BOUNDARY AND HOLDER REGULARITIES OF DOUADY-EARLE EXTENSIONS AND EIGENVALUES OF LAPLACE OPERATORS ACTING ON RIEMANN SURFACES

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ABSTRACT OF THE DISSERTATION

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Douady-Earle extensions of homeomorphisms of the unit circle are of particular interest in understanding contractibility and complex structures of Teichmueller and assymptotic Teichmueller spaces. Motivated by questions in analysis and partial differential equations, one can ask how regular the Douady-Earle extensions can be on the closed unit disk if one puts sufficient regularity on the circle homeomorphisms to start with. In part I of this thesis, we prove that Douady-Earle extensions of Holder continuous circle homeomorphisms are Holder continuous with the same Holder exponent, and Douady-Earle extensions of circle diffeomorphisms are diffeomorphisms of the closed unit disk. Eigenvalues of Laplace operators on Riemannian manifolds are widely studied by differential geometers. But when the manifold is a hyperbolic Riemann surface, the problem becomes more special, because the collar lemma and the minimax principles allow us to construct functions which produce lower and upper bounds on eigenvalues on that Riemann surface. In part II of this thesis, we show, using the minimax principles, given any small positive number ϵ and given any big natural number k, we can construct a Riemann surface whose k-th eigenvalue is less than ϵ. The result was first proved by

Burton randol, here we provide a much simpler and geometric proof

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Dedication

The thesis is dedicated to my parents, to my Father for inspiring me to like Mathematics at a young age, and my Mother for her constant encouragement in my academic career.

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Part I

Regularities of Douady-Earle extensions

Chapter 1

Introduction and Preliminaries

In this chapter, we will define and give examples for some necessary concepts, which will be needed later to understand the main mathematical results of our topic. In particular, we will be defining several types of mappings: holomorphic, conformal, quasiconformal, quasisymmetric, harmonic mappings between open subsets of the complex plane, and whenever necessary, between Riemann surfaces. In all the upcoming defintions, assume that U, V are connected open subsets of the complex plane, unless otherwise mentioned. Also, we will use the words maps and homeomorphims synonymously.

Definition (Holomorphic functions). A function f from an open subset U of $\mathbb{C}$ to $\mathbb{C}$ is called holomorphic at a point $a \in U$ if for the limit $\lim_{z \to a} \frac{f(z)-f(a)}{z-a}$ exists. We call the limit $f'(a)$ to be the derivative of f and call the function f to be complex-diffrentiable or holomorphic at a. If f is holomorphic at every $a \in U$, we call f to be holomorphic in U.

Definition (Conformal maps and conformal automorphisms): A map $f : U \to V$ is called conformal if f is a homeomorphism, f is holomorphic and $f'(b)$ is never zero at any b in U. In this case, we also say, U, V are biholomorphic. If $U = V$, we say f is a conformal automorphism of U. One can prove that when $U = \mathbb{D}$, conformal automorphisms of $U = \mathbb{D}$ are precisely the maps of the form $f(z) = e^{i\theta} \frac{z-a}{1-\bar{a}.z}$

Definition (Riemann surface). A topological 2 (real) dimensional manifold whose transitions functions are conformal between open subsetf of the complex plane. So, a Riemann surface is a 1-dimensional complex manifold.

Definition (Quasiconformal homeomormorphisms). A map $f : U \to V$ is called quasiconformal if:

a) f has distributional derivatives which are locally square-integrable on U, i.e. $f \in$

$L^2_{loc}(U)$.

b) $|f_{\bar{z}|} \leq k|f_z|$ almost everywhere for some $0 < k < 1$.

Note that, a conformal homeomorphism, by defintition is a quasiconformal one, since f is infinitely differentiable, satisfying condition a), and by Cauchy-Riemann equation for holomorphicity, $f_{\bar{z}} = 0$ for all z, satisfying condition b). From a geometric point of view, a conformal homeomorphism sends infinitesinally small circles to infinitesimally small circles, but (strongly) differentiable quasiconformal homeomorphisms sends infinitesinally small circles to infinitesimally small ellipses with bounded eccentricities, so geometrically, they are the next best kind of maps we can get after conformality.

Also note that, the above conditions a) and b) being chart-independent, we can define a quasiconformal map between Riemann surfaces just by defining it to be quasiconformal between any two local charts of the surfaces.

Definition (Quasisymmetric homeomorphisms). A map $f : \mathbb{R} \to \mathbb{R}$ or $f : \mathbb{S}^1 \to \mathbb{S}^1$ is called quasisymmetric if given four distinct points a, b, c, d in $\mathbb{S}^1$ or $\mathbb{R}$ with cross-ratio 1, the cross-ratio of $f(a), f(b), f(c), f(d)$ is uniformly bounded.

Definition (Extension operators on homeomorphims of the unit circle $\mathbb{S}^1$): Any operator E from the space of homeorphisms of $\mathbb{S}^1$ to homeomorphims of $\overline{\mathbb{D}}$ will be called an extension operator. there are several such operators, namely the complex harmonic extension, the Beurling-Ahlfors extension, the Douady-Earle extension, among many others. For their definitions and a brief account, please see the next chapter.

Definition (Conformally natural extension operators). An extension operator E from the space of homeomorphims of $\mathbb{S}^1$ to the space of homeomorphisms of $\overline{\mathbb{D}}$ is called conformally natural if for every homeomorphism f of $\mathbb{S}^1$, and for every pair of conformal automorphisms A, B of $\mathbb{D}$, we have $E(A \circ f \circ B) = A \circ E(f) \circ B$. The property of conformal naturality is crucial in proving the contractibility of Teichmueller spaces, which is the space of quasiconformally equivalent Riemann surfaces modulo isotopy. The best know example of such an operator is the Douady-Earle extension operator, whose Holder and boundary regularity is the main topic of our work. For a brief introduction to Douady-Earle extenions, please consult chapter 2, and a for a detiled account, please consult the original paper by Profs. Adrien Douady and Clifford Earle

[8].

Chapter 2

Introduction to different extensions of circle homeomorphims and their properties

The problem of finding quasiconformal extension to $\overline{\mathbb{D}}$ of quasi-symmetric homeomorphism of the unit circle $\mathbb{S}^1$ has been studied for a long time. The first work in this direction was done by Beurling and Ahlfors in [3], where they constructed an explicit example of such a homeomorphism. While Beurling-Ahlfors extensions have the advantages of being very explicit, and hence its regularity properties can be easily investigated, they are not conformally natural, which is a very useful feature of Douady-Earle extensions. But Douady-Earle extensions are harder to define and so their regularity properties are not so easy to understand. Both of the above extension operators send quasisymmetric circle maps to quasiconformal maps of $\mathbb{D}$. Another extension operator that is studied widely in analysis and potential theory is complex harmonic extension, which sends homeomorphims of circle to homeomorphisms of the closed disk, but it is not conformally natural, and does not send quasisymmetric maps to quasiconformal maps in general.

2.1 The Beurling-Ahlfors extension

Let $h : \mathbb{R} \to \mathbb{R}$ be an orientation-preserving homeomorphism of the real line. Define $F = F(h) : \mathbb{H} \to \mathbb{H}$ by

$$F(x + iy) = u(x, y) + iv(x, y),$$

where:

$$u(x,y) = \frac{1}{2y}\int_{-y}^{y} h(x+t)dt,$$

$$v(x,y) = \frac{1}{2y}\int_{0}^{y} (h(x+t) - h(x-t))dt.$$

It is clear that $v(x,y) \geq 0$, and $v(x,y) \to 0$ as $y \to 0$. Moreover, $u(x,0) = h(x)$, so $F(h)$ is a continuous extension of h to the upper half plane $\mathbb{H}$. The oparator F has the following properties:

1) If h is a homeomorphims of $\mathbb{R}$, $F(h)$ is a homeomorphism of $\overline{\mathbb{H}}$.

2) If h is a C^1 diffeomorphism of $\mathbb{R}$, then $F(h)$ is a C^1 diffeomorphim of $\overline{\mathbb{H}}$.

3) If h is M-quasisymmetric, then $F(h)$ is K-quasiconformal, with K depending only on M.

4) As we mentioned before F is not conformally natural extension.

2.2 The complex harmonic extension

The complex harmonic extension can be defined for any continuous function on the circle that takes value in the complex plane. More precisely, Let $f : \mathbb{S}^1 \to \mathbb{C}$ be continuous. Then its complex harmonic extension $H(f)$ is the solution to the Dirichlet problem of finding the unique harmonic function with the prescribed boundary value f. More explicitly, it is given by: $H(f)(z) := \int_{\mathbb{S}^1} f(t)p(z,t)|dt|$, where $p(z,t) = \frac{1}{2\pi}\cdot\frac{1-|z|^2}{|z-t|^2}$ is the Poisson kernel. The extension $H(f)$ satisfies the following properties:

1) Poisson kernel itself being harmonic function, $H(f)$ is also harmonic.

2) If f is a homeomorphism of $\mathbb{S}^1$, then $H(f)$ is a homeomorphism of $\overline{\mathbb{D}}$.

3) $H(f)$ is always real-analytic in $\mathbb{D}$.

4) $H(f)$ is not necessarily quasiconformal, even if f is quasisymmetric, in fact $H(f)$ is quasiconformal if and only if The Hilbert transform of the derivative of f is in $L^\infty(\mathbb{S}^1)$.

5) H is only conformally natural from the right hand side, i.e. $H(f \circ B) = H(f) \circ B$ for every conformal automorphism B of $\mathbb{D}$.

2.3 The Douady-Earle extension

Finally, we come to the most important extension operator, namely the Douady-earle Extension operator. Douady-Earle extensions or the barycentric extensions of circle homeomorphisms are indispensable tools for studying the topological and complex structure of the Teichmueller space and the asymptotic Teichmueller space of a Riemann surface, which are the spaces of possible quasiconformally equivalent complex structures on that Riemann surface modulo isotopy fixing the ideal boundary of that Riemann surface [13]. In particular, they are used to show that the Teichmueller spaces are all contractible [8], to obtain alternate characterizations of equivalence in the definitions of asymptotic Teichmueller spaces ([11], [10], [13]). But it is quite interesting that the definitions of Douady-Earle extensions do not involve any of the above spaces. Below, we briefly point out the definition, following [8], of the Douady-earle extension $\Phi(f)$ of a given homeomorphism f of $\mathbb{S}^1$. For a non-atomic probability measure μ on S^1, the vector field V_μ given by:

$$V_\mu(w) = \int_{S^1} \frac{t-w}{1-\bar{w}t} d\mu(t)$$

has a unique zero in $\mathbb{D}$ and hence the conformal barycenter $B(\mu)$ of the measure μ is defined to be the unique zero of the vector field V_μ. Next, the Douady-Earle/Barycentric extension $\Phi(f)$ of f is defined to be : $\Phi(f)(z) = B(f_*\eta_z)$, where η_z is the harmonic measure on S^1 with respect to $z \in \mathbb{D}$ and $f_*\eta_z$ is its pull back by f. Stated more explicitly, $\Phi(f)(z)$ is the unique w such that the integral $\int_{S^1} \frac{f(t)-w}{1-\bar{w}.f(t)} p(z,t)|dt| = 0$, where $p(z,t) = \frac{1}{2\pi} . \frac{1-|z|^2}{|z-t|^2}$ is the Poisson kernel which appears in the context of the complex harmonic extension of a continuous function defined on S^1.

Douady-Earle extension has the following important properties:

(1) Conformal naturality: $\Phi(A \circ f \circ B) = A \circ \Phi(f) \circ B \; \forall A, B \in G = Aut(\mathbb{D})$.

(2) Continuity: let $f_n \to f$ uniformly on $\mathbb{S}^1$, then $\Phi(f_n)$ and all its derivatives converge

pointwise to those of $\Phi(f)$.

(3) $\Phi(f)$ is continuous upto the boundary of $\mathbb{D}$, in fact it is a homeomorphism of $\overline{\mathbb{D}}$.

(4) $\Phi(f)|_{\mathbb{D}}$ is real-analytic diffeomorphism of $\mathbb{D}$.

(5) If f is quasisymmetric (respectively, symmetric), then $\Phi(f)$ is quasiconformal (respectively, assymptotically conformal).

(6) If f is the restriction of a biholomorphic automorphism F of $\mathbb{D}$, then $\Phi(f) = F$. In particular, $\Phi(Id) = Id$.

(7) $\Phi(f)$ is bi-Lipchitz with respect to the Poincare (=hyperbolic) metric on $\mathbb{D}$.

For all the above properties, consult [8] and [11].

In the next two chapters, which form the main part of this thesis, we will prove several boundary regularity properties of Douady-Earle extensions.

Chapter 3

Douady-Earle extensions of Holder continuous and Lipchitz continuous circle homeomorphisms

3.1 Introduction

In this chapter, we prove that Douady-Earle extension $\Phi(f)$ of a Holder continuous circle homeomorphism f is Holder continuous with the same Holder exponent. We also provide some boundary estimates along the radial direction of $\Phi(f)$ when f is Lipchitz. As will be clear later in the chapter, our considerations are highly motivated by the same regularity questions for complex harmonic extensions, which exhibits the same boundary regularity if one starts with Holder continuous or Lipchitz continuous circle maps.

Given any operator sending homeomorphisms of $\mathbb{S}^1$ to homeomorphisms of $\overline{\mathbb{D}}$, it is natural to ask the question: if we put further regularity conditions on the first one, whether we can expect the same for the second one? This question has been largely treated by analysts (see [21]). For complex harmonic extensions, where it is well-known that the complex harmonic extension of a $C^{k,\alpha}$ circle map is also $C^{k,\alpha}, 0 < \alpha < 1$. The corresponding theorem is called **Kellog's theorem**, and a proof of this theorem for $k = 0$ is given in the next section, which closely follows the one given in [21]. In this paper, we prove the corresponding result for Douady-Earle extensions:

Theorem 1. Let $0 < \alpha < 1$ and f be a $\mathcal{C}^{0,\alpha}$-homeomorphism of the unit circle $\mathbb{S}^1$. Then the Douady-Earle extension $\Phi(f)$ is also $\mathcal{C}^{0,\alpha}$-homeomorphism of the unit disk $\mathbb{D}$.

Theorem 2. Let f be a Lipchitz (i.e. $\mathcal{C}^{0,1}$) homeomorphism of $\mathbb{S}^1$. Then $|\Phi(f)(r\zeta) - \Phi(f)(\zeta)| \leq M(1-r)ln(\frac{1}{1-r})$ as $r \to 1-, \zeta \in \mathbb{S}^1$.

3.2 $C^{0,\alpha}$-Holder continuity of the harmonic extensions of $C^{0,\alpha}$ map on the circle

It is a popular theorem among mathematicians working in the areas of partial differential equations and potential theory that the complex harmonic extension of, i.e., the solution to the Dirichlet problem with $C^{k,\alpha}$-boundary data is also $C^{k,\alpha}, 0 < \alpha < 1$. In order to prove the main theorem, we need only the case $k = 0$, which is the following:

Theorem 3 (Kellog's theorem). Let $f : \mathbb{S}^1 \to \mathbb{C}$ be $\mathcal{C}^{0,\alpha}(\mathbb{S}^1)$. Then its complex harmonic extension, i.e. Poisson integral $H(f)(z) := \int_{\mathbb{S}^1} f(t)p(z,t)|dt|$ is $\mathcal{C}^{0,\alpha}(\mathbb{D})$.

For a proof of the above, we introduce the notion of *modulus of continuity*. Let the function ϕ be uniformly continuous on a connected set $A \subset \mathbb{C}$. Then its *modulus of continuity* is defined by

$$\omega(\delta) = \omega(\delta, \phi, A) = sup\{|\phi(z_1) - \phi(z_2)| : z_1, z_2 \in A, |z_1 - z_2| \leq \delta\} \geq 0.$$

This is an increasing continuous function with $\omega(0) = 0$. If A is convex it is easy to see that

$$\omega(n\delta) \leq n\omega(\delta), n \in \mathbb{N} \tag{1}$$

If ϕ is analytic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$ then the modulus of continuity in $\overline{\mathbb{D}}$ and $\mathbb{S}^1$ are essentially the same because

$$\omega(\delta, \phi, \mathbb{S}^1) \leq \omega(\delta, \phi, \overline{\mathbb{D}}) \leq 3\omega(\delta, \phi, \mathbb{S}^1) \tag{2}$$

for $\delta \leq \pi/2$.

The function ϕ is called *Dini-continuous* if

$$\int_0^\pi \frac{\omega(\theta)}{\theta} d\theta < \infty.$$

The limit π could be replaced by any positive constant. For a *Dini-continuous* function ϕ and $0 < \delta < \pi$, we define:

$$\omega^*(\delta) = \omega^*(\delta, \phi, A) = \int_0^\delta \frac{\omega(\theta)}{\theta} d\theta + \delta \int_\delta^\pi \frac{\omega(\theta)}{\theta^2} d\theta \tag{3}$$

Proposition 1. Let ϕ be 2π-periodic and Dini-continuous in $\mathbb{R}$. Then we have the following:

(a) $$g(z) := \frac{i}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \phi(\theta) d\theta, z \in \mathbb{D}$$

has a continuous extension to $\overline{\mathbb{D}}$.

(b) $$|g'(z)| \leq \frac{2}{\pi} \frac{\omega(1-r)}{1-r} + 2\varphi \int_{1-r}^\pi \frac{\omega(\theta)}{\theta^2} d\theta \leq 2\pi \frac{\omega^*(1-r)}{1-r} \ \forall |z| \leq r.$$

(c) $$|g(z_1) - g(z_2)| \leq 20\omega^*(\delta) \ \forall |z_1 - z_2| \leq \delta < 1.$$

Note that, the above proposition implies **Kellog's theorem**, since $f : \mathbb{S}^1 \to \mathbb{S}^1$ is 2π-periodic and α-Holder continuous implies that the function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(\theta) = \tilde{f}(\theta) - 2\pi$ is 2π-periodic and α-Holder continuous on $\mathbb{R}$. So by $c), g$ is α-Holder continuous on $\mathbb{D}$ [since $\omega^*(\delta) = O(\delta^\alpha)$ for α-Holder continuous maps on $\mathbb{R}$. Hence $H(f) = Re(g)$ is also α-Holder continuous. So the proof of Kellog's theorem boils down to proving the above proposition.

Proof of proposition 1. It follows from the definition of g that

$$(4) \qquad g'(z) = \frac{i}{2\pi} \int_0^{2\pi} \frac{e^{i\theta}}{(e^{i\theta} - z)^2} \phi(\theta) d\theta, z \in \mathbb{D}.$$

Since the above integral vanishes for a constant function ϕ, substituting $\theta = \nu + \tau$ shows

$$g'(re^{i\nu}) = \frac{i}{\pi} \int_{-\pi} \pi \frac{e^i \tau - i\nu}{|e^{i\tau} - r|^2} d\tau.$$

Next, consider the integral over the intervals $[0, 1-r]$ and $[1-r, \pi]$ separately and use the fact that $|e^{i\theta} - r| \geq max\{\theta/\pi, 1-r\}$ to get the inequality b)

The second one follows from (see(2))

$$\frac{1}{3}\omega(\delta) \leq \omega(\delta/3) \leq \int_{\delta/3}^{\delta} \frac{\omega(\theta)}{\theta} d\theta \leq \int_0^{\delta} \frac{\omega(\theta)}{\theta} d\theta.$$

The estimates hold also for $|z| < r$ by the maximum principle.

Let $z_1 = r_1\zeta_1, z_2 = r_2\zeta_2, \zeta_1, \zeta_2 \in \mathbb{S}^1$ with $|z_1 - z_2| \leq \delta$ and put $r = 1 - \delta$. Integrating the estimate from b) over $[z_1, z_2]$, we obtain

$$|g(z_1) - g(z_2)| \leq 2\pi\omega^*(\delta)$$

for $r_1 \leq r, r_2 \leq r$.

Suppose now that either $r_1 > r$ or $r_2 > r$. From the estimate b) we see that

$$\begin{aligned} & |g(r\zeta_j) - g(r_j\zeta_j)| \\ & \leq \int_r^{r_j} |g'(\xi r_j)| d\xi \\ & \leq \frac{2}{\pi} \int_0^{\delta} \frac{\omega(x)}{x} dx + 2\pi \int_0^{\delta} (\int_x^{\pi} \frac{\omega(\theta)}{\theta^2} d\theta dx. \end{aligned}$$

Exchanging the order of integration we obtain that the last term is equal to

$$2\pi \int_0^{\delta} \frac{\omega(\theta)}{\theta} d\theta + 2\pi\delta \int_{\delta}^{\pi} \frac{\omega(\theta)}{\theta^2} d\theta$$

Thus, $|g(r\zeta_j) - g(r_j\zeta_j)| \leq 7\omega^*(\delta)$ by (3) (definition of ω^*).

Furthermore, $|g(r\zeta_1) - g(r\zeta_2)| \leq 2\pi\omega^*(\delta)$ by (4). It follows that (c) holds in all cases. This finishes the proof of **Kellog's theorem**.

3.3 Proof of Theorem 1

Let $H(f)(z) := \int_{\mathbb{S}^1} f(t)p(z,t)|dt|$ denote the complex harmonic extension of the circle homeomorphism f, and let, as usual, $\Phi(f)$ denote the Douady-Earle extension of f. Our way to get the Holder regularity of $\Phi(f)$ is to compare $\Phi(f)$ with the harmonic extension $H(f)$ and show that the difference is also Holder continuous. So, we begin with the following lemmas.

Lemma 1.

$$|\Phi(f)(z) - H(f)(z)| \leq \int_{\mathbb{S}^1} |f(t) - f(s)|p(z,t)|dt| \forall s \in \mathbb{S}^1, z \in \mathbb{D}.$$

Proof of Lemma 1.

We write $\Phi = \Phi(f), H = H(f)$ for short. Note that

$$\Phi(z) - f(t) - \frac{f(t) - \Phi(z)}{1 - \Phi(\bar{z})f(t)}.f(t)\Phi(\bar{z}) = \frac{\Phi(z) - f(t)}{1 - \Phi(\bar{z})f(t)}.$$

Multiplying both sides by the Poisson kernel $p(z,t)$ and integrating w.r.t. $|dt|$, we obtain

$$\begin{aligned}\Phi(z) - H(z) - \int_{\mathbb{S}^1} \frac{f(t) - \Phi(z)}{1 - \Phi(\bar{z})f(t)} f(t)\Phi(\bar{z})p(z,t)|dt| \\ = -\int_{\mathbb{S}^1} \frac{f(t) - \Phi(z)}{1 - \Phi(\bar{z})f(t)} f(t)p(z,t)|dt| = 0.\end{aligned}$$

Therefore,

$$\Phi(f)(z) - H(f)(z) = \int_{\mathbb{S}^1} \frac{f(t) - \Phi(f)(z)}{1 - \Phi(\bar{f})(z) f(t)} (f(t) - f(s)) \Phi(\bar{f})(z) p(z,t) |dt|.$$

Taking the aboslute value sign on both sides, we get

$$|\Phi(f)(z) - H(f)(z)| \leq \int_{\mathbb{S}^1} |f(t) - f(s)| p(z,t) |dt|.$$

This finsihes the proof of Lemma 1.

Lemma 2. For $0 < \alpha < 1, |z-s|^\alpha \leq \int_{\mathbb{S}^1} |t-s|^\alpha p(z,t)|dt| \leq C.|z-s|^\alpha$,, where C is a positive constant C is independent of $z \in \mathbb{D}$ and $s \in \mathbb{S}^1$.

Proof of Lemma 2. Call $G_s(z) = |z-s|^\alpha, H_s(z) = \int_{\mathbb{S}^1} |t-s|^\alpha p(z,t)|dt|$. Next, consider the Laplacian of $G_s - H_s$ with respect to z, i.e., for $\Psi(z) = G_s(z) - H_s(z), \Delta\Psi \geq 0$, and that $\Psi = 0$ on $\mathbb{S}^1$. Now maximum principle applied to Ψ immediately implies the left hand side of the inequality.

To prove the right hand side, which is important for our purpose, we first note that:the function on $\mathbb{S}^1$ defined by $t \mapsto |t-1|^\alpha$ is $\mathcal{C}^{0,\alpha}$ on $\mathbb{S}^1$. So by Theorem 3 (Kellog's theorem), its complex harmonic extension is also $\mathcal{C}^{0,\alpha}$ on $\mathbb{D}$.

Therefore, $\int_{\mathbb{S}^1} |t-1|^\alpha p(z,t)|dt| \leq C|z-1|^\alpha$. So, we have

$$\begin{aligned}
\int_{\mathbb{S}^1} |t-s|^\alpha p(z,t)|dt| &= \int_{\mathbb{S}^1} |\frac{t}{s} - 1|^\alpha p(z,t)|dt| \\
&= \int_{\mathbb{S}^1} |\frac{t}{s} - 1|^\alpha p(z,t)|dt| \\
&= \int_{\mathbb{S}^1} |t-1|^\alpha p(\frac{z}{s}, t)|dt| \\
&\leq C|\frac{z}{s} - 1|^\alpha \\
&= C|z-s|^\alpha.
\end{aligned}$$

This finishes the proof of lemma 2.

With this lemma, we can finally finish the proof of the Theorem 1.

Proof of Theorem 1. Using lemmas 1 and 2, and Theorem 3 (Kellog's theorem)

$$\begin{aligned}
&|\Phi(f)(z) - \Phi(f)(s)| \\
&\leq |\Phi(f)(z) - H(f)(z)| + |H(f)(z) - H(f)(s)| \\
&\leq C|z-s|^{\alpha} + K|z-s|^{\alpha} \\
&= (C+K)|z-s|^{\alpha},
\end{aligned}$$

where C, K are positive constants independent of $z \in \mathbb{D}$ and $s \in \mathbb{S}^1$. This finishes the proof of Theorem 1.

3.4 Proof of Theorem 2

Again, we denote $\Phi(f)$ by Φ, $H(f)$ by H, and let $\zeta \in \mathbb{S}^1$. Then

$$\begin{aligned}
&|\Phi(r\zeta) - \Phi(\zeta)| \\
&\leq |\Phi(r\zeta) - H(r\zeta)| + |H(r\zeta) - H(\zeta)| \\
&\leq \int_{\mathbb{S}^1} |f(t) - f(\zeta)| p(r\zeta, t)|dt| + \int_{\mathbb{S}^1} |f(t) - f(\zeta)| p(r\zeta, t)|dt| \\
&\leq K_1 \int_{\mathbb{S}^1} |t-\zeta| p(r\zeta, t)|dt| + K_2 \int_{\mathbb{S}^1} |t-\zeta| p(r\zeta, t)|dt| \\
&= K \int_{\mathbb{S}^1} |t-\zeta| p(r\zeta, t)|dt|.
\end{aligned}$$

The proof of theorem 2 will be completed if we can show that

$$\int_{\mathbb{S}^1} |t-\zeta| p(r\zeta, t)|dt| \leq M(1-r)ln(\frac{1}{1-r}) \qquad \text{as } r \to 1- . \tag{5}$$

Proof of (5):

$$\begin{aligned}
&\int_{\mathbb{S}^1} |t-\zeta| p(r\zeta,t)|dt| = \int_{\mathbb{S}^1} |t-\zeta| \frac{1-r^2}{|r\zeta-t|^2}|dt| \\
&= \int_{\mathbb{S}^1} |\zeta-1|^2 \frac{1-r^2}{|r\zeta-t|^2}|dt| \\
&\leq 2(1-r)\int_{\mathbb{S}^1} \frac{t-\zeta}{|r\zeta-t|^2}|dt| \\
&= 2(1-r)\int_{\mathbb{S}^1} \frac{t-\zeta}{|r\zeta-t|}\frac{1}{|r\zeta-t|.|dt|} \\
&\leq 4(1-r)\int_{\mathbb{S}^1} \frac{|dt|}{|r\zeta-t|}.
\end{aligned}$$

Note that, to prove (5), it is enough to assume $\zeta = 1$. So (5) will be proved if we can show that

$$I(r) := \int_{\mathbb{S}^1} \frac{|dt|}{|r-t|} = O(ln\frac{1}{1-r}).$$

$$\begin{aligned}
I(r) &= \int_{-\pi}^{\pi} (1+r^2-2rcos\theta)^{-1/2}d\theta, t = e^{i\theta} \\
&= \int_{-\pi}^{\pi} [(1-r)^2 + 2r(1-cos\theta)]^{-1/2}
\end{aligned}$$

Choose and fix once and for all $\theta_0 \in (0, \pi/4)$ so that

$$1 - cos\theta = 2sin^2\frac{\theta}{2} \geq \frac{\theta^2}{4}\forall\theta \in (0,\theta_0)$$

.

Then we get

$$\begin{aligned}
&\int_{-\pi}^{\pi} [(1-r)^2 + 2r(1-cos\theta)]^{-1/2} \\
&= \int_{-\theta_0}^{\theta_0} [(1-r)^2 + 2r(1-cos\theta)]^{-1/2}d\theta + \int_{[-\pi,\pi]\setminus[-\theta_0,\theta_0]} (1+r^2-2rcos\theta)^{-1/2}d\theta \\
&= 2\int_{0}^{\theta_0} [(1-r)^2 + 2r(1-cos\theta)]^{-1/2}d\theta + O(\frac{1}{\sqrt{r}})
\end{aligned}$$

Choosing r sufficiently close to 1, the above becomes

$$
\begin{aligned}
&\leq 2\int_0^{1-r}[(1-r)^2+2r.\frac{1}{4}\theta^2]^{-1/2}d\theta+2\int_{1-r}^{\theta_0}[(1-r)^2+2r.\frac{1}{4}\theta^2]^{-1/2}d\theta+O(\frac{1}{\sqrt{r}})\\
&\leq 2.\int_0^{1-r}\frac{1}{1-r}d\theta+2\int_{1-r}^{\theta_0}\frac{1}{2}r.\frac{d\theta}{\theta}\\
&=2+rln\theta_0+\frac{1}{2}r\ln\frac{1}{1-r}\\
&\leq 2+\frac{1}{2}ln\frac{1}{1-r}=O(\ln\frac{1}{1-r})\qquad \text{as } r\to 1-\,.
\end{aligned}
$$

Hence

$$
I(r):=\int_{\mathbb{S}^1}\frac{|dt|}{|r-t|}=O(ln\frac{1}{1-r}),
$$

which proves (5), and hence Theorem 2.

Chapter 4

C^1 boundary-regularity of Douady-Earle extensions of C^1 circle diffeomorphims

4.1 Motivation and main theorems

The Douady-Earle extension $\Phi(f)$ has several intersting regularity properties. $\Phi(f)$ is real-analytic on $\mathbb{D}$ with non-singular Jacobian at every point on $\mathbb{D}$. Thus, $\mathbb{D}$ is a diffeomorphism of $\mathbb{D}$. But one can only expect $\Phi(f)$ is continuous upto and at any boundary point p since f is a homeomorphism of $\mathbb{S}^1$. It gives rise to a question: what regularity does $\Phi(f)$ have at a boundary point p if f is differentiable at p or $\mathcal{C}^1$ in a neighborhood of p on $\mathbb{S}^1$? Earle showed in [9] the following result provided that f is differentiable at p and $f'(p) \neq 0$.

Earle's theorem (Existence of angular derivative). Let f be an orientation-preserving homeomorphism of $\mathbb{S}^1$. If f is differentiable at a point $p \in \mathbb{S}^1$ and $f'(p) \neq 0$, then the difference quotient $\frac{\Phi(f)(z)-\Phi(f)(p)}{z-p}$ converges to $f'(p)$ as a point z in $\mathbb{D}$ converges to p non-tangentially. Furthermore, $\frac{\partial}{\partial z}\Phi(f)(z)$ and $\frac{\partial}{\partial \bar{z}}\Phi(f)(z)$ converge to $f'(p)$ and 0 respectively as z approaches p non-tangentially.

In the previous theorem, by a *non-tangential limit* of an expression involving z a z approaching p, we mean that for any small positive real number ϵ, the expression has a limit as along as $z \in \mathbb{D} \cap \Omega_\epsilon$ and approaches p, where Ω_ϵ is the region in the complex plane bounded by two rays emanating from p and having angle $\frac{\pi}{2} - \epsilon$ with the radius through p. By definition, the existence of the limits of the three expressions and the pattern of the limiting values in Earles Theorem entitle the map $\Phi(f)$ to have the *angular derivative* at $p \in \mathbb{S}^1$ [9].

In [9], Earle also pointed out that there is an analogous result for the extensions of orientation-reversing homeomorphisms of $\mathbb{S}^1$. Since this paper considers extensions of all homeomorphisms of $\mathbb{S}^1$, it is helpful to know the exact statement of the analogue, which is as follows.

Analogue of Earle's Theorem. Let f be an orientation-reversing homeomorphism of $\mathbb{S}^1$. If f is differentiable at a point $p \in \mathbb{S}^1$ and $f'(p) \neq 0$, then the difference quotient $\frac{\Phi(f)(z)-\Phi(f)(p)}{z-p}$ converges to $f'(p)$ as a point z in $\mathbb{D}$ converges to p non-tangentially. Furthermore, $\frac{\partial}{\partial z}\Phi(f)(z)$ and $\frac{\partial}{\partial \bar{z}}\Phi(f)(z)$ converge to $f'(p)$ and 0 respectively as z approaches p non-tangentially.

Earles Theorem and its analogue invite curiosities to study the limits of those three expressions as z approaches p tangentially or what more can imply the existence of such limits. In this chapter, we first show that those expressions do have limits as z approaches p provided that f is $\mathcal{C}^1$ in a neighborhood of p on $\mathbb{S}^1$ and $f'(p) \neq 0$.

Theorem 1 (Differentiability at a boundary point). Let f be a homeomorphism of $\mathbb{S}^1$ and $p \in \mathbb{S}^1$. Assume that f is $\mathcal{C}^1$ in a neighborhood of a point $p \in \mathbb{S}^1$ and $f'(p) \neq 0$.
Then if f is orientation-preserving, then the difference quotient $\frac{\Phi(f)(z)-\Phi(f)(p)}{z-p}$ converges to $f'(p)$ as a point z in $\mathbb{D}$ converges to p non-tangentially. Furthermore, $\frac{\partial}{\partial z}\Phi(f)(z)$ and $\frac{\partial}{\partial \bar{z}}\Phi(f)(z)$ converge to $f'(p)$ and 0 respectively as z approaches p.
If f is orientation-reversing, then the difference quotient $\frac{\Phi(f)(z)-\Phi(f)(p)}{z-p}$ converges to $f'(p)$ as a point z in $\mathbb{D}$ converges to p. Furthermore, $\frac{\partial}{\partial z}\Phi(f)(z)$ and $\frac{\partial}{\partial \bar{z}}\Phi(f)(z)$ converge to 0 and $f'(p)$ respectively as z approaches p.
It implies the following theorem.

Theorem 2 (Global $\mathcal{C}^1$ diffoemorphism). For any $\mathcal{C}^1$ diffeomorphism f of $\mathbb{S}^1$, the Douady-Earle extension $\Phi(f)$ of f is a $\mathcal{C}^1$ diffeomorphism of $\mathbb{D}$.

Here by a $\mathcal{C}^1$ diffeomorphism F of $\mathbb{D}$ we mean it is a homeomorphism of $\overline{\mathbb{D}}$ and a diffeomorphism of $\mathbb{D}$, and furthermore it has a $\mathcal{C}^1$-diffeomorphic extension on an open neighborhood of $\overline{\mathbb{D}}$ in $\mathbb{R}^2$.

Remark 1(1) Let f be an orientation-preserving $\mathcal{C}^1$ diffeomorphism of $\mathbb{S}^1$. Then f is symmetric. Using the Beurling-Ahlfors extension of f and a result in [7], one concludes that $\Phi(f)$ is an asymptotically conformal homeomorphism of $\mathbb{D}$. Note that our Theorem 1 does recover this fact in the case when f is an orientation-preserving $\mathcal{C}^1$ diffoemorphism of $\mathbb{S}^1$.

(2) Growing out of ideas in [1] and unpublished ideas of Milnor, an effective algorithm for finding the images of points under $\Phi(f)$, called the MAY iterator, was formally introduced in [1]. With availability of computer facilities, one may use the MAY iterator to explore properties of Douady-Earle extensions of circle homeomorphisms of different regularities. For example, one may verify Earle's theorem and Theorem 1 numerically with some examples.

In [9], Earle proved his theorem on the existence of angular derivative by arranging arguments and reasonings with maps defined on $\mathbb{D}$ and $\mathbb{S}^1$. After understanding his proof, we asked ourselves how the proof would go if we considered the corresponding maps defined on the upper half plane $\mathbb{H}$ and the extended real line $\mathbb{R}$. The click of this idea leads to a very short proof of Earles Theorem. From there, we developed our Theorem 1.

4.2 Proof of the main theorems

Because of the conformal naturality, we introduce the following normalization to the map f considered in Theorem 1. Pre-composed and post-composed by rotations around the origin, we may assume that $p = 1$ and $f(1) = 1$. Then followed by postcomposition by a translation preserving $\mathbb{S}^1$ and fixing 1, we may assume that f fixes 1 too. Finally, post-composed by a hyperbolic transformation fixing 1 and 1, we may assume that $f(1) = 1$ (resp. $f(-1) = -1$) if the derivative of f at 1 is positive (resp. negative). In summary, we may assume that $p = 1, f(1) = 1,\ f(1) = 1$, and $f'(1) = 1$ or 1 (corresponding to an orientation preserving or reversing homeomorphism f).

In addition to the conformal naturality, a functional property of Douady-Earle extensions of circle homeomorphisms developed by Douady and Earle in [5] is another important key to develop proofs of our theorems.

Proposition 1. (Douady-Earle [8]) Let $\mathcal{H}(\mathbb{S}^1)$ (respectively, $\mathcal{H}(\mathbb{D})$) be the space of homeomorphisms of $\mathbb{S}^1$ (respectively, $\mathbb{S}^1$) equipped with $\mathcal{C}^0$-topology, and $\mathcal{D}(\mathbb{D})$ the space of diffeomorphisms of the open unit disk with $\mathcal{C}^\infty$-topology. Then the map $\Phi : \mathcal{H}(\mathbb{S}^1) \to \mathcal{D}(\mathbb{D}) \cap \mathcal{H}(\mathbb{D})$ is continuous.

By arranging arguments and reasonings with corresponding maps defined on the extended real line $\mathbb{R}$ and the upper half plane $\mathbb{H}$, we are able to provide quite simple proofs of Theorem 1, and Earles Theorem and its analogue. In order to do so, we conjugate f and its Douady-Earle extension $\Phi(f)$ by a Mobius transformation between the unit disk $\mathbb{D}$ and the upper half plane $\mathbb{H}$. That is, let $h(z) = i\frac{z1}{z+1}$, $\tilde{f} = h \circ f \circ h^1$, and $\Phi(\tilde{f}) = h \circ \Phi(f) \circ h^1$. Then $\tilde{f}$ is a homeomorphism of $\mathbb{R}$, and $\tilde{f}(0) = 0$ and $\tilde{f}'(0) = 1$ or -1. We also call $\Phi(\tilde{f})$ the Douady-Earle extension of $\tilde{f}$. Note also that $h(0) = i$, and we will view i as a center of the hyperbolic plane $\mathbb{H}$. Based on these settings, we need the following corollary of the previous proposition.

Corollary 1. Let $\mathcal{H}(\mathbb{R})$ be the space of homeomorphisms of $\mathbb{R}$ with the topology of uniform convergence of homeomorphisms on compact subsets of $\mathbb{R}$, and let $\mathcal{D}(\mathbb{H})$ be the space of diffeomorphisms of $\mathbb{H}$ with the topology of uniform convergence of homeomorphisms and all derivatives on compact subsets of $\mathbb{H}$. Then with respect to these two topologies, $f \to \Phi(f) = h \circ \Phi(h^{-1} \circ f \circ h) \circ h^{-1}$ is continuous.

The proof of Theorem 1 is reduced to show the following two propositions.

Proposition 2. Let f be a homeomorphism of $\mathbb{R}$, and assume that f is $\mathcal{C}^1$ in a neighborhood of 0, $f(0) = 0$ and $f'(0) = 1$. Let F denote the Douady-Earle extension $\Phi(f)$ of f. Then as a point z of $\mathbb{H}$ approaching 0,

$$\lim_{z\to 0} \frac{F(z)}{z} = 1, \lim_{z\to 0} \frac{\partial F(z)}{\partial z} = 1, \lim_{z\to 0} \frac{\partial F(z)}{\partial \bar{z}} = 0.$$

Proof. For brevity of notation, we use $\epsilon(t)$ to denote a quantity that approaches 0 as an involved real variable t goes to 0.

Let b be a positive real number and $m_b(z) = bz$. Given any real number a, let $t_a(z) = z + a$. We first show

$$F(a+bi) = f(a) + bi + \epsilon(|a|+b)b, \tag{1}$$

where $\epsilon(|a|+b)$ goes to 0 as $|a|+b$ approaches 0. By the conformal naturality,

$$\begin{aligned}
&\frac{F(a+bi)-a}{b}\\
&= \frac{\Phi(f)(a+bi)-a}{b}\\
&= \frac{\Phi(f)\circ t_a \circ m_b(i) - f(a)}{b}\\
&= m_{1/b}\circ t_{-f(a)} \circ \Phi(f)\circ t_a \circ m_b(i)\\
&= \Phi(m_{1/b}\circ t_{-f(a)} \circ f \circ t_a \circ m_b)(i).
\end{aligned}$$

Let

$$f_{(a,b)} = m_{1/b}\circ t_{-f(a)} \circ f \circ t_a \circ m_b.$$

Then

$$\frac{F(a+bi)-a}{b} = \Phi(f_{(a,b)})(i).$$

Now we rewrite the boundary map $f_{(a,b)}$ as

$$f_(a,b)(x) = \frac{f(a+bx) - f(a)}{b} = f'(a)x + \epsilon(|bx|)x. \tag{2}$$

Since f is $\mathcal{C}^1$ in a neighborhood of 0 and $f'(0) = 1$, $f_{(a,b)}$ converges to the identity map Id uniformly on every compact subset of $\mathbb{R}$ as a and b go to 0. By Corollary 1, as a and b approach 0, we obtain that $\Phi(f_{(a,b)})(i)$ converges to $\Phi(Id)(i) = i$,and $\frac{\partial}{\partial z}(\Phi(f_{(a,b)})(i)$ and $\frac{\partial}{\partial \bar{z}}(\Phi(f_{(a,b)})(i)$ converge to $\frac{\partial}{\partial z}(\Phi(Id))(i) = 1$ and $\frac{\partial}{\partial \bar{z}}(\Phi(Id))(i) = 0$ respectively. It is clear that $\Phi(f_(a,b))(i)$ converging to i as a and b approaching 0 implies the estimate (1). Furthermore, applying the chain rule of taking derivatives to the right side of $\Phi(f_{(a,b)})(z) = \frac{F(a+bz)-f(a)}{b}$, we obtain: $\frac{\partial}{\partial z}\Phi(f_{(a,b)})(z) = \frac{\partial}{\partial z}F(a+bz)$ and $\frac{\partial}{\partial \bar{z}}\Phi(f_{(a,b)})(z) = \frac{\partial}{\partial \bar{z}}F(a+bz)$.
Now letting $z = i$, we can see that: $\frac{\partial}{\partial z}F(a+bi)$ and $\frac{\partial}{\partial \bar{z}}F(a+bi)$ converege to 1 and 0 respectively as the same as $\frac{\partial}{\partial z}\Phi(f_{(a,b)})(i)$ and $\frac{\partial}{\partial \bar{z}}\Phi(f_{(a,b)})(i)$ when a and b go to 0.
It remains to show that $\frac{F(a+bi)}{a+bi}$ converges to 1 as $a + bi$ goes to 0. Using again the assumption of f in a neighborhood of 0, the estimate of $F(a+bi)$ in (1) can be further expressed as:

$$F(a+bi) = a + \epsilon(a)a + bi + \epsilon(|a|+b)b = a + bi + \epsilon(a)a + \epsilon(|a|+b)b. \tag{3}$$

Thus,

$$\frac{F(a+bi)}{a+bi} = 1 + \frac{\epsilon(a)a + \epsilon(|a|+b)b}{a+bi}.$$

Clearly,

$$|\frac{\epsilon(a)a + \epsilon(|a|+b)b}{a+bi}| = \frac{\epsilon(a)a + \epsilon(|a|+b)b}{\sqrt{a^2+b^2}} \leq |\epsilon(a)| + |\epsilon(|a|+b)|,$$

which converges to 0 as $a+bi$ approaches 0. By letting $z=a+bi$, we have shown that $\lim_{z\to 0}\frac{F(z)}{z}=1$. We complete the proof.

Proposition 3. Let f be a homeomorphism of $\mathbb{R}$, and assume that f is $\mathcal{C}^1$ in a neighborhood of 0, $f(0)=0$ and $f'(0)=1$. Let F denote the Douady-Earle extension $\Phi(f)$ of f. Then as a point z of H approaches 0,

$$\lim_{z\to 0}\frac{F(z)}{\bar{z}}=-1, \lim z\to 0\frac{\partial F(z)}{\partial \bar{z}}=-1, \lim_{z\to 0}\frac{\partial F(z)}{\partial z}=0.$$

Proof. Under the same notation introduced in the proof of the previous proposition, one can see that the expression of the boundary map $f_{(a,b)}$ given in (2) now implies that $f_{(a,b)}$ converges to Id uniformly on every compact subset of $\mathbb{R}$ as a and b approach 0. One extra work to complete the proof of this proposition is to find the Douady-Earle extension of Id. Let $f=Id$ and $h(z)=-i\frac{z-1}{z+1}$. Then $g(z)=h^{-1}\circ f\circ h(z)\ \frac{1}{z}=\bar{z}$, where $z\in\mathbb{S}^1$. Using the definition of the conformal barycenter, one can easily verify that $\Phi(g)(0)=0$.

Using the conformal naturality, we have shown that $\Phi(g)(z_0)=\bar{z_0}$ for any $z_0\in\mathbb{D}$. It follows that $\Phi(f)(z)=h\circ\Phi(g)\circ h^{-1}=-\bar{z}$ for each z in the upper half plane $\mathbb{H}$.

The rest of the proof of this proposition follows the exact same strategies to complete the proof of the previous proposition.

The proofs of Earles theorem and its analogue are reduced to show the following two propositions.

Proposition 4. Let f be a homeomorphism of $\mathbb{R}$, and assume that $f(0)=0$ and

$f'(0) = 1$. Let F denote the Douady-Earle extension $\Phi(f)$ of f. Then given any real number $M > 0$, if a point $z = a + bi$ of $\mathbb{H}$ approaching 0 with $|a/b| \leq M$, then

$$\lim_{z\to 0} \frac{F(z)}{z} = 1, \lim_{z\to 0} \frac{\partial F(z)}{\partial z} = 1, \lim_{z\to 0} \frac{\partial F(z)}{\partial \bar{z}} = 0.$$

Proof. Let us use the same notation as introduced in the proof of Proposition 2. Using the expression (2) of $f_{(a,b)}$, we rewrite

$$\begin{aligned} & f_{(a,b)}(x) \\ & = \frac{f(a+bx) - f(a)}{b} \\ & = \frac{[a + bx + \epsilon(a+bx)(a+bx)] - [a + \epsilon(a).a]}{b} \\ & = x + \epsilon(|a+bx|)(a/b + x)\epsilon(|a|)a/b. \end{aligned}$$

Now we can see that if both a and $b > 0$ approach 0 with $|a/b| \leq M$, then $f(a, b)$ converges to the identity map on every compact subset of $\mathbb{R}$. The rest of the proof is exactly same as presented in the proof of the previous proposition with $z = a + bi$ approaching 0 in the upper half plane arbitrarily replaced by $z = a + bi$ approaching 0 with $|a/b|M$. To save space, we skip it.

Proposition 5. Let f be a homeomorphism of $\mathbb{R}$, and assume that $f(0) = 0$ and $f'(0) = 1$. Let F denote the Douady-Earle extension $\Phi(f)$ of f. Then given any real number $M > 0$, if a point $z = a + bi$ of $\mathbb{H}$ approaching 0 with $|a/b| \leq M$, then

$$\lim_{z\to 0} \frac{F(z)}{\bar{z}} = -1, \lim_{z\to 0} \frac{\partial F(z)}{\partial \bar{z}} = -1, \lim_{z\to 0} \frac{\partial F(z)}{\partial z} = 0.$$

Proof. The proof follows the exact same strategies to show the previous proposition after rewriting the boundary map $f_{(a,b)}$ in this case as follows:

$$\begin{aligned}
&f_{(a,b)}(x)\\
&= \frac{f(a+bx)-f(a)}{b}\\
&= \frac{[-(a+bx)+\epsilon(a+bx)(a+bx)]-[-a+\epsilon(a).a]}{b}\\
&= -x+\epsilon(|a+bx|)(a/b+x)\epsilon(|a|)a/b.
\end{aligned}$$

Proof of Theorem 2. For each point z on the extended complex plane, let z be the mirror image of z with respect to $\mathbb{S}^1$, that is, $z^\star = \frac{1}{\bar{z}}$ Then extend $F = \Phi(f)$ to the exterior of the unit disk by defining $F(z) = (F(z^\star))^\star$ for each z outside the unit disk $\mathbb{D}$. Using Theorem 1, one can see that the extended map F is a diffeomorphism of the extended complex plane.

Part II

On the Eigenvalues of Laplace Operator acting on Riemann surfaces

Chapter 5

Introduction, preliminaries and brief history

A closed hyperbolic surface is a compact surface without boundary whose sectional curvature is -1. One can show, by considering the Euler characteristic of the surface and using Gauss-Bonnet theorem for closed surface that a closed orinted surface can support a complete Riemannian metric with constant Gaussian curvature -1 (called the hyperbolic metric on the surface) if and only if its genus is greater than or equal to 2. In this paper, we will be primarily concerned with hyperbolic metrics.

Laplacian on M is a linear operator acting on the space of all smooth functions on M, defined by $\Delta f = div(\nabla f)$. Let λ_n be the n-th eigenvalue of the Laplace operator. For a closed oriented surface, it is known that, the spectrum, i.e. the set of all the eigenvalus of the Laplacian is always discrete, countable and the eigenvalues are non-negative [6]. So we can talk about the n-th eigenvalue of Laplacian operator. Some authors define the Laplacian operator as $\Delta f = -div(\nabla f)$, so for them the eigenvalues would be non-positive. Eigenvalues of the Laplace operator has been an area of continuous study and research, because of its obvious connection with physics and other areas in mathematics. There have been much reserach on the upper bound on the eigenvalues and also on whether we can produce closed surfaces with small eigenvalues. Good references for the geometry and speactra for closed hyperbolic surfaces are the books by P. Buser[4] and I. Chavel[6], among others. Another good reference for both compact and non-compact Riemann surfaces is the book by Nicolas Bergeron [5]. I will generalize a result mentioned in [5].

Some of the key theorems on bounds on eigenvalues on Riemann surfaces has been proved by P. Buser [4]. He proved, using the minimax principle in [4] that for any closed oriented Riemann surface of genus g, the $(4g-2)$-nd eigenvalue $\lambda_{4g-2} \geq 0.25$. He also showed that $\forall n \in \mathbb{N}$, and $\forall \epsilon > 0$, there exists a closed Riemann surface S with $\lambda_n(S) \leq 0.25 + \epsilon$. The number 0.25 is not special in this case: it is also proved by Buser that $\forall \epsilon > 0$, there exists a closed Riemann surface X so that $\lambda_{2g-3} < \epsilon$. So we can also ask the question is $2g-3$ optimal in this case? The answer is no, since in [11], B. Randol has proved that $\forall \epsilon > 0$, and $\forall n \in \mathbb{N}$, there exists a closed Riemann surface X with $\lambda_n(X) < \epsilon$. His proof however used the very sophisticated techniques from Selberg Trace Formula, which is also immensely important in Number Theory. In [20], I gave a very simple and geometric proof, depending solely on the minimax principle, of the following theorem.

Chapter 6

Statement and Proof of the main theorem

6.1 Minimax Principles

We will start with the minimax principle. Let $W^{1,2}(M)$ denote the Sobolev space of functions on M whose first order distributional derivatives exist on M and are (globally) square-integrable on M.

Theorem 1 (Minimax principle). Let $f_1, f_2, ...f_{k+1}$ be continuous functions on M such that they lie in the Sobolev space $W^{1,2}(M)$ and assume that volume of (support of $f_i \cap$ support of $f_j) = 0 \forall 1 \leq j \leq k+1$. Then the k-th eigenvalue λ_k of M satisfes the upper bound

$$\lambda_k \leq max_{1 \leq i \leq k+1} \frac{(||\nabla f_i||_2)^2}{(||f_i||_2)^2}.$$

For a proof of minimax principle, please see Peter Buser's book [4] or in Issac Chavel's book [6]. Let us remark here that some authors also use the symbol $H^1(M)$ or $W^1(M)$ in stead of $W^{1,2}(M)$ to denote the corresponding Sobolev spaces.

Our main theorem, Theorem 2 is actually a generalization of the following theorem.

Theorem 1 (Bergeron). Given any connected, closed, hyperbolic surface M, and given any $\epsilon > 0$, there exists a finite cover $\tilde{M}$ of M such that its 1-st eigenvalue satisfies

$$\lambda_1(M) < \epsilon.$$

To prove his theorem, Bergeron used the following (technical) lemma which we will generalize as well

Lemma 1. Let M be a closed hyperbolic surface such that $M = A \cup B$ where A and B are two connected compact sets satisfying $A \cap B = \bigcup_{i=1}^{n} \gamma_i$, where γ_i's are simple closed geodesics in M. Let $l(\gamma_i)$ denote the length of γ_i and let

$$h = \frac{\sum_{i=1}^{n} l(\gamma_i)}{minimum\{area(A), area(B)\}}.$$

Let $\eta > 0$ be a positive number such that the η -neighborhood of every γ_i is embedded in M. Then there exists a constant $C(\eta)$, depending only on η, such that the first positive eigenvalue $\lambda_1(M)$ satisfies

$$\lambda_1(M) \leq C(\eta)(h + h^2).$$

The proof of Theorem 1 follows from lemma 1.

Finally, we state the main theorem of this paper.

Theorem 2 (Main Theorem). Given any connected, closed, hyperbolic surface M, given any natural number n, and given any $\epsilon > 0$, there exists a finite cover $\tilde{M}$ of M such that its n-th eigenvalue satisfies

$$\lambda_n(\tilde{M}) < \epsilon.$$

Proof of the main theorem will follow from the following lemma, which is a generalization of lemma 1, mentioned before.

Lemma 2. Let $\tilde{M}$ be a closed hyperbolic surface such that $\tilde{M} = \bigcup_{i=1}^{n+1} A_i$ and $A_1 \cap A_2 = \gamma_1, A_2 \cap A_3 = \gamma_2, ..., A_n \cap A_{n+1} = \gamma_n, A_{n+1} \cap A_1 = \gamma_{n+1}$, where A_i's are closed subsets of $\tilde{M}$ and γ_i's are pairwise disjoint simple closed geodesics in $\tilde{M}$, and $A_i \cap A_j = \emptyset \forall j \geq i+2$ except that $A_1 \cap A_{n+1} = \gamma_{n+1}$. Further assume that areas of all A_i and lengths of all the γ_i 's are equal, and that the η neighborhood of each γ_i is embedded in $\tilde{M}$. Then we have

$$\lambda_n(\tilde{M}) \leq C(\eta)(h + h^2),$$

where $C(\eta)$ is a positive constant depending only on η, and $h = (n+1).\frac{l(\gamma_1)}{area(A_1)}$.

Proof of Lemma 2. We will use the minimax principle [6] to prove the lemma. We will produce $(n+1)$ functions $g_1, g_2, \cdots g_{n+1}$ on $\tilde{M}$ such that

$$\frac{(||\nabla g_i||_2)^2}{(||g_i||_2)^2} \leq C(\eta)(h + h^2).$$

Define for t small positive

$$A_i(t) = \{z \in A_i : dist(z, \gamma_i) \leq t\}.$$

So $A_i(t)$ is a half-collar around the simple closed geodesic γ_i.

Next, define the functions $f_i : \tilde{M} \to \mathbb{R}$ by :

$$f_i(z) = \begin{cases} \frac{1}{t}\ \mathrm{dist}(z, \gamma_i) & \text{if } z \in A_i(t) \\ 1 & \text{if} z \in A_i \backslash A_i(t) \\ 0 & \text{if} z \in \tilde{M} \backslash A_i \end{cases}$$

$$(||\nabla f_i||_2)^2 = \frac{1}{t^2} area(A_i(t))$$

$$(||f_i||_2)^2 \geq area(A_i \backslash (A_i(t)))$$

It is clear that

$$f_i \in C^0(M) \cap W^{1,2}(M) \forall 1 \leq i \leq n.$$

Now

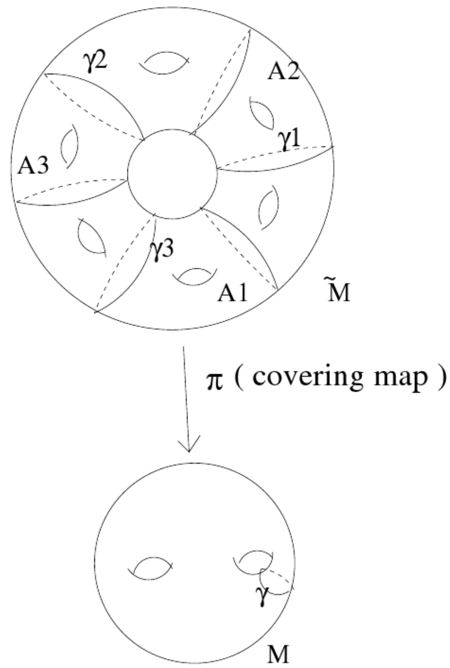
$$
\begin{aligned}
&\frac{(||\nabla f_i||_2)^2}{(||f_i||_2)^2} \\
&\leq \frac{1}{t^2}\frac{area(A_i(t))}{area(A_i) - area(A_i(t))} \\
&\leq \frac{1}{t^2}\frac{l(\gamma_i)sinh(t)}{area(A_i) - l(\gamma_i)sinh(t)} \\
&\leq \frac{1}{t^2}\frac{1}{n+1}\frac{harea(A_i)sinh(t)}{area(A_i) - \frac{1}{n+1}harea(A_i)sinh(t)} \\
&\leq \frac{1}{t^2}\frac{1}{n+1}\frac{hsinh(t)}{1 - \frac{1}{n+1}hsinh(t)} \\
&\leq \frac{hsinh(t)}{t^2(1 - sinh(t))} \\
&\leq \frac{h}{t(1 - sinh(t))} \\
&\leq \frac{2h}{\eta}\frac{1}{1 - sinh(t)} \\
&\leq \frac{2}{\eta}h(1+h) \\
&= C(\eta)h(1+h), \text{ where } C(\eta) = \tfrac{2}{\eta}.
\end{aligned}
$$

This completes the proof of the lemma 2.

6.2 Proof of the main theorem 2

Since genus of M is ≥ 2, there exists a simple closed geodesic γ embedded in M such that $M \backslash \gamma$ is connected. Take η positive such that η -neighborhood of γ is embedded in M. Fix η once and for all. Now, for each natural number N, construct a cover $\tilde{M}$ of M of degree $(n+1)N$ in the following way : Take $(n+1)N$ copies of $M \backslash \gamma$ and join them in a cyclical way, i.e. each copy of $M \backslash \gamma$ is joined to two other and different copies of $M \backslash \gamma$. Then there exists $(n+1)$ lifts of γ cutting $\tilde{M}$ into $(n+1)$ pieces $A_1, A_2, ...A_{n+1}$; each one formed by N fundamental domains for the action for the covering $\tilde{M}$ such that $A_1 \cap A_2 = \gamma_1, A_2 \cap A_3 = \gamma_2, ..., A_n \cap A_{n+1} = \gamma_n, A_{n+1} \cap A_1 = \gamma_{n+1}$,such that each A_i is a union of N copies of $M \backslash \gamma$,which is a disjoint union except for a set of measure zero. Then for each i, area(A_i)=N.area(M). (see below the corresponding figure of the

$(2+1).2 = 6$-fold cover of M for $n = 2, N = 2$).



A picture of the covering surface when n = 2, N = 2.

Then, by the previous lemma,

$$\lambda_n(\tilde{M}) \leq C(\eta)[\frac{l(\gamma)}{N \times area(M)} + (\frac{l(\gamma)}{N \times area(M)})^2] \to 0$$

as $n \to \infty$. This proves the main theorem of this chapter. □

6.3 Application of the main theorem

We define the quantity l_n already defined in [SWY].

Fix a closed hyperbolic surface M. For a fixed n, let C stand for any (finite) collection of simple closed geodesics in M such that the complement of the union of the geodesics in C is a disjoint union of $(n+1)$ components. Let $\mathcal{C}_n$ denote the family of all such C's. Let $l(C)$ denote the sum of lengths of all the geodesics in C. Let l_n denote the infimum of all $l(C)$, where C varies in $\mathcal{C}_n$. It is easy to show directly using the definition that, $l_n \leq l_{n+1}$.

In [24] and [7], it is shown that λ_n is related to a geometric quantity l_n, where l_n is the quantity defined in the definition 4.1. Then the results of the papers above show us that

Theorem 4.2([24],[7]) With l_n defined as above, we have $C_1(g).l_n \leq \lambda_n \leq C_2(g).l_n \forall 1 \leq n \leq (2g-3)$, where $C_1(g), C_2(g)$ are constants depending on only the genus g of the surface.

For a proof, see [24] or [7]. Here we can easily prove as a

Corollary of the main theorem. In the theorem 3.1 above, we cannot make $C_1(g)$ independent of g.

Proof of corollary. For a large positive integer N, the n-th eigenvalue of $\tilde{M}$ that we just constructed is arbitrarily close to zero but $l_n(\tilde{M}) \geq l_1(\tilde{M}) \geq l_1(M)$; since the image of any family of geodesics that cut $\tilde{M}$ into two pieces cut M into two pieces as well, and the image of any geodesic in the family cutting $\tilde{M}$ has the same length of its image, and two geodesics in $\tilde{M}$ could be identified in M . But $l_1(M)$ is a fixed positive number since M is fixed once and for all. So $C_1(g)$ cannot be made independent of g : note that the genus of the covering surfaces go to infinity as ϵ is made arbitrarily smaller. $\square$

Remark. A possible way to prove the dependence of $C_2(g)$ on the genus g is to construct a sequence of hyperbolic surfaces from M with n-th eigenvalues going to ∞ and their l_n's being less than or equal to that of M.

Part III

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