Note on Calculus

一个短篇

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目录

1 重积分

1.1 二重积分

Definition 1.1

$$\iint\limits_{D} f(x,y)\,\mathrm{d}\sigma = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i)\sigma_i$$

1.1.1 直角坐标系

$$\mathrm{d}\sigma = \mathrm{d}x\,\mathrm{d}y$$

Formula 1.1 先横切再竖切

$$D := \{(x,y) \mid y \in [c,d], x \in [\varphi_1(y), \varphi_2(y)]\}$$

$$\iint\limits_{D} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{c}^{d} \left[\int_{\varphi_1(y)}^{\varphi_2(y)} f(x,y)\,\mathrm{d}x \right] \mathrm{d}y$$

Also represented as

$$\int_{c}^{d} \mathrm{d}y \int_{\varphi_1(y)}^{\varphi_1(y)} f(x,y)\,\mathrm{d}x$$

Formula 1.2 先竖切再横切

$$D := \{(x,y) \mid x \in [a,b], y \in [\varphi_1(x), \varphi_2(x)]\}$$

$$\iint\limits_{D} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{a}^{b} \mathrm{d}x \int_{\varphi_1(x)}^{\varphi_1(x)} f(x,y)\,\mathrm{d}y$$

1.1.2 极坐标系

Formula 1.3

$$\iint\limits_{D} f(\rho\cos\theta, \rho\sin\theta)\rho\,\mathrm{d}\rho\,\mathrm{d}\theta = \int_{\alpha}^{\beta} \mathrm{d}\theta \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} f(\rho\cos\theta, \rho\sin\theta)\rho\,\mathrm{d}\rho$$

1.2 三重积分

Definition 1.2 三重积分

$$\iiint\limits_{\Omega} f(x,y,z)\,\mathrm{d}v = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i)\Delta v_i$$

1.2.1 直角坐标系

$$\iiint\limits_{\Omega} f(x, y, z)\,\mathrm{d}v = \iiint\limits_{\Omega} f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Formula 1.4 投影穿线法：　（先对 z 积分，再对 x, y 做二重积分的处理）

将 封闭区域 Ω 投影至 xOy 面上得到封闭面 D_{xy}，因此：

$$\Omega := \left\{ (x, y, z) \mid z_1(x, y) \le z \le z_2(x, y), (x, y) \in D_{xy} \right\}$$
$$D_{xy} := \left\{ (x, y) \mid y_1(x) \le y \le y_2(x), a \le x \le b \right\}$$

则

$$\iiint\limits_{\Omega} f(x, y, z)\,\mathrm{d}v = \int_a^b \mathrm{d}x \int_{y_1(x)}^{y_2(x)} \mathrm{d}y \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\,\mathrm{d}z$$

Formula 1.5 投影切面法：

记 l 为 Ω 在 z 轴上的投影，　D_z 为 Ω 在 $z = z$ 的截面：

$$\Omega := \left\{ (x, y, z) \mid x, y \in D_z, a \le z \le b \right\}$$
$$\iiint\limits_{\Omega} f(x, y, z)\,\mathrm{d}v = \int_a^b \mathrm{d}z \iint\limits_{D_x} f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y$$

1.2.2 柱坐标系
圆柱，圆锥，旋转体

Formula 1.6

$$\iiint\limits_{\Omega} f(\rho\cos\theta, \rho\sin\theta, z)\rho\;\mathrm{d}\rho\;\mathrm{d}\theta\;\mathrm{d}z = \int_\alpha^\beta \mathrm{d}\theta \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} \rho\,\mathrm{d}\rho \int_{z_1}^{z_2} f(x, y, z)\,\mathrm{d}z$$

1.2.3 球坐标系
积分区域与球有关

Definition 1.3

在球面坐标系中，球半径设为 r，r 与 z 轴的夹角设为 φ，r 在 xoy 面上的投影距离 x 轴的夹角设为 θ，有：

$$\begin{cases} z = r\cos\varphi \\ x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \end{cases}$$

体积元 $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r^2\sin\varphi\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\theta$

Formula 1.7

$$I = \iiint\limits_{\Omega} f(r\sin\varphi\cos\theta, r\sin\varphi\sin\theta, r\cos\varphi)\, r^2\sin\varphi\; \mathrm{d}r\; \mathrm{d}\varphi\; \mathrm{d}\theta$$

$$= \int_{\theta_1}^{\theta_2} \mathrm{d}\theta \int_{\varphi_1}^{\varphi_2} \mathrm{d}\varphi \int_{r_1}^{r_2} F(r, \varphi, \theta)\, r^2\sin\varphi\; \mathrm{d}r$$

2 曲线积分和曲面积分

2.1 曲线积分

2.1.1 对弧长的曲线积分

Definition 2.1

$$\int_L f(x, y)\, \mathrm{d}s = \lim_{\lambda \to 0} \sum f(\xi_i, \eta_i)\Delta s_i$$

Formula 2.1

有参数方程:

$$L := \begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases} \qquad (\alpha \le t \le \beta)$$

所以:

$$\int_L f(x, y)\, \mathrm{d}s = \int_\alpha^\beta f[\varphi(t), \psi(t)]\sqrt{\varphi'^2(t) + \psi'^2(t)}\, \mathrm{d}t \qquad (\alpha < \beta)$$

2.1.2 对坐标的曲面积分

Definition 2.2

$$\int_L F(x, y) \cdot \mathrm{d}\boldsymbol{r} = \int_L P(x, y)\, \mathrm{d}x + Q(x, y)\, \mathrm{d}y$$

$$= \lim_{\lambda \to 0} \sum_{i=1}^n [Q(\xi_i, \eta_i)\Delta y_i + P(\xi_i, \eta_i)\Delta x_i]$$

Formula 2.2 有参数方程

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}$$

则

$$I = \int_L P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y$$

$$= \int_\alpha^\beta \{P[\varphi(t),\psi(t)]\varphi'(t) + Q[\varphi(t),\psi(t)]\psi'(t)\}\,\mathrm{d}t$$

2.1.3 格林公式

Formula 2.3

$$\oint_L P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = \pm\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y$$

2.2 曲面积分

2.2.1 对面积的曲面积分

Formula 2.4

$$\iint_\Sigma f(x,y,z)\,\mathrm{d}S$$

$$= \iint_{D_{xy}} f[x,y,z(x,y)]\sqrt{1 + z_x^2 + z_y^2}\,\mathrm{d}x\,\mathrm{d}y$$

2.2.2 对坐标的曲面积分

Formula 2.5

$$\iint_\Sigma f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \pm\iint_{D_{xy}} f[x,y,z(x,y)]\,\mathrm{d}x\,\mathrm{d}y$$

2.2.3 高斯公式

Formula 2.6

$$\oiint_\Sigma P\,\mathrm{d}y\,\mathrm{d}z + Q\,\mathrm{d}z\,\mathrm{d}x + R\,\mathrm{d}x\,\mathrm{d}y = \iiint_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\mathrm{d}v$$