}$$

Since f is C^1 in a neighborhood of $\vec{0}$ in $\mathbb{R}^n$ and $Df(\vec{0})$ is non-singular, $f_{(\vec{a},b)}$ converges to the linear map $Df(\vec{0}) : \vec{x} \mapsto Df(\vec{0})\vec{x}$ uniformly on every compact subset of $\mathbb{R}^n$ as $\vec{a} \to \vec{0}$ and $b \to 0$. From Corollary 2, it follows that as $\vec{a} \to \vec{0}$ and $b \to 0$, $\Phi(f_{(\vec{a},b)})(\vec{e}_{n+1})$ converges to $\Phi(Df(\vec{0}))(\vec{e}_{n+1})$ and $D\Phi(f_{(\vec{a},b)})(\vec{e}_{n+1})$ converges to $D\Phi(Df(\vec{0}))(\vec{e}_{n+1})$. The first convergence implies the designed estimate (4). Using the assumption of f in a neighborhood of $\vec{0}$ in $\mathbb{R}^n$, this estimate can be further written as

$$\begin{aligned} & F(\vec{a} + b\vec{e}_{n+1}) \\ = \;& f(\vec{0}) + Df(\vec{0})\vec{a} + b\Phi(Df(\vec{0}))(\vec{e}_{n+1}) + \epsilon(||\vec{a}||)||\vec{a}|| + \epsilon(||\vec{a}|| + b)b \\ = \;& F(\vec{0}) + L(\vec{a} + b\vec{e}_{n+1}) + \epsilon(||\vec{a}||)||\vec{a}|| + \epsilon(||\vec{a}|| + b)b, \end{aligned}$$

where L is an $(n+1) \times (n+1)$ matrix with the upper left $n \times n$ square block equal to $Df(\vec{0})$, the most right column equal to $\Phi(Df(\vec{0}))(\vec{e}_{n+1})$, and all other entries equal to 0. That is,

$$L = \begin{bmatrix} Df(\vec{0}) & \\ 0 \cdots 0 & \end{bmatrix} \Phi(Df(\vec{0}))(\vec{e}_{n+1}) \Bigg]. \tag{6}$$

Since $\Phi(Df(\vec{0}))(\vec{e}_{n+1})$ is a vector in the upper half space $\mathbb{H}^{n+1}$, the linear map L maps $\mathbb{H}^{n+1}$ into itself. One can easily check that

$$\epsilon(||\vec{a}||)||\vec{a}|| + \epsilon(||\vec{a}|| + b)b = \epsilon(\sqrt{||\vec{a}||^2 + b^2})\sqrt{||\vec{a}||^2 + b^2}.$$

Thus, F is differentiable at $\vec{0}$ and $DF(\vec{0}) = L$. Since $\Phi(Df(\vec{0}))(\vec{e}_{n+1})$ is a vector in the half space of $\mathbb{R}^{n+1}$ with positive last entry, the determinant of L has the same sign as $Df(\vec{0})$.

It remains to show that $DF(\vec{x})$ converges to $DF(\vec{0})$ as a point $\vec{x}$ of $\mathbb{H}^{n+1}$ approaches $\vec{0}$. By the conformal naturality,

$$\Phi(f_{(\vec{a},b)})(\vec{x}) = \frac{F(\vec{a}+b\vec{x}) - f(\vec{a})}{b}.$$

Using the chain rule of differentiation to the right hand side of the previous expression, we obtain

$$D\Phi(f_{(\vec{a},b)})(\vec{x}) = DF(\vec{a}+b\vec{x}).$$

Now by letting $\vec{x} = \vec{e}_{n+1}$,

$$D\Phi(f_{(\vec{a},b)})(\vec{e}_{n+1}) = DF(\vec{a}+b\vec{e}_{n+1}).$$

We have shown that $D\Phi(f_{(\vec{a},b)})(\vec{e}_{n+1})$ converges to $D\Phi(Df(\vec{0}))(\vec{e}_{n+1})$ as $\vec{a} \to \vec{0}$ and $b \to 0$. Therefore, by letting $\vec{x} = \vec{a} + b\vec{e}_{n+1}$, we conclude that $DF(\vec{x})$ converges to $D\Phi(Df(\vec{0}))(\vec{e}_{n+1})$ as $\vec{x}$ goes to $\vec{0}$ in $\mathbb{H}^{n+1}$. Using Lemma 1, we can also conclude that F is differentiable at $\vec{0}$ with $DF(\vec{0}) = D\Phi(Df(\vec{0}))(\vec{e}_{n+1})$.

Finally, by the uniqueness of the differentiability of F at $\vec{0}$, we conclude

$$DF(\vec{0}) = L = D\Phi(Df(\vec{0}))(\vec{e}_{n+1}). \tag{7}$$

□

Remark 3. Using the same notation introduced in the previous proof, we can see that in the case of dimension one, the linear map $Df(0)$ is

$$Df(0) : \mathbb{R} \to \mathbb{R} : x \mapsto f'(0)x.$$

If $f'(0) > 0$ (respectively, $f'(0) < 0$), then $\Phi(Df(0))$ is

$$\Phi(Df(0)) : \mathbb{H} \to \mathbb{H} : z \mapsto f'(0)z \;\; \text{(respectively, } \; f'(0)\bar{z}).$$

Thus

$$\Phi(Df(0))(\vec{e}_2) = \begin{pmatrix} 0 \\ f'(0) \end{pmatrix} \;\; \text{(respectively,} \begin{pmatrix} 0 \\ -f'(0) \end{pmatrix}).$$

Therefore

$$L = D\Phi(Df(0))(\vec{e}_2) = \begin{bmatrix} f'(0) & 0 \\ 0 & f'(0) \end{bmatrix} \;\; \text{(respectively,} \begin{bmatrix} f'(0) & 0 \\ 0 & -f'(0) \end{bmatrix}).$$

Clearly, L represents a conformal (respectively, anti-conformal) automorphism of $\mathbb{H}$ in this case. But in higher dimensional cases, the first expression (6) of L tells us that L is not conformal (respectively, anti-conformal) if $Df(\vec{0})$ is not so.

The proof of Theorem 3 is reduced to show the following proposition.

Proposition 8. *Let f be a homeomorphism of $\mathbb{R}^n$, and assume that $f(\vec{0}) = \vec{0}$, f is differentiable at $\vec{0}$ and $Df(\vec{0})$ is non-singular. Let F denote the Douady-Earle extension $\Phi(f)$ of f and $\vec{a} + b\vec{e}_{n+1}$ represent a point $\mathbb{H}^{n+1}$ with $\vec{a}$ a point in $\mathbb{R}^n$ (equivalent to a point in $\mathbb{R}^{n+1}$ with the last entry to be 0. Then for any positive M, if a point $\vec{x} = \vec{a} + b\vec{e}_{n+1}$ of $\mathbb{H}^{n+1}$ approaches $\vec{0}$ with $\frac{||\vec{a}||}{b} \leq M$, then*

$$F(\vec{x}) = L(\vec{x}) + o(||\vec{x}||),$$

where $L = D\Phi(Df(0))(\vec{e}_{n+1})$, and furthermore, $DF(\vec{x})$ converges to L.

Proof. Under the same notation used in the proof of the previous proposition, we rewrite

$$\begin{aligned} f_{(\vec{a},b)}(\vec{x}) &= \frac{f(\vec{a}+b\vec{x})-f(\vec{a})}{b} \\ &= \frac{[Df(\vec{0})(\vec{a}+b\vec{x})+\epsilon(||\vec{a}+b\vec{x}||)||\vec{a}+b\vec{x}||]-[Df(\vec{0})\vec{a}+\epsilon(||\vec{a}||)||\vec{a}||]}{b} \\ &= Df(\vec{0})\vec{x}+\epsilon(||\vec{a}+b\vec{x}|)||\vec{a}/b+\vec{x}||-\epsilon(||\vec{a}||)||\vec{a}||/b. \end{aligned}$$

Clearly, if both $||\vec{a}||$ and $b>0$ approach 0 with $||\vec{a}||/b \leq M$, then $f_{(\vec{a},b)}$ converges to the linear map $Df(\vec{0}) : \mathbb{R}^n \to \mathbb{R}^n$ on every compact subset of $\mathbb{R}^n$. The rest of the proof is exactly same as presented in the proof of the previous proposition with $\vec{x} = \vec{a} + b\vec{e}_{n+1}$ approaching $\vec{0}$ in the upper half space $\mathbb{H}^{n+1}$ arbitrarily replaced by $\vec{x} = \vec{a} + b\vec{e}_{n+1}$ approaching $\vec{0}$ with $||\vec{a}||/b \leq M$. Again to save space, we skip it. □

Proof of Theorem 5. Similar to the reflection construction used in the proof of Theorem 2, we can extend $\Phi(f)$ to a continuous map F of the extended $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1} \cup \{\infty\}$. Using our Theorem 4, we conclude that F is a C^1 smooth map from $\mathbb{R}^{n+1} \cup \{\infty\}$ to itself. □

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Department of Mathematics, Brooklyn College of CUNY, Brooklyn, NY 11210, and, Ph.D. Program in Mathematics, Graduate Center of CUNY, 365 Fifth Avenue, New York, NY 10016

E-mail address: junhu@brooklyn.cuny.edu or JHu1@gc.cuny.edu

Department of Mathematics, Rutgers University, New Brunswick, NJ 08854

E-mail address: susovan97@gmail.com