Linear Attention and Beyond

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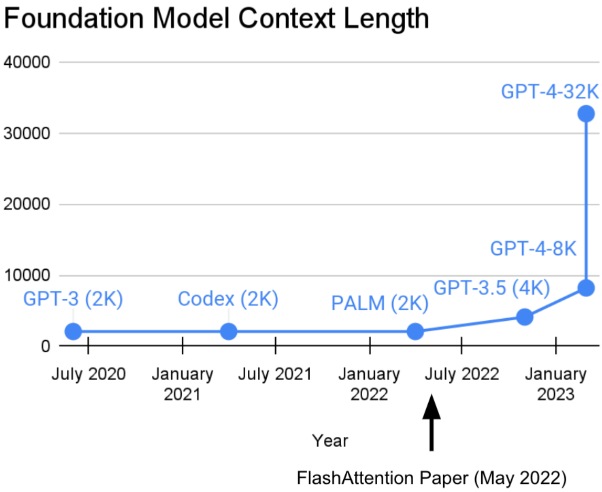
Introduction

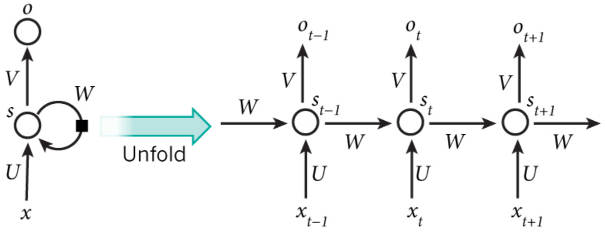
Foundation Model's Context Length is growing rapidly



Issues with Transformers

- Training: quadratic time complexity
  - Expensive for long sequence modeling (e.g., video or DNA modeling)
- Inference: linear memory complexity
  - Requires storing KV cache for each token
  - High memory burden.

Revisiting RNNs



- Training: linear complexity, however, traditional RNNs are not parallelizable.
- Inference: constant memory

Modern linear recurrent models

Use linear recurrence for parallel training

- Gated linear RNNs (HGRN, Griffin, ...)
- State-space models (S4, Mamba, ...)
- Linear attention (RetNet, GLA, xLSTM, DeltaNet, ...)

Modern linear recurrent models

Use linear recurrence for parallel training

- Gated linear RNNs (HGRN, Griffin, ...)
- State-space models (S4, Mamba, ...)
- **Linear attention (RetNet, GLA, xLSTM, DeltaNet, ...)**

Linear attention is the focus of this talk.

MiniMax-01 (MiniMax et al. 2025) used hybrid attention: **7/8** linear attention layers + **1/8** softmax attention layers, with simple linear attention using data-independent decay: Lightning-Attention (Qin et al. 2024b).

Linear attention

Softmax attention

Attention:

$$\text{Parallel training}: \quad \mathbf{O} = \mathrm{softmax}(\mathbf{Q}\mathbf{K}^\top \odot \mathbf{M})\mathbf{V} \quad \in \mathbb{R}^{L\times d}$$

$$\text{Iterative inference}: \quad \mathbf{o_t} = \sum_{j=1}^{t} \frac{\exp(\mathbf{q}_t^\top \mathbf{k}_j)}{\sum_{l=1}^{t} \exp(\mathbf{q}_t^\top \mathbf{k}_l)} \mathbf{v}_j \quad \in \mathbb{R}^{d}$$

where $\mathbf{M} \in \mathbb{R}^{L\times L}$ is the casual mask:

$$\mathbf{M}_{i,j} = \begin{cases} -\infty & \text{if } j > i \\ 1 & \text{if } j \leq i \end{cases}$$

Linear attention = standard attention - softmax

Linear attention (Katharopoulos et al. 2020):

$$\text{Parallel training}: \quad \mathbf{O} = \cancel{\text{softmax}}(\mathbf{Q}\mathbf{K}^\top \odot \mathbf{M})\mathbf{V} \quad \in \mathbb{R}^{L \times d}$$

$$\text{Iterative inference}: \quad \mathbf{o_t} = \sum_{j=1}^{t} \frac{\cancel{\exp}(\mathbf{q}_t^\top \mathbf{k}_j)}{\cancel{\sum_{l=1}^{t} \exp(\mathbf{q}_t^\top \mathbf{k}_l)}} \mathbf{v}_j \quad \in \mathbb{R}^d$$

where $\mathbf{M}$ is the causal mask for linear attention:

$$\mathbf{M}_{i,j} = \begin{cases} 0 & \text{if } j > i \\ 1 & \text{if } j \le i \end{cases}$$

Equivalent View: Matrix-Valued Hidden States

$$
\begin{aligned}
\mathbf{o_t} &= \sum_{j=1}^{t} (\mathbf{q}_t^\top \mathbf{k}_j)\mathbf{v}_j \\
&= \sum_{j=1}^{t} \mathbf{v}_j(\mathbf{k}_j^\top \mathbf{q}_t) \quad \mathbf{k}_j^\top \mathbf{q}_t = \mathbf{q}_t^\top \mathbf{k}_j \in \mathbb{R} \\
&= \underbrace{(\sum_{j=1}^{t} \mathbf{v}_j \mathbf{k}_j^\top)}_{\mathbf{S}_t \in \mathbb{R}^{d \times d}} \mathbf{q}_t \qquad \text{By associativity}
\end{aligned}
$$

Linear attention = Linear RNN + matrix-valued hidden states

Let $\mathbf{S}_t = \sum_{j=1}^{t} \mathbf{v}_j \mathbf{k}_j^\top \in \mathbb{R}^{d \times d}$ be the matrix-valued hidden state, then:

$$
\begin{aligned}
\mathbf{S}_t &= \mathbf{S}_{t-1} + \mathbf{v}_t \mathbf{k}_t^\top \quad && \in \mathbb{R}^{d \times d} \\
\mathbf{o}_t &= \mathbf{S}_t \mathbf{q}_t && \in \mathbb{R}^{d}
\end{aligned}
$$

- Linear attention implements **elementwise linear recurrence**.
- Linear attention has a **matrix-valued hidden state**, significantly increasing the state size.

Challenges in training: the parallel form

$$\mathbf{O} = (\mathbf{Q}\mathbf{K}^\top \odot \mathbf{M})\mathbf{V} \in \mathbb{R}^{L\times d}$$

The time complexity is still quadratic in sequence length, which is problematic for long sequences.

Challenges in training: the recurrent form

$$\mathbf{S}_t = \mathbf{S}_{t-1} + \mathbf{v}_t \mathbf{k}_t^\top \quad \in \mathbb{R}^{d \times d}$$
$$\mathbf{o}_t = \mathbf{S}_t \mathbf{q}_t \quad \in \mathbb{R}^{d}$$

Poor GPU utilization due to:

- Sequential computation limits parallelization opportunities across the sequence dimension.
- Rank-1 outer product updates and matrix-vector multiplications are not optimized for GPU tensor cores, which are designed for dense matrix-multiply operations (typically at least 16x16 matrix sizes).

Chunkwise parallel form

Chunkwise Form:

- Interpolates between recurrent and parallel forms.
- Splits a sequence of length L into L/C chunks of size C:
  - When $C = 1$, it reduces to the recurrent form.
  - When $C = L$, it reduces to the parallel form.
- **Key Property:** Chunkwise form is **NOT an approximation**—it computes the exact same output as the original formulation.

Chunkwise parallel form

Chunkwise form computes only the **last hidden state** per chunk.
Output is derived from:

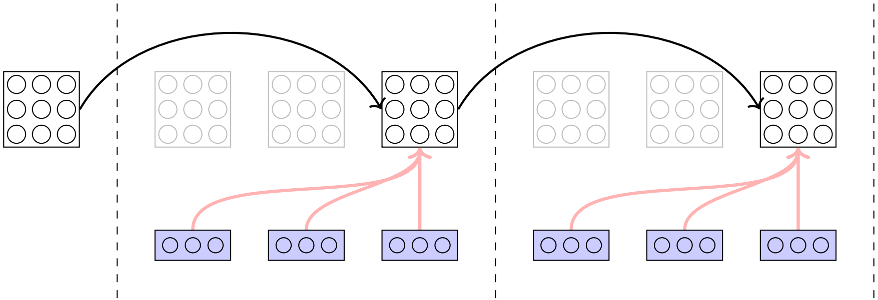
- **Recurrent Form:** Historical context across chunks.
- **Parallel Form:** Local context within a chunk.

Notations

$$
\begin{aligned}
\mathbf{S}_{[i]} &:= \mathbf{S}_{iC} \in \mathbb{R}^{d \times d} && \text{the last hidden state of chunk } i, \\
\mathbf{Q}_{[i]} &= \mathbf{Q}_{iC+1:(i+1)C} \in \mathbb{R}^{C \times d} && \text{the query block of chunk } i.
\end{aligned}
$$

We define $\mathbf{K}_{[i]}, \mathbf{V}_{[i]}, \mathbf{O}_{[i]}$ in a similar way.

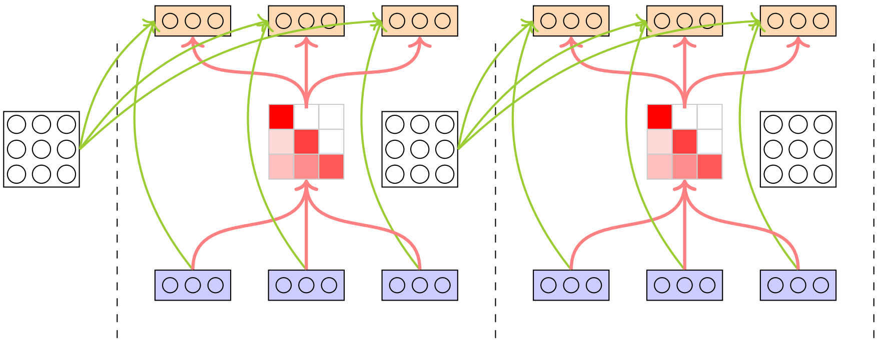
Sequential Chunk-Level State Passing:



$$\mathbf{S}_{[t+1]} = \underbrace{\mathbf{S}_{[t]}}_{\mathbb{R}^{d\times d}} + \boxed{\underbrace{\mathbf{V}_{[t]}^{\top}}_{\mathbb{R}^{d\times C}} \underbrace{\mathbf{K}_{[t]}}_{\mathbb{R}^{C\times d}}} \quad \in \mathbb{R}^{d\times d}$$

Computational Complexity: $\mathcal{O}(Cd^2)$ per chunk and $\mathcal{O}(Ld^2)$ for the entire sequence.

Parallel Output Computation:



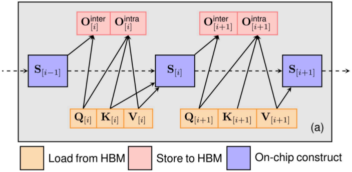
$$\mathbf{O}_{[t]} = \underbrace{\mathbf{Q}_{[t]}\mathbf{S}_{[t]}^{\top}}_{\text{inter-chunk:}\mathbf{O}_{[t]}^{\text{inter}}} + \underbrace{(\mathbf{Q}_{[t]}\mathbf{K}_{[t]}^{\top} \odot \mathbf{M})\mathbf{V}_{[t]}}_{\text{intra-chunk:}\mathbf{O}_{[t]}^{\text{intra}}} \in \mathbb{R}^{C \times d}$$

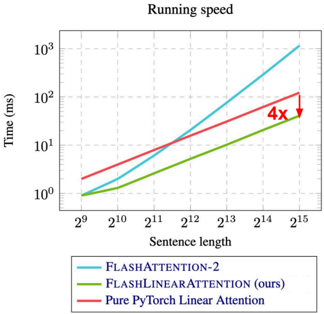
Computational Complexity: $\mathcal{O}(C^2 d + C d^2)$ per chunk.
$\mathcal{O}(L d^2 + L C d)$ for the entire sequence.

Chunkwise parallel form

- Total complexity: $\mathcal{O}(Ld^2 + LdC)$, achieving subquadratic complexity in sequence length when C is small.
- Practical settings: C is typically set to $\{64, 128, 256\}$.
- Extensibility: Can be generalized to linear attention with decay or delta rule (to be discussed later).
- Adoption: The de facto standard for training modern linear attention models, including:
  - Mamba2, Based, GLA, DeltaNet, Lightning Attention, mLSTM, and others.

Flash linear attention





I/O optimization significantly improves the wall-clock time.

Flash linear attention

The Flash Linear Attention library provides hardware-efficient implementation of various linear attention models.

- RetNet, GLA, Based, HGRN2, RWKV6, GSA, Mamba2, DeltaNet, Gated DeltaNet, RWKV7 ...

Summary

- Linear attention = Softmax attention - softmax.
- Linear attention = Linear RNN + matrix-valued hidden state.
- The chunkwise parallel form is more hardware-friendly than the recurrent and parallel forms.
- Flash Linear Attention is an I/O-aware implementation of the chunkwise parallel form.

Linear attention with decay

Linear attention is not enough

$$
\begin{aligned}
\mathbf{S}_t &= \mathbf{S}_{t-1} + \mathbf{v}_t \mathbf{k}_t^\top && \in \mathbb{R}^{d \times d} \\
\mathbf{o}_t &= \mathbf{S}_t \mathbf{q}_t && \in \mathbb{R}^{d}
\end{aligned}
$$

Vanilla linear attention performs poorly in language modeling.

- Only stores key-value pairs in memory.
- Has no mechanism to forget old memories.
- Leads to memory saturation and degradation in performance since the memory size is fixed.

Linear attention with data-independent decay

$$\mathbf{S}_t = \gamma \mathbf{S}_{t-1} + \mathbf{v}_t \mathbf{k}_t^\top \qquad \in \mathbb{R}^{d \times d}$$
$$\mathbf{o}_t = \mathbf{S}_t \mathbf{q}_t \qquad \in \mathbb{R}^{d}$$

- γ is a constant exponential decay factor $0 < \gamma < 1$.
- This mechanism weighs recent tokens more than distant tokens, and language modeling has a strong recency bias.
- Works well in practice: RetNet (Sun et al. 2023), Lightning Attention (Qin et al. 2024b).

Linear attention with data-dependent decay

$$\mathbf{S}_t = \gamma_t \mathbf{S}_{t-1} + \mathbf{v}_t \mathbf{k}_t^\top \qquad \in \mathbb{R}^{d \times d}$$

$$\mathbf{o}_t = \mathbf{S}_t \mathbf{q}_t \qquad \in \mathbb{R}^{d}$$

- $\gamma_t \in (0, 1)$ is a data-dependent decay term that is a function of $\mathbf{x}_t$.
- Enables dynamic control of memory retention/forgetting based on input data.
- Examples: Mamba2 (Dao and Gu 2024), mLSTM (Beck et al. 2024), Gated Retention (Sun et al. 2024b).

The parallel form for linear attention with decay

$$\mathbf{S}_t = \gamma_t \mathbf{S}_{t-1} + \mathbf{v}_t \mathbf{k}_t^\top \qquad \in \mathbb{R}^{d \times d}$$

$$\mathbf{o}_t = \mathbf{S}_t \mathbf{q}_t \qquad \in \mathbb{R}^{d}$$

Linear attention with decay has the following parallel form:

$$\mathbf{O} = (\mathbf{Q}\mathbf{K}^\top \odot \mathbf{D})\mathbf{V} \in \mathbb{R}^{L \times d}$$

$$\mathbf{D}_{i,j} = \begin{cases} \prod_{m=j+1}^{i} \gamma_m & \text{if } i > j \\ 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

The duality between recurrent and parallel forms is referred to as state space duality (SSD) in Mamba2 (Dao and Gu 2024).

Linear attention with more fine-grained decay

$$\mathbf{S}_t = \mathbf{G}_t \odot \mathbf{S}_{t-1} + \mathbf{v}_t \mathbf{k}_t^\top \qquad \in \mathbb{R}^{d \times d}$$

$$\mathbf{o}_t = \mathbf{S}_t \mathbf{q}_t \qquad \in \mathbb{R}^{d}$$

Condition for the matmul-based (chunkwise) parallel form (Yang et al. 2023):

$$\mathbf{G}_t = \boldsymbol{\beta}_t \boldsymbol{\alpha}_t^\top \in \mathbb{R}^{d \times d}, \quad \boldsymbol{\alpha}_t, \boldsymbol{\beta}_t \in \mathbb{R}^d$$

- Mamba-1 (Gu and Dao 2023) does not conform to this structure, preventing the use of tensor cores.
- It is common to set $\boldsymbol{\beta}_t = \mathbf{1}$ in practice, examples: GLA (Yang et al. 2023), RWKV6 (Peng et al. 2024), GSA (Zhang et al. 2024), HGRN2 (Qin et al. 2024a).

Summary

- Language modeling has a strong recency bias.
- Decay helps bridge the perplexity gap between linear and softmax attention.
- Fine-grained decay enhances performance but poses scaling challenges.
- Outer-product structure enables efficient chunkwise training with fine-grained decay.

Toward more expressive update rule: the delta rule

Fast weight programming

Before diving into the delta rule, let's first review fast weight programming, which will help us understand the delta rule and other test-time-trainers (TTT, Titans, Mesa layer, etc.).

> "Fast weights provide a neurally plausible way of implementing the type of temporary storage that is required by working memory, while slow weights capture more permanent associations learned over many experiences." – Geoffrey Hinton

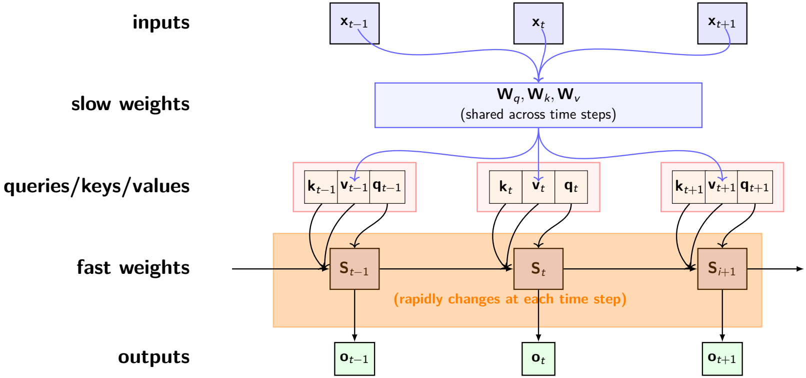
Recall the memory readout in linear attention:

$$\mathbf{o}_t = \mathbf{S}_t \mathbf{q}_t$$

We can think of the recurrent hidden state $\mathbf{S}_t$ as a fast weight matrix that maps input $\mathbf{q}_t$ to output $\mathbf{o}_t$ and is updated as it goes:

$$\mathbf{S}_t = \mathbf{S}_{t-1} + \mathbf{v}_t \mathbf{k}_t^\top$$

Linear attention is secretly a fast weight programmer



- **Fast Weight**: $\mathbf{S}_t$ maps $\mathbf{q}_t$ to $\mathbf{o}_t$, updated dynamically during inference for rapid adaptation.
- **Slow Weight**: $\mathbf{W}_q$, $\mathbf{W}_k$, and $\mathbf{W}_v$ are fixed during inference and only updated during training (e.g., via gradient descent).

The choice of update rule

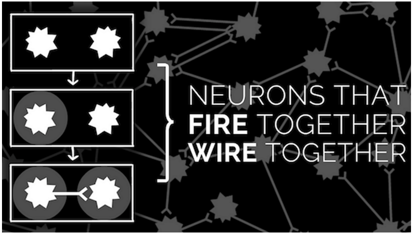


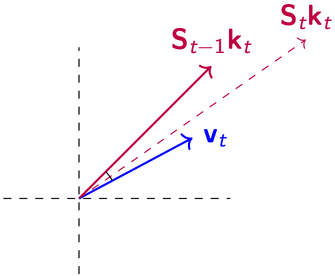
Figure: The principle of Hebbian learning.

- Hebbian update rule: $\mathbf{S}_t = \mathbf{S}_{t-1} + \mathbf{v}_t \mathbf{k}_t^\top$
- Delta rule: $\mathbf{S}_t = \mathbf{S}_{t-1} - \beta_t \left(\mathbf{S}_{t-1} \mathbf{k}_t - \mathbf{v}_t \right) \mathbf{k}_t^\top$
- ...

Both Hebbian and delta update rules can be regarded as optimizing online learning objective via test-time SGD.

Linear Attention: Test-Time Objective

Maximize alignment
$= \text{Minimize angle difference} + \text{enlarge } |\mathbf{Sk}|$

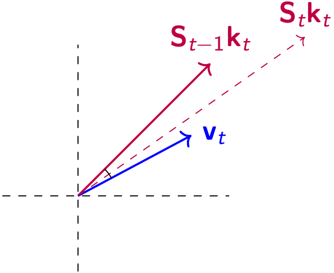


Objective: $\mathcal{L}_t(\mathbf{S}) = -\langle \mathbf{S}\mathbf{k}_t, \mathbf{v}_t \rangle$

SGD update: $\mathbf{S}_t = \mathbf{S}_{t-1} - \beta_t \nabla \mathcal{L}_t(\mathbf{S}_{t-1}) = \mathbf{S}_{t-1} + \beta_t \mathbf{v}_t \mathbf{k}_t^\top$

Linear Attention: Test-Time Objective

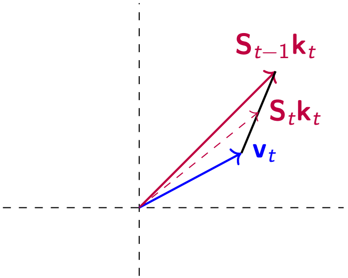
Maximize alignment
$= \text{Minimize angle difference} + \text{enlarge } |\mathbf{Sk}|$



- Linear attention may favor increasing $|\mathbf{Sk}|$ over angle alignment, leading to numerical instabilities.
- Mamba2 uses decay ($\mathbf{S}_t = \alpha_t \mathbf{S}_{t-1} + \mathbf{v}_t\mathbf{k}_t^\top$) to control $|\mathbf{Sk}|$, stabilizing the training process.

DeltaNet: Test-Time Objective

Directly minimize Euclidean distance



Objective: $\mathcal{L}_t(\mathbf{S}) = \frac{1}{2}\|\mathbf{S}\mathbf{k}_t - \mathbf{v}_t\|^2$

SGD update: $\mathbf{S}_t = \mathbf{S}_{t-1} - \beta_t \nabla \mathcal{L}_t(\mathbf{S}_{t-1}) = \mathbf{S}_{t-1} - \beta_t(\mathbf{S}_{t-1}\mathbf{k}_t - \mathbf{v}_t)\mathbf{k}_t^\top$

- Better numerical stability: the norm of $\mathbf{S}_t$ is controlled.
- Better in-context associative recall: directly optimizes key-value association prediction (Liu et al. 2024)

In-context associative recall

Multi-Query Associative Recall (MQAR, Arora et al. 2023)

A synthetic benchmark for testing in-context associative recall. **Example:**

- Given key-value pairs: “A 4 B 3 C 6 F 1 E 2”
- Query: “A ? C ? F ? E ? B ?”
- Expected output: “4, 6, 1, 2, 3”

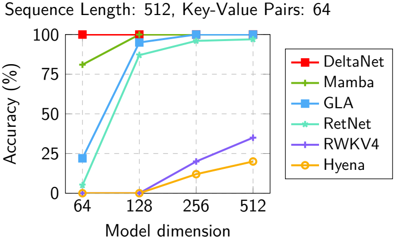
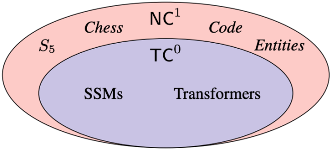


Figure: Accuracy (%) on MQAR. DeltaNet achieves the perfect recall.

Transformers and SSMs in TC^0



- Transformers are in TC^0.
- Linear RNNs with diagonal transition matrices (e.g., $\mathbf{S}_t = \mathbf{S}_{t-1}\,\mathrm{diag}(\boldsymbol{\alpha}_t) + \mathbf{v}_t\mathbf{k}_t^\top$ in GLA) fall under TC^0.
- Nonlinear RNN or linear RNN with data-dependent nondiagonal transition matrices could achieve expressiveness beyond TC^0 (Merrill, Petty, and Sabharwal 2024).

DeltaNet's expressiveness

$$
\begin{aligned}
\mathbf{S}_t &= \mathbf{S}_{t-1} - \beta_t \left(\mathbf{S}_{t-1}\mathbf{k}_t - \mathbf{v}_t\right) \mathbf{k}_t^\top \\
&= \mathbf{S}_{t-1}\left(\mathbf{I} - \beta_t \mathbf{k}_t \mathbf{k}_t^\top\right) + \beta_t \mathbf{v}_t \mathbf{k}_t^\top
\end{aligned}
$$

DeltaNet uses Generalized Householder (GH) transition matrices, which are both data-dependent and nondiagonal, making it possible to achieve expressiveness beyond TC^0.

DeltaNet's expressiveness

$$\mathbf{S}_t = \mathbf{S}_{t-1} \underbrace{(\mathbf{I} - \beta_t \mathbf{k}_t \mathbf{k}_t^\top)}_{\text{GH transition}} + \beta_t \mathbf{v}_t \mathbf{k}_t^\top = \sum_{i=1}^{t} \left(\beta_i \mathbf{v}_i \mathbf{k}_i^t \underbrace{\prod_{j=i+1}^{t} (\mathbf{I} - \beta_j \mathbf{k}_j \mathbf{k}_j^\top)}_{\text{cumulative GH products}} \right)$$

Key Properties:

- **Expressiveness**: When allowing negative eigenvalues in GH matrices (Grazzi et al. 2024), the cumulative products of GH matrices can represent *any* matrix with Euclidean norm < 1.
- **Complexity Class**: Cumulative products of general matrices cannot be computed in TC^0 (Mereghetti and Palano 2000).
- **Conclusion**: DeltaNet with negative eigenvalues has expressiveness beyond TC^0, strictly exceeding SSMs and Transformers.

State tracking performance

	Parity	Mod. Arithm. (w/o brackets)	Mod. Arithm. w/ brackets)
Transformer	0.022	0.031	0.025
mLSTM	0.087 (0.04)	0.040 (0.04)	0.034 (0.03)
sLSTM	**1.000** (1.00)	**0.787** (1.00)	**0.173** (0.57)
Mamba $[0,1]$	0.000	0.095	0.092
Mamba $[-1,1]$	**1.000**	**0.241**	**0.136**
DeltaNet $[0,1]$	0.017	0.314	0.137
DeltaNet $[-1,1]$	**1.000**	**0.971**	**0.200**

Figure: State tracking performance comparison (source: Grazzi et al. 2024). $[0,1]$ and $[-1,1]$ denotes the ranges of eigenvalues for each model's transition matrix.

- Allowing negative eigenvalues could boost state tracking performance for both Mamba and DeltaNet.
- DeltaNet achieves superior performance due to its richer expressiveness.

DeltaNet: Chunkwise Parallel Training

$$
\begin{aligned}
\mathbf{S}_t &= \mathbf{S}_{t-1}\left(\mathbf{I} - \beta_t \mathbf{k}_t \mathbf{k}_t^\top\right) + \beta_t \mathbf{v}_t \mathbf{k}_t^\top \\
&= \mathbf{S}_{t-1} + \underbrace{\beta_t(\mathbf{v}_t - \mathbf{S}_{t-1}\mathbf{k}_t)}_{\text{defined as } \mathbf{u}_t}\mathbf{k}_t^\top \\
&= \mathbf{S}_{t-1} + \mathbf{u}_t \mathbf{k}_t^\top \\
&= \sum_{i=1}^{t} \mathbf{u}_i \mathbf{k}_i^\top
\end{aligned}
$$

Once “pseudo-values” $\mathbf{u}_t$ are computed, DeltaNet can be trained using the same kernel as linear attention.

DeltaNet: Chunkwise Parallel Training

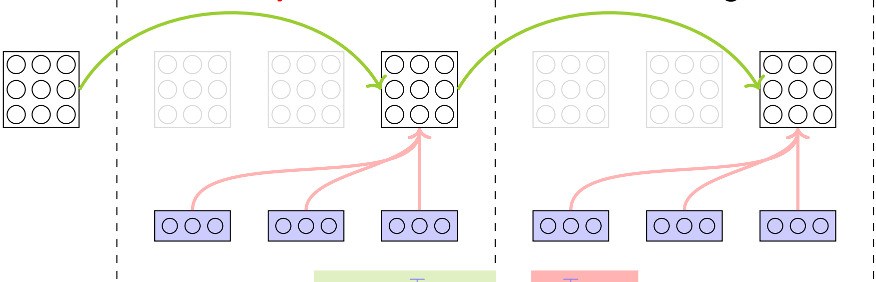
$$\begin{aligned}\mathbf{S}_t &= \mathbf{S}_{t-1}\left(\mathbf{I} - \beta_t \mathbf{k}_t \mathbf{k}_t^\top\right) + \beta_t \mathbf{v}_t \mathbf{k}_t^\top \\ &= \sum_{i=1}^{t}\left(\beta_i \mathbf{v}_i \mathbf{k}_i^\top \underbrace{\prod_{j=i+1}^{t}\left(\mathbf{I} - \beta_j \mathbf{k}_j \mathbf{k}_j^\top\right)}_{\mathbf{P}_j^t}\right)\end{aligned}$$

Using the WY representation (Bischof and Loan 1985):

$$\mathbf{P}_1^t = \mathbf{I} - \sum_{i=1}^{t} \mathbf{w}_i \mathbf{k}_i^\top.$$

Key Insight: The cumulative product $\prod$ becomes a cumulative sum $\sum$, enabling efficient matrix-multiply-based training.

Sequential Chunk-Level State Passing:



$$\boldsymbol{S}_{[t+1]} = \boldsymbol{S}_{[t]} \left(\boldsymbol{I} - \boldsymbol{W}_{[t]}^{\top} \boldsymbol{K}_{[t]}\right) + \boldsymbol{U}_{[t]}^{\top} \boldsymbol{K}_{[t]}$$

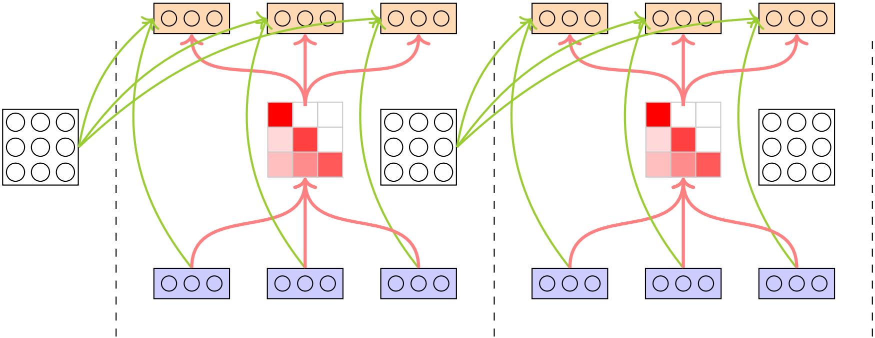
Using hardware-friendly UT transform (Joffrain et al. 2006):

$$\mathbf{T}_{[t]} = \left(\mathbf{I} + \operatorname{tril}(\operatorname{diag}(\beta_{[t]})\mathbf{K}_{[t]}\mathbf{K}_{[t]}^{\top}, -1)\right)^{-1} \operatorname{diag}\left(\beta_{[t]}\right) \quad \in \mathbb{R}^{C \times C}$$

$$\mathbf{W}_{[t]} = \mathbf{T}_{[t]}\mathbf{K}_{[t]}, \quad \mathbf{U}_{[t]} = \mathbf{T}_{[t]}\mathbf{V}_{[t]} \quad \in \mathbb{R}^{C \times d}$$

See https://sustcsonglin.github.io/blog/2024/deltanet-2/ for details.

Parallel Output Computation:



$$\mathbf{O}_{[t]} = \boxed{\mathbf{Q}_{[t]}\mathbf{S}_{[t]}^{\top}} + \boxed{\left(\mathbf{Q}_{[t]}\mathbf{K}_{[t]}^{\top} \odot \mathbf{M}\right)\left(\mathbf{U}_{[t]} - \mathbf{W}_{[t]}\mathbf{S}_{[t]}^{\top}\right)}$$

Compared to vanilla linear attention, the "pseudo-values" need to be adjusted by the historical context: $\mathbf{W}_{[t]}\mathbf{S}_{[t]}^{\top}$.

Gated DeltaNet

Enhancing DeltaNet with a Mamba2-like gating mechanism could boost performance on real-world tasks.

$$\mathbf{S}_t = \mathbf{S}_{t-1}\left(\alpha_t(\mathbf{I} - \beta_t \mathbf{k}_t \mathbf{k}_t^\top)\right) + \beta_t \mathbf{v}_t \mathbf{k}_t^\top$$

Model	**ppl** ↓	**LM-eval** ↑	**Recall** ↑	**Long** ↑
Mamba1	17.92	53.12	21.0	14.6
Mamba2	16.56	54.89	29.8	13.5
DeltaNet	17.72	52.14	26.2	13.6
Gated DeltaNet	**16.42**	**55.32**	**30.6**	**16.6**

Table: Performance comparison of different 1.3B models trained on 100B tokens. Source: Yang, Kautz, and Hatamizadeh 2024.

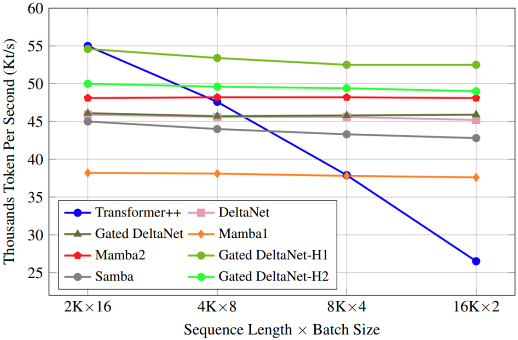


Figure: End-to-end training throughput comparison for different 1.3B models on a single H100. Source: Yang, Kautz, and Hatamizadeh 2024.

Combining DeltaNet and Mamba2's chunkwise algorithms for hardware-efficient training:

- Pure Gated DeltaNet is slightly slower than Mamba2.
- Hybridizing sliding window attention with Gated DeltaNet (i.e., Gated DeltaNet-H1) improves throughput.

DeltaNet's chunkwise algorithm can be extended to diagonal-plus-low-rank transition matrices:

$$\mathbf{S}_t = \mathbf{S}_{t-1}(\mathbf{D}_t + \boldsymbol{\alpha}_t \boldsymbol{\beta}_t^\top) + \mathbf{v}_t \mathbf{k}_t^\top, \quad \mathbf{D}_t \in \mathbb{R}^{d \times d}, \quad \boldsymbol{\alpha}_t, \boldsymbol{\beta}_t \in \mathbb{R}^d$$

- **RWKV-7** employs this linear recurrence, showing effectiveness.
- A fast implementation is available in the flash-linear-attention library (https://github.com/fla-org/flash-linear-attention/blob/main/fla/ops/rwkv7/chunk.py).

DeltaProduct

Generalizing the DeltaNet by performing multiple gradient descent steps (i.e., n_h) per token:

$$\begin{aligned}
\mathbf{S}_{t,j} &= \mathbf{S}_{t,j-1} - \beta_{t,j}\nabla\mathcal{L}_{t,j}(\mathbf{S}_{t,j-1}) \\
&= \left(\mathbf{I} - \beta_{t,j}\mathbf{k}_{t,j}\mathbf{k}_{t,j}^{\top}\right)\mathbf{S}_{t,j-1} + \beta_{t,j}\mathbf{v}_{t,j}\mathbf{k}_{t,j}^{\top}
\end{aligned}$$

where $\mathbf{S}_{t,0} = \mathbf{S}_{t-1}$ and $\mathbf{S}_{t,n_h} = \mathbf{S}_{t}$. This results in a **high-rank** recurrent updates

$$\begin{aligned}
\mathbf{S}_{t} &= \mathbf{S}_{t-1}\mathbf{A}_{t} + \mathbf{B}_{t} \\
\mathbf{A}_{t} &= \prod_{j=1}^{n_h}\left(\mathbf{I} - \beta_{t,j}\mathbf{k}_{t,j}\mathbf{k}_{t,j}^{\top}\right) \\
\mathbf{B}_{t} &= \sum_{j=1}^{n_h}\beta_{t,j}\mathbf{v}_{t,j}\mathbf{k}_{t,j}^{\top}\prod_{l=1}^{j-1}\left(\mathbf{I} - \beta_{t,l}\mathbf{k}_{t,l}\mathbf{k}_{t,l}^{\top}\right)
\end{aligned}$$

where both the transition matrix $\mathbf{A}_t$ and the input $\mathbf{B}_t$ are rank-n_h matrices.

DeltaProduct

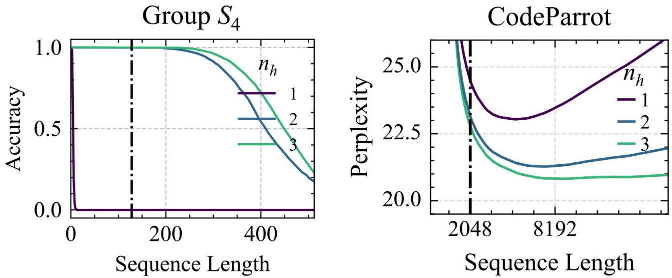


Figure: This figure is copied from Siems et al. 2025. (*Left*) DeltaProduct$_{n_h}$ learns higher-order permutation groups like S_4 in one layer, while DeltaNet ($n_h = 1$) is limited to S_2. (*Right*) Length extrapolation of DeltaProduct improves significantly with higher n_h.

Summary

- Linear attention and DeltaNet are fast weight programmers with different test-time SGD weight updates.
- DeltaNet has strictly more expressive power than Mamba/GLA while maintaining efficient parallelization
- DeltaNet's chunkwise algorithm could be generalized to diagonal-plus-low-rank transition matrices
- Gating and high-rank updates further enhance DeltaNet's performance

Beyond linear regression objective

Beyond linear regression objective

DeltaNet optimizes the online linear regression loss:

$$\mathcal{L}_t(\mathbf{S}) = \frac{1}{2}\|\mathbf{S}\mathbf{k}_t - \mathbf{v}_t\|^2$$

- This optimization objective assumes linear relationships in historical data dependencies
- However, generative AI tasks involve complex, nonlinear dependencies
- A linear regression loss may be insufficient to capture these rich patterns.

Going beyond online linear regression objective

TTT (Sun et al. 2024a) extends this to a nonlinear regression loss:

$$\mathcal{L}_t(\mathbf{S}) = \frac{1}{2}\|f_{\mathbf{S}}(\mathbf{k}_t) - \mathbf{v}_t\|^2$$

where $f_{\mathbf{S}}$ is a nonlinear transformation parameterized by $\mathbf{S}$.
Examples:

- TTT-linear:

$$f_{\mathbf{S}}(x) = \mathrm{LN}(\mathbf{S}x) + x,$$

- TTT-MLP:

$$f_{\mathbf{S}}(x) = \mathrm{LN}(\mathrm{MLP}_{\mathbf{S}}(x)) + x$$

where LN denotes layer normalization.

Beyond Linear Regression Objective

The nonlinear loss induces a nonlinear recurrence, posing challenges for parallelization.

Solution: Mini-batch Gradient Descent

- Minibatch size aligns with chunk size.
- Each token within a chunk is treated as an independent training example for parallel processing.
- Sequential dependencies are preserved via a lightweight linear recurrence within chunks.

This approach essentially combines intra-chunk linear recurrence with inter-chunk nonlinear recurrence.

Beyond linear regression objective

TTT (Sun et al. 2024a) extends this to a nonlinear regression loss:

$$\mathcal{L}_t(\mathbf{S}) = \frac{1}{2}\|f_{\mathbf{S}}(\mathbf{k}_t) - \mathbf{v}_t\|^2$$

where $f_{\mathbf{S}}$ is a nonlinear transformation parameterized by $\mathbf{S}$.

- Titans (Behrouz, Zhong, and Mirrokni 2024) further enhances TTT by integrating momentum and weight decay into the mini-batch SGD update.

Summary

- **Modern RNNs through the lens of online learning**:
  - (Decaying/Gated) Linear attention: Negative inner-product loss.
  - (Gated) DeltaNet: Linear regression loss.
  - TTT & Titans: Nonlinear regression losses.
- **Gradient-based optimization techniques prove valuable**:
  - Weight decay enables effective forgetting (e.g., Mamba2, Gated DeltaNet).
  - Momentum improves performance (e.g., Titans).
- **Efficient hardware utilization via**:
  - Chunkwise training for linear attention.
  - Hybrid linear/nonlinear approaches across chunks (e.g., TTT & Titans).
- **Promising future**: Bridging optimization and RNN architectures.

Thanks!

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