CONSTRUCTION OF A CLOSED HYPERBOLIC SURFACE OF ARBITRARILY SMALL EIGENVALUE OF PRESCRIBED SERIAL NUMBER

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Abstract. In this paper we construct, for given any small positive number ϵ and given natural number n, and given any closed hyperbolic surface M, a closed hyperbolic covering surface $\tilde{M}$, such that its n-th eigenvalue is less than ϵ. An application of this result will also be discussed. The main result follows from the techniques used in B.Randol's paper in 1974 [Ran]. Here I give a new and geometric proof of the main result.

1. Introduction and Preliminaries

A closed hyperbolic surface is a compact surface without boundary whose Gaussian (or sectional) curvature is -1.

One can show that, by considering the Euler characteristic of the surface and using Gauss-Bonnet theorem for closed surface that closed orinted surface can support a complete Riemannian metric with constant Gaussian curvature -1 (called the hyperbolic metric on the surface) if and only if its genus is greater than or equal to 2. In this paper, we will be primarily concerned with hyperbolic metrics.

Laplacian on M is the linear operator acting on the space of all smooth functions on M, defined by $\Delta f = div(\nabla f)$ and let λ_n be the n-th eigenvalue of the Laplace operator. For a closed oriented surface, it is known that, the spectrum, i.e. the set of all the eigenvalus of the Laplacian is always discrete, countable and the eigenvalues are non-negative[Ch], so we can talk about the n-th eigenvalue of Laplacian operator. Some authors define the Laplacian operator as $\Delta f = -div(\nabla f)$, so for them the eigenvalues would be non-positive. Eigenvalues of the Laplace operator has been an area of continuous study and research, because of its obvious connection with Physics and other areas in Mathematics. There have been much reserach on the upper bound on the eigenvalues and also on whether we can produce closed surfaces with small eigenvalues. Good references for the geometry and speactra for closed hyperbolic surfaces are the books by P. Buser[Bu] and I. Chavel[Ch], among others. Another good reference for both compact and non-compact Riemann surfaces is the book by Nicolas Bergeron[Bergeron]. I will generalize a result mentioned in [Bergeron].

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The main result, theorem 3.4, follows from the techniques involving Selberg's trace formula used by Prof. Burton Randol's 1974 paper "Small Eigenvalues of the Laplace Operator on Compact Riemann Surfaces"[Ran]. He also mentioned the result in Chapter 11 of Issac Chavel's book "Eigenvalue in Riemannian Geometry". Here I am giving an alternative proof of his result using more geometric techniques and elementary methods.

2. Organization of the paper

In section 3, I will state the the minimax principle (theorem 3.1) used in the context of eigenvalues, and the result mentioned in [Bergeron],the lemma used there to prove that result, which is crucial in the proof of the lemma. Then in the same section, I will state the main theorem 3.4 ,which is a generalization of the result in [Bergeron], and the lemma 3.5 used to prove the main theorem, which generalizes the previous lemma 3.3, and hinges on the minimax principle. I will give a complete proof of the lemma and the main theorem.

In section 4, I will mention an application of the main theorem to a result of R.Schoen, S. Wolpert and S. T. Yau [SWY], whose another proof is given in [DPRS]. I will state the relevant defnitions.

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3. Statement and Proof of the main thorem

We will start with the minimax principle. Let $W^{1,2}(M)$ denote the Sobolev space of functions on M whose first order distributional derivatives exist on M and are (globally) square-integrable on M.

Theorem 3.1 (Minimax principle) :

Let $f_1, f_2, ...f_{k+1}$ be continuous functions on M such that they lie in the Sobolev space $W^{1,2}(M)$ and assume that volume of (support of $f_i \cap$ support of f_j) $= 0 \forall 1 \leq j \leq k+1$. Then the k-th eigenvalue λ_k of M satisfes the upper bound:

$$\lambda_k \leq max_{1 \leq i \leq k+1} \frac{(||\nabla f_i||_2)^2}{(||f_i||_2)^2}$$

For a proof of Minimax principle, please see Peter Buser's book [Bu] or in Issac Chavel's book [Ch]. Let us remark here that some authors also use the symbol $H^1(M)$ or $W^1(M)$ in stead of $W^{1,2}(M)$ to denote the corresponding Sobolev spaces.

Next, let us state the theorem from [Bergeron] that we intend to generalize.

Theorem 3.2 (Bergeron):

Given any connected, closed, hyperbolic surface M, and given any $\epsilon > 0$, there exists a finite cover $\tilde{M}$ of M such that its 1-st eigenvalue $\lambda_1(M) < \epsilon$

To prove his theorem, [Bergeron] used the following (technical) lemma which we will generalize as well :

Lemma 3.3 :

Let M be a closed hyperbolic surface such that $M = A \cup B$ where A and B are two connected compact sets satisfying $A \cap B = \bigcup_{i=1}^{n} \gamma_i$, where γ_i's are simple closed geodesics in M.Let $l(\gamma_i)$ denote the length of γ_i and let :

$$h = \frac{\sum_{i=1}^{n} l(\gamma_i)}{minimum\{area(A), area(B)\}}$$

Let $\eta > 0$ be a positive number such that η -neighborhood of every γ_i is embedded in M. Then there exists a constant $C(\eta)$, depending only on η such that the first positive eigenvalue $\lambda_1(M)$ satisfies : $\lambda_1(M) \leq C(\eta)(h + h^2)$.

The proof of Theorem 3.2 [Bergeron] follows from lemma 3.3.

Finally, we state the main theorem of this paper :

Theorem 3.4. (Main Theorem):

Given any connected, closed, hyperbolic surface M, given any natural number n ; and given any $\epsilon > 0$; there exists a finite cover $\tilde{M}$ of M such that its n-th eigenvalue $\lambda_n(\tilde{M}) < \epsilon$.

Proof of the main theorem will follow from the following lemma :

Lemma 3.5 :

Let $\tilde{M}$ be a closed hyperbolic surface such that $\tilde{M} = \bigcup_{i=1}^{n+1} A_i$ and $A_1 \cap A_2 = \gamma_1, A_2 \cap A_3 = \gamma_2, ..., A_n \cap A_{n+1} = \gamma_n, A_{n+1} \cap A_1 = \gamma_{n+1}$, where A_i's are closed subsets of $\tilde{M}$ and γ_i 's are pairwise disjoint simple closed geodesic in $\tilde{M}$, and $A_i \cap A_j = \emptyset \forall j \geq i+2$ except that $A_1 \cap A_{n+1} = \gamma_{n+1}$. Further assume that areas of all A_i and lengths of all the γ_i 's are equal,and that η neighborhood of each γ_i is embedded in $\tilde{M}$. Then we have : $\lambda_n(\tilde{M}) \leq C(\eta)(h + h^2)$, where $C(\eta)$ is a positive constant depending only on η, and $h = (n+1).\frac{l(\gamma_1)}{area(A_1)}$.

Proof:

We will use the minimax principle [Ch] to prove the lemma. We will produce $(n+1)$ functions $g_1, g_2, ...g_{n+1}$ on $\tilde{M}$ such that $\frac{(||\nabla g_i||_2)^2}{(||g_i||_2)^2} \leq C(\eta)(h + h^2)$.

Define for t small positive,

$A_i(t) = \{z \in A_i : dist(z, \gamma_i) \leq t\}$

So $A_i(t)$ is a half-collar around the simple closed geodesic γ_i.

Next, define the functions $f_i : \tilde{M} \to \mathbb{R}$ by :

$$f_i(z) = \begin{cases} \frac{1}{t}.dist(z, \gamma_i) & \text{if } z \in A_i(t) \\ 1 & \text{if } z \in A_i \backslash A_i(t) \\ 0 & \text{if } z \in \tilde{M} \backslash A_i. \end{cases}$$

Then , $(||\nabla f_i||_2)^2 = \frac{1}{t^2}.area(A_i(t))$

And, $(||f_i||_2)^2 \geq area(A_i \backslash (A_i(t)))$

It is clear that $f_i \in C^0(M) \cap W^{1,2}(M) \forall 1 \leq i \leq n$. Now,

$\frac{(||\nabla f_i||_2)^2}{(||f_i||_2)^2}$

$\leq \frac{1}{t^2} \cdot \frac{area(A_i(t))}{area(A_i) - area(A_i(t))}$

$\leq \frac{1}{t^2} \cdot \frac{l(\gamma_i).sinh(t)}{area(A_i) - l(\gamma_i).sinh(t)}$

$\leq \frac{1}{t^2} \cdot \frac{1}{n+1} \cdot \frac{h.area(A_i)sinh(t)}{area(A_i) - \frac{1}{n+1}.h.area(A_i)sinh(t)}$

$\leq \frac{1}{t^2} \cdot \frac{1}{n+1} \cdot \frac{h.sinh(t)}{1 - \frac{1}{n+1}.h.sinh(t)}$

$\leq \frac{h.sinh(t)}{t^2(1 - sinh(t))}$

$\leq \frac{h}{t(1 - sinh(t))}$

$\leq \frac{2h}{\eta} \cdot \frac{1}{(1 - sinh(t))}$

$\leq \frac{2}{\eta}.h(1+h)$

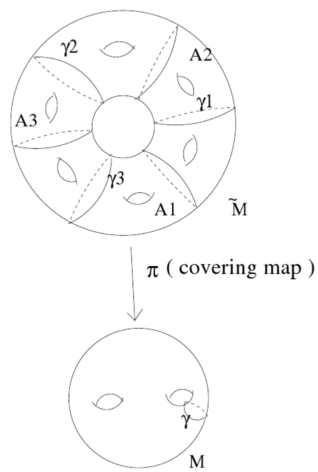
$=$ C$(\eta).h(1+h)$, where $C(\eta) = \frac{2}{\eta}$

This completes the proof of the lemma 3.5.

Proof of the main theorem 3.4 :

Proof of the main theorem :

As genus of M is ≥ 2, there exists a simple closed geodesic γ embedded in M such that $M \backslash \gamma$ is connected. Take η positive such that η -neighborhood of γ is embedded in M. Fix η once and for all. Now, for each natural number N, construct a cover $\tilde{M}$ of M of degree $(n+1)N$ in the following way : Take $(n+1)N$ copies of $M \backslash \gamma$ and join them in a cyclical way, i.e. each copy of $M \backslash \gamma$ is joined to two other and different copies of $M \backslash \gamma$. Then there exists $(n+1)$ lifts of γ cutting $\tilde{M}$ into $(n+1)$ pieces $A_1, A_2, \ldots .A_{n+1}$; each one formed by N fundamental domains for the action for the covering $\tilde{M}$ such that $A_1 \cap A_2 = \gamma_1, A_2 \cap A_3 = \gamma_2, \ldots, A_n \cap A_{n+1} = \gamma_n, A_{n+1} \cap A_1 = \gamma_{n+1}$,such that each A_i is a union of N copies of $M \backslash \gamma$,which is a disjoint union except for a set of measure zero. Then for each $i, area(A_i) = N.area(M)$. (see the corresponding figure of the $(2+1).2 = 6$-fold covering surface of M for $n = 2, N = 2$ below).



A picture of the covering surface when n = 2, N = 2.

Then, by the previous lemma,

$\lambda_n(\tilde{M}) \leq C(\eta).[\frac{l(\gamma)}{N.area(M)} + (\frac{l(\gamma)}{N.area(M)})^2] \to 0$ as $n \to \infty$

This proves the main theorem of the article. $\square$

4. Application of the main theorem

Let us first define some quanitities already defined in [SWY],[DPRS].

We define the quantity l_n already defined in [SWY].

Definition 4.1 :

Fix a closed hyperbolic surface M. For fixed n let C stand for any (finite) collection of simple closed geodesics in M such that the complement of the union of the geodesics in C is a disjoint union of $(n+1)$ components. Let $\mathcal{C}_n$ denote the family of all such C's.Let $l(C)$ denote the sum of lengths of all the geodesics in C. Let l_n denote the infimum of all $l(C)$ where C varies in $\mathcal{C}_n$ .It is easy to show directly using the definition that, $l_n \leq l_{n+1}$.

In [SWY], [DPRS], it is shown that λ_n is related to a geometric quantity l_n, where l_n is the quantity defined in the definition 4.1. Then the results of the papers above show us that :

Theorem 4.2 (SWY, DPRS) :

With l_n defined as above, we have $C_1(g).l_n \leq \lambda_n \leq C_2(g).l_n \forall 1 \leq n \leq (2g-3)$, where $C_1(g), C_2(g)$ are constants depending on only the genus g of the surface.

For a proof, see [SWY] or [DPRS]. Here we can easily prove as a

Corollary of the main theorem :

In the theorem 3.1 above, we cannot make $C_1(g)$ independent of g.

Proof of corollary: For large N, the n-th eigenvalue of $\tilde{M}$ that we just constructed is arbitrarily close to zero but $l_n(\tilde{M}) \geq l_1(\tilde{M}) \geq l_1(M)$; since image of any family of geodesics that cut $\tilde{M}$ into two pieces cut M into two pieces as well, and the image of any geodesic in the family cutting $\tilde{M}$ has the same length of its image, and two geodesics in $\tilde{M}$ could be identified in M . But $l_1(M)$ is a fixed positive number since M is fixed once and for all. So $C_1(g)$ cannot be made independent of g : note that the genus of the covering surfaces go to infinity as ϵ is made arbitrarily smaller. □

A way to prove the dependence of $C_2(g)$ on the genus g could be to construct a sequence of hyperbolic surfaces from M with n-th eigenvalues going to ∞ and their l_n being less than or equal to that of M.

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