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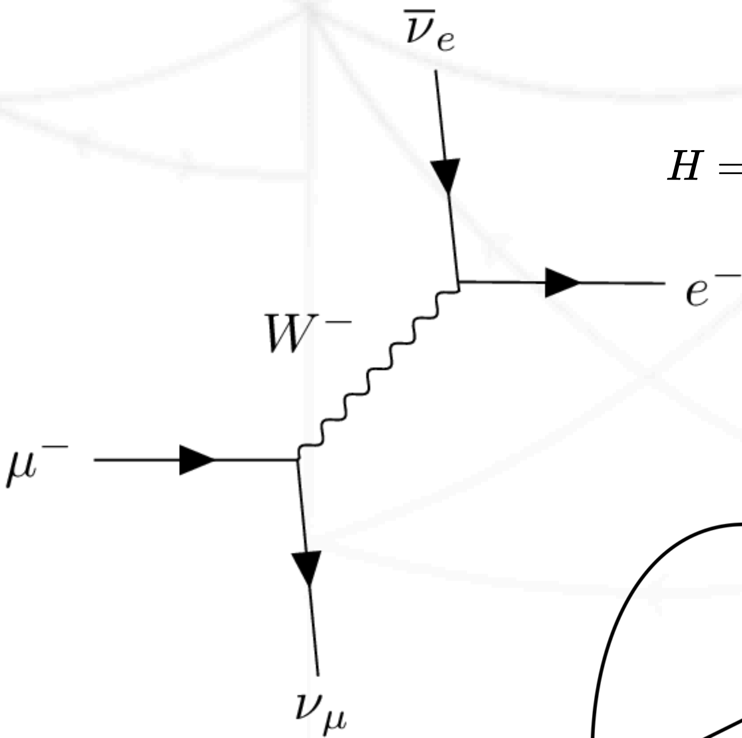
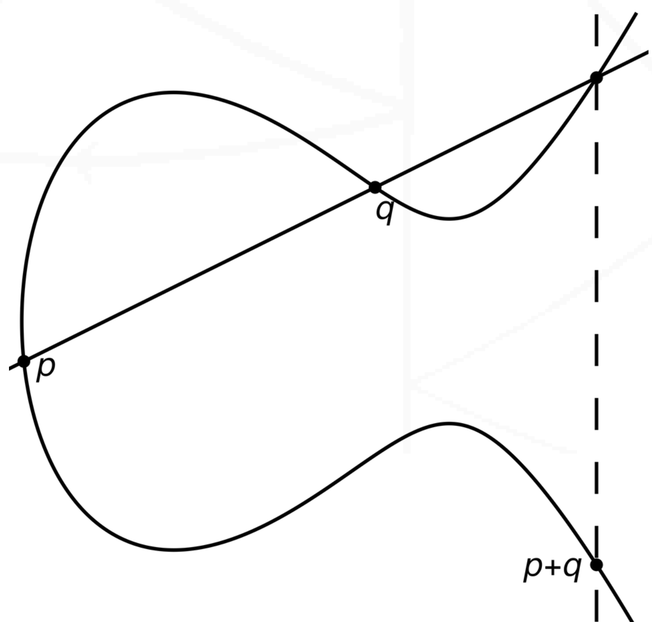
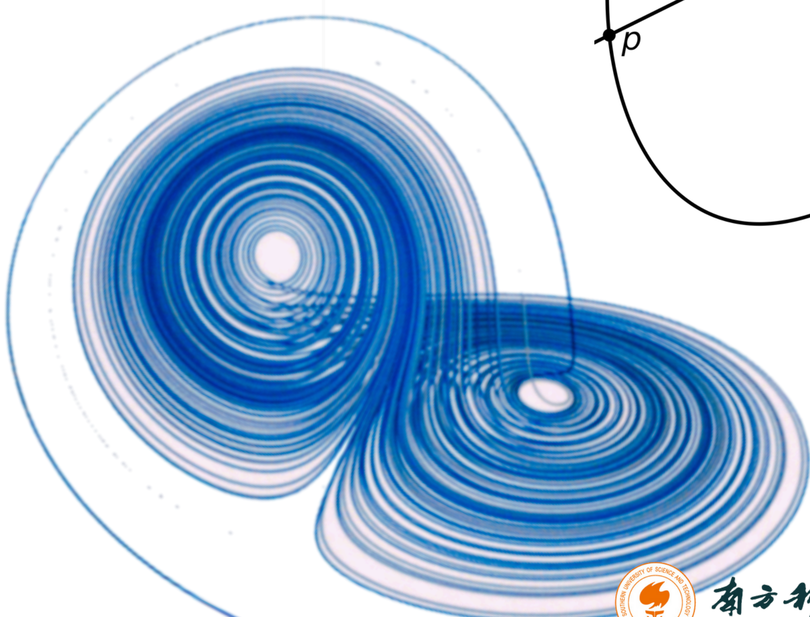
CHARTER ISSUE



$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 q^2 + \frac{1}{3} g q^3$$





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CLUB OF MATHEMATICS AND PHYSICS
数学物理社

前言

大家好，我是《Naïve》数学物理社社刊的主编。因为是本刊物的创刊号，我想向读者们介绍一下数学物理社创立本刊物的初衷和设想。创立本刊物的想法以及行动实质上在将近两年前就已经存在了，但是由于主编以及决定投稿创刊号的社员们种种安排的变动，创刊号的出版工作时至今年才完成。在创刊设想的初期，彼时数学物理社的活跃成员们正苦于为数学物理社寻找除了讲座和讨论班以外的、受众更广而有趣的活动形式。中国科学技术大学数学系的《蛙鸣》杂志为我们提供了非常好的灵感，我们意识到大家平日许多交流的内容可以整理成文并发表出来。后来在与物理系和数学系的老师交流时，双方都提到了系内早就有创建类似校内刊物的想法；同时，物理系的老师也提到校内渐渐缺乏同学之间对课业之外的学科内容的交流氛围。其实学生内部缺乏交流并不完全是功利主义、动力缺失此类问题：由于数学物理社物理系成员多为更加了解高能物理领域的同学，主编总是会听到其他学习凝聚态物理较多的同学“感觉内容差别太大，聊不进去”的声音。其实正如中山大学叶鹏老师所说：“哪有什么高能不高能的!”，这两个领域的前沿也逐渐在发生融合，如超导理论领域应用规范-引力对偶等。我们希望这本刊物能为促进同学们之间的交流尽一份来自学生自身的力量。同时，作为数学物理社的成员，大家都意识到数学系与物理系这两门有着极深渊源的学科之间的交流是大有裨益的。在理学院的 M2002 教室内，常能看到数学系与物理系的同学在交换双方的认识：学习动力系统的同学可以与物理系的同学讨论熵的定义与性质、学习广义相对论的同学可以与学习黎曼几何的同学交换直观⋯⋯作为一本融合数学与物理内容的杂志，我们也希望能让两个系的同学们欣赏双方独有的思想和美感。

原则上为了能让更多同学体验到数学和物理的美感，也因为创刊时提出的“降低同学们对数学物理不必要的戒心与恐惧，给学科‘黑话’祛魅”

的目的，我们是更支持投稿不需要过多前置知识，哪怕当科普阅读也能有所收获的稿件的。但是由于创刊号我们能投稿的同学较为缺乏，出于刊物容量考虑，本期内部分稿件（尤其是物理）需要较多的前置知识；同时由于如上所述，社内多数同学都学习高能物理，本期没有凝聚态物理相关的稿件，也让我们觉得非常惭愧和遗憾。我们希望以后能接收到有余力和想法的同学们的积极投稿。我们欢迎包括但不限于对某个概念的介绍与讲解、优秀文章翻译乃至科幻小说的投稿！

作为校内刊物，我们支持中文或者英文的投稿。但由于本次投稿的同学们都相对更适应英文学术写作，本次投稿的内容都是英文。我们明白这会造成一些阅读上的困难。若有改善的建议或觉得有不当之处，欢迎读者们指正！

最后祝大家学习数学物理的旅途愉快！

目录

作者：林宏博

前置知识：线性代数、多元微积分、分析力学（可选）

根据爱因斯坦等效原理的表述，广义相对论即所谓“局部洛伦兹”时空的理论；换言之，广义相对论时空都局部相似于狭义相对论。因而理解狭义相对论的表述对理解广义相对论是至关重要的。然而许多狭义相对论及场论教材都会或多或少地夹杂电动力学等内容。这些内容虽同样重要，但对于过渡到广义相对论以领会其精神却稍显冗余。在本文中，作者将会从少量公设中快速建立狭义相对论的核心表述，并对时空的等距变换、协变性等在广义相对论中处于核心地位的概念作简要的梳理。

作者：赵云柯

前置知识：广义相对论

尽管爱因斯坦给出的等效原理与广义协变性为描述引力理论提供了重要的框架，这些思想仅仅限制了引力的表述形式，即几何，以及其他场与引力耦合的方式。对于一个完整的引力理论，尚缺乏引力自身动力学的描述。在本文中，作者将从引力的经典极限出发，给出引力的动力学应当满足的约束条件。最终，读者将了解到爱因斯坦引力为满足该约束条件的最简单的引力动力学。

作者：祁琼宇

前置知识：量子场论

在量子场论中，散射矩阵为处于所有理论核心地位的算符。这是因为在现实的高能物理实验中，其范式为对撞机进行的散射实验，而散射振幅为其唯一的可测量量。因此，建立对散射的描述对理解量子场论不可或缺。在含有相互作用的量子场论中，这一过程需要理解诸多细节。譬如：假定稳定的单粒子态（即不存在自然衰变），若该理论含有相互作用，应当如何囊括粒子的自相互作用以建立一个自洽的入射态描述？在本文中，作者将梳理建立量子场论中散射描述的过程。

FROM COMPLEX ANALYSIS TO THE GROUP STRUCTURE ON ELLIPTIC CURVE

JINGHUA XI

Abstract. This note aims to derive the group structure on elliptic curves using elementary tools such as Liouville's theorem and argument principle.

1. Lattice and Elliptic functions

Let $w_1, w_2 \in \mathbb{C}$ such that $\Im(w_1/w_2) > 0$. We define the *lattice* associated with w_1, w_2 as $L \coloneqq w_1\mathbb{Z} \oplus w_2\mathbb{Z} \subset \mathbb{C}$.

Associated to this lattice L, we define a *complex torus* as the quotient space $\mathbb{C}/L$. This space inherits the natural structure of a Riemann surface. Within this framework, a meromorphic function on the complex torus can be naturally identified with a meromorphic function on $\mathbb{C}$ possessing a double period. These are termed elliptic functions.

Specifically, an elliptic function associated with lattice L is defined as a meromorphic function f on $\mathbb{C}$ satisfying $f(x) = f(y)$ for any $x, y \in \mathbb{C}$ such that $x - y \in L$. We denote the set of all such functions by $\mathcal{E}_L$.

Remark 1.1. The set $\mathcal{E}_L$ forms a field closed under differentiation.

We introduce some notations initially. Let Π represent the parallelogram in $\mathbb{C}$, defined as $\{(sw_1, tw_2) : s, t \in [0, 1]\}$. For a given $\alpha \in \mathbb{C}$, consider a function f from the set $\mathcal{E}_L$, which exhibits neither zeros nor poles on the boundary $\partial(\alpha+\Pi)$, denoted as $\alpha+\Pi = \{\alpha+z : z \in \Pi\}$. The existence of such α is guaranteed, as zeros and poles in Π are discrete and thus finite, given Π's compactness.

Lemma 1.2. *For any $f \in \mathcal{E}_L$ that is holomorphic on $\partial(\alpha + \Pi)$, we have $\int_{\partial(\alpha+\Pi)} f = 0$.*

Proof. The integral $\int_{\partial(\alpha+\Pi)} f$ can be decomposed into the sum of integrals over each side of the parallelogram:

$$\int_{\alpha}^{\alpha+w_1} f + \int_{\alpha+w_1}^{\alpha+w_1+w_2} f + \int_{\alpha+w_1+w_2}^{\alpha+w_2} f + \int_{\alpha+w_2}^{\alpha} f.$$

Since f is elliptic, by change of variable

$$\int_{\alpha+w_1}^{\alpha+w_1+w_2} f(z)dz = \int_{\alpha}^{\alpha+w_2} f(z+w_1)d(z+w_1) = \int_{\alpha}^{\alpha+w_2} f(z)dz.$$

Likewise,

$$\int_{\alpha+w_2}^{\alpha+w_1+w_2} f(z)dz = \int_{\alpha}^{\alpha+w_1} f(z)dz.$$

This implies $\int_{\partial(\alpha+\Pi)} f = 0$. □

Key words and phrases. elliptic function, elliptic curve.

Proposition 1.3. *For any elliptic function $f \in \mathcal{E}_L$, there are equal numbers of zeros and poles, counted with multiplicities, within the parallelogram Π.*

Proof. It suffices to verifies this in $\alpha + \Pi$, since f is elliptic. Given f has neither zeros nor poles on $\partial(\alpha + \Pi)$, the Argument Principle implies

$$\#Z(f) - \#P(f) = \int_{\partial(\alpha+\Pi)} \frac{f'}{f}$$

, where $\#Z(f), \#P(f)$ denote zeros and poles of f up to multiplicities in Π respectively. Notice $\frac{f'}{f} \in \mathcal{E}_L$ thus it follows that

$$\#Z(f) - \#P(f) = \int_{\partial(\alpha+\Pi)} \frac{f'}{f} = 0$$

by Lemma1.2. $\square$

Proposition 1.4. *For every elliptic function $f \in \mathcal{E}_L$, the sum $\sum_{z\in\alpha+\Pi} ord_z(f) \cdot z$ is an element of the lattice L, where $ord_z(f)$ represents the order of f at the point z.*

Proof. By the generalized argument principle, we have

$$\sum_{z\in\alpha+\Pi} \mathrm{ord}_z(f) \cdot z = \frac{1}{2\pi i} \int_{\partial(\alpha+\Pi)} \frac{f'}{f} z dz.$$

We select α such that f does not vanish on the boundary. For integrations, consider

$$\int_{\alpha+w_1}^{\alpha+w_1+w_2} \frac{f'}{f} z dz = \int_{\alpha}^{\alpha+w_2} \frac{f'}{f}(t + w_1)dt = \int_{\alpha}^{\alpha+w_2} \frac{f'}{f} t dt + w_1 \int_{\alpha}^{\alpha+w_2} \frac{f'}{f} t dt.$$

Similarly,

$$\int_{\alpha+w_2}^{\alpha+w_1+w_2} \frac{f'}{f} z dz = \int_{\alpha}^{\alpha+w_1} \frac{f'}{f} t dt + w_2 \int_{\alpha}^{\alpha+w_1} \frac{f'}{f} t dt.$$

This implies

$$\sum_{z\in\alpha+\Pi} \mathrm{ord}_z(f) \cdot z = \frac{1}{2\pi i} \int_{\partial(\alpha+\Pi)} \frac{f'}{f} z dz = w_1 \int_{\alpha}^{\alpha+w_2} \frac{f'}{f} t dt + w_2 \int_{\alpha}^{\alpha+w_1} \frac{f'}{f} t dt.$$

On the other hand,

$$\frac{1}{2\pi i} \int_{\alpha}^{\alpha+w_1} \frac{f'}{f} dz = \frac{1}{2\pi i} \int_{f\gamma} \frac{dz}{z} \in \mathbb{Z}$$

and similarly

$$\frac{1}{2\pi i} \int_{\alpha}^{\alpha+w_2} \frac{f'}{f} dz \in \mathbb{Z}.$$

Consequently,

$$\sum_{z\in\alpha+\Pi} \mathrm{ord}_z(f) \cdot z \in L.$$

$\square$

Remark 1.5. Proposition 1.4 will be interpreted as a weighted average of zeros and poles of a given elliptic function.

2. Example of elliptic function: Weierstrass p-function

Define Weierstrass function

$$\wp(z) = \frac{1}{z^2} + \sum_{w \in L \setminus \{0\}} \frac{1}{(w-z)^2}.$$

It is evidently even. To see it is elliptic, consider

$$\wp'(z) = \sum_{w \in L} \frac{-2}{(z-w)^3},$$

which is elliptic and odd. Therefore,

$$\wp(z + w_1) = \wp(z) + \int_0^{w_1} \wp' = \wp(z) + \int_{-w_1/2}^{w_1/2} \wp' = \wp(z).$$

Likewise, we have

$$\wp(z + w_2) = \wp(z).$$

To sum up, $\wp \in \mathcal{E}_L$.

Remark 2.1.
This function is convergent; we will leave the details to readers. See also [2][Chapter 9].

Remark 2.2. Given $R > 0$, for $|z| < R$, consider

$$\wp(z) = \frac{1}{z^2} + \sum_{0<|w|<2R} + \sum_{|w|>2R} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right).$$

Then the second term is holomorphic and the first term exhibits poles of second degree at those lattice points in the disc $B(0, R)$. Therefore, $\wp$ has poles at L.

Next, we deduce an essential property between $\wp$ and $\wp'$.
To this end, let's revoke the identity

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1) x^k.$$

Thus, we have

$$\begin{aligned}
\wp(z) &= \frac{1}{z^2} + \sum_{w \in L^*} \left(\frac{1}{w^2} \left(\sum_{k=0}^{\infty} (k+1) \left(\frac{z}{w} \right)^k - \frac{1}{w^2} \right) \right) \\
&= \frac{1}{z^2} + \sum_{w \in L^*} \sum_{k=1}^{\infty} (k+1) \left(\frac{z^k}{w^{k+2}} \right) \\
&= \frac{1}{z^2} + \sum_{k=1}^{\infty} G_{k+2} (k+1) z^k \\
&= \frac{1}{z^2} + \sum_{k=1}^{\infty} G_{2k+2} (2k+1) z^{2k}.
\end{aligned}$$

Here, we denote by $G_k \coloneqq \sum_{w \in L \setminus \{0\}} \frac{1}{w^k}$.

Proposition 2.3.
$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3,$$
where $g_2 = 40G_4$ *and* $g_3 = 140G_6$.

Proof. Notice
$$\begin{aligned}
\wp &= \frac{1}{z^2} + 3G_4 z^2 + 5G_6 z^4 + \dots, \\
\wp^2 &= \frac{1}{z^4} + 6G_4 + 10G_6 z^2 + \dots, \\
\wp^3 &= \frac{1}{z^6} + \frac{9G_4}{z^2} + 15G_6 + \dots, \\
\wp' &= -\frac{2}{z^3} + 6G_4 z + 20G_6 z^3, \\
(\wp')^2 &= \frac{4}{z^6} - \frac{24G_4}{z^2} - 80G_6 + \dots.
\end{aligned}$$
Therefore, we find that
$$\begin{aligned}
(\wp')^2 - 4\wp^3 + 60G_4\wp &= \left(\frac{4-4}{z^6}\right) + \left(\frac{-24G_4 - 36G_4 + 60G_4}{z^2}\right) + (-80G_6 - 60G_6) + \dots \\
&= -140G_6 + \dots.
\end{aligned}$$
Since $(\wp')^2 - 4\wp^3 + 60G_4\wp$ is a holomorphic elliptic function, thus constant, it follows that
$$(\wp')^2 - 4\wp^3 + 60G_4\wp = -140G_6 = -g_3$$
by Liouville's theorem. □

Rational Parametrization and Group Structure

One can use the Weierstrass p-function to parametrize elliptic curves as follows:
$$\varphi : \mathbb{C}/L \to X(g_2(L), g_3(L)), \quad w \mapsto \begin{cases} [\wp(w) : \wp'(w) : 1], & w \notin L, \\ [0:1:0], & w \in L, \end{cases}$$
where $X(a, b)$ denotes the zero set of $F(X, Y, Z) = ZY^2 - 4X^3 + aXZ^2 + bZ^3$ in $\mathbb{P}^2$.

Fact: If $a^3 - 27b^2 \neq 0$, then there exists a lattice L with $g_2(L) = a$ and $g_3(L) = b$.

Proposition 2.4. *The map* φ *is an isomorphism between* $\mathbb{C}/L$ *and* $X(g_2(L), g_3(L))$.

Proof. Proof details can be adapted as needed. Here we focus on the lemmas and propositions formatting.

Lemma 2.5. *If* $\wp(w_1) = \wp(w_2)$*, then either* $w_1 - w_2$ *or* $w_1 + w_2$ *belongs to* L.

Lemma 2.6. *Let* X *be a compact Riemann surface and* $f \in \mathcal{M}(X)$*. If* f *is non-constant, then it is surjective.*

Injectivity: Suppose $w_1, w_2 \in \mathbb{C}/L$ with $\varphi(w_1) = \varphi(w_2)$. If $\wp(w_1) = \wp(w_2)$, then either $w_1 - w_2$ or $w_1 + w_2 \in L$. Assume $w_1 + w_2 \in L$. Since $\wp'$ is odd, $\wp'(w_1) = -\wp'(-w_1) = -\wp'(w_2)$. If this term is non-zero, it contradicts $\varphi(w_1) = \varphi(w_2)$. If this term is zero, then $2w_1 \in L$, which implies $w_1 = w_2$.

Surjectivity: Since $\wp$ is surjective onto $\mathbb{P}^1$, for any point $[x : y : 1] \in X(a, b) \cap U_2$, there exists a $w \in \mathbb{C}/L$ such that $\wp(w) = x$. By the formulas $(\wp')^2 = 4\wp^3 - a\wp - b$ and

$y^2 = 4x^3 - ax - b$, it follows that either $(\wp(w), \wp'(w))$ or $(\wp(-w), \wp'(-w))$ equals (x, y), ensuring surjectivity.

Holomorphicity: For $w \notin L$, choose any neighborhood W excluding the origin with chart α, and let U_2 be the standard coordinate neighborhood with chart π_2. The composition $\pi_2 \circ \varphi \circ \alpha^{-1} = (\wp(\cdot), \wp'(\cdot))$ is holomorphic. Similarly, for $w = 0$, choose a neighborhood W without zeros of $\wp'$, with chart β and using neighborhood U_0, we find

$$\pi_0 \circ \varphi \circ \beta^{-1} = (\frac{\wp(\cdot)}{\wp'(\cdot)}, \frac{1}{\wp'(\cdot)})$$

is also holomorphic. This confirms the holomorphicity of φ. □

3. Rational Parametrization and Group Structure

One can use the Weierstrass p-function to parametrize elliptic curves as follows:

$$\varphi : \mathbb{C}/L \to X(g_2(L), g_3(L)), \quad w \mapsto \begin{cases} [\wp(w) : \wp'(w) : 1], & w \notin L, \\ [0 : 1 : 0], & w \in L, \end{cases}$$

where $X(a, b)$ denotes the zero set of $F(X, Y, Z) = ZY^2 - 4X^3 + aXZ^2 + bZ^3$ in $\mathbb{P}^2$.

Fact: If $a^3 - 27b^2 \neq 0$, then there exists a lattice L with $g_2(L) = a$ and $g_3(L) = b$.

Proposition 3.1. *The map φ is an isomorphism between $\mathbb{C}/L$ and $X(g_2(L), g_3(L))$.*

Proof details can be adapted as needed. Here we focus on the lemmas and propositions formatting.

Lemma 3.2. *If $\wp(w_1) = \wp(w_2)$, then either $w_1 - w_2$ or $w_1 + w_2$ belongs to L.*

Proof. Suppose $w_1 - w_2 \notin L$. Consider the function $f(w) = \wp(w) - \wp(w_1) \in \mathcal{E}_L$. It has poles of second order at L. Therefore $f(w)$ has exactly two simple zeros at w_1, w_2 by Lemma 1.3. We can find $\alpha \in \mathbb{C}$ such that $\alpha + \Pi$ containing 0 and $\partial(\alpha + \Pi) \cap P(f) =$. We may choose $w_1', w_2' \in \alpha + \Pi$ where $w_1' - w_1, w_2' - w_2 \in L$. Revoking Lemma 1.4, we have that

$$\sum_{z \in \alpha + \Pi} ord_z(f) \cdot z = w_1' + w_2' - 2 \cdot 0 \in L$$

if $f \in \mathcal{E}_L$. That is $w_1 + w_2 \in L$. □

Lemma 3.3. *Let X be a compact Riemann surface and $f \in \mathcal{M}(X)$, the meromorphic function on X. If f is non-constant, then it is surjective.*

Proof. Confer [1][Theorem 2.7]. □

Proof of Proposition 3.1. **Injectivity:** Suppose $w_1, w_2 \in \mathbb{C}/L$ with $\varphi(w_1) = \varphi(w_2)$. If $\wp(w_1) = \wp(w_2)$, then either $w_1 - w_2$ or $w_1 + w_2 \in L$ by Lemma 3.2. Assume $w_1 + w_2 \in L$. Since $\wp'$ is odd, $\wp'(w_1) = -\wp'(-w_1) = -\wp'(w_2)$. If this term is non-zero, it contradicts $\varphi(w_1) = \varphi(w_2)$. If this term is zero, then $2w_1 \in L$, which implies $w_1 = w_2$.

Surjectivity: By Lemma 3.3, non-constant $\wp$ is surjective onto $\mathbb{P}^1$. Thus for any point $[x : y : 1] \in X(a, b) \cap U_2$, there exists a $w \in \mathbb{C}/L$ such that $\wp(w) = x$. By the formulas $(\wp')^2 = 4\wp^3 - a\wp - b$ and $y^2 = 4x^3 - ax - b$, it follows that either $(\wp(w), \wp'(w))$ or $(\wp(-w), \wp'(-w))$ equals (x, y), ensuring surjectivity.

Holomorphicity: For $w \notin L$, choose any neighborhood W excluding the origin with chart α, and let U_2 be the standard coordinate neighborhood with chart π_2. The composition

$\pi_2 \circ \varphi \circ \alpha^{-1} = (\wp(\cdot), \wp'(\cdot))$ is holomorphic. Similarly, for $w = 0$, choose a neighborhood W without zeros of $\wp'$, with chart β and using neighborhood U_0, we find

$$\pi_0 \circ \varphi \circ \beta^{-1} = (\frac{\wp(\cdot)}{\wp'(\cdot)}, \frac{1}{\wp'(\cdot)})$$

is also holomorphic. This confirms the holomorphicity of φ. □

For the moment, we are trying to transfer the Abelian group structure from $(\mathbb{C}/L, +)$ to $X(a, b)$.

Lemma 3.4. *If $u, v, w \in \mathbb{C}$ with $u + v + w \in L$ and $\wp(u) \neq \wp(v)$, then*

$$\det \begin{pmatrix} 1 & \wp(u) & \wp'(u) \\ 1 & \wp(v) & \wp'(v) \\ 1 & \wp(w) & \wp'(w) \end{pmatrix} = 0.$$

Proof. If $\wp(u) \neq \wp(v)$, then there exists a unique pair $(a, b) \in \mathbb{C}^2$ such that

$$\begin{cases} b + a\wp'(u) = \wp(u), \\ b + a\wp'(v) = \wp(v). \end{cases} \tag{3.1}$$

On one hand, u, v are two distinct roots of the elliptic function $f = b + a\wp'(u) - \wp(u)$. This function, having poles of multiplicity three at L, must also have an extra simple zero.

Let us find $\alpha \in \mathbb{C}$ such that $\alpha + \Pi$ contains 0 and $\partial(\alpha + \Pi) \cap P(f) = \emptyset$. Choose $u', v' \in \alpha + \Pi$ where $u' - u, v' - v \in L$. Referring to the lemma above, we have $\sum_{z \in \alpha + \Pi} \mathrm{ord}_z(f) \cdot z = 0$ if $f \in \mathcal{E}_L$. If w' is the third zero of f, then $u' + v' + w' + 3 \cdot 0 = 0$. Hence, $w' = -(u' + v')$ and thus $w' - w \in L$. This results in the system

$$\begin{cases} b + a\wp'(u) = \wp(u), \\ b + a\wp'(v) = \wp(v), \\ b + a\wp'(-(u + v)) = \wp(-(u + v)), \end{cases} \tag{3.2}$$

being linearly dependent, or equivalently,

$$\det \begin{pmatrix} 1 & \wp(u) & \wp'(u) \\ 1 & \wp(v) & \wp'(v) \\ 1 & \wp(u + v) & -\wp'(u + v) \end{pmatrix} = 0.$$

□

To transition the group structure from $\mathbb{C}/L$ to $X(a, b)$, we define the operation $p + q = \varphi(\varphi^{-1}(p) + \varphi^{-1}(q))$ with the identity element $o \coloneqq \varphi^{-1}(0) = [0 : 1 : 0]$. Notice, the inverse of $p = [\wp(w), \wp'(w) : 1]$ is given by $-p = [\wp(-w), \wp'(-w) : 1] = [\wp(w), -\wp'(w) : 1]$.

Remark 3.5. We reinterpret the lemma in a more geometric manner: if $p + q + r = o$, then they are collinear. This confirms the group structure on $X(a, b)$ as dispicted in Figure 3.1.

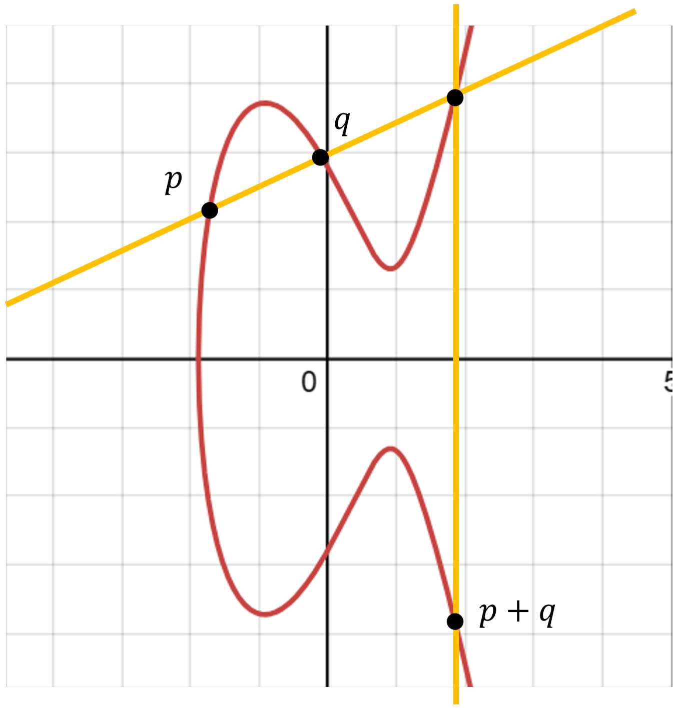

FIGURE 3.1. Group law on elliptic curve

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A BRIEF INTRODUCTION TO HYPERBOLIC DYNAMICS

DAVID ZHU

Abstract. 在统计物理学以及一些信息类学科中, 遍历性 (ergodicity) 是一个重要假设. 数学家们已经构造出了很多遍历的模型, 但是在实验中, 我们实际上是使用一系列遍历模型去拟合所研究问题. 因此, 数学家们非常关心具有鲁棒性的遍历模型, 也就是说, 关心微小扰动不影响遍历性的模型. 这些具有鲁棒性的遍历模型被称为稳定遍历的 (stably ergodic). 本文将介绍的双曲动力系统 (hyperbolic dynamical system) 就是一类具有稳定遍历性的系统. 最后我们将通过 Smale 马蹄模型直观展示双曲动力系统轨道的混沌程度.

“お願い! 要我怎么做你们才愿意学动力系统? 只要是我能做的, 我什么都愿意做! ”

注: 本文初版写于 2023 年底至 2024 年初. 当时作者刚看完 MyGo (不包含其续作, Ave Mujica 这种成绩, 使人汗颜), 略上头, 所以有了如下的开头. 初版里面有很多繁琐的技术性细节, 本版进行了适当删节. 如果读者感兴趣, 可以查阅参考书籍.

1. Hyperbolicity

As in linear algebra, take some matrices A with eigenvalues λ and μ s.t. $0 < \lambda < 1 < \mu$. Let E_λ and E_μ be the eigenspaces of λ and μ, respectively. For any nonzero vector $v_\lambda \in E_\lambda$, $A^n v_\lambda = \lambda^n v_\lambda$, so $\|A^n v_\lambda\| \to 0$ as n iterates to infinity, that is, eigenvectors of $\lambda < 1$ contract to 0. Similarly, eigenvectors of $\mu > 1$ expand to ∞. We can extend this phenomenon to compact smooth manifolds.

1.1. **Basic Definitions.** Let M be a compact smooth manifold. A C^1-diffeomorphism $f : M \to M$ is called ***hyperbolic*** (双曲的) if there are $0 < \lambda < 1 < \mu$, $C > 0$, and families of linear subspaces $E^s(p) \subset T_pM$ and $E^u(p) \subset T_pM$ for every $p \in M$, such that

(1) $T_pM = E^s(p) \oplus E^u(p)$,
(2) $\|df_p^n v^s\| \leqslant C\lambda^n \|v^s\|$ for every $v^s \in E^s(p)$ and $n \in \mathbb{N} \cup \{0\}$,
(3) $\|df_p^{-n} v^u\| \leqslant C\mu^{-n} \|v^u\|$ for every $v^u \in E^u(p)$ and $n \in \mathbb{N} \cup \{0\}$,
(4) $df_p E^s(x) = E^s(f(p))$ and $df_p E^u(p) = E^u(f(p))$.

The subspace $E^s(p)$ is called the ***stable subspace*** (稳定子空间) at p; the subspace $E^u(p)$ is called the ***unstable subspace*** (不稳定子空间) at p. The eigenvalues of df_p are not on the unit circle of $\mathbb{C}$. The spaces $E^s(p)$ and $E^u(p)$ may not be (but not far from) eigenspaces of

df_p. The ***stable*** and ***unstable manifolds*** of $p \in M$ are defined by

$$\mathcal{W}^s(p) = \{q \in M : \ d(f^n(p), f^n(q)) \to 0 \text{ as } n \to \infty\},$$
$$\mathcal{W}^u(p) = \{q \in M : \ d(f^{-n}(p), f^{-n}(q)) \to 0 \text{ as } n \to \infty\},$$

respectively.

1.2. **Basic Properties.** These properties are highly nontrivial, so we omit the proofs. It is essential to build up the geometric intuition. The first theorem illustrates that the dynamics of a hyperbolic differomorphism is similar to that of their differential (a linear transformation).

Theorem 1.1 (Hartman-Grobman, Theorem 2.13 of [Wen15])**.** *Suppose p is a fixed point of a C^1 hyperbolic diffeomorphism f. Then there is a neighborhood of p in M and V of 0 in T_pM and a homeomorphism $h : U \to V$ s.t. $df_p \circ h = h \circ f$.*

The next theorem shows that the stable and unstable manifolds are locally embedded discs (graphs of some uniformly Lipschitz continuous functions) tangent to the stable and unstable subspaces, respectively.

Theorem 1.2 (Hadamard-Perron, Theorem 6.2.8. of [KH95] and Corollary 5.6.6 of [BS02])**.** *The stable and unstable manifolds are immersed C^1 invariant submanifolds of M and are locally embedded unit discs in corresponding dimensions. Moreover, $T_p\mathcal{W}^s(p) = E^s(p)$ and $T_p\mathcal{W}^u(p) = E^u(p)$.*

The manifolds are immersed but not (globally) embedded because they might be dense in M. Now for any point $p \in M$, we can define its stable (unstable) subspaces and manifolds. But what are the relations of those subspaces and manifolds among different points on M?

Theorem 1.3 (Theorem 5.20 of [BP20])**.** *If f is a $C^{1+\alpha}$ hyperbolic differomorphism (df Hölder continuous), then E^s and E^u are Hölder continuous f-invariant subbundles of the tangent bundle TM with coefficient β s.t. $TM \cong E^s \oplus E^u$. Here*

$$\beta = \frac{\ln \mu - \ln \lambda}{\ln a - \ln \lambda}\alpha$$

where $\lambda < 1 < \mu$ are the hyperbolic coefficients and $a > \max_{x \in M} \|df_x\|^{1+\alpha}$.

$\mathcal{W}^s$ and $\mathcal{W}^u$ are transversely absolutely continuous foliations (explain later).

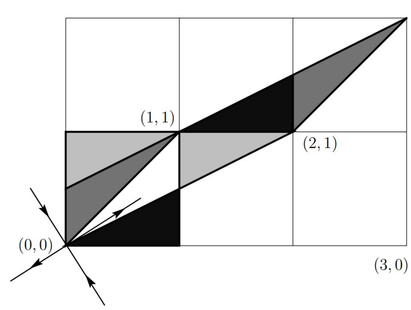

Figure 1.1. The Arnold cat map

1.3. **Hyperbolic Toral Automorphisms.** For example, let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ act on the torus $\mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ by

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y & \mod 1 \\ x + y & \mod 1 \end{pmatrix}.$$

The eigenvalues of A are $\lambda = \frac{3-\sqrt{5}}{2}$ and $\mu = \frac{3+\sqrt{5}}{2}$. Then for any $p \in \mathbb{T}^2$, $E^s(p)$ and $E^u(p)$ are eigenspaces of λ and μ, respectively. That is,

$$E^s(p) = \mathrm{Span} \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix} \text{ and } E^u(p) = \mathrm{Span} \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}.$$

Finally, the stable and unstable manifolds are given by $\mathcal{W}^s(p) = \pi(E^s(p))$ and $\mathcal{W}^u(p) = \pi(E^u(p))$, respectively.

The set $\mathrm{Per}(A)$ of periodic points of A is dense in $\mathbb{T}^2$.

2. Ergodicity and Hyperbolicity

Figure 2.1. Boltzmann, Ehrenfest, Poincaré, and Birkhoff

L. E. Boltzmann (1844–1906) and his student P. Ehrenfest (1880–1933) once tried to base statistical physics entirely on mechanics. They put forward the "ergodic hypothesis" (遍历性假说), which can be expressed as: "For an isolated conservative mechanical system, any initial condition can be sufficiently close to almost every point on the energy surface after a long enough time."

2.1. **Basic Definitions.** Let f be a volume-preserving map on a probability space (X, μ), i.e., $\mu(A) = \mu(f^{-1}A)$ for any measurable $A \subset X$ (usually equivalent to Lebesgue, so we say "volume"; the physics background of the volume-preserving property comes from the Liouville theorem, see [Kar07, Section 3.2]). A measurable function $\varphi : X \to \mathbb{R}$ is ***essentially f-invariant*** if $\varphi(f^n(x)) = \varphi(x)$ for μ-a.e. points of X and any $n \in \mathbb{N}$. A measurable set A is ***essentially f-invariant*** if its characteristic function χ_A is essentially f-invariant or equivalently the iterates of A only differ by a null set.

A volume-preserving map f is ***ergodic*** (遍历的) if any essentially f-invariant measurable set has either measure 0 or full measure.

Let (X, μ) be a Borel probability space, the ***support*** (支集) of μ, denoted by $\mathrm{supp}\mu$, is the smallest full measure closed subset in X. We then obtain a result that Boltzmann and Ehrenfest wanted.

Theorem 2.1 (Transitivity, Proposition 4.3.4 of [BS02]). *Let (X, μ) be a compact metric Borel probability space, f a continuous volume-preserving map. If f is ergodic, then the orbit of μ-a.e. point is dense in* $\mathrm{supp}\mu$. *The converse may not hold.*

Proof. Let $U \neq \emptyset$ be an open subset in suppμ, then $\mu(U) > 0$. By the ergodicity, the essentially f-invariant set $\bigcup_{n=0}^{\infty} f^{-n}(U)$ has full measure, thus the forward orbit of μ-a.e. point in X visits U. Since X is a compact metric space, it is separable and has a countable topological basis $\{U_i\}_{i\in\mathbb{N}}$. Notice that the forward orbit of μ-a.e. point visits U_i, for all i, it follows that the set of points whose forward orbit visits every U_i has full measure in X. □

By the Krylov-Bogolyubov theorem [BS02, Theorem 4.6.1], there is always some invariant measures under the hypotheses of Theorem 2.1. In fact, all the invariant Borel probability measures form a compact convex set with respect to the weak* topology whose extreme points are exactly the ergodic measures. The converse of Theorem 2.1 fails if the system (X, f) has more then one ergodic measures and the given invariant measure μ is not an extreme point. For a counterexample, see for instance [Fur61].

Theorem 2.2 (Proposition 4.3.1. of [BS02])**.** *Let f be a volume-preserving map on a probability space (X,μ). If $0 < p \leqslant \infty$, then f is ergodic $\Longleftrightarrow$ every essentially f-invariant function $\varphi \in L^p_\mu(X)$ is μ-a.e. constant.*

Proof. "$\Leftarrow$" For any essentially f-invariant measurable set A, $\mu(A) \leqslant 1$, then $\chi_A \in L^p_\mu(X)$ for $0 < p \leqslant \infty$. By the essential f-invariance of χ_A, we obtain A is a null or full set. Then f is ergodic.

Conversely, if f is ergodic, let $\varphi \in L^p_\mu(X)$ be an essentially f-invariant function. For any $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, let

$$X_k^n = \left\{ x \in X : \frac{k}{2^n} \leqslant \varphi(x) < \frac{k+1}{2^n} \right\},$$

then $f^{-1}(X_k^n)\Delta X_k^n \subset \{x \in X : \varphi \circ f(x) \neq f(x)\}$, which implies $\mu(f^{-1}(X_k^n)\Delta X_k^n) = 0$. By the ergodicity, $\mu(X_k^n) = 0$ or 1. For every n, $X = \coprod_{k\in\mathbb{Z}} X_k^n$, then there exists a unique k_n s.t. $\mu(X_{k_n}^n) = 1$. Let $Y = \bigcap_{n=1}^{\infty} X_{k_n}^n$, then $\mu(Y) = 1$ and φ is constant on Y, i.e., φ is μ-a.e. constant. □

2.2. **The Birkhoff Theorem.** G. D. Birkhoff (1884–1944) was an American mathematician. His ergodic theorem discovered in 1931 transformed the Maxwell–Boltzmann Kinetic Theory of gases into a rigorous principle through the use of measure theory.

Let $\mathcal{O}(q_1, ..., p_s)$ be any observable mechanical quantity. We may consider $\mathcal{O}$ as a function $X \to \mathbb{R}$. Moreover, since the generalized positions q_i and momentum p_i change with time, we define a flow $f : X \times [0, \infty) \to X$ by $f(x, t) = f^t(x)$, where $x = (q_1, ..., p_s)$ is an abbreviation. Notice that the observing times are discrete, suppose $\mathbb{N}$. So f can also be regarded as a continuous map.

Theorem 2.3 (The Birkhoff Ergodic Theorem, Theorem 4.5.5 of [BS02])**.** *Let f be a volume-preserving map in a Borel probability space (X,μ), and let $\mathcal{O} \in L^1_\mu(X)$. Then the time average*

$$\langle \mathcal{O} \rangle (x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathcal{O}(f^k(x))$$

exists for μ-a.e. $x \in X$ and satisfies

1. *Integrability: $\langle \mathcal{O} \rangle \in L^1_\mu(X)$,*
2. *φ-invariance: $\langle \mathcal{O} \rangle (f^n(x)) = \langle \mathcal{O} \rangle (x)$ for all $n \in \mathbb{N}$ and x for which $\langle \mathcal{O} \rangle$ is defined,*
3. *$\displaystyle\int_X \langle \mathcal{O} \rangle \, d\mu = \int_X \mathcal{O} d\mu$. In particular, if f is ergodic, then the time average $\langle \mathcal{O} \rangle$ is μ-a.e. constant and equals to the space average $\int_X \mathcal{O} d\mu$.*

(4) *If f is invertible, the backward time average is μ-a.e. equal to the forward time average.*

The physics background of Theorem 2.3 can be found in [Kar07, Section 3.1]. The proof of the Birkhoff Ergodic Theorem is nothing more than real analysis and is quite annoying. But if you want to study dynamical systems, it is better to go through the proof once by yourselves.

It is vital to emphasize that the ergodic hypothesis, is flawed.

(i) The system may be far from ergodicity. The orbits can be very large but do not pass a full measure subset. Indeed, it is a hot issue to find ergodic (or further mixing, unique ergodic, stably ergodic,...) systems in mathematics.
(ii) Even if the system is ergodic, the certainty of the result is only demonstrated for exceedingly long time intervals, and no assurance as to the pertinence of the result with respect to the short time intervals involved in actual experimentation is provided. When I asked my friends in the physics department about this, they told me "宏观小, 微观大".
(iii) Even if the system is ergodic, we do not know its robustness (鲁棒性, 稳定性). There exists some small errors in experiments. We want to maintain the ergodicity of the system after some error occurs. That is why we study the stable ergodicity.

Physicists also put forward ***Boltzmann's assumption of equal a priori equilibrium probabilities*** (a corollary of the ergodic hypothesis which has good physics). We know that in one orbit, the density is invariant. What about different orbits? For example, in ***microcanonical ensemble*** (微正则系综), for fixed E, the central postulate of statistical mechanics is that the equilibrium probability distribution is given by

$$f_E(q,p) = \frac{1}{\Omega(E)}\chi(H(q,p)=E) \tag{2.1}$$

where Ω is the normalization factor and χ the characteristic function on the level set $H(q,p) = E$. It is sometimes more convenient to define the microcanonical ensembles by requiring $E - \Delta E \leqslant H(q,p) \leqslant E + \Delta E$ for some small enough ΔE [Kar07, Section 4.2].

2.3. **The Hopf Argument.** Yau Shing-Tung: "I always like to learn more details about it (the Hopf argument) [Wil24]". The following result is proved by D. V. Anosov (1936–2014) and Ya. G. Sinai (1935–) in the 1960s.

Theorem 2.4 (The Hopf Argument, [AS67])**.** *A $C^{1+\alpha}$ hyperbolic diffeomorphism preserving a smooth probability measure (equivalent to Lebesgue) is ergodic.*

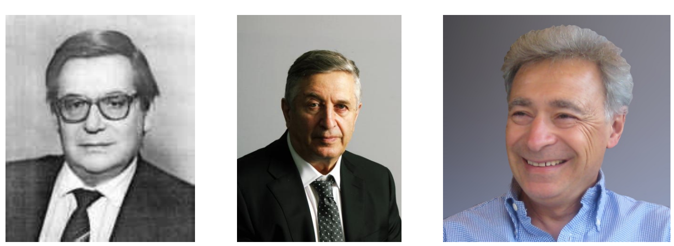

FIGURE 2.2. Anosov, Sinai, and Pesin

Sketch of the Proof. We apply Theorem 2.3 on the stable (unstable) manifolds. For any measurable $\mathcal{O} : M \to \mathbb{R}$, the forward Birkhoff time average

$$\langle \mathcal{O} \rangle^s (x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathcal{O}(f^k(x))$$

is a.e. constant on each stable manifold; the backward Birkhoff time average

$$\langle \mathcal{O} \rangle^u (x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathcal{O}(f^{-k}(x))$$

is a.e. constant on each unstable manifold. Theorem 2.3 implies that $\langle \mathcal{O} \rangle^s = \langle \mathcal{O} \rangle^u$ μ-a.e.. We simplify our notation by $\langle \mathcal{O} \rangle = \langle \mathcal{O} \rangle^s = \langle \mathcal{O} \rangle^u$ on a full measure set of M. If we can show that $\langle \mathcal{O} \rangle$ is μ-a.e. constant on M, then let $\mathcal{O}$ be a characteristic function, we obtain f to be ergodic. We know (1) in $\mathbb{R}^2$, the product of two null sets is also a null set (or more generally, the Fubini theorem); (2) the local stable and unstable manifolds form a local product structure. Can we have something similar to the Fubini theorem in a foliation box?

We next introduce some measures on the stable leaves. Let $\mathcal{F}^s \subset M$ be a foliation box of local stable manifolds $\mathcal{W}^s_{\text{loc}}(x)$ (you can regard $\mathcal{F}^s$ as some special open set and $\mathcal{W}^s_{\text{loc}}(x)$ as some discs). The restriction of the Riemannian metric in $\mathcal{W}^s_{\text{loc}}(x)$ induces a volume $\mathfrak{m}^s_x$ called the ***leaf volume on the stable manifold***. If μ is smooth, it induces a unique conditional measure $\mu^s(\cdot|x)$ [PU10, Section 1.6] that behaves almost the same as the leaf volume $\mathfrak{m}^s_x$ on $\mathcal{W}^s_{\text{loc}}(x)$. Similar for the unstable cases.

For x_1 and x_2 close enough, $x_2 \in \mathcal{W}^s(x_1)$. Let $h^s : \mathcal{W}^u_{\text{loc}}(x_1) \to \mathcal{W}^u_{\text{loc}}(x_2)$ be the holonomy along stable leaves. To apply the Fubini Theorem, we need some uniform $K > 0$ independent of the choice of $x_1 \in M$ such that for any small enough τ

$$K^{-1} \leqslant \frac{\mathfrak{m}^u_{x_2}(h^s(\mathbb{B}^u(x_1,\tau)))}{\mathfrak{m}^u_{x_1}(\mathbb{B}^u(x_1,\tau))} \leqslant K$$

where $\mathbb{B}^u(x,\tau)$ is a disc in $\mathcal{W}^u(x)$ centering at x of radius τ and $\mathfrak{m}^u_x$ the leave volume on $\mathcal{W}^u_{\text{loc}}(x)$. Letting $\tau \to 0$, we have

$$K^{-1} \leqslant \operatorname{Jac}(h^s, x_1) \leqslant K$$

We now try to approximate the holonomy map h^s such that the Jacobian is more computable. By the Whitney embedding theorem, we can embed M into a Euclidean space. Define $\pi_n : f^n\mathcal{W}^u_{\text{loc}}(x_1) \to f^n\mathcal{W}^u_{\text{loc}}(x_2)$ as follows: for $y \in \mathcal{W}^u_{\text{loc}}(x_1)$, $\pi_n(y)$ is the unique intersection point of the affine plane $f^n(y)+E^s(y)\oplus(T_yM)^\perp$ with $f^n\mathcal{W}^u(x_2)$ that is the closest to $f^n\circ h^s(y)$ along $f^n\mathcal{W}^u(x_2)$. Check that $f^{-n}\circ\pi_n\circ f^n \to h^s$ as $n\to\infty$ and

$$\begin{aligned}\operatorname{Jac}(f^{-n}\circ\pi_n\circ f^n, x_1) = &\left(\prod_{k=0}^{n-1}\left(\operatorname{Jac}(f, f^{k-n}\circ\pi_n\circ f^n(x_1))|_{E^u}\right)^{-1}\right)\\ &\times \operatorname{Jac}(\pi_n, f^n(x_1)) \times \left(\prod_{k=0}^{n-1}\operatorname{Jac}(f, f^k(x_1))|_{E^u}\right)\end{aligned}$$

Since $f^n\mathcal{W}^u_{\text{loc}}(x_1)$ and $f^n\mathcal{W}^u_{\text{loc}}(x_2)$ get closer and closer as $n\to\infty$, we have $\operatorname{Jac}(\pi_n, f^n(x_1)) \to \operatorname{Id}$ and

$$\operatorname{Jac}(h^s, x_1) = \prod_{k=0}^{\infty}\frac{\operatorname{Jac}(f, f^k(x_1))|_{E^u}}{\operatorname{Jac}(f, f^k(x_2))|_{E^u}}$$

By Theorem 1.3, we may bound the Jacobian of f on the unstable direction by

$$\begin{aligned}\|\log \operatorname{Jac}(f, f^k(x_1))|_{E^u} - \log \operatorname{Jac}(f, f^k(x_2))|_{E^u}\| &\leqslant C_1 \cdot \operatorname{dist}\left(f^k(x_1), f^k(x_2)\right)^\beta \\ &\leqslant C_1 \lambda^{\beta k} \cdot \operatorname{dist}(x_1, x_2)^\beta\end{aligned}$$

So now for each $n \in \mathbb{N}$ and

$$\begin{aligned}\prod_{k=0}^{n} \frac{\operatorname{Jac}(f, f^k(x_1))|_{E^u}}{\operatorname{Jac}(f, f^k(x_2))|_{E^u}} &\leqslant C_2 \exp \sum_{k=0}^{n} \|\log \operatorname{Jac}(f, f^k(x_1))|_{E^u} - \log \operatorname{Jac}(f, f^k(x_2))|_{E^u}\| \\ &\leqslant C_1 C_2 \cdot \operatorname{dist}(x_1, x_2)^\beta \exp \sum_{k=0}^{n} \lambda^{\beta k} < \infty\end{aligned}$$

So $\operatorname{Jac}(h^s, x_1)$ is below from ∞. Symmetrically, we obtain $\operatorname{Jac}(h^s, x_1)$ is bounded above 0. By some technical discussions, we obtain:

Theorem 2.5 (Absolute Continuity)**.** *Once the leaf volume $\mathfrak{m}_x^u$ of some local unstable leaf $\mathcal{W}_{\mathrm{loc}}^u(x)$ to a foliation box $\mathcal{F}^s$ of local stable manifolds satisfies $\mathfrak{m}_x^u(\mathcal{F}^s \cap \mathcal{W}_{\mathrm{loc}}^u(x)) > 0$, then it also holds for any other local unstable leaves.*

Then the Hopf argument is proved. □

For more details of Theorem 2.4, see (1) a summer school course of ICTP by Jana Rodriguez-Hertz from Sustech [RH18]; and (2) a lecture of CMSA/Tsinghua Math-Science Literature Lecture by Amie Wilkinson from University of Chicago [Wil24].

2.4. **Stable Ergodicity of Hyperbolic Dynamical Systems.** Hyperbolic diffeomorphisms are robust, that is, the set of Anosov diffeomorphisms of a given compact manifold M is open in the space of all C^1-diffeomorphisms $\operatorname{Diff}^1(M)$ [BS02, Corollary 5.5.2]. And by Theorem 2.4, we know that a $C^{1+\alpha}$ volume-preserving hyperbolic diffeomorphism is stably ergodic, i.e., it is in the $C^{r \geqslant 2}$-interior of the set of all volume-preserving ergodic diffeomorphisms.

2.5. **Nonuniform Hyperbolicity.** The (uniform) hyperbolic dynamical systems have plenty of good properties. However, (uniform) hyperbolity is an extremely strong condition that imposes strict restrictions on the topology of the background space. In the 1970's, Yakov Pesin generalized this setting into the ***nonuniform hyperbolicity***. See for instance [BP20].

Under the assumption of good regularity and hyperbolicity, the complexity (metric entropy $h_\mu(f)$) of the system with respect to the volume μ is given by the mean total expansion of the system with respect to μ:

$$h_\mu(f) = \int_M \sum_{i:\ \chi_i(x) > 0} k_i(x) \chi_i(x) d\mu(x) = \int_M \ln \left| \operatorname{Jac}(df|_{E^u(x)}) \right| d\mu(x)$$

where $\chi_i(x)$ is the ith Lyapunov exponent at x (average growth rate of df along the orbit of x) and $k_i(x)$ is the corresponding multiplicity.

3. Finding a Horseshoe on the Beaches of Rio

Maybe it is a bit tiring when coming through the mathematical discussions above. So now we let us take a break. I will introduce the story of how Stephen Smale (1930–) found the horseshoes [Sma98] which play an essential role in dynamics.

Smale was born in Flint, Michigan and entered the University of Michigan in 1948. Initially, he was a good student, placing into an honors calculus sequence taught by Bob Thrall and


earning himself A's. However, his sophomore and junior years were marred with mediocre grades, mostly B's, C's and even an F in nuclear physics. Smale obtained his Bachelor of Science degree in 1952. Despite his grades, with some luck, Smale was accepted as a graduate student at the University of Michigan's mathematics department. Yet again, Smale performed poorly in his first years, earning a C average as a graduate student. When the department chair, Theophil H. Hildebrandt (1888–1980), threatened to kick Smale out, he began to take his studies more seriously.

Smale finally finished his PhD thesis in 1956, under Raoul Bott (1923–2005), beginning his career as an instructor at the University of Chicago. During that summer, Smale, with his wife, Clara, attended in Mexico City a conference with the world's notables in topology present and giving lectures. There he met a Brazilian, Elon L. Lima (1929–2017), who was writing a thesis in topology at the University of Chicago where Smale was about to take up the position of instructor and they became good friends. A couple of years later, Lima introduced Smale to Maurício M. Peixoto, a young visiting professor from Brazil. Peixoto was working in the subject of differential equations or dynamics and showed Smale some beautiful results. Before long Smale had proved some theorems in dynamics.

In December, 1959, Lima and Peixoto invited Smale to Rio de Janeiro. So Smale and his family left the US and flew down to Rio. In front of their apartment, away from the hill, lay the famous beach of Copacabana – the "beach office" of Stephen Smale. Smale would spend his mornings on that wide, beautiful, sandy beach, swimming and body surfing. Also, he took a pen and paper and would work on mathematics. In a typical afternoon Smale would take a bus to IMPA (Instituto Nacional de Matemática, Pura e Aplicada, 巴西国家数学与应用数学研究所) and soon be discussing topology with Lima or dynamics with Peixoto, or be browsing in the library.

"Mathematics research typically doesn't require much – a pad of paper and a ballpoint pen, library resources, and colleagues to query. I was satisfied. Especially enjoyable were the times spent on the beach. My work was mostly scribbling down ideas and trying to see how arguments could be put together. I would sketch crude diagrams of geometric objects flowing through space, and try to link the pictures with formal deductions. Deep in this kind of thinking and writing on a pad of paper, I was not bothered by the distractions of the beach. It was good to be able to take time off from the research to swim." [Sma98]

At that time, as a topologist, Smale prided himself on a paper that he had just published in dynamics. He was delighted with a conjecture in that paper which had as a consequence that "chaos doesn't exist"! This euphoria was soon shattered by a letter from Norman Levinson (1912–1975). Levinson stated an earlier result of his which effectively contained

Figure 3.1. Beach of Copacabana in Rio de Janeiro

a counterexample to Smale's conjecture. Smale worked day and night to try to resolve the challenge to his beliefs that the letter posed. He eventually convinced himself that indeed Levinson was correct. But while learning that, Smale discovered the horseshoe.

"Sometimes a horseshoe is considered an omen of good luck The horseshoe I found on the beach of Rio certainly seemed to have such a property." [Sma98]

In that spring of 1960 Smale was motivated by the problems of topology, and most of all driven by the great unsolved problem posed by Poincaré. Now on those beaches, within two months of finding the horseshoe, Smale found to his amazement an idea which seemed to succeed provided him returned to Poincare's original assertion and then restricted the dimension to 5 or more. In fact the idea not only led to a solution of Poincaré's Conjecture in dimensions greater than 4, but it gave rise to a large number of other nice results in topology. It was for this work that Smale received the Fields Medal.

In 1960, Smale received a Sloan Research Fellowship and was appointed to the Berkeley mathematics faculty, moving to a professorship at Columbia the following year. In 1964 he returned to a professorship at Berkeley, where he has spent the main part of his career. In 1966, having travelled to Moscow under an NSF (National Science Foundation, 美国国家科学基金会) grant to accept the Fields Medal, Smale held a press conference there to denounce the American position in Vietnam, Soviet intervention in Hungary, and Soviet maltreatment of intellectuals, see for instance [Sma66]. After his return to the US, he was unable to renew the grant. At one time he was subpoenaed by the HUAC (House Un-American Activities Committee, 众议院非美活动委员会). Smale became a professor emeritus at Berkeley in 1995 and took up a post as professor at the City University of Hong Kong. He also amassed over the years one of the finest private mineral collections in existence. Many of Smale's mineral specimens can be seen in the book – "*The Smale Collection: Beauty in Natural Crystals*".

4. Smale Horseshoes and Poincaré Homoclinic Tangles

In this section, we will show how chaotic can an orbit of a hyperbolic diffeomorphism be.

FIGURE 3.2. Smale came to Sustech in June 2018
https://newshub.sustech.edu.cn/en/html/201807/7663.html

4.1. **Coding – The Most Chaotic System.** Consider the sequence spaces

$$\Sigma_2 := \{\omega = (\omega_n)_n : \ \omega_n = 0, 1, \ \forall \ n \in \mathbb{Z}\} = \{0, 1\}^{\mathbb{Z}},$$

and $\sigma : \Sigma_2 \to \Sigma_2$, the ***shift map*** (移位映射) of Σ_2, is defined by $(\omega_n)_n \mapsto (\omega_{n+1})_n$. Fix k integers $n_1, ..., n_k \in \mathbb{Z}$ and numbers $\alpha_1, ..., \alpha_k \in \{0, 1\}$, we call the set

$$C^{n_1,...,n_k}_{\alpha_1,...,\alpha_k} := \{\omega = (\omega_n)_n \in \Sigma_2 : \ \omega_{n_i} = \alpha_i \text{ for } i = 1, ..., k\}$$

a ***cylinder*** (柱集). An alternative way to define the topology in Σ_2 is by declearing that all cylinders are open sets and that they form a subbasis for the topology. The shift map σ is continuous with respect to this topology.

4.2. **Smale Horseshoes – Mathematical Descriptions.** Consider a rectangle $\Delta = \mathbb{D}^s \times \mathbb{D}^u \subset \mathbb{R}^s \oplus \mathbb{R}^u$, where $\mathbb{D}^s \subset \mathbb{R}^s$ and $\mathbb{D}^u \subset \mathbb{R}^u$ are disks. We denote by $\pi_s : \mathbb{R}^{s+u} \to \mathbb{R}^s$ and $\pi_u : \mathbb{R}^{s+u} \to \mathbb{R}^u$ the canonical projections. We say that a C^1-map $f : \Delta \to \mathbb{R}^{s+u}$ has a ***horseshoe*** if there are $0 < \lambda < 1 < \mu$ such that

(1) f is diffeomorphic onto its image;
(2) $f(\Delta) \cap \Delta$ has two components Δ_0 and Δ_1;
(3) if $x \in \Delta$ and $f(x) \in \Delta_i$, then the sets $G^s_i(x) = f^{-1}(\mathbb{D}^s \times \{\pi^u(x)\}) \cap \Delta_i$ and $G^u_i(x) = f(\{\pi_s(x) \cap \mathbb{D}^u\}) \cap \Delta_i$ are connected, and the restrictions $\pi_s|_{G^s_i(x)}$ and $\pi_u|_{G^u_i(x)}$ are one-to-one;
(4) if $x, f(x) \in \Delta$, then $\|df_x v\| \geqslant \mu\|v\|$ for every v near the unstable direction, and the inverse $df^{-1}_{f(x)}$ $\|df^{-1}_{f(x)} v\| \geqslant \lambda^{-1}\|v\|$ for every v near the stable direction.

The intersection $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(R)$ is called a ***horseshoe***. The condition (4) above is not very mathematical since it originally contains some definitions beyond our discussions, see for instance [BS02, Page 125].

For more intuitions, one may imagine the process of making 兰州拉面. As a cook stretches (in the unstable direction) and folds the noodles, the noodles become thinner (in the stable direction).

Theorem 4.1 (Smale)**.** *The restriction $f|_\Lambda$ is topologically conjugate to the full two-sided shift σ in the space Σ_2, i.e., there is a homeomorphism $\xi : \Sigma_2 \to \Lambda$ such that $f \circ \xi = \xi \circ \sigma$.*

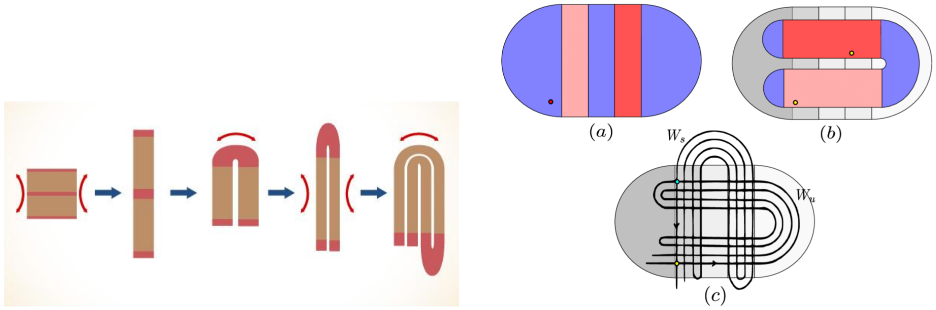

FIGURE 4.1. Figures of the Smale horseshoes

FIGURE 4.2. 兰州拉面制作

Proof. The idea comes from [Wen15]. We just need to show that $\xi : \Sigma_2 \to \Lambda$ by

$$\boldsymbol{\omega} = (\omega_n)_n \mapsto \bigcap_{n \in \mathbb{Z}} f^n(\Delta_{\omega_n})$$

is a homeomorphism. □

4.3. **Poincaré Homoclinic Tangles.** Let p be a hyperbolic periodic point of a diffeomorphism $f : M \to M$. A point $q \neq p$ is called ***homoclinic*** (同宿) for p if $q \in \mathcal{W}^s(p) \cap \mathcal{W}^u(p)$; it is called ***transverse homoclinic*** (横截同宿) for p if in addition $\mathcal{W}^s(p)$ and $\mathcal{W}^u(p)$ intersect transversely at q. For example, consider $\mathbf{0} = (0)_n \in \Sigma_2$, then any $\boldsymbol{\omega}$ with finite terms equal to 1 is a homoclinic point of $\mathbf{0}$.

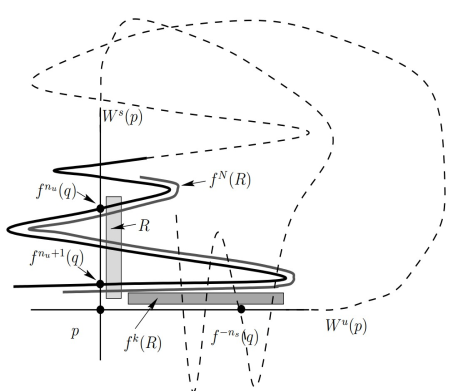

FIGURE 4.3. The Poincaré homoclinic tangle

The following theorem is much beyond our discussions. It can be proved by the λ-Lemma (or some called the Inclination Lemma) [BS02, Lemma 5.7.2].

Theorem 4.2 (Theorem 5.8.3 of [BS02])**.** *Let p be a fixed point of a hyperbolic diffeomorphism $f : M \to M$, and let q be a transverse homoclinic point of p. Then for every $\varepsilon > 0$ the union of the ε-neighborhoods of p and the orbit of q contains a horseshoe of f.*

Recall our favorite example $(\mathbb{T}^2, A)$ with $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Note that $p = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a fixed point and $q = \begin{pmatrix} \frac{\sqrt{5}}{5} \\ \frac{5-\sqrt{5}}{10} \end{pmatrix}$ a transverse homoclinic point of p, then for every $\varepsilon > 0$, the union of the ε-neighborhood of p and the ε-neighborhood of the orbit of q contains a horseshoe of some interate of A. The figure above is so called a ***Poincaré homoclinic tangle***.

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C-REPRESENTATIONS

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Abstract. In this article, for a local field K with absolute Galois group G, we will establish an isomorphism between $H^1_{\text{cont}}(G, \mathrm{GL}_d(C))$ and $H^1_{\text{cont}}(\Gamma, \mathrm{GL}_d(K_\infty))$, by which we get a bijection between C-representations of G and K_∞-representations of Γ, up to isomorphism.

1. Introduction

This article aims to introduce the theory of Tate and Sen to classify C-representations, involving an "*almost étale descent*" and a "*decompletion*". To make the proof strict and complete, the preliminaries are quite complex, including ramification theory, p-adic Lie groups and Sen's filtration theorem. Some of these are too far from our topic, and without specific motivation, some results may seem confusing, thus I would like to put all such things in the appendix. As a result, the appendix has grown to be quite large, even longer than the main body of the text. I hope readers will understand this approach, as the content in the appendix is not unimportant. On the contrary, to truly grasp Tate and Sen's theory, it is essential to master these concepts.

For readers who wish to first gain a general understanding of Tate and Sen's theory, they can simply read the main body, where they will learn about the method of classifying C-representations and understand the conditions (such as the TS1, TS2, and TS3 conditions), along with the underlying properties of p-adic fields that these conditions depend on, such as the ramification of their Galois groups or their structure as p-adic Lie groups.

For those who wish to study Tate and Sen's theory in greater depth, they can begin by reading the main text and, when the need arises to supplement their knowledge in order to prove certain properties, refer to the appendix.

I hope this approach will make the reading experience clearer and more enjoyable for the readers.

In this note, let K be a p-adic field and we write $v = v_K$ for abbreviation. Let $C := \hat{\bar{K}}$. Let $H = G_{K_\infty} = \mathrm{Gal}(\bar{K}/K_\infty)$, $\Gamma = \mathrm{Gal}(K_\infty/K) \cong \mathbb{Z}_p$. Let γ be a topological generator of Γ, and let $\Gamma_m = \Gamma^{p^m}$. Then Γ_m is topologically generated by $\gamma_m = \gamma^{p^m}$. For a matrix $M = (m_{ij}) \in M_{r\times s}(C)$, let $v(M) = \mathbf{min}(v(m_{ij}))$.

We study C-representations of $G_K = \mathrm{Gal}(\bar{K}/K)$ by the following two theorems:

Theorem 1.1. *(Almost étale descent)*
The inflation map gives a bijection

$$j : H^1_{cont}(\Gamma, GL_d(\hat{K_\infty})) \xrightarrow{\sim} H^1_{cont}(G, GL_d(C)).$$

Theorem 1.2. *(Decompletion)*
The inclusion $GL_d(K_\infty) \hookrightarrow GL_d(\hat{K}_\infty)$ induces a bijection

$$\eta : H^1_{cont}(\Gamma, GL_d(K_\infty)) \xrightarrow{\sim} H^1_{cont}(\Gamma, GL_d(\hat{K}_\infty))$$

Then by the following well-known result, we have established a bijection between isomorphism classes of C-representations of G and K_∞-representations of Γ.

Theorem 1.3. *Let G be a topological group and B be a topological commutative ring equipped with a continuous action of G compatible with the structure of ring, that is, for all $g \in G$, and $b_1, b_2 \in B$,*

$$g(b_1 + b_2) = g(b_1) + g(b_2),\ g(b_1 b_2) = g(b_1)g(b_2).$$

Suppose d is a positive integer Then there is a bijection between the set of equivalence classes of free B-representations of G of rank d and $H^1_{cont}(G, GL_d(B))$. Moreover, A free B-representation X is trivial if and only if it corresponds to the distinguished point in $H^1_{cont}(G, GL_d(B))$.

2. Almost étale descent

Recall that $H^1_{\mathrm{cont}}(G, M) = \frac{\{f : G \to M | f \text{ continuous, } f(g_1 g_2) = g_1 f(g_2) + f(g_1)\}}{\{s_a = (g \mapsto g \cdot a - a) | a \in M\}}$. For such an 1-cocycle U, denote by $U_\sigma \in M$ the value of U at $\sigma \in G$.

Lemma 2.1. *Let H_0 be an open subgroup of H and U be a continuous 1-cocycle of H_0 with values in $GL_d(C)$ such that $v(U_\sigma - 1) \geq a$ for a constant $a > 0$ for all $\sigma \in H_0$. Then there exists a matrix $M \in GL_d(C)$ with $v(M - 1) \geq a/2$, such that*

$$v(M^{-1} U_\sigma \sigma(M) - 1) \geq a + 1,\ \forall \sigma \in H_0.$$

Proof. Fix an open normal subgroup H_1 of H_0 such that $v(U_\sigma - 1) \geq a + 1 + a/2$ for $\sigma \in H_1$, which is possible by continuity. By Corollary f.2 (condition TS1), we can find $\alpha \in C^{H_1}$, such that

$$v(\alpha) \geq -a/2,\ \sum_{\tau \in H_0/H_1} \tau(\alpha) = 1.$$

Let $S \subset H_0$ be a set of representatives of H_0/H_1. Let

$$M_S - 1 = \sum_{\sigma \in S} \sigma(\alpha) U_\sigma$$

be the corresponding Poincaré series, we have $M_S - 1 = \sum_{\sigma \in S} \sigma(\alpha)(U_\sigma - 1)$. Hence

$$v(M_S - 1) \geq a + 1 > a/2,$$

and moreover the sequence

$$M_S^{-1} = \sum_{n=0}^{+\infty} (1 - M_S)^n$$

converges, and $M_S \in \mathrm{GL}_d(C)$. We also see that $v(M_S) = v(M_S^{-1}) = 0$. We claim that M_S is the matrix we need.

If $\tau_1 \in H_1$, then $U_{\sigma\tau} - U_\sigma = U_\sigma(\sigma(U_\tau) - 1)$. If $S' \subset H_0$ is another set of representatives of H_0/H_1, then for any $\sigma' \in S'$, there exists a unique $\sigma \in S$ and $\tau_\sigma \in H_1$ such that $\sigma' = \sigma\tau_\sigma$. So we get

$$M_S - M_{S'} = \sum_{\sigma \in S} \sigma(\alpha)(U_\sigma - U_{\sigma\tau_\sigma}) = \sum_{\sigma \in S} \sigma(\alpha) U_\sigma (1 - \sigma(U_{\tau_\sigma}))$$

and

$$v(M_S - M_{S'}) \geq a + 1 + a/2 - a/2 = a + 1.$$

For any $\tau \in H_0$,

$$U_\tau \tau(M_S) = \sum_{\sigma \in S} \tau\sigma(\alpha) U_\tau \tau(U_\sigma) = M_{\tau S}.$$

Thus

$$M_S^{-1} U_\tau \tau(M_S) = 1 + M_S^{-1}(M_{\tau S} - M_S),$$

with $v(M_S^{-1}(M_{\tau S} - M_S)) \geq a + 1$. The claim is proved. □

Corollary 2.2. *Under the same hypothesis as the above lemma, there exists $M \in GL_d(C)$ such that*

$$v(M-1) \geq a/2, \;\; M^{-1} U_\sigma \sigma(M) = 1$$

for all $\sigma \in H_0$.

Proof. Suppose M_1 is constructed for U_σ and a, for any positive integer i, repeat the lemma and suppose M_i is the matrix constructed for $V_\sigma = (M_1 \cdots M_{i-1})^{-1} U_\sigma (M_1 \cdots M_{i-1})$ and $a + i - 1$. V_σ is indeed an 1-cocycle since

$$\begin{aligned} V_{\sigma\tau} &= (M_1 \cdots M_{i-1})^{-1} U_\sigma \sigma(U_\tau)(M_1 \cdots M_{i-1}) \\ &= (M_1 \cdots M_{i-1})^{-1} U_\sigma (M_1 \cdots M_{i-1})(M_1 \cdots M_{i-1})^{-1} \sigma(U_\tau)(M_1 \cdots M_{i-1}) \\ &= (M_1 \cdots M_{i-1})^{-1} U_\sigma (M_1 \cdots M_{i-1}) \sigma((M_1 \cdots M_{i-1})^{-1} U_\tau (M_1 \cdots M_{i-1})) \\ &= V_\sigma \sigma(V_\tau). \end{aligned}$$

Thus $v(M_i - 1) \geq a/2$ and $v(M^{-1} U_\sigma \sigma(M) - 1) \geq a + i$. Take $M^{(n)} = \prod_{i=1}^n M_i$, then

$$v(M^{(n)} - M^{(n-1)}) = v(M_n(M_{n-1} - 1)) = v(M_{n-1} - 1) \geq a + n - 1.$$

Thus we may take $M = \lim_{n \to \infty} M^{(n)}$, which is the matrix we need. □

Proposition 2.3. $H^1_{cont}(H, GL_d(C)) = 1.$

Proof. We need to show that any given cocycle U on H with values in $\mathrm{GL}_d(C)$ is trivial. Pick $a > 0$, by continuity, we can choose an open normal subgroup H_0 of H such that $v(U_\sigma - 1) > a$ for any $\sigma \in H_0$. By Corollary 2.2, there exists $M \in \mathrm{GL}_d(C)$ such that $M^{-1} U_\sigma \sigma(M) = 1$, for all $\sigma \in H_0$, i.e. $U_\sigma|_{H_0} = M(\sigma(M))^{-1}$ is trivial on H_0. By the inflation-restriction exact sequence

$$1 \to H^1_{\mathrm{cont}}(H/H_0, \mathrm{GL}_d(C^{H_0})) \to H^1_{\mathrm{cont}}(H, \mathrm{GL}_d(C)) \to H^1_{\mathrm{cont}}(H_0, \mathrm{GL}_d(C)),$$

$U_\sigma \in H^1_{\mathrm{cont}}(H/H_0, \mathrm{GL}_d(C^{H_0}))$. By Hilbert's theorem 90,

$$H^1_{\mathrm{cont}}(H/H_0, \mathrm{GL}_d(C^{H_0})) = H^1_{\mathrm{cont}}(\mathrm{Gal}(C^{H_0}/C^H), \mathrm{GL}_d(C^{H_0})) = 1,$$

we complete the proof. □

Proof of Theorem 1.1. Consider the exact inflation-restriction exact sequence

$$1 \to H^1_{\mathrm{cont}}(\Gamma, \mathrm{GL}_d(C^H)) \to H^1_{\mathrm{cont}}(G, \mathrm{GL}_d(C)) \to H^1_{\mathrm{cont}}(H, \mathrm{GL}_d(C)),$$

the result follows from that $C^H = \hat{K_\infty}$. □

3. Decompletion

In Appendix F, we defined Tate's normalized trace map $R_r(x) : \widehat{K_\infty} \to K_r$ for every $r \in \mathbb{N}$. By Corollary F.6 and Proposition F.7 (conditions TS2 and TS3), there exists constant c_1 and c_2 independent of r, such that

$$v(R_r(x)) \geq v(x) - c_1, \ \forall x \in \widehat{K_\infty}$$

and

$$v((\gamma_r - 1)^{-1}(x)) \geq v(x) - c_2, \ \forall x \in X_r = \{x \in \widehat{K_\infty} | R_r(x) = 0\}.$$

Lemma 3.1. *Given $\delta > 0$, $b \geq 2c_1 + 2c_2 + \delta$, $b' \geq b$ and $r \geq 0$, suppose $U = 1 + U_1 + U_2$ with*

$$U_1 \in M_d(K_r), \ v(U_1) \geq b - c_1 - c_2$$

and

$$U_2 \in M_d(\widehat{K_\infty}), \ v(U_2) \geq b' \geq b,$$

then there exists $M \in GL_d(\widehat{K_\infty})$ with $v(M) \geq b - c - d$, such that

$$M^{-1}U\gamma_r(M) = 1 + V_1 + V_2,$$

with

$$V_1 \in M_d(K_r), \ v(V_1) \geq b - c_1 - c_2$$

and

$$V_2 \in M_d(\widehat{K_\infty}), \ v(V_2) \geq b' + \delta,$$

Proof. Write $U_2 = R_r(U_2) + (1 - \gamma_r)V$ for some $V \in \mathrm{M}_d(\widehat{K_\infty})$, then

$$v(R_r(U_2)) \geq v(U_2) - c_1$$

and

$$\begin{aligned} v(V) = v((1 - \gamma_r)^{-1}(U_2 - R_r(U_2))) &\geq v(U_2 - R_r(U_2)) - c_2 \\ &\geq \min\{v(U_2), v(R_r(U_2))\} - c_2 \geq v(U_2) - c_1 - c_2 \geq b - c_1 - c_2. \end{aligned}$$

Hence

$$\begin{aligned} (1 + V)^{-1}U\gamma_r(1 + V) &= (1 - V + V^2 - \cdots)(1 + U_1 + U_2)(1 + \gamma_r(V)) \\ &= 1 + U_1 + R_r(U_2) + \text{terms of degree} \geq 2. \end{aligned}$$

Let $V_1 = U_1 + R_r(U_2) \in \mathrm{M}_d(K_r)$, W be the terms of degree ≥ 2. then

$$\begin{aligned} v(W) &\geq 2\min\{v(R_r(U_2)), v(U_2), v(V)\} \\ &\geq 2(v(U_2) - c_1 - c_2) \geq 2b' - 2c_1 - 2c_2 \geq b' + \delta. \end{aligned}$$

We can just take $M = 1 + V$ and $V_2 = W$. □

Corollary 3.2. *Keep the same hypotheses as in Lemma 3.1, then there exists $M \in GL_d(\widehat{K_\infty})$ with $v(M - 1) \geq b - c_1 - c_2$ such that $M^{-1}U\gamma_r(M) \in GL_d(K_r)$.*

Proof. By the lemma, we have M_1, $V_1^{(1)}$ and $V_2^{(2)}$ such that

$$M_1^{-1}U\gamma_r(M_1) = 1 + V_1^{(1)} + V_2^{(1)},$$

with

$$V_1^{(1)} \in \mathrm{M}_d(K_r), \ v(V_1^{(1)}) \geq b - c_1 - c_2$$

and

$$V_2^{(1)} \in \mathrm{M}_d(\widehat{K_\infty}),\ v(V_2^{(1)}) \geq b' + \delta.$$

Use the lemma again, we have

$$M_2^{-1} M_1^{-1} U \gamma_r (M_1 M_2) = 1 + V_1^{(2)} + V_2^{(2)},$$

with

$$V_1^{(2)} \in \mathrm{M}_d(K_r),\ v(V_1^{(2)}) \geq b + \delta - c_1 - c_2$$

and

$$V_2^{(2)} \in \mathrm{M}_d(\widehat{K_\infty}),\ v(V_2^{(2)}) \geq b' + 2\delta.$$

Repeat the lemma as the above and take the limit, finally we will have

$$M^{-1} U \gamma_r (M) = (M_1 M_2 \cdots)^{-1} U \gamma_r (M_1 M_2 \cdots) = 1$$

since $v(V_1^{(n)})$ and $v(V_2^{(2)}) \to \infty$ as $n \to \infty$, where the sequences $V_1^{(n)}$ and $V_2^{(n)}$ are defined similarly. □

Lemma 3.3. *Suppose $B \in M_{d\times s}(\widehat{K_\infty})$ is a matrix, and there exists $V_1 \in GL_d(K_i)$ and $V_2 \in GL_s(K_i)$ such that for some $r \geq i$,*

$$v(V_1 - 1) > c_2,\ v(V_2 - 1) > c_2,\ \gamma_r(B) = V_1 B V_2,$$

then $B \in M_{d\times s}(K_i)$.

Proof. Take $T = B - R_i(B)$, it suffices to show that $T = 0$. Note that T has entries in $X_i = (1 - R_i)\widehat{K_\infty}$, and R_i is K_i-linear and commutes with γ_r, thus

$$\gamma_r(T) - T = V_1 T V_2 - T = (V_1 - 1) T V_2 + V_1 T (V_2 - 1) - (V_1 - 1) T (V_2 - 1).$$

Hence, $v(\gamma_r(T) - T) > v(T) + c_2$. On the other hand, by proposition F.7,

$$v_K(T) = v_K((\gamma_r - 1)^{-1}(\gamma_r - 1)(T)) \geq v_K(\gamma_r(T) - T) - c_2 > v(T),$$

thus we must have $v(T) = \infty$ and hence $T = 0$. □

Proof of Theorem 1.2. Firstly we show the injectivity. Suppose U, U' are cocycles in $H^1_{\mathrm{cont}}(\Gamma, \mathrm{GL}_d(K_\infty))$ which are cohomologous in $H^1_{\mathrm{cont}}(\Gamma, \mathrm{GL}_d(\widehat{K_\infty}))$, that is, there exists $M \in \mathrm{GL}_d(\widehat{K_\infty})$, such that $M^{-1} U_\sigma \sigma(M) = U'_\sigma$. In particular, $\gamma_r(M) = U_{\gamma_r}^{-1} M U'_{\gamma_r}$. Let r be sufficiently large so that $V_1 = U_{\gamma_r}^{-1}$ and $V_2 = U'_{\gamma_r}$ satisfies the condition in Lemma 3.3. Then we get $M \in \mathrm{GL}_d(K_\infty)$, whence U and U' are cohomologous in $H^1_{\mathrm{cont}}(\Gamma, \mathrm{GL}_d(K_\infty))$.

We now prove the surjectivity. Given U, a cocycle in $H^1_{\mathrm{cont}}(\Gamma, \mathrm{GL}_d(\widehat{K_\infty}))$, by continuity there exists $r > 0$ such that for all $\sigma \in \Gamma_r$, $v(U_\sigma - 1) \geq 2c_1 + 2c_2$. Let $U_1 = 0$ and $U_2 = U_\sigma$, then $U = 1 + U_{\gamma_r}$ satisfies the condition in Corollary 3.2. Hence there exists $M \in \mathrm{GL}_d(\widehat{K_\infty})$, such that $U'_{\gamma_r} = M^{-1} U_{\gamma_r} \gamma_r(M) \in \mathrm{GL}_d(K_r)$. Put $U'_\sigma = M^{-1} U_\sigma \sigma(M)$, then

$$U_\sigma \sigma(U_{\gamma_r}) = U_{\sigma\gamma_r} = U_{\gamma_r \sigma} = U_{\gamma_r} \gamma_r(U_\sigma),$$

which gives

$$\gamma_r(U_\sigma) = U_{\gamma_r}^{-1} U_\sigma \sigma(U_{\gamma_r}).$$

By Lemma 3.3 again, we have $U_\sigma \in \mathrm{GL}_d(K_\infty)$, which completes the proof. □

Appendix A. Hilbert's Theorem 90

Lemma A.1. *(Dedekind's Lemma)*
Let G be a group and K be a field, $\tau_1, \cdots, \tau_n$ be distinct characters from G to K^*. Then the τ_i are linearly independent over K.

Proof. Suppose that the lemma is false. Choose k minimal (relabeling thre τ_i if necessary) so that there are $c_i \in$K with $\sum_i c_i\tau_i(g) = 0$ for all $g \in G$. Then all $c_i \neq 0$. Since $\tau_1 \neq \tau_2$, there is an $h \in G$ with $\tau_1(h) \neq \tau_2(h)$. We have $\sum_{i=1}^k (c_i\tau_1(h))\tau_i(g) = 0$, and

$$\sum_{i=1}^k c_i\tau_i(hg) = \sum_{i=1}^k (c_i\tau_1(h))\tau_i(g) = 0, \forall g \in G.$$

Substracting gives $\sum_{i=1}^k (c_i(\tau_1(h) - \tau_i(h))\tau_i(g) = 0$ for all g. This is an expression involving $k-1$ of the τ_i with not all of the coefficients zero. This contradicts the minimality of k. $\square$

Theorem A.2. *(Normal basis theorem)*
Let L/K be a finite Galois extension, $G = Gal(L/K)$, then there exsits $x \in L$, such that $\{g(x)\}_{g\in G}$ forms a K-basis for L.

Proof. Let $n = [L : K]$. Suppose $L = K(\alpha)$, $f \in K[x]$ be the minimal polynomial of α. Suppose $G = \{\sigma_1, \cdots, \sigma_n\}$, and let $\alpha_i = \sigma_i(\alpha)$. Let $g(x) = \frac{f(x)}{(x-\alpha)f'(\alpha)}$ and $g_i(x) = \sigma_i(g(x)) = \frac{f(x)}{(x-\alpha_i)f'(\alpha_i)}$. Then

$$g_i(x)g_j(x) \equiv 0 \ (\mathrm{mod} f(x)),\ \forall i \neq j.$$

Consider the equation

$$g_1(x) + \cdots + g_n(x) - 1 = 0,$$

Since $g_1(x) + \cdots + g_n(x)$ is a polynomial of degree at most $n-1$, and $x = \alpha_1, \cdots, \alpha_n$ are solutions to the equation, the left side must be identically 0. Multiplying the equation above by $g_i(x)$, we know that

$$(g_i(x))^2 \equiv (g_i(x)) \ (\mathrm{mod} f(x)).$$

Now let's show that $D(x) = \det(\sigma_i\sigma_j(g(x))) \neq 0$. In fact,

$$\begin{aligned}
D(x)^2 &= \det((\sigma_i\sigma_j(g(x)))(\sigma_j\sigma_i(g(x)))) \\
&= \det(\sum_k (\sigma_k\sigma_i(g(x)))(\sigma_k\sigma_j(g(x)))) \\
&= \det(\sum_k (\sigma_k(g_i(x)))(\sigma_k(g_j(x)))) \\
&= \det(\sum_k \sigma_k(g_i(x)g_j(x))) \\
&\equiv \det(\sum_k \sigma_k(\delta_{ij})) \\
&= 1 \ (\mathrm{mod} f(x)).
\end{aligned}$$

Let $a = g(\alpha)$, we claim that it is as desired. In fact, for any $x_1, \cdots, x_n \in K$, suppose $x_1\sigma_1(a) + \cdots + x_n\sigma_n(a) = 0$, let σ_i acts on the left hand side, we have a linear system of n

equations

$$\sigma_i\sigma_1(g(a))x_1 + \cdots \sigma_i\sigma_n(g(a))x_n = 0,$$

whose determinant $D(a) \neq 0$. Thus $x_1 = \cdots = x_n = 0$. $\square$

Theorem A.3. *(Hilbert's Theorem 90)*
Let K be a field and L be a Galois extension of K, finite or not, then:
1). $H^n(L/K, L) = 0, \forall n \geq 1$;
2). $H^1(L/K, L^\times) = 1$;
3). $H^1(L/K, GL_n(L)) = 1$, $\forall n \geq 1$.

Proof. It suffices to show the finite extension case.
(1) follows from Theorem A.2, since by the theorem we know that $L = K \otimes_{\mathbb{Z}} \mathbb{Z}[G]$ is an induced module.
(2). Let $s \mapsto a_s$ be an 1-cocycle, then $a_{st} = s \cdot a_t \cdot a_s$. For $c \in L$, define the corresponding "Poincaré series" by

$$b := \sum_{s \in G} a_s \cdot s(c).$$

Notice that

$$\begin{aligned} s(b) =& \sum_{t \in G} s \cdot a_t \cdot st(c) \\ =& \sum_{t \in G} a_{st} a_s^{-1} \cdot st(c) \\ =& a_s^{-1} \sum_{t \in G} a_{st} \cdot st(c) \\ =& a_s^{-1} \sum_{t \in G} a_t \cdot t(c) \\ =& b a_s^{-1}, \end{aligned}$$

i.e.

$$a_s = b \cdot (s(b))^{-1}$$

is a coboundary.
(3). Similarly, let $s \mapsto a_s$ be a 1-cocycle. $\forall c \in \mathrm{GL}_n(L)$, the corresponding Poincaré series is

$$b = \sum_{s \in G} a_s \cdot s(c),$$

and we have that

$$s(b) = a_s^{-1} \cdot b.$$

To show that $s \mapsto a_s$ is a coboundary, it suffices to find c such that b is invertible. Now for every $x \in L^n$, let $b(x) = \sum_{s \in G} a_s \cdot s(x)$.

Claim: $b(x)$ generate K^n as x runs through L^n.

In fact, for if the linear form $u : L^n \to L$ vanishes on all the $b(x)$, then for every $h \in L^*$,

$$0 = u(b(hx)) = \sum_{s \in G} s(h) \cdot u(a_s \cdot s(x)).$$

Consider the RHS as a linear relation among $s(h)$. Consider $s : L^* \to L^*$ as a character, then by Dedekin's lemma, $s(h)$'s are linearly independent, and hence the equality implies that the coefficients $u(a_s \cdot s(x)) = 0$. Since a_s are invertible, and x is arbitrary in L^n, this implies that $u = 0$, thus we have the claim.

With this point settled, let $x_1, \cdots, x_n$ be vectors in L^n such that $y_i = b(x_i)$ are linearly independent over L. Let c be the matrix of the map that sends the canonical basis e_i to the x_i. Notice that

$$\begin{aligned} b(e_i) =& (\sum_{s\in G} a_s \cdot s(c))e_i \\ =& \sum_{s\in G} a_s \cdot s(c)e_i \\ =& \sum_{s\in G} a_s \cdot s(c) \cdot s(e_i) \\ =& \sum_{s\in G} a_s \cdot s(c \cdot e_i) \\ =& \sum_{s\in G} a_s \cdot s(x_i) \\ =& y_i, \end{aligned}$$

we conclude that b is invertible. □

Appendix B. Ramifications Subgroups and Herbrand Functions

B.1. Finite Extensions.

Lemma B.1. *Let K be a local field and L be a finite extension of K, let $G = Gal(L/K)$. Let $s \in G$, x be an element of $\mathcal{O}_L$ that generate $\mathcal{O}_L$ as an $\mathcal{O}_K$-algebra. Then the followings are equivalent:*
1). s operate trivially on $\mathcal{O}_L/\mathfrak{p}_L^{i+1}$;
2). $v_L(s(a) - 1) \geq i + 1,\ \forall a \in \mathcal{O}_L$;
3). $v_L(s(x) - x) \geq i + 1$.

Definition B.2. For each integer $i \geq -1$, let G_i be the set of $s \in G$ satisfying the above conditions.

It is clear that $G_{-1} = G$, G_0 is the inertia subgroup of G and the G_i are normal subgroup of G. Moreover, the G_i form a decreasing sequence, and $G_i = 1$ for i sufficiently large.

Proposition B.3. *Let $H \leq G$ be a subgroup of G, then $H_i = G_i \cap H,\ \forall i \geq -1$.*

Proposition B.4. *If H is a normal subgroup of G, then for every $\sigma \in G/H$,*

$$i_{G/H}(\sigma) = \frac{1}{e_{L/K'}} \sum_{s \text{ represents } \sigma} i_G(s).$$

Proof. Let $K' = L^H$. For $\sigma = 1$, both sides are equal to ∞. Suppose that $\sigma \neq 1$. Let x (resp. y) be an $\mathcal{O}_K$-generator of $\mathcal{O}_L$ (resp. $\mathcal{O}_{K'}$). By definition, $i_{G/H} = v_{K'}(\sigma(y) - y) = \frac{1}{e_{L/K'}} v_L(\sigma(y) - y)$ and $i_G(s) = v_L(s(x) - x)$. If we choose an element $s \in G$ representing σ,

then the other representatives have the form st, $t \in H$. Hence it all comes down to showing that the elements
$$a = s(y) - y \quad \text{and} \quad b = \prod_{t \in H} (st(x) - x)$$
generate the same ideal in $\mathcal{O}_L$.

Let $f \in \mathcal{O}_{K'}[X]$ be the minimal polynomial of x over K', then $f(X) = \prod_{t \in H}(X - t(x)) = \sum c_i x^i$ for some $c_i \in \mathcal{O}_{K'}$, and $s(f)(X) = \prod_{t \in H}(X - st(x)) = \sum s(c_i)x^i$. Since for every i, $v_{K'}(s(c_i) - c_i) \geq i_H(s) \geq v_{K'}(s(y) - y)$, $s(y) - y$ divides coefficients of $s(f) - f$, and hence $a = s(y) - y$ divides $s(f)(x) - f(x) = s(f)(x) = b$.

Conversely, write $y = g(x)$ as a polynomial of x, $g \in \mathcal{O}_K[X]$. The polynomial $g(X) - y \in \mathcal{O}_{K'}[X]$ has x as a root, therefore $g(X) - y = f(X)h(X)$ for some $h \in \mathcal{O}_{K'}[X]$. Transform this equation by s an substitute by x for X in the result, one gets
$$a = y - s(y) = s(f)(x)s(h)(x),$$
in particular, $b = s(f)(x)$ divides a. □

Corollary B.5. *If $H = G_j$ for some integer $j \geq 0$, then $(G/H)_i = G_i/H$ for $i \leq j$ and $(G/H)_i = 1$ for $i > j$.*

Definition B.6. For any real number≥ 0, define $G_u := G_{\lfloor u \rfloor}$. Put
$$\Phi(u) := \int_0^u [G_0 : G_t]^{-1} dt,$$
where for $-1 \leq u \leq 0$, $[G_0 : G_u] = [G_u : G_0]^{-1}$.

Proposition B.7. *For $0 \leq m \leq u \leq m + 1$,*
$$\Phi(u) = \frac{1}{g_0}(g_1 + \cdots + g_m + (u - m)g_{m+1}), \text{ with } g_i = |G_i|.$$

Proposition B.8. *The function $\Phi : [-1, +\infty) \to [-1, +\infty)$ is continuous, piecewise linear, increasing and concave.*

Proposition B.9. $\Phi(u) = \frac{1}{g_0} \sum_{s \in G} min\{i_G(s), u + 1\} - 1.$

Definition B.10. Let $\Psi : [-1, +\infty) \to [-1, +\infty)$ be the inverse of Φ. Put $G^v := G_u$, where $u = \Psi(v)$.

Proposition B.11. $\Psi(v) = \int_0^v [G^0 : G^w] dw.$

Theorem B.12. *If H is a normal subgroup of G, then $(G/H)_v = G_u H/H = G_u/H_u$, where $v = \Phi_{L/K'}(u)$*

Proof. For every $s' \in G/H$, we choose a preimage $s \in G$ of maximal value $i_G(s)$ and show that
$$i_{G/H}(s') - 1 = \Phi_{L/K'}(i_G(s) - 1).$$
Let $m = i_G(s)$. If $t \in H$ belongs to H_{m-1}, then $i_G(t) \geq m$, and $i_G(st) \geq m$, and so that $i_G(st) = m$. If $t \notin H_{m-1}$, then $i_G(t) < m$ and $i_G(st) = i_G(t)$. In both cases we therefore find that $i_G(st) = \min\{i_G(t), m\}$. Thus
$$\begin{aligned} i_{G/H}(s') &= \frac{1}{e_{L/K'}} \sum_{t \in H} i_G(st) = \frac{1}{e_{L/K'}} \sum_{t \in H} \min\{i_G(t), m\} \\ &= \Phi_{L/K'}(m - 1) + 1 = \Phi_{L/K'}(i_G(s) - 1) + 1. \end{aligned}$$

With this point settled,

$$\begin{aligned} s' \in G_u H/H &\Leftrightarrow i_G(s) - 1 \geq u \\ &\Leftrightarrow \Phi_{L/K'}(i_G(s) - 1) \geq \Phi_{L/K'}(u) \\ &\Leftrightarrow i_{G/H}(s') \geq v \Leftrightarrow s' \in (G/H)_v. \end{aligned}$$

□

Proposition B.13. *If K'/K is a Galois subextension of L/K, then*

$$\Phi_{L/K} = \Phi_{K'/K} \circ \Phi_{L/K'}, \text{ and } \Psi_{L/K} = \Psi_{L/K'} \circ \Psi_{K'/K}.$$

Proof. From the above theorem, we obtain that $G_u/H_u = (G/H)_v$ with $v = \Phi_{L/K'}(u)$. Thus

$$\frac{1}{e_{L/K}}|G_u| = \frac{1}{e_{K'/K}}|(G/H)_v|\frac{1}{e_{L/K'}}|H_u|.$$

Since $\Phi'_{-}(u) = \frac{1}{[G_0:G_u]}$, the equation is equivalent to

$$\Phi'_{L/K}(u) = \Phi'_{K'/K}(v)\Phi'_{L/K'}(u) = (\Phi_{K/K'} \circ \Phi_{L/K'})'(u).$$

As $\Phi_{L/K}(0) = \Phi_{K'/K} \circ \Phi_{L/K'}(0)$, it follows that $\Phi_{L/K} = \Phi_{K'/K} \circ \Phi_{L/K}$. □

Theorem B.14. *If H is a normal subgroup of G, then $(G/H)^v = G^v H/H$*

Proof.

$$(G/H)^v = (G/H)_{\Psi_{K'/K}(v)} = G_{\Psi_{L/K'}(\Psi_{K'/K}(v))} H/H = G_{\Psi_{L/K}(v)} H/H = G^v H/H.$$

□

B.2. **Infinite Extensions.**

Definition B.15. Let L/K be an infinite Galois extension of local fields with Galois group $G = \mathrm{Gal}(L/K)$. Then G^v, the ramification groups in upper numbering of G is defined by

$$G^v := \varprojlim_{L'/K \text{ finite Galois inside } L} \mathrm{Gal}(L'/K)^v.$$

Herbrand's Theorem remains true:

Proposition B.16. *Let L/K be an infinite Galois extension with group G. If H is a closed normal subgroup of G , corresponding to the invariant field $L^G =: K'$, then:*
1). If H is also open in G, then $G^v \cap H = H^{\Psi_{K'/K}(v)}$;
2). In general, $(G/H)^v = G^v H/H$.

Definition B.17. An Galois extension L/K is called an arithmetically profinite extension and in abbreviation APF if for any $v \geq -1$, G^v is an open subgroup of $G = \mathrm{Gal}(L/K)$.

If L/K is APF, then we can define

$$\Psi_{L/K}(v) := \begin{cases} \displaystyle\int_0^v [G^0 : G^w] dw, \text{ if } v \geq 0; \\ v, \text{ if -1} \leq \text{v} \leq 0. \end{cases}$$

As in the finite extension case, Ψ is a homeomorphism of $[-1, +\infty)$ to itself which is continuous, piecewise linear, increasing and concave and satisfies $\Psi(0) = 0$. Let $\Phi_{L/K}$ be the inverse function of Ψ. One can then define the ramification group G_u in lower numbering by $G_u := G^{\Phi(u)}$. Further, if the extension K'/K is APF and L/K is finite, then the transitive formulas $\Phi_{L/K} = \Phi_{K'/K} \circ \Phi_{L/K'}$ and $\Psi_{L/K} = \Psi_{L/K'} \circ \Psi_{K'/K}$ still hold.

B.3. **Structure of Ramification Subgroups.** In this subsection, let L/K be a finite Galois extension of local fields with Galois group G. Let $k(L)$ be the residue field of L, and let π be a uniformizer of L.

Proposition B.18. *For any i, the map $G_i \to U_L,\ s \mapsto s(\pi)/\pi$ induces by passage to the quotient an injective homomorphism*

$$\theta_i : G_i/G_{i+1} \to U_L^i/U_L^{i+1}.$$

This homomorphism is independent of the choice of uniformizer π.

Proof. If π' be another uniformizer, then $\pi' = u\pi$ for some $u \in U_L$. For any $s \in G_i$,

$$s(u) \equiv u \pmod{\mathfrak{p}_L^{i+1}}, \quad \text{i.e.} \quad s(u)/u \equiv 1 \pmod{U_L^{i+1}},$$

whence

$$s(\pi')/\pi' = (s(\pi)/\pi) \cdot (s(u)/u) \equiv s(\pi)/\pi \pmod{U_L^{i+1}},$$

which shows that θ_i does not depend on the choice of π. On the other hand, let $s, t \in G_i$, one can write

$$st(\pi)/\pi = (s(t(\pi))/t(\pi)) \cdot (t(\pi)/\pi) \equiv (s(\pi)/\pi) \cdot (t(\pi)/\pi) \pmod{U_L^{i+1}},$$

which shows that θ_i is an injective homomorphism. □

Since for $i \geq 1$, the group U_L^i/U_L^{i+1} is cannonically isomorphic to the group $\mathfrak{p}_L^i/\mathfrak{p}_L^{i+1}$, and non-cannonically isomorphic to the additive group of the residue field $k(L)$, we can draw the following conclusions to the quotients G_i/G_{i+1}.

Proposition B.19. *The group G_0/G_1 is cyclic. Moreover, if the characteristic of $k(L)$ is 0, then $G_1 = 0$ and G_0 is cyclic.*

Proof. The first assertion follows easily from the fact that the multiplicative group of a finite field is cyclic. If the characteristic of $k(L)$ is 0, then U_L^i/U_L^{i+1} has no nontrivial finite subgroup, hence the quotient G_i/G_{i+1} must be 1. Then the second assertion follows from that $G_i = 1$ for i large enough. □

Proposition B.20. *If the characteristic of $k(L)$ is $p \neq 0$, the quotient $G_i/G_{i+1},\ i \geq 1$, are abelian groups, and are direct products of cyclic groups of order p. The group G_1 is a p-group.*

B.4. **A Lemma: An Upper Bound For u_G.**

Lemma B.21. *For any finite Galois extension L/K of p-adic fields L/K with Galois group G, $u_G \leq e_L/(p-1)$.*

Write $u = u_G$. To deal with the lemma, we first suppose that L/K is totally ramified and cyclic of order p. This means that $e_L = p \cdot e_K$, and allows to choose a generator σ of G. We first prove the following supplementary results:

Lemma B.22. *For any $a \in L$, $v_L((\sigma - 1)a) \geq v_L(a) + u$, and the equality holds if and only if $v_L(a)$ is prime to p.*

Proof. Let π be a uniformizer for L. Since L/K is totally ramified, $\{1, \pi, \pi^2, \cdots, \pi^{p-1}\}$ forms a K-basis for L, and $\mathcal{O}_L = \mathcal{O}_K[\pi]$. Since σ generates G and by the definition of u, we have $(\sigma - 1)\pi = s\pi^{u+1}$ for some unit s. Thus $v((\sigma - 1)\pi^i) = i + u$, $\forall i \geq 1$. For any $a \in L$,

$a = \sum_{i=0}^{p-1} \lambda_i \pi^i$ for some $\lambda_i \in K$. Because of $e_L = p \cdot e_K$, $p | v(\lambda_i)$, $v(\lambda_i) + i$ are distinct for $i = 0, 1, \cdots, p-1$. Thus

$$v(a) = \min_{i \geq 0} \{v(\lambda_i) + i\}$$
$$v((\sigma - 1)(a)) = \min_{i \geq 1} \{v(\lambda_i) + i + u\}$$

If $v(a) = v(\lambda_n) + n$ for some $n \neq 0$, then $v((\sigma-1)a) = v(\lambda_n) + n + l = v(a) + l$; otherwise, $v(a) = v(\lambda_0) \leq \min_{i \geq 1}\{v(\lambda_i) + i\} = v((\sigma - 1)a) - u$, and the result follows. ☐

Let $P \in \mathbb{Z}[T]$ be the polynomial determined by

$$1 + T + \cdots + T^{p-1} = (T-1)^{p-1} + pP(T).$$

If we substitute $T = \sigma$, then we have

$$\mathrm{Tr}_{L/K}(a) = (\sigma - 1)^{p-1}(a) + pP(\sigma)(a).$$

Lemma B.23. *$P(\sigma)$ does not change the valuation of a. In particular, we have $v_L(pP(\sigma)(a)) = e_L + v_L(a)$.*

Proof. We note that $P(1) = 1$. Write $P(T) = \sum_{i=1}^{p-1} c_i T^i$ and $\sum_{i}^{p-1} c_i = 1$. We have

$$\frac{P(\sigma)(a)}{a} = \sum_{i=1}^{p-1} c_i \frac{\sigma^i(a)}{a}.$$

Since we are working within G_1, $\frac{\sigma^i(a)}{a} \equiv 1 \pmod{\pi}$. Thus

$$\frac{P(\sigma)(a)}{a} \equiv c_1 + \cdots + c_{p-1} = 1 \pmod{\pi},$$

and the result follows. ☐

Proof of Lemma B.21 We claim that we can assume L/K to be totally ramified and cyclic of degree p. Firstly, because $\frac{e_L}{p-1} > 0$, we work inside of G_1, which is the Galois group of a totally ramified subextension. Secondly, the last non-trivial ramification group G_u is an abelian p-group by results from the previous subsection, and thus has a subgroup $H \cong \mathbb{Z}/p\mathbb{Z}$. This group is the Galois group of some intermediate field, and since $H_r = G_r \cap H$ for every r, $u_H = u$. Thus we only need to consider H, which is the case as desired.

Now if u is divisible by p, choose $a \in L$ such that $v_L(a)$. By Lemma B.22, $v_L((\sigma - 1)^{p-1}(a)) = (p-1)u + 1$, which is not divisible by p, neither is $v_L(pP(\sigma)(a)) = e_L + 1$. However, we have $p | v_L(\mathrm{Tr}_{L/K}(a)) = p v_K(\mathrm{Tr}_{L/K}(a))$. By the relation $(\sigma - 1)^{p-1}(a) + pP(\sigma)(a) = \mathrm{Tr}_{L/K}(a)$, we must have $e_L + 1 = v_L((\sigma-1)^{p-1}(a)) = v_L(pP(\sigma)(a)) = (p-1)u + 1$.

On the other hand, if u is prime to p, choose $a \in L$ such that $v_L(a) = u$. Then $v_L((\sigma)^{p-1}(a)) = (p-1)u + u = pu$ and $v_L(pP(\sigma)(a)) = e_L + u$. Since $p | v_L(\mathrm{Tr}_{L/K}(a)) = p v_K(\mathrm{Tr}_{L/K}(a))$ still holds, we conclude that $pu < e_L + u$. ☐

Appendix C. Different

Let L/K be a finite separable extension of local fields.

Definition C.1. The different $\mathfrak{D}_{L/K}$ of L/K is the inverse of the dual $\mathcal{O}_K$-module of $\mathcal{O}_L$ to the trace map inside L, i.e.

$$\mathfrak{D}_{L/K}^{-1} := \{x \in L \mid \mathrm{Tr}(xy) \in \mathcal{O}_K \text{ for all } y \in \mathcal{O}_L\}.$$

Proposition C.2. *Let* $\mathfrak{b}$ *(resp.* $\mathfrak{b}$*) be a fractional ideal of* K *(resp.* L*), then*

$$Tr(\mathfrak{b}) \subset \mathfrak{a} \Leftrightarrow \mathfrak{b} \subset \mathfrak{a}\mathfrak{D}_{L/K}^{-1}.$$

Corollary C.3. *Let* $M \supset L \supset K$ *be finite separable extensions. Then*

$$\mathfrak{D}_{M/K} = \mathfrak{D}_{M/L} \cdot \mathfrak{D}_{L/K}.$$

Corollary C.4. *Let* L/K *be a finite extension of* p*-adic fields with ramification index* e*. Let* $\mathfrak{D}_{L/K} = \mathfrak{m}_L^m$*,* $m \in \mathbb{N}$*. Then for any integer* n*,*

$$Tr(\mathfrak{m}_L^n) = \mathfrak{m}_K^r,$$

where $r = \lfloor \frac{m+n}{e} \rfloor$

Proof. Since the trace map is $\mathcal{O}_K$-linear, $\mathrm{Tr}(\mathfrak{m}_L^n)$ is an ideal in $\mathcal{O}_K$. Thus $\mathrm{Tr}(\mathfrak{m}_L^n) \subset \mathfrak{m}_K^r$ if and only if $\mathfrak{m}_L^n \subset \mathfrak{m}_K^r \mathfrak{D}_{L/K}^{-1} = \mathfrak{m}_L^{er-m}$ if and only if $r \leq \frac{m+n}{e}$. □

Lemma C.5. *(Euler)*
Let $x \in \mathcal{O}_L$ *such that* $L = K[x]$*, let* $f(X)$ *be the minimal polynomial of* x *over* K*. Let* $n = deg(f)$*, then*

$$Tr(x^i/f'(x)) = \begin{cases} 0, & \text{if } i = 0, 1, \cdots, n-2; \\ 1, & \text{if -}i = n-1. \end{cases}$$

Proposition C.6. *Let* $x \in \mathcal{O}_L$ *such that* $L = K[x]$*, let* $f(X)$ *be the minimal polynomial of* x *over* K*. Then* $\mathfrak{D}_{L/K} = (f'(x))$*.*

Proposition C.7. *Let* L/K *be a finite Galois extension of local fields with Galois group* G*, then*

$$v_L(\mathfrak{D}_{L/K}) = \sum_{s \neq 1} i_G(s) = \sum_{i=0}^{\infty} (|G_i| - 1)$$

$$= \int_{-1}^{\infty} (|G_u| - 1) du = |G_0| \int_{-1}^{\infty} (1 - |G^v|^{-1}) dv. \tag{C.1}$$

Thus

$$v_K(\mathfrak{D}_{L/K}) = \int_{-1}^{\infty} (1 - |G^v|^{-1}) dv.$$

Proof. Let x be a generator of $\mathcal{O}_L$ over $\mathcal{O}_K$ and let f be its minimal polynomial. Then $\mathfrak{D}_{L/K}$ is generated by $f'(x)$ by the above proposition. Thus

$$v_L(\mathfrak{D}_{L/K}) = v_L(f'(x)) = \sum_{s \neq 1} v_L(x - s(x)) = \sum_{s \neq 1} i_G(s).$$

The second and the third equalities of (C.1) is easy. For the last equality,

$$\int_{-1}^{\infty} (1 - |G^v|^{-1}) dv = \int_{-1}^{\infty} (1 - |G_u|)^{-1} \Phi'(u) du = \frac{1}{|G_0|} \int_{-1}^{\infty} (|G_u| - 1) du.$$

□

Corollary C.8. *Let* $L \supset M \supset K$ *be finite Galois extensions of local fields, then*

$$v_K(\mathfrak{D}_{L/M}) = \int_{-1}^{\infty} (\frac{1}{|Gal(M/K)^v|} - \frac{1}{|Gal(L/K)^v|}) dv.$$

Appendix D. p-adic Lie theory

D.1. **Pro-p Groups.** I will not prove most of the theorems in this section, as long as they seem convincing.

Definition D.1. A profinite group G is finitely generated if there is an (algebraically) finitely generated set whose closure is G.

Definition D.2. A profinite group G is a pro-p group if and only if every quotient G/N by an open normal subgroup is p-group

Proposition D.3. *A profinite group G is a pro-p group if and only if G is (topologically and algebraically) isomorphic to a projective limit of finite p-groups.*

Lemma D.4. *Let G be a pro-p group and g be an element of G. Let (a_i), (b_i) be two sequences of integers that converge to the same limit in $\mathbb{Z}_p$, then the sequence (g^{a_i}) and (g^{b_i}) both converge in G, and their limits are equal.*

Definition D.5. Let G be a pro-p group, $g \in G$, $\lambda \in \mathbb{Z}_p$, then

$$g^\lambda := \lim_{n\to\infty} g^{a_n},$$

where (a_n) is a sequence of integers converging to λ in $\mathbb{Z}_p$.

Definition D.6. The Frattini subgroup $\Phi(G)$ of a profinite group G is the intersection of all maximal open subgroup of G.

Proposition D.7. *For a pro-p group G, $\Phi(G) = \overline{G^p[G,G]}$.*

Proposition D.8. *A pro-p group G is finitely generated if and only if $\Phi(G)$ is open in G. In this situation, $\Phi(G) = G^p[G,G]$.*

Definition D.9. The lower p-series for a pro-p group G is defined as

$$G = P_0(G) \supset P_1(G) \supset \cdots \supset P_n(G) \supset \cdots,$$

where

$$P_{n+1}(G) = \overline{P_n(G)[P_n(G), G]}.$$

When it is not leading to confusion, we write G_n for $P_n(G)$. If G is finitely generated, then the groups G_n form a basis of the topology of G.

Lemma D.10. *In a pro-p group G, we have $[G_n, G_m] \subset G_{n+m+1}$ for all $n, m \in \mathbb{N}$.*

Theorem D.11. *If G is finitely generated pro-p group, then every subgroup of finite index in G is open.*

Proposition D.12. *Every algebraic homomorphism from a finitely generated pro-p group G to a profinite group is continuous. In particular, any automorphism is also a topological automorphism.* ***The topology of G is thus completely determined by its algebraic structure.***

D.2. **Uniform Groups.** For a topologically finitely generated group G, let $d(G)$ denote the minimal cardinality of a generating set.

Proposition D.13. *Let G be a profinite group. Then*

$$\begin{aligned} rk(G) := & \sup\{d(H) \mid H \textit{ is a closed subgroup of } G\} \\ = & \sup\{d(H) \mid H \textit{ is a closed subgroup of } G, \textit{ and } d(H) < \infty\} \\ = & \sup\{d(H) \mid H \textit{ is a open subgroup of } G\} \\ = & \sup\{d(H) \mid H \textit{ is a normal open subgroup of } G\}. \end{aligned}$$

The number $rk(G)$ is called the rank of G.

We wish to have $rk(H) < rk(G)$ for subgroups $H \subset G$. In fact, this property holds if G is a powerful group.

Definition D.14. A pro-p group G is powerful if p is odd and $G/\overline{G^p}$ is abelian or if $p = 2$ and $G/\overline{G^4}$ is abelian.

Proposition D.15. *A pro-p group is powerful if and only if it is the projective limit of a surjective system finite powerful p-groups.*

One of the most important properties of finitely generated powerful groups is that every element of G^p is already a p-th power:

Proposition D.16. *If G is a finitely generated powerful group, then*

$$G^p = \{x \in G \mid x = g^p \textit{ for some } g \in G\}.$$

and by induction, this carries on to subgroups of higher powers of p:

Theorem D.17. *Let G be a finitely generated powerful p-group.*
1). For each n, we have

$$G_{n+1} = G_n^p = \{x \in G \mid x = g^{p^{n+1}} \textit{ for some } g \in G\} = \Phi(G_n);$$

2). For each n, the p-th power map $x \mapsto x^p$ induces a surjective homomorphism from G_n/G_{n+1} onto G_{n+1}/G_{n+2};
3). If H is a closed subgroup, we have $d(H) \leq d(G)$, and $rk(H) = d(G)$

Proposition D.18. *Let G be a pro-p group. Then G has finite rank if and only if G is finitely generated and has a powerful open subgroup.*

Definition D.19. A pro-p group G is uniform if G is powerful, finitely generated and all successive quotient G_n/G_{n+1} have the same size.

By Lemma D.10, each of these quotients is an abelian p-group; by 1). of Theorem D.17, theses quotients are annihilated by p-th powers. Thus $G_n/G_{n+1} \cong (\mathbb{Z}/p\mathbb{Z})^d$ for some d, and the p-th power map $x \mapsto x^p$ induces the an isomorphism $\mathrm{sh} : G_n/G_{n+1} \to G_{n+1}/G_{n+2}$.

Example. Let $\Gamma = \mathrm{GL}_d(\mathbb{Z}_p)$, Γ_n be the congruence subgroup of Γ on the level n, i.e.

$$\Gamma_n = \{\gamma \in \Gamma \mid \gamma \equiv \mathrm{I}_d \bmod p^n\} = \ker(\Gamma \to \mathrm{GL}_d(\mathbb{Z}/p^n\mathbb{Z})).$$

One can show that Γ_1 is a pro-p group. Moreover, set $G = \Gamma_1$ if p is odd and $G = \Gamma_2$ if $p = 2$, G can be shown to be a uniform group with $G_n = \Gamma_{n+1}$ if p is odd and $G_n = \Gamma_{n+2}$ if $p = 2$. $d(G) = d^2$.

Proposition D.20. *Let G be a finitely generated powerful pro-p group, then G_n is uniform for all sufficiently large n.*

Corollary D.21. *A pro-p group of finite rank has a characteristic uniform subgroup, i.e. a uniform subgroup invariant under any automorphism of G.*

Proposition D.22. *Let G be a powerful finitely generated pro-p group with $d(G) = d$. Then the following are equivalent:*
1).G is uniform;
2).$d(G_n) = d(G) = d$ for all n;
3).$d(H) = d(G)$ for every powerful open subgroup H of G.

Definition D.23. Let G be a pro-p group of finite rank, the dimension of G is

$$\dim(G) = d(H),$$

where H is any open uniform subgroup of G. For a finite group G, put $\dim G = 0$.

Proposition D.24. *Let G be a uniform group and $(a_1, \cdots, a_d)$ a set of topological generator with $d = d(G) = \dim(G)$, there is a homeomorphic mapping*

$$\phi : \mathbb{Z}_p^d \to G,\ (\lambda_1, \cdots, \lambda_d) \mapsto (a_1^{\lambda_1}, \cdots, a_d^{\lambda_d}).$$

Theorem D.25. *A finitely generated powerful pro-p group is uniform if and only if it is torsion free.*

Theorem D.26. *Let G be a uniform group amd $N \subset G$ be a closed, normal subgroup. If G/N is uniform, then N is uniform as well. Moreover,*

$$\dim(G) = \dim(N) + \dim(G/N).$$

D.3. **Powerful Lie Algebras.**

Definition D.27. Let R be a commutative ring. A R-Lie algebra is an R-module $\mathfrak{g}$ with a bilinear mapping

$$(\ ,\ ) : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

that satisfies: 1). $(a, a) = 0$, $\forall\ a \in \mathfrak{g}$;
2). $((a, b), c) + ((b, c), a) + ((c, a), b) = 0$.
A Lie ideal $\mathfrak{a} \subset \mathfrak{g}$ is an R-submodule that is closed under the Lie bracket. A morphism φ of Lie algebra is a morphism of R-modules that satisfies $\varphi((a, b)) = (\varphi(a), \varphi(b))$.

Definition D.28. A Lie algebra $\mathfrak{g}$ over $\mathbb{Z}_p$ is powerful if there is an isomorphism $\phi : \mathfrak{g} \to \mathbb{Z}_p^d$ of $\mathbb{Z}_p$ modules for some d and

$$(\mathfrak{g}, \mathfrak{g}) \subset \begin{cases} p\mathfrak{g},\ p > 2; \\ 4\mathfrak{g},\ p = 2. \end{cases}$$

In the following, we will show that the category of uniform groups are equivalent to the category of finitely generated powerful $\mathbb{Z}_p$ Lie algebras.

Let G be a uniform pro-p group. We first define a structure of Lie algebra on G. By 1). of Theorem D.17, the map

$$G \to G_n, x \mapsto x^{p^n}$$

is a homeomorphism, with inverse map written as p^n-th roots. For each n, these define a new group structure on G:

$$+_n : G \times G \to G,\ (x, y) \mapsto x +_n y = (x^{p^n} y^{p^n})^{p^{-n}}.$$

One can show that these operations are compatible with higher powers and invariant under multiplicative "disturbances" from within G_n. As a result, we can let n go to infinity:

Proposition D.29. *G becomes an abelian group under the operation $x +_G y = \lim_{n\to\infty} x +_n y$. For all $x, y \in G$, we have*
1). If $xy = yx$, then $x +_G y = xy$;
2). For each integer m, $mx = x^m$;
3). For each $n \geq 0$, $p^n G = G_n$;
4). If $x, y \in G_n$, then $x +_G y \equiv xy \pmod{G_{n+1}}$.

In fact, under this new-defined addition $+_G$, G is a $\mathbb{Z}_p$-free module:

Proposition D.30. *With the original topology, $(G, +_G)$ is a uniform pro-p group of dimension $d = d(G)$. G has the structure of free $\mathbb{Z}_p$-module on the basis of the topological generator $\{a_1, \cdots, a_d\}$, and there is an isomorphism of $\mathbb{Z}_p$-modules*

$$\psi : G \to \mathbb{Z}_p,\ \lambda_1 a_1 + \cdots + \lambda_d a_d \mapsto (\lambda_1, \cdots, \lambda_d).$$

Since the original group $(G, \cdot)$ is not necessarily abelian, the information about non-commutativity is lost, which should be transferred onto $(G, +_G)$ by defining a Lie bracket: By Lemma D.10, the commutator $[x^{p^n}, y^{p^n}]$ of p^n-th powers is contained in G_{2n+1} for each n. This allow to extract p^{2n}-th roots and to set $(x, y)_n = [x^{p^n}, y^{p^n}]^{p^{-2n}}$ (which lies in pG if $p > 2$ and $4G$ if $p = 2$). By similar arguments as for the addition "$+_G$", the sequence $(x, y)_n$ can be shown to be Cauchy. We thus define $(x, y)_G = \lim_{n\to\infty}(x, y)_n$. Thus we conclude:

Proposition D.31. *If $(G, \cdot)$ is a uniform pro-p-group, then $(G, +_G, ()_G)$ defined as above is a finitely generated powerful $\mathbb{Z}_p$ Lie algebra.*

Conversely, put

$$L(X) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} X^n,\ E(X) = \sum_{n=0}^{\infty} \frac{1}{n!} X^n$$

be formal power series in $\mathbb{Q}_p\langle\langle X\rangle\rangle$, then on any normed $\mathbb{Q}_p$-algebra A, they define analytic maps

$$\exp : \hat{A}_0 \to 1 + \hat{A}_0,\ \log : 1 + \hat{A}_0 \to \hat{A}_0,$$

where

$$A_0 = \begin{cases} \{x \in A \mid \|x\| \leq p^{-1}\},\ p \neq 2; \\ \{x \in A \mid \|x\| \leq 2^{-2}\},\ p = 2. \end{cases}$$

Put $P(X, Y) = E(X)E(Y) - 1$, and $\Phi(X, Y) = (L \circ P)(X, Y)$, i.e. $\Phi(x, y) = \log(\exp x \cdot \exp y)$ as long as it converges.[1] Under this "simulated addition", $\mathfrak{g}$ is transferred to a uniform pro-p group:

Proposition D.32. *Let $\mathfrak{g}$ be a powerful $\mathbb{Z}_p$ Lie algebra. Then the operation*

$$\star : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},\ x \star y := \Phi(x, y),$$

makes $\mathfrak{g}$ into a uniform pro-p group.

As a result, we achieve the main result of this section:

[1]Some readers may feel confused about why $\Phi(x, y)$ is not necessarily $x + y$ nor $\Phi(y, x)$. In fact, this is because the map exp and log are not necessarily homomorphism since $\mathfrak{g}$ is not commutative.

Theorem D.33. *There is an equivalence of categories between uniform pro-p groups and powerful $\mathbb{Z}_p$-Lie algebras:*

Uniform pro-p groups		*Finitely generated Powerful $\mathbb{Z}_p$-Lie algebras*
G	$\rightarrow$	$L_G = (G, +_G, ()_G)$
$(\mathfrak{g}, \star)$	$\leftarrow$	$\mathfrak{g}$

D.4. **Associated $\mathbb{Z}_p$ algebra:** $\log(G)$**.**

Definition D.34. The limit

$$\mathbb{Z}_p[\![G]\!] = \varprojlim \mathbb{Z}_p[G/N]$$

is called the completed group algebra of G (or also Iwasawa algebra of G), where N runs through open normal subgroups of G.

Proposition D.35. *Let G be a finitely generated pro-p group. There exists a norm $\|\cdot\|$ on $\mathbb{Z}_p[G]$ such that the corresponding completion $\widehat{\mathbb{Z}_p[G]}$ is topologically isomorphic to $\mathbb{Z}_p[\![G]\!]$.*

In fact, let $I = (G-1)\mathbb{Z}_p[G]$ be the augmentation ideal of $\mathbb{Z}_p[G]$ and let $J = I + p\mathbb{Z}_p[G]$, then the norm is precisely the J-adic norm, defined by

$$\|\alpha\| = p^{-k}, \text{ if } 0 \neq \alpha \in J^k - J^{k+1} \text{ and } \|0\| = 0.$$

One can show that the topology on $\mathbb{Z}[G]$ given by the norm induces on G the original topology of G.

Proposition D.36. *Let G be a uniform pro-p group, then the norm $\|\cdot\|$ can be uniquely extended to a norm on $\mathbb{Q}_p[G]$. Moreover, each $g \in G$ satisfies $\|g-1\| \leq p^{-1}$.*

Let $A = \mathbb{Q}_l[G]$. The inequality above guarantees that $G \subset A_0$. Put $\mathcal{L} = \log(G)$.

Lemma D.37. *For $g, h \in G$ and $\lambda \in \mathbb{Z}_p$, we have*

$$\log g + \log h = \log(g +_G h),\ \lambda \log g = \log g^{\lambda} \quad \textit{and}$$

$$[\log g, \log h] = \log(g, h)_G.$$

The following is an immediate consequence:

Proposition D.38. $(\mathcal{L}, +, [\ \ ,\ \ ])$ *is a $\mathbb{Z}_p$-Lie subalgebra of $\widehat{\mathbb{Q}_p[G]}$. It is denoted by* $\log(G)$ *and called the associated $\mathbb{Z}_p$ Lie algebra of G. It is isomorphic to $(G, +_G, (\ )_G)$ under the analytic map* $\log$ *as Lie algebra. Both algebras are free $\mathbb{Z}_p$-module of rank d.*

We conclude this subsection with a result that clarifies the role of Lie ideals:

Proposition D.39. *Let G be a uniform group, and let $\mathcal{L} = \log(G)$ be the associated $\mathbb{Z}_p$ Lie algebra. Then the followings are equivalent:*
1). $I \subset \mathcal{L}$ is a Lie ideal such that $\mathcal{L}/I$ is torsion free;
2). $N = \exp(I)$ is uniform, closed and normal in G, and G/N is uniform.

D.5. p**-adic Lie Groups.**

Definition D.40. A topological group G is a Lie group if G has the structure as an analytic manifold with the group operation being analytic functions. G is a p-adic Lie group if the manifold is p-adic analytic.

We have a characterisation of p-adic Lie groups by uniform groups:

Theorem D.41. *Let G be a topological group. Then G is a p-adic group if and only if G contains an open subgroup which is a uniform pro-p group.*

Proposition D.42. *For a topological group G, the following are equivalent:*
1). G is a compact p-adic Lie-group;
2). G contains an open normal uniform pro-p subgroup of finite index;
3). G is a profinite group containing an open subgroup which is a pro-p group of finite rank.

Proposition D.43. *Let G be a p-adic Lie group. There exists an integer d such that any open pro-p subgroup of G has finite rank and (algebraic) dimension d. Moreover, every chart in an atlas of G has (analytic) dimension d.*

In the following, we list some properties of p-adic Lie groups.

Proposition D.44. *Let G_1 and G_2 be p-adic or real analytic groups. Then every continuous homomorphism $G_1 \to G_2$ is analytic.*

Proposition D.45. *Let G be a p-adic Lie group. Then:*
1). Any closed subgroup H is a p-adic Lie group, and the inclusion map $H \hookrightarrow G$ is analytic;
2). Any quotient G/H with a normal sungroup N is a p-adic Lie group again. The projection $G \to G/N$ is analytic.

Proposition D.46. *Let G be a Hausdorff topological group, and N a closed normal subgroup. If both N and G/N are p-adic Lie group, then G is a p-adic Lie group as well.*

Appendix E. Sen's Filetration Theorem

For a group G, A filtration (G_n) is uniformly equivalent in scaling s to another filtration (G^m) if $n \cdot s + O(1) = m$, i,e. $G^{ns+c} \subset G_n \subset G^{ns-c}$ for some constant c.[2]

A filtration $(G(n))$ on a p-adic analytic group G is called a Lie filtration if it agrees with the lower p-series on some open uniform subgroup $H \subset G$, that is, there exists $r \in \mathbb{N}$ such that $P_n(H) = G(n+r)$ for all $n \geq 0$.

The main result of this section is to prove

Theorem E.1. *(Sen) Let L/K be a totally ramified extension of local fields in characteristic 0. Let $e = e_K = v_K(p)$ be the absolute ramification index of K. Assume that the Galois group $G = G(L/K)$ is p-adic analytic with $\dim G > 0$. Let $G(n)$ be a filtration on G that is uniformly equivalen t in scaling 1 to some Lie filtration, then $G(n)$ is uniformly equivalent in scaling e to the filtration of ramification subgroups (G^n), i.e. there exists a constant $c > 0$ such that*

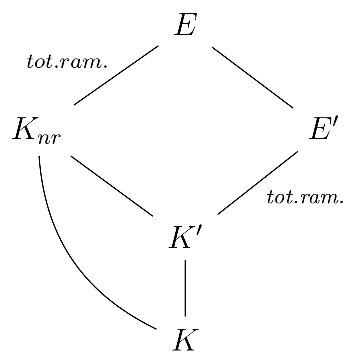
$$G^{ne+c} \subset G(n) \subset G^{ne-c}$$

for all n sufficiently large.

[2]The expression "equivalent" comes from the fact that both filtrations are topologically equivalent. We emphasize that "uniform equivalence" is an equivalence relation between filtrations if and only if we deal with scaling 1.

E.1. **Small Groups and Non-small Groups.** In this subsection, we will define a Galois group of a finite totally ramified extension L/K to be small or not, characterized by the properties in Proposition F.7. Then we will draw some conclusions on the last jump of the ultimate jump of ramification group u_G and v_G. We will omit the proofs of some results from the local class field theory.

Lemma E.2. *Let K be a complete local field. Let K_{nr} be the maximal unramified extension of K, and let E be a finite extension of K_{nr} of degree n. Then there exist a finite subextension K' of K_{nr} and an extension E'/K' of degree n, linearly disjoint from K_{nr} over K', such that $E = E'K_{nr}$. The extension E'/K' is then totally ramified, and E can be identified with E'_{nr}. Moreover, if E/K is separable (resp. Galois), then E'/K' can be chosen likewise.*



Lemma E.3. *Assume L/K is a totally ramified finite Galois extension with Galois group G. There is a complete non-archimedean field extension L'/K' with the same Galois group G such that the residue field of K' is algebraically closed and the ramification groups of L/K and L'/K' coincide.*

Lemma E.4. *Suppose K is a complete discrete valuation field with quasi-finite residue field. Let L/K be an abelian extension with Galois group G, then the image of U_K^n under the reciprocity map $K^\times \to G$ is dense in $(G)^n$.*

In the following, let K be a local field of characteristic 0. Let L be a totally ramified Galois extension of K with Galois group G. Let $e = e_G = v_K(p)$. If G is finite, put

$$v_G = \inf\{v \mid v \geq 0,\ G^v = 1\}$$

and

$$u_G = \inf\{u \mid u \geq 0,\ G_u = 1\}.$$

Then

$$u_G = \Psi_{L/K}(v_G) \geq |G| v_G.$$

Lemma E.5. *Let $U = U_K$. Then*

$$\begin{cases} U_v^p \subset U_{pv} \ \textit{if}\ v \leq \dfrac{e_G}{p-1} \\ U_v^p = U_{v+e_G},\ \textit{if}\ v \geq \dfrac{e_G}{p-1} \end{cases}$$

Proof. Let $(1+x)$ be an element of U_v, and assume that $v(x) = r \geq v$. For $p \geq 2$,

$$(1+x)^p = 1 + px + (p(p-1)/2)x^2 + \cdots + px^{p-1} + x^p.$$

Note that:

$$\begin{aligned}
&v(px) = e + r;\\
&v((p(p-1)/2)x^2) = e + 2r;\\
&\cdots\\
&v(px^{p-1}) = e + (p-1)r;\\
&v(x^p) = pr.
\end{aligned}$$

We conclude that

$$v((1+x)^p - 1)\begin{cases} = \min\{e+r, e+2r, \cdots, e+(p-1)r, pr\} = \min\{e+r, pr\}, \text{ if } e+r \neq pr;\\ \geq v(x^p + px), \text{ otherwise.}\end{cases}$$

This distinction holds as well for $p = 2$, so we remove the above restriction on p.

When $r \leq \frac{e}{p-1}$, $pr \leq e + r$, and hence $v((1+x)^p - 1) = pr \geq pv$, so we derive $U_v^p \subset U_{pv}$ in the first case. If conversely $r \geq \frac{e}{p-1}$, we have $U_v^p \subset U_{e+v}$. It remains to show that $U_{e+r} \subset U_v^p$ in this case.

In fact, the number p splits as $p = \gamma\pi^e$, so the element x can uniquely be written as $x = \alpha\gamma^{-1}\pi^r$ with a unit α, and thus

$$\begin{aligned}
(1+x)^p \equiv& 1 + p\alpha\gamma^{-1}\pi^r\\
\equiv& 1 + \alpha\pi^{e+r} \bmod \pi^{r+e+1}.
\end{aligned}$$

The above relation induces an isomorphism $U_{v+e}/U_{v+e+1} \cong U_v^p/U_{v+e+1}$, and it follows that for any positive integer n,

$$U_{v+e}/U_{v+e+n} \cong U_v^p/U_{v+e+n}.$$

Since K is complete, we conclude the proof by taking limit. $\square$

Theorem E.6. *For any $r \geq 0$, the relative Artin map $(\quad, L/K)$ maps the unit groups U_K^v onto $G(L/K)^v$.*

Proposition E.7. *Suppose G is a finite abelian p-group. Then*

$$\begin{cases}(G^v)^p \subset G^{pv} \text{ if } v \leq \dfrac{e_G}{p-1}\\ (G^v)^p = G^{v+e_G}, \text{ if } v \geq \dfrac{e_G}{p-1}\end{cases}$$

Proof. By Lemma E.2, we can assume that the residue field k is algebraically closed. In this case, one can always find a quasi-finite field k_0, such that k is the algebraic closure of k_0. Regard $K_0 = W(k_0)[\frac{1}{p}]$ as a subfield of K. By Lemma E.1, one can find a finite extension K_1 of K_0 inside K and a finite totally ramified extension L_1 of K_1, such that $L_1K = L$. Moreover, $\mathrm{Gal}(L_1/K_1) = \mathrm{Gal}(L/K)$ and their ramification groups coincide since L_1 and K_1

are linearly disjoint over K_1.

$$
\begin{tikzcd}
& L \arrow[dl, dash, "tot.ram."'] \arrow[dr, dash] & \\
K \arrow[dr, dash] \arrow[ddr, dash, bend right] & & L_1 \arrow[dl, dash, "\text{tot.ram.}"] \\
& K_1 \arrow[d, dash] & \\
& K_0 &
\end{tikzcd}
$$

As the residue field of K_1 is a finite extension of k_0, it is quasi-finite. The proposition is now reduced to the case that the residue field k is quasi-finite, following from Lemma E.4, Lemma E.5 and Theorem E.6. □

Corollary E.8. *Suppose G is a finite abelian Galois p-group and denote $G[n]$ for the n-torsion subgroup of G. If $v_G \leq \frac{p}{p-1}e$, then $v_G \geq p^m v_{G/G[p^m]}$ for all $m \geq 1$. If $v_G > \frac{p}{p-1}e$, then $v_G = v_{G/G[p]} + e$.*

Proof. If $v_G \leq \frac{p}{p-1}e$, put $t_m := p^{-m}v_G$, $t_m \leq \frac{1}{p-1}e$. For any $\epsilon > 0$, $(G^{t_m+\epsilon})^{p^m} \subset G^{p^m t_m + \epsilon p^m} = G^{v_G+\epsilon p^m} = 1$, hence $G^{t_m+\epsilon} \subset G[p^m]$. Thus $t_m \geq v_{G/G[p^m]}$.

If $v_G > \frac{p}{p-1}e$, put $t = v_G - e$, for any $\epsilon > 0$, $(G^{t+\epsilon})^p = G^{t+e+\epsilon} = 1$. Thus $v_G = v_{G/G[p]} + e$. □

We call a finite abelian Galois p-group G **small** if $v_G \leq \frac{p}{p-1}e$, or equivalently, if $(G^x)^p \subset G^{px}$ for all $x \geq 0$.

Lemma E.9. *If G is small, then for every $m \geq 1$,*

$$u_G \geq p^{m-1}(p-1)[G[p^m] : G[p]]u_{G/G[p]}.$$

Proof. For every $\epsilon > 0$, we have

$$\begin{aligned} u_G = \Psi(v_G) &= \int_0^{v_G} [G : G^t]dt \geq \int_{p^{-1}v_G+\epsilon}^{v_G} [G : G^t]dt \\ &= (v_G - p^{-1}v_G - \epsilon)[G : G^{p^{-1}v_G+\epsilon}] \geq (\frac{p-1}{p}v_G - \epsilon)[G : G[p]]. \end{aligned}$$

The last inequality holds since $(G^{p^{-1}v_G+\epsilon})^p = 1$. Then by Corollary E.7,

$$u_G \geq \frac{p-1}{p}v_G[G : G[p]] \geq (p-1)p^{m-1}[G : G[p]]v_{G/G[p^m]}.$$

Since $u_{G/G[p^m]} \leq |G/G[p^m]|v_{G/G[p^m]} = [G : G[p^m]]v_{G/G[p^m]}$, we have the desired result. □

We now suppose G is a p-adic Lie group of dimension $d > 0$ with a Lie filtration $\{G(n)\}$. We may assume that $G(0) = G$ and $G(1)$ is an open uniform pro-p subgroup of G, then

$$G(n) = G(n+1)^{p^{-1}} = \{s \in G \mid s^p \in G(n+1)\}.$$

For $n \geq 1$, we denote

$$\Psi_n = \Psi_{G/G(n)},\ v_n = v_{G/G(n)},\ u_n = u_{G/G(n)} = \Psi_n(v_n),\ e_n = e_{G(n)}.$$

Proposition E.10. *For each $n \geq 1$ we have $G^v \cap G(n) = G(n)^{\Psi_n(v)}$ for $v \geq 0$. In particular,*

$$G^v = G(n)^{u_n+(v-v_n)[G:G(n)]}, \text{ for } v \geq v_n, \tag{E.1}$$

i.e.

$$G^{v_n+te} = G(n)^{u_n+te_n}, \ t > 0. \tag{E.2}$$

As a consequence, for $n, r \geq 1$,

$$v_{G(n)/G(n+r)} = u_n + (v_{n+r} - v_n)[G : G(n)] \tag{E.3}$$

Proof. The first equality follows from Proposition B.16. For $v > v_n$, $(G/G(n))^v = G^v G(n)/G(n) = 1$, hence $G^v \subset G(n)$. And compute that

$$\Psi_n(v) = \Psi_n(v_n) + \int_{v_n}^{v} [G/G(n) : (G/G(n))^t]dt = \Psi_n(v_n) + [G : G(n)](v - v_n),$$

thus

$$G^v \cap G(n) = G^v = G(n)^{u_n+(v-v_n)[G:G(n)]}, \text{ for } v > v_n$$

i.e.

$$G^{v_n+te} = G(n)^{u_n+tv_K(p)[G:G(n)]} = G(n)^{u_n+tv_{K_n}(p)} = G(n)^{u_n+te_n}, \text{ for } t > 0.$$

Finally, let $v = v_{G(n)/G(n+r)}$, v is characterized by the fact that $G(n)^v \not\subset G(n+r)$ and $G(n)^{v+\epsilon} \subset G(n+r)$ for any $\epsilon > 0$. But $x = v_{n+r}$ is characterized by the fact that $F^x \not\subset G(n+r)$ and $G^{x+\epsilon} \subset G(n+r)$ for any $\epsilon > 0$. Thus (E.3) follows from (E.1). □

Corollary E.11. *Suppose there are $n, m \geq 1$, such that $A = G(n)/G(n+m)$ is abelian. Then*
1). If A is small, then $v_{n+1} - v_n < \frac{e}{p^{m-2}(p-1)}$;
2). If A is not small, then $v_{n+m} - v_{n+m-1} = e$.

Proof. Note that

$$A/A[p^{m-1}] = \frac{G(n)/G(n+m)}{(G(n)/G(n+m))[p^{m-1}]} = \frac{G(n)/G(n+m)}{G(n+1)/G(n+m)} = G(n)/G(n+1),$$

and similarly

$$A/A[p] = G(n)/G(n+m-1).$$

For case (1), since A is small by assumption, $v = v_{G(n)/G(n+m)} \leq \frac{p}{p-1}e_n$. From Corollary E.8, we derive $v \geq p^{m-1}v_{G(n)/G(n+1)}$. Now we apply (E.3) to substitute the expression $v_{G(n)/G(n+m)}$. This yields the inequalities

$$\frac{p}{p-1}e_n \geq v \geq p^{m-1}(u_n + (v_{n+1} - v_n)[G : G(n)]).$$

The result follows by $u_n > 0$ and $e_n = e[G : G(n)]$.

For (2), Corollary E.8 yields $v = v_{G(n)/G(n+m-1)} + e_n$. Use (E.3) again, then we get

$$(v_{n+m} - v_n)[G : G(n)] = (v_{n+m-1} - v_n)[G : G(n)] + e_n,$$

and the result follows directly. □

Lemma E.12. *Let H be a finite p-group, but not necessarily abelian. Assume that H is provided with the lower numbering from some Galois extension of p-adic fields L/K. Let $H[p]$ be the set of elements annihilated by p. Any subgroup A that contains $H[p]$ has the last jump as H as H, i.e. $u_A = u_H$.*

Proof. Since $A_u = H_u \cap A$, it is clear that $u_A \leq u_H$. Conversely, if $A_{u_H} = 1$, $H_{u_H} \cap H[p] \subset H_{u_H} \cap A = 1$. However, by proposition $B.20$, H_{u_H} is annihilated by p, i.e. $H_{u_H} \subset H[p]$, contradicting, and hence $u_A \geq u_H$. □

Lemma E.13. *Let $A = G(n)/G(n+m)$ with $m \geq 1$. We have:*
1). $u_{n+m} = u_A$;
2). $u_{n+1} = u_{G(n)/G(n+1)} = u_{A/A[p^{m-1}]}$.

Proof. Set $H = G/G(n+m)$ and note that $H[p] = G(n+m-1)/G(n+m)$. Then $u_{n+m} = u_H = u_A$. The second equality follows that $A/A[p^{m-1}] = G(n)/G(n+1)$. □

E.2. **The First Inclusion.**

Lemma E.14. *Assume that $A = G(n)/G(n+m)$ is abelian and small. For $m \geq 2$, we have*

$$\frac{u_{n+m}}{e_{n+m}} \geq (p-1)p^{m-(d+2)} \cdot \frac{u_{n+1}}{e_{n+1}}.$$

Proof. We have $u_{n+m} = u_A$ by the lemma above, and $e_{n+m} = e_{n+1} \cdot [G_{n+1} : G_{n+m}]$ since we are dealing with a totally ramified extension. This gives

$$\frac{u_{n+m}}{e_{n+m}} = p^{-(m-1)d} \cdot u_A \cdot \frac{1}{e_{n+1}}. \tag{E.4}$$

By Lemma E.8,

$$u_A \geq (p-1)p^{m-2} \cdot [A[p^{m-1}] : A[p]] \cdot u_{A/A[p^{m-1}]} = (p-1)p^{m-2}p^{(m-2)d}u_{n+1}.$$

Substite into (E.4), we conclude the proof. □

Proposition E.15. *Assume that there is $m \geq \dim(G) + 3 = d + 3$ such that $G(n)/G(n+m)$ is abelian for n sufficiently large, then not all of these latter quotients are small.*

Proof. We deduce from Lemma E.12 that if all of these quotients are small, we would have

$$\frac{u_{n+m}}{e_{n+m}} \geq (p-1)p\frac{u_{n+1}}{e_{n+1}}.$$

By Lemma B.21, $u_k/e_k \leq \frac{1}{p-1}$ for every k, we get a contradiction

$$\frac{1}{p-1} \geq \frac{u_{n+m}}{e_{n+m}} \geq p.$$

□

Proposition E.16. *Suppose there are integers n_0 such and $m \geq 2$ such that $G(n_0)/G(n_0+m)$ is non-small and abelian. Let $n \geq n_0 + m$. If each quotient $G(n)/G(n+2)$ is abelian, then non of them is small, and $v_{n+1} = v_n + e$*

Proof. Since $G(n_0)/G(n_0+m)$ is non-small, case 2). of Corollary E.11 implies $v_{n_0+m} = v_{n_0+m-1} + e$. If the quotient $G(n_0+m-1)/G(n_0+m+1)$ were small, we would have the contradiction $v_{n_0+m+1} - v_{n_0+m} < \frac{e}{p^{m-2}(p-1)} < e$. Thus $G(n_0+m-1)/G(n_0+m+1)$ is non-small, and hence $v_{n_0+m+1} - v_{n_0+m} = e$. Again, this implies that $G(n_0+m)/G(n_0+m+2)$ is non-small, and $v_{n_0+m+2} - v_{n_0+m+1} = e$. We thus get the assertion by induction. □

Corollary E.17. *There exists an integer n_1 such that $v_n = v_{n_1} + (n-n_1)e$ for $n > n_1$. It follows that there is a constant c such that $G^{ne+c} \subset G(n)$.*

Proof. We need to show that the requirement of Proposition E.16 are fulfilled. In fact, by Lemma D.10, the quotient $G(n)/G(n+m)$ are abelian for $m \leq n+1$ and arbitraty n. Set $m = d+3$. Since there are enough non-small quotients (Proposition E.15), there exists an abelian non-small group $G(n_0)/G(n_0+m)$ with $n_0 \geq d+2$. Set $n_1 = n_0 + m$ for brevity. Then by Proposition E.16, $v_n = v_{n_1} + (n - n_1)e$. It follows that $v_n < ne + v_{n_1}$, thus $(G/G(n))^{ne+v_{n_1}} = G^{ne+v_{n_1}}G(n)/G(n) = 0$, i.e. $G^{ne+v_{n_1}} \subset G(n)$. $\square$

E.3. **The Second Inclusion.** Rather than in the non-abelian group G, we wish to work within its associated $\mathbb{Z}_p$-Lie algebra $\mathcal{L} = \log(G)$. We aim to relate the sequence $G(ne+c)$ to some projective system within $\mathcal{L} = \varprojlim \mathcal{L}/p^n\mathcal{L}$.

Recall that there is an equivalence of categories between powerful Lie algebras and uniform groups (cf. Theorem D.33). Under this equivalence, p-th powers in G correspond to multiplication with p in L , giving $G(n) = \exp(p^n\mathcal{L})$ and $log(G(n)) = p^n\mathcal{L}$. We point out again that the analytic map log is not a group homomorphism, but only maps $G(n)$ onto $p^n\mathcal{L}$ as sets. However, the shift isomorphism sh : $G(n)/G(m) \cong G(n+1)/G(m+1),\ g \mapsto g^p$ corresponds to an isomorphism sh : $p^n\mathcal{L}/p^m\mathcal{L} \to p^{n+1}\mathcal{L}/p^{m+1}\mathcal{L},\ x \mapsto p \cdot x$. With this notation, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{L}/p^{n+1}\mathcal{L} \arrow[rr, "\pi"] \arrow[d, "\exp\circ\ \mathrm{sh}^{n+1}"] & & \mathcal{L}/p^n\mathcal{L} \arrow[d, "\exp\circ\ \mathrm{sh}^{n}"] \\
G(n+1)/G(2n+1) \arrow[r, "\pi"] & G(n+1)/G(2n+1) \arrow[r, "\mathrm{sh}^{-1}"] & G(n)/G(2n)
\end{tikzcd}$$

where π is the projection onto the smaller quotients.

Let $n_1 = n_0 + m = n_0 + d + 3$, $c = v_{n_1}$ as defined in the proof of Corollary E.17. Then for every $n \geq n_1$, $G^{ne+c} \subset G(n)$, and hence the quotient $G^{ne+c}G(2n+1)/G(2n+1)$ is abelian by Lemma D.10. The desired projective system is constructed as the following:

Proposition E.18. *For each $n \geq n_1$, let M_n denote the quotient*

$$M_n = sh^{-n}(\log \frac{G^{ne+c}G(2n)}{G(2n)}) \subset \frac{\mathcal{L}}{p^n\mathcal{L}}.$$

Then the quotients (M_n) form a projective system with surjective transition maps. We denote the limit by $M = \varprojlim_{n\geq n_1} M_n \subset \mathcal{L}$

We need the following result:

Lemma E.19. *Let n_1, c be as above, $n \geq n_1$. Then we have*

$$(G^{ne+c})^pG(2n+1) = G^{(n+1)e+c}G(2n+1).$$

Proof. Recall that $c = v_n - (n - n_1)e$ for $n \geq n_1$. By (E.2),

$$G^{ne+c} = G^{v_n+n_1e} = G(n)^{u_n+n_1e_n}.$$

We set $\gamma = u_n + n_1 e_n > \frac{e_n}{p-1}$, then

$$\begin{aligned}(G^{ne+c})^p G(2n+1)/G(2n+1) &= (G(n)^\gamma)^p G(2n+1)/G(2n+1)\\ &= (G(n)^\gamma G(2n+1)/G(2n+1))^p\\ &= ((G(n)/G(2n+1))^\gamma)^p\\ &= (G(n)/G(2n+1))^{\gamma+e_n}\\ &= G(n)^{u_n+(n_1+1)e_n} G(2n+1)/G(2n+1)\\ &= G^{v_n+(n_1+1)e_n} G(2n+1)/G(2n+1),\end{aligned}$$

which completes the proof. □

Proof of Proposition E.18

$$\begin{aligned}M_{n+1} &= \mathrm{sh}^{-n-1}(\log \frac{G^{(n+1)e+c}G(2n+2)}{G(2n+2)})\\ &\xrightarrow{\exp\circ\ \mathrm{sh}^{n+1}} \frac{G^{(n+1)e+c}G(2n+2)}{G(2n+2)} \xrightarrow{\pi} \frac{G^{(n+1)e+c}G(2n+1)}{G(2n+1)}\\ &= \frac{(G^{ne+c})^p G(2n+1)}{G(2n+1)} \underset{\sim}{\xrightarrow{\mathrm{sh}^{-1}}} \frac{(G^{ne+c})^p G(2n)}{G(2n)}\\ &\xrightarrow{\mathrm{sh}^{-n}\circ\log} \mathrm{sh}^{-n}(\log \frac{G^{ne+c}G(2n)}{G(2n)}) = M_n.\end{aligned}$$

All the maps above are onto, thus we have the proposition. □

Now we reach the point to prove the second inclusion of Sen's Theorem:

Proposition E.20. *Assume that $p^{n_2}\mathcal{L} \subset M$ for some n_2, and let $n > n_2$. We then have $G(n+n_2) \subset G^{ne+c}$.*

Proof. With $n > n_2$, the assumption implies

$$\frac{p^{n_2}\mathcal{L}}{p^n\mathcal{L}} \subset \frac{M + p^n\mathcal{L}}{p^n\mathcal{L}} = M_n.$$

We apply $\exp\circ \mathrm{sh}^n$ to translate this into $G(n+n_2)/G(2n) \subset G^{ne+c}G(2n)/G(2n)$, which gives

$$G(n+n_2) \subset G^{ne+c}G(2n).$$

Any $g \in G(n+n_2)$ can thus be written as $g = g_0 b_0$, with $g_0 \in G^{ne+c}$ and $b_0 \in G(2n)$. By the same argument (e.g. with $n' = 2n - n_2$ in place of n), we see that this last factor admits a representation as $b_0 = g_1 b_1$ with $g_1 \in G^{ne+c}$ and $b_1 \in G(2n+1)$. We thus have $g = (g_0 g_1) b_1$. Repeated Performance gives a sequence (h_m) with components

$$h_m = \prod_{i=1}^m g_i$$

that lie within G^{ne+c}, and a sequence (b_m) with $b_m \in G(2n+m)$, such that $g = h_m b_m$ for every m. The (h_m) converge towards g, for

$$h_m \cdot g^{-1} = h_m \cdot (h_m b_m)^{-1} = h_m b_m^{-1} h_m^{-1}$$

is an element of $G(2n+m)$ ude to the normality of the $G(n)$. The group $G(ne+c)$ is complete as closed subset of G and thus contains the limit g. □

Proof of Theorem E.1 By a suitably redefining the constant c, Proposition E.17 and Proposition E.20 together gives the whole inclusions $G^{ne+c} \subset G(n) \subset G^{ne-c}$. □

Appendix F. Totally Ramified $\mathbb{Z}_p$-extensions

In this section, we suppose that K_∞/K is totally ramified. In fact, this assumption is not strict because in general we know that K_∞/K is almost totally ramified (i.e. totally ramified after some finite extension of K).

F.1. **TS1.**

Proposition F.1. *If L is a finite extension of K_∞, then $Tr_{L/K_\infty}(\mathcal{O}_L) \supset \mathfrak{m}_{K_\infty}$.*

Proof. It suffices to show for the case L/K is Galois, since then for every $x \in \mathfrak{m}_{K_\infty}$, there exists $y \in M$, where M is a normal closure of L over K, such that $\mathrm{Tr}_{M/K}(y) = \mathrm{Tr}_{L/K}(\mathrm{Tr}_{M/L}(y))$. Take $L_0 = L^\Gamma$, then $L_0 \cap K_\infty = K_n$ for some $n \in \mathbb{N}$. Replace K by K_n, we may assume that $K_\infty \cap L_0 = K$. Put $L_m = L_0 K_m$, $\forall m \in \mathbb{N}$, then by Corollary C.8,

$$v_K(\mathfrak{D}_{L_m/K_m}) = \int_{-1}^{\infty} (|\mathrm{Gal}(K_m/K)^v|^{-1} - |\mathrm{Gal}(L_m/K)^v|^{-1}) dv.$$

Suppose that $\mathrm{Gal}(L_0/K)^v = 1$ for $v \geq h$, then $\mathrm{Gal}(L/K)^v \subset \Gamma$ and $\mathrm{Gal}(L_m/K)^v = \mathrm{Gal}(K_m/K)^v$ for $v \geq h$. We have

$$v_K(\mathfrak{D}_{L_m/K_m}) \leq \int_0^h |\mathrm{Gal}(K_m/K)^v|^{-1} dv = \frac{1}{|\mathrm{Gal}(K_m/K)^0|}\Psi(h) \to 0$$

as $m \to \infty$. Thus $v_K(\mathfrak{D}_{L/K_\infty}) = 0$, $\mathfrak{D}_{L/K_\infty} = 1$; as a result, by Corollary C.4,

$$\mathrm{Tr}(\mathfrak{m}_L^n) = \mathfrak{m}_K^r \supset \mathfrak{m}_K$$

since $r = \lfloor \frac{0+1}{e_{L/K_\infty}} \rfloor = 0 < 1$. □

Corollary F.2. *For any $a > 0$, there exists $x \in L$, such that*

$$v_K(x) > -a, \text{ and } Tr_{L/K_\infty}(x) = 1. \tag{F.1}$$

Proof. For any $a > 0$, there exists $\alpha \in \mathcal{O}_L$, such that $v_K(\mathrm{Tr}_{L/K_\infty}(\alpha))$ is less than a. Let $x = \frac{\alpha}{\mathrm{Tr}_{L/K_\infty}(\alpha)}$, then x satisfies (D.1). □

Clearly this proposition and the corollary are still true if replacing K_∞ by any field M such that $K_\infty \subset M \subset L$. (D.1) is called the almost étale condition.

F.2. **TS2.**

Proposition F.3. *There is a constant c such that*

$$v(\mathfrak{D}_{K_n/K}) = en + c + p^{-n} a_n,$$

where a_n is bounded.

Proof. By Herbrand's theorem (Proposition B.16), $\mathrm{Gal}(K_n/K)^v = (\Gamma/\Gamma_n)^v = \Gamma^v\Gamma_n/\Gamma_n = \Gamma_i/\Gamma_n$ for $v_i < v \le v_{i+1}$, $i \le n$ and 1 if $i > n$. By Proposition E.16, there exsits an integer n_0, such that $v_{n+1} - v_n = e$ for $n > n_0$. Thus

$$\begin{aligned}
v(\mathfrak{D}_{K_n/K}) &= \int_{-1}^{\infty}(1 - |\mathrm{Gal}(K_n/K)^v|^{-1})dv \\
&= \sum_{i=1}^{n}\int_{v_i}^{v_{i+1}}(1-p^{i-n})dv \\
&= \sum_{i=1}^{n_0}\int_{v_i}^{v_{i+1}}(1-p^{i-n})dv + \sum_{i=n_0+1}^{n}\int_{v_i}^{v_{i+1}}(1-p^{i-n})dv \\
&= \sum_{i=1}^{n} e(1-p^{i-n}) + c' \qquad (\text{where } c' = \sum_{i=1}^{n_0}\int_{v_i}^{v_{i+1}}(1-p^{i-n})dv - \sum_{i=1}^{n_0} e(1-p^{i-n})) \\
&= en + c' - p^{-n}\sum_{i=1}^{n} ep^i \\
&= en + c - p^{-n}\frac{e}{1-p} \qquad (\text{where } c = c' + \frac{e}{1-p})
\end{aligned}$$

□

Corollary F.4. *There is a constant c which is independent of n, such that for all $x \in K_n$,*

$$v_K(p^{-n}\,Tr_{K_n/K}(x)) \ge v_K(x) - c.$$

Proof. By the above proposition, $v_K(\mathfrak{D}_{K_{n+1}/K_n}) = e + p^{-n}b_n$ with $p^{-n}b_n$ bounded. Let $\mathcal{O}_n$ be the ring of integers of K_n and $\mathfrak{m}_n$ its maximal ideal. Suppose $\mathfrak{D}_{K_{n+1}/K_n} = \mathfrak{m}_{n+1}^d$, where $d = p^{n+1}e + pb_n$. By Corollary C.4, we have

$$\mathrm{Tr}_{K_{n+1}/K_n}(\mathfrak{m}_{n+1}^i) = \mathfrak{m}_{\mathfrak{n}}^{\mathfrak{j}},$$

where $j = \lfloor \frac{i+d}{p} \rfloor$. Thus

$$v_{K_n}(\mathrm{Tr}_{K_{n+1}/K_n}(x)) = \lfloor \frac{v_{K_{n+1}}(x) + d}{p} \rfloor,$$

hence

$$v_K(p^{-1}\mathrm{Tr}_{K_{n+1}/K_n}(x)) = \frac{1}{p^n}\lfloor \frac{p^{n+1}v_K(x) + p^{n+1}e + pb_n}{p} \rfloor - e \ge v_K(x) + p^{-n}b_n \ge v_K(x) + p^{-n}a,$$

where a is a bound of b_n, which is independent of n. Then by induction

$$\begin{aligned}
v_K(p^{-n}\mathrm{Tr}_{K_n/K}(x)) &= v_K(p^{-n}\mathrm{Tr}_{K_n/K_{n-1}} \circ \mathrm{Tr}_{K_{n-1}/K_{n-2}} \circ \cdots \circ \mathrm{Tr}_{K_1/K}(x)) \\
&\ge v_K(p^{-n+1}\mathrm{Tr}_{K_{n-1}/K_{n-2}} \circ \cdots \circ \mathrm{Tr}_{K_1/K}(x)) - p^{-n}a \\
&\ge \cdots \ge v_K(x) - a(1 + p + \cdots + p^{-n}) \ge v_K(x) - 2a.
\end{aligned}$$

□

Definition F.5. For $n \ge 0$, Tate's normalized trace map $R_n : K_\infty \to K_n$ is the map

$$R_n(x) = p^{-m}\mathrm{Tr}_{K_{n+m}/K_n}(x), \text{ if } x \in K_{n+m}.$$

By the transitive properties of the trace map, one can easily see the definition is independent of the choice of m. Denote $R_0 =: R(x)$.

Corollary F.4 is restated as:

Corollary F.6. *There is a constant c such that for all $x \in K_\infty$,*

$$v_K(R(x)) \geq v_K(x) - c.$$

Proposition F.7. *There exists a constant $d > 0$ such that for all $x \in K_\infty$,*

$$v_K(x - R(x)) \geq v_K(\gamma x - x) - d.$$

Proof. We prove by induction on $n \geq 1$, the inequality

$$v_K(x - R(x)) \geq v_K(\gamma x - x) - c_n, \text{ if } x \in K_n \tag{F.2}$$

with $c_1 = e$. $c_{n+1} = c_n + ap^{-n}$ for some constant $a > 0$.

For $x \in K_{n+1}$,

$$px - \mathrm{Tr}_{K_{n+1}/K_n}(x) = px - \sum_{i=0}^{p-1} \gamma_n^i x = \sum_{i=1}^{p-1}(x - \gamma_n^i x) = \sum_{i=1}^{p-1}(1 + \gamma_n + \cdots + \gamma_n^{i-1})(1 - \gamma_n)x,$$

thus

$$v_K(x - p^{-1}\mathrm{Tr}_{K_{n+1}/K_n}(x)) \geq v_K(\gamma_n x - x) - e.$$

In particular, let $c_1 = e$, this shows (F.2) holds for the case $n = 0$.

Now suppose for every $x \in K_n$, (F.2) holds. Write

$$\begin{aligned} v_K(x - R(x)) &= v_K(x - p^{-1}\mathrm{Tr}_{K_{n+1}/K_n}(x) + p^{-1}\mathrm{Tr}_{K_{n+1}/K_n}(x) - R(x)) \\ &= \min\{v_K(x - p^{-1}\mathrm{Tr}_{K_{n+1}/K_n}(x)), v_K(p^{-1}\mathrm{Tr}_{K_{n+1}/K_n}(x)) - R(x)\}. \end{aligned}$$

The first term is calculated as above. As for the second term, notice that

$$R(\mathrm{Tr}_{K_{n+1}/K_n}(x)) = p^{-n}\mathrm{Tr}_{K_n/K}\mathrm{Tr}_{K_{n+1}/K_n}(x) = pR(x)$$

and

$$(\gamma - 1)\mathrm{Tr}_{K_{n+1}/K_n}(x) = \mathrm{Tr}_{K_{n+1}/K_n}(x)(\gamma x - x)$$

We have

$$\begin{aligned} v_K(\mathrm{Tr}_{K_{n+1}/K_n}(x) - pR(x)) &= v_K(\mathrm{Tr}_{K_{n+1}/K_n}(x) - R(\mathrm{Tr}_{K_{n+1}/K_n}(x))) \\ &\geq v_K((\gamma - 1)(\mathrm{Tr}_{K_{n+1}/K_n}(x))) - c_n \end{aligned}$$

by the hypothesis. By Corollary C.4, we compute that

$$v_K((\mathrm{Tr}_{K_{n+1}/K_n}(x))) = \frac{1}{p^n}\lfloor \frac{p^{n+1}v_K(x) + p^{n+1}e + pb_n}{p} \rfloor \geq v_K(x) + p^{-n}b_n + e,$$

where b_n is a bounded sequence. Thus

$$\begin{aligned} v_K((\gamma - 1)(\mathrm{Tr}_{K_{n+1}/K_n}(x))) - c_n &= v_K((\mathrm{Tr}_{K_{n+1}/K_n}((\gamma - 1)x))) - c_n \\ &\geq v_K(\gamma x - x) + p^{-n}b_n + e - c_n \end{aligned}$$

As a result, let a be a bound of b_n, then

$$\begin{aligned} v_K(x - R(x)) &= \min\{v_K(x - p^{-1}\mathrm{Tr}_{K_{n+1}/K_n}(x)), v_K(p^{-1}\mathrm{Tr}_{K_{n+1}/K_n}(x)) - R(x)\} \\ &\geq \min\{v_K(\gamma_n x - x) - e, v_K(\gamma x - x) + p^n b_n - c_n\} \\ &\geq v_K(\gamma x - x) - c_{n+1}, \end{aligned}$$

which completes the proof. $\square$

By Corollary F.6, the linear operator R_n is continuous on K_∞ for each n. Therefore, it can be extended to $\widehat{K_\infty}$ by continuity. Denote

$$X_n := \{x \in \widehat{K_\infty} \mid R_n(x) = 0\}.$$

Proposition F.8. *For each n, X_n is a closed subspace of $\widehat{K_\infty}$. Moreover,*
1). $\widehat{K_\infty} = K_n \oplus X_n$;
2). The operator $\gamma_n - 1$ is bijective on X_n and has a continuous inverse such that

$$v_K((\gamma_n - 1)^{-1}(x)) \geq v_K(x) - d$$

for $x \in X_n$;
3). If $\lambda \in U_K^1$ is not a root of unity, then $\gamma_n - \lambda$ has a continuous inver on $\widehat{K_\infty}$.

Proof. It suffices to prove the case $n = 0$.

1). follows immediately from the fact that $R = R \circ R$ is idempotent.

2). For $m \in \mathbb{N}$, let $K_{m,0} = K_m \cap X_0$, then $K_m = K \oplus K_{m,0}$ and X_0 is the completion of $K_{\infty,0} = \cup_m K_{m,0}$. Note that $K_{m,0}$ is a finite dimensional K-vector space, the operator $\gamma - 1$ is injective on $K_{m,0}$ and hence bijective on $K_{m,0}$ and on $K_{\infty,0}$. $v_K((\gamma-1)^{-1}(y)) = v_K((\gamma-1)^{-1}(\gamma-1)(x))$ for some x with $(\gamma - 1)(x) = y$. By Proposition F.7,

$$v_K((\gamma - 1)^{-1}(x)) = v_K(x) \geq v_K((\gamma - 1)(x)) - d = v_K(y) - d.$$

Hence $(y - 1)^{-1}$ extends by continuity to X_0 and the inequality still holds.

3). The bijectivity of $\gamma - \lambda$ is obvious from definition. Note that

$$\gamma - \lambda = (\gamma - 1)(1 - (\gamma - 1)^{-1}(\lambda - 1)),$$

it suffices to show that $1 - (\gamma - 1)^{-1}(\lambda - 1)$ has a continuous inverse. If $v_K((\lambda - 1)) > d + 1$ for the d in Proposition F.7, then $v_K((\gamma - 1)^{-1}(\lambda - 1)(x)) \geq v_K((\lambda - 1)(x)) - d > 1$ in X_0 and

$$(1 - (\gamma - 1)^{-1}(\lambda - 1))^{-1} = \sum_{k \geq 0}((\gamma - 1)^{-1}(\lambda - 1))^k$$

is the continuous. In general, as d is not changed if replacing K by K_n, we can assume that $v_K(\lambda^{p^n} - 1) > d$ for $n \gg 0$. Then $\gamma^{p^n} - \lambda^{p^n}$ has a continuous inverse in X, and so does $\gamma - \lambda$. □

ONE STEP BEFORE GENERAL: SR FOR GR LEARNERS

ALEX LIN

Abstract. This article provides a quick review and derivation of special relativity formalism, with minimum postulations of constant light speed and invariance of physical laws. Omitted most calculations, the author hopes to offer an inspiring insight into all relativistic theories' core philosophy.

1. Postulations and Motivation

Reviewing the history of the theory of relativity, one of the largest motivations is to explain a theoretical and experimental result: the light speed appears the same to everyone no matter how fast the frames of reference are moving concerning each other. The most famous among which may be the Michelson-Morley experiment[2]. It is a natural result of the Maxwell equations, however, if you want the law of electromagnetism to remain consistent for any frame of reference. Nevertheless, many who were deeply convinced of the law of Galilean transformation deemed such a result absurd.

$$\begin{cases} \mathbf{r}' = \mathbf{r} - \mathbf{v}t \\ t' = t \end{cases}$$

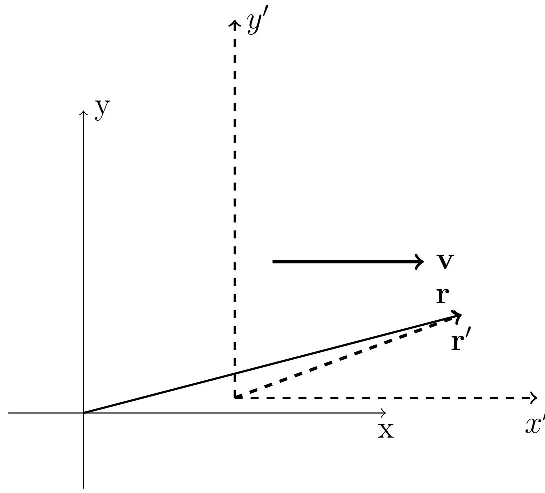


Figure 1.1. Galilean Transformation

Despite many physicists trying to fix the gap between the experiment results and their Galilean beliefs, it eventually turned out that electromagnetism and the whole universe should run under a new law of transformation. We now call it the Lorentz transformation, the transformation law of the relativity theory.

With all that being said, you probably have noticed that one of the most important postulations in special relativity is that the light speed does not change when we try to describe the physics observed by different frames of reference. We also hope that the physics won't change, so someone at a different speed will not see a Cthulic world filled with unspeakable objects describing the same things as we see.

I have been very ambiguous about frames of reference and spacetime coordinates. You may refer to Einstein's original papers for more rigorous discussions. I will only offer some intuition on them. We are tackling the concept of time and space themselves, so we must carefully discard some of our old thoughts. Luckily, that also means we are granted the freedom to define them. Before we talk about space, we need to admit that we can always measure the time. Note that we are not stating that the time is globally defined for anyone, but that you have a measurable quantity by watching things evolve, by your watch, or clock, or whatsoever, let's denote it by t. It is nothing as hard to admit as the foundation of quantum mechanics, just an action that you are doing every day. Now, remember that nature assures us that the speed of light is always c. You may wonder how we could ever know that something is of constant velocity before we know what space is, and that's exactly the point. We can use the light to define the space! Let's say a boring man sends many pulses of light to many mirrors, and measures the time till the light comes back. By seeing the difference in the roundabout times of different mirrors, the boring man, as long as he's not only boring but dumb, will know something about those mirrors being different, which we call distance. He can do more experiments to figure out such quantities have 3 degrees of freedom when the direction is included, and possess some particular topology and geometry, known as 3 dimensional Euclidean space, but that part is no different from Newtonian mechanics. The point is, since physics should not change, everyone is technically able to do the same thing, so we defined a uniform concept of space for everyone to describe the law of physics unambiguously. It is still the space and time we know and love, at least from the aspect of measurement.

We also have to define referencing frames. It may require another pile of words to be completely rigorous, so let me be intuitive and simplistic. Note that we constructed a fictitious observer (i.e. the boring man) when defining spacetime coordinates. We will call spacetime coordinates constructed by one such observer the referencing frame. However, we have to restrict the observers to be at **constant velocity**. *With respect to whom?*, you may ask. Well, we don't know. The only thing we can firmly conclude is that the existence of acceleration **WILL** change the law of physics. Or equivalently, the observers can experimentally figure out whether they are accelerated. Therefore, all referencing frames are classified into disjoint subsets using acceleration as an equivalent relation. And all changes of referencing frame we conduct in this article will not deal with acceleration. We are only dealing with transformations within one particular subset, in accordance with the postulation of unchanging law of physics.[1]

Phew, what a stinky junk of words! Luckily, we can reinterpret all the above arguments using simple maths.

Definition 1.1 (Spacetime Coordinates)**.** A **spacetime coordinates** is a real vector $\mathbf{x} \in \mathbb{R}^4$, whose component is interpreted as follows, after fixing a (constant velocity) referencing frames and assuming that the observer is at $(0, 0, 0, 0)$:

- x^0 component describes the time measured by the observer timed by c

[1]It is possible to understand accelerating frames with a few more arguments, but it is best understood with some principles of the General Relativity.

- other three components $\mathbf{r} \equiv (x^1, x^2, x^3)$ describes the direction and the distance at which a beam of light from the origin will reach in $\dfrac{x^0}{c}$

Now we are a bit clearer on those concepts, and it's time to draw our first important conclusion from our populations.

Lemma 1.2 (Linearity of Lorentz Transformation)**.** *A valid transformation of the referencing frame must be affine. That is, there exists a matrix* Λ *and a vector* $\mathbf{a}$ *independent of* $\mathbf{x}$*, such that*

$$\mathbf{x} \mapsto \Lambda(\mathbf{v})\,\mathbf{x} + \mathbf{a} \tag{1.1}$$

Proof. We expect the spacetime to be homogeneous, from which we may conclude that the velocity of a particle remains constant. After a change of referencing frame, the spacetime should still stay homogeneous, otherwise, we would find some referencing frames special in which the spacetime is homogenous. Hence, consider a particle moving at constant velocity $\tilde{\mathbf{u}}$, it should remain at constant velocity after transformation.[2] Let the transformation be general $\tilde{\mathbf{x}} \mapsto \mathbf{x}(\tilde{\mathbf{x}}, t)$ $\tilde{t} \mapsto t(\tilde{\mathbf{x}}, t)$

$$\begin{cases} \tilde{u}^i = \dfrac{d\tilde{x}^i}{d\tilde{t}} \\ dx^i = \displaystyle\sum_{j=1}^{3} \frac{\partial x^i}{\partial \tilde{x}^j} d\tilde{x}^j + \frac{\partial x^i}{\partial \tilde{t}} d\tilde{t} \\ dt = \displaystyle\sum_{j=1}^{3} \frac{\partial t}{\partial \tilde{x}^j} d\tilde{x}^j + \frac{\partial t}{\partial \tilde{t}} d\tilde{t} \end{cases} \implies u^i = \frac{dx^i}{dt} = \frac{\displaystyle\sum_{j=1}^{3} \frac{\partial x^i}{\partial \tilde{x}^j} \tilde{u}^j + \frac{\partial x^i}{\partial \tilde{t}}}{\displaystyle\sum_{j=1}^{3} \frac{\partial t}{\partial \tilde{x}^j} \tilde{u}^j + \frac{\partial t}{\partial \tilde{t}}}$$

if u^i is to be constant, then all the partial derivatives between $\tilde{\mathbf{x}}, \tilde{t}$ and $\mathbf{x}, t$ has to be independent of coordinates. Therefore, the transformation between $\tilde{\mathbf{x}}, \tilde{t}$ and $\mathbf{x}, t$ has to be affine. □

The transformation Λ may depend on some quantities $\mathbf{v}$ that characterize the frame of reference. It will eventually turn out to be the velocity of the frame of reference. And since the translation $\mathbf{a}$ only represents the choice of the origin, we often discard it.

2. Einstein Summation Convention - Part I

I want to pause here and introduce part of a handy way of writing expressions in special relativity, the famous Einstein Summation Convention.

In the following texts, two identical indices of vectors (or tensors that we will meet later) appearing in a product implies a summation. If the index is a Greek alphabet, like μ or ν, then they run from 0 to 3, corresponding to the time and the 3 space components respectively. If the index is just a normal English alphabet like a or b, then they only run over 1 to 3, or only the space components.

It's better understood with examples:

$$x^\mu y^\mu z^\nu = \sum_{\mu=0}^{3} x^\mu y^\mu z^\nu$$

[2]This is more of a plausibility argument. One can consider this as an aspect of covariance.

$$x^a y^a z^b = \sum_{a=1}^{3} x^a y^a z^b$$

They will prove very useful.

3. Isometry and the Spacetime Interval

Finally, we're ready to dive into derivations of the mathematical formalism of the theory. Let's start with a very simple thought experiment utilizing the unchanging speed of light.

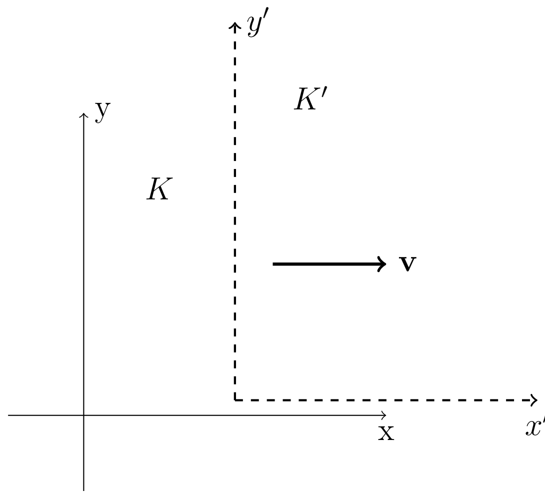


Figure 3.1. The Two Moving Frames

Imagine that there is two observers K and K', with K' moving with respect to K at velocity $\mathbf{v}$, then K' will see K going with $-\mathbf{v}$. Assume that K sees a light traveling. Let's mark the spacetime coordinates of the subsequent events that the light passes a point in an infinitesimal interval dt. The events' coordinates are (t_0, x_0, y_0, z_0) and $(t_0 + dt, x_0 + dx, y_0 + dy, z_0 + dz)$. Of course, since the light is at the speed of light, those infinitesimal intervals should satisfy:

$$(c\,dt)^2 = dx^2 + dy^2 + dz^2 \implies ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = 0$$

Let's wrap those intervals for convenience:

$$\begin{cases} dx^0 = c\,dt \\ dx^1 = dx \\ dx^2 = dy \\ dx^3 = dz \end{cases}$$

and define a matrix

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Then the constraint becomes very tight and clean

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = 0 \tag{3.1}$$

Just don't care for now why we have indices up and down.

Now in K', the light also exists, and K' is also able to observe the events that K observed. Do you remember our postulations? The two events we constructed above are related to two events in K' by a linear transformation Λ, so do those intervals:

$$(dx')^\mu = \Lambda^\mu{}_\nu dx^\nu$$

But the light in K' should also travel at the speed of light! So $(dx')^\mu$ again have to satisfy the constraint (3.1). Therefore we get

$$\eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\alpha\beta} \Lambda^\alpha{}_\mu dx^\mu \Lambda^\beta{}_\nu dx^\nu = 0$$

Since the direction of light is arbitrary, a very strong restriction is posed on Λ:

$$\eta_{\mu\nu} = \kappa \eta_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu$$

We know what this is. If we view $\eta_{\mu\nu}$ as a bilinear form, then Λ is an isometry of η only if the annoying factor $\kappa = 1$. It is! Let's approach this in a way similar to Landau's[1]. First, κ should only depend on the relative velocity $\mathbf{v}$ of K' because we expect our law of transformation to work everywhere in the universe. Second, the dependency on $\mathbf{v}$ can also only depend on the length of $\mathbf{v}$, because we expect there's no special direction in the universe. Consider a subsequent change of frames of reference from K to K' to K'' with $\mathbf{v}_2$, $\mathbf{v}_{23}$ and $\mathbf{v}_3$ corresponding to the speed between K and K', K' and K'', and K and K''. An easy calculation reveals that $\kappa(v_2)\kappa(v_{23}) = \kappa(v_3)$, or $\kappa(v_{23}) = \dfrac{\kappa(v_3)}{\kappa(v_2)}$. Note that v_{23} must be a function of the angle between v_2 and v_3, say θ (because that's already true in our daily life), so the left side is dependent on θ but the right side is not! The only possibility, then, is that

$$\kappa(v_{23}) = \frac{\kappa(v_3)}{\kappa(v_2)} = const$$

Therefore $\kappa = 1$

Eventually, we get what a change in the referencing frame means in special relativity.

Theorem 3.1 (Lorentz Transformation). *A valid relativistic change in referencing frames Λ is an isometry of η*

$$\eta_{\mu\nu} = \eta_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \tag{3.2}$$

Such Λs form a group, the *Lorentz group* $Iso(\eta)$.

3.1. **Structure of the Lorentz Group.** We discovered the Lorentz group by inspecting restrictions on possible relativistic frame transformations. Yet, the Lorentz group can be further divided into four parts, in which only one represents true changes in referencing frames. They are worth discussing, because of their close relations to the CPT symmetry to be studied in Quantum Field Theory. The physically possible Lorentz transformations are the elements of the connected subgroup of the Lorentz group, the connected component that contains the identity. It won't be too hard to understand this result, because the identity transformation has to be one physical transformation, and other transformations should be continuously related to it using $\mathbf{v}$ and Euler angles as parameters. This connected subgroup called the *proper, orthochronous* Lorentz group, is what we usually refer to as the Lorentz group. The proper, orthochronous Lorentz group elements have positive determinant $\det \Lambda > 0$ and preserve the sign of x^0 when transforming timelike x^μ.

The other three components of the Lorentz group are, naturally, those that don't have a positive determinant or don't preserve the sign of x^0. Interestingly, they are related to the proper, orthochronous Lorentz group in an obvious way.

- Define $T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. The proper, orthochronous Lorentz transformations composed with T no longer have positive determinants nor do they preserve the sign of x^0. T is called the time-reversal operator, and the resulting components are called the proper, non-orthochronous Lorentz transformations
- Define $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. The proper, orthochronous Lorentz transformations composed with P no longer have positive determinants but they preserve the sign of x^0. P is called the parity operator, and the resulting components are called the improper, orthochronous Lorentz transformations
- The proper, orthochronous Lorentz transformations composed with PT have positive determinants but fail to preserve the sign of x^0

So the complete Lorentz group, generated from the proper, orthochronous Lorentz group with $\{I, T, P, PT\}$, are physical referencing frame transformations accompanied by flipping of axes. They have far-reaching consequences, but the story goes in Quantum Field Theory. You will find a detailed analysis from Weinberg[3].

As a consequence of the restriction (3.2), you notice that ds^2, called the *spacetime interval* by physicists, is invariant under changes of the referencing frames. As an extension to the invariance of ds^2, quantities in the form $\eta_{\mu\nu}dx^\mu dy^\nu$ remain invariant under any changes of the referencing frame, no matter what values dx^μ and dy^ν take. Such quantities are called *Lorentz scalar* by physicists. This is our first glimpse into *Lorentz covariance*, we will talk about it further in the next section.

4. Lorentz Covariance

In physics, especially in the context of relativity, you often hear physicists mumbling something like "This equation is invariant under changes in frames of reference". That's actually a jargon meaning "both sides of the equation transform in the same way". Well, what does that mean then? In this case, we're actually talking about *Lorentz covariance*.

To start with, let's consider what consequences can a change of referencing frame cause on an equation. Consider, now, three equations of vector components that transform with respect to (1.1).

$$\begin{cases} x^\mu y^\nu = u^\mu v^\nu \\ x^\mu y^\mu = u^\mu v^\mu \\ x^\mu y^\nu \eta_{\mu\nu} = u^\mu v^\nu \eta_{\mu\nu} \end{cases}$$

And let's see how they act under $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$.

The first one is easy, the left side becomes $\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta x^\alpha y^\beta$ and the right side becomes $\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta u^\alpha v^\beta$. Since we know $x^\alpha y^\beta = u^\alpha v^\beta$, the two sides are exactly the same, just as before. Therefore, Lorentz transform doesn't change this particular equality.

How about the second one? This equation involves an implicit summation and therefore is an equation of quantities that need not be indexed. It looks very similar to the dot product of two vectors, and you may want to call it a scalar, but it will fail you soon. Why? Let's see how they transform! The left side, $\Lambda^\mu{}_\alpha \Lambda^\mu{}_\beta x^\alpha y^\beta$. And the right side, $\Lambda^\mu{}_\alpha \Lambda^\mu{}_\beta u^\alpha v^\beta$. A complete mess! Now we have mixed components of x y u and v everywhere in our equation. You can immediately feel that this equality no longer holds. And you are feeling it right. Just in case you are a bit doubtful, let's check it. Take $x = y = \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}^T$ and $u = v = \begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix}^T$. You can easily verify the equation before the transformation holds. And take a very simple non-trivial Λ, say

$$\Lambda = \begin{pmatrix} \cosh\beta & \sinh\beta & 0 & 0 \\ \sinh\beta & \cosh\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with β being an arbitrary non-zero real number. I will leave it to you to verify it satisfies (3.2). Now see what the equation becomes. The left side: $\cosh\beta^2 + \sinh\beta^2$, the right side: 1. They don't agree! Does that mean we shall never have dot products when trying to construct equations invariant under Lorentz transforms? Of course not. Things will become clear when we step toward the next equation.

The third equation, if you have read the previous section carefully, is clearly invariant. That's because the product in the form $x^\mu y^\nu \eta_{\mu\nu}$ never changes its value under Lorentz transforms. Let's check that again.

$$x^\mu y^\nu \eta_{\mu\nu} \mapsto \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \eta_{\alpha\beta} x^\mu y^\nu = \eta_{\mu\nu} x^\mu y^\nu$$

We applied (3.2) in the equality. And you find that nothing changed! Indeed, that's why we call Λ an isometry. Therefore, $x^\mu y^\nu \eta_{\mu\nu}$ somehow defines a new dot product, just like the regular dot product is invariant under orthogonal transformations. No wonder we call it Lorentz scalar!

Let's push this result further. Whenever you want to multiply and sum indices and keep that invariant, you'd better sandwich an η into the product. Then when a Lorentz transform occurs, (3.2) comes to rescue. Since such operation happens so often, we introduce a new notion:

$$x_\mu = \eta_{\mu\nu} x^\nu$$

then, quantities in the form $x^\mu y_\mu$, one index up and one index down is Lorentz invariant. You now know why we have indices up and down. This equation is also referred to as *lowering the index*.

It's convenient to discuss how x_μ transforms under Lorentz transforms. Assume $x^\nu \mapsto \Lambda^\nu{}_\mu x^\mu$, $x_\mu \mapsto \Lambda^\nu{}_\alpha x^\alpha \eta_{\mu\nu} = (\Lambda^{-1})^\nu{}_\mu x_\nu$. So vectors with lowered indices transform with inverses. We can also define a matrix $g^{\mu\nu}$ that lift up the indices:

$$x^\mu = g^{\mu\nu} x_\nu$$

Then $x^\mu = g^{\mu\nu}\eta_{\nu\sigma} x^\sigma$. Therefore $g^{\mu\nu}\eta_{\nu\sigma} = \delta^\mu_\sigma$. Hence $g^{\mu\nu}$ is the inverse of $\eta_{\mu\nu}$. It's obvious that $g^{\mu\nu}$ has the same components as $\eta_{\mu\nu}$, and let's redefine $\eta^{\mu\nu} \equiv g^{\mu\nu}$.

After all those discussions, you must have noticed that the key is how those quantities transform! That's why physicists classify different quantities by their properties under Lorentz transforms:

(1) Scalars. Quantities that never change under Lorentz transforms. We've seen them a lot in this section.
(2) Vectors. Quantities that transform as $x^\nu \mapsto \Lambda^\nu{}_\mu x^\mu$, just like what we used to call a vector.
(3) Co-Vectors. Quantities that transforms as $x_\mu \mapsto \left(\Lambda^{-1}\right)^\nu{}_\mu x_\nu$. They go the reversed way of a vector.
(4) Tensors. Quantities that some parts of it transform like a vector and other parts of it transform like a co-vector. $T^{\mu\nu}{}_{\sigma\lambda} \mapsto \Lambda^\mu{}_\rho \Lambda^\nu{}_\tau \left(\Lambda^{-1}\right)^\gamma{}_\sigma \left(\Lambda^{-1}\right)^\theta{}_\lambda T^{\rho\tau}{}_{\gamma\theta}$. We conventionally write such quantities with indices transforming like vectors with indices up, and those like co-vectors indices down.

It's time to update our summation convention!

4.1. **Einstein Summation Convention - Part II.** Apart from the rules of inserting sums mentioned below, we also follow two rules of thumb to keep our equations Lorentz invariant.

- Summation of products (we will call such action *contraction*) must be conducted in pairs: one up and one down.
- Indices not contracted (*floating indices*) on both sides must match. That is, there must be exactly the same floating indices on both sides, and their up/down position must match correspondingly.

We apply those rules because they ensure we have the same type of tensors on two sides and the result of the summation is still a tensor. Therefore, they will transform in the same way and keep the equality. Their mathematical backends are tensor algebra and dual spaces, we'll talk about them later in general relativity.

5. Relativistic Dynamics

Before we can make our way to general relativity, we need a quick review of dynamics in special relativity, as they will serve as building blocks to general relativity. I will omit all reasons why the dynamics have to be this way, but focus on the result.

5.1. **The Light Cone.** We start by feeling how the concept of time changes in special relativity. We do it by acting the orthochronous Lorentz group onto some fixed spacetime points and observing their orbits. This equivalently gives out all possible spacetime coordinates of that event seen by other observers. We will classify the spacetime point x^μ by its Lorentz length $x^\mu x_\mu$. You can think of this quantity as the spacetime interval between the observer and the event.

- $x^\mu x_\mu > 0$: We call such a spacetime point *timelike*. When the orthochronous Lorentz group acts on the point, it generates one branch of a hyperboloid with two sheets ($D = 3$) with the point on it.
- $x^\mu x_\mu < 0$: Points in this way are *spacelike*. They generate hyperboloids with one sheet ($D = 3$) with an orthochronous Lorentz group.
- $x^\mu x_\mu = 0$: Points in this way are *lightlike* or *null*. They generate a cone ($D = 3$) with an orthochronous Lorentz group.

I made the geometrical interpretation in spacetime dimension $D = 3$ because it's easy to imagine, and I will give a picture in $D = 2$ because it's easy to draw. Nothing gets special when $D = 4$.

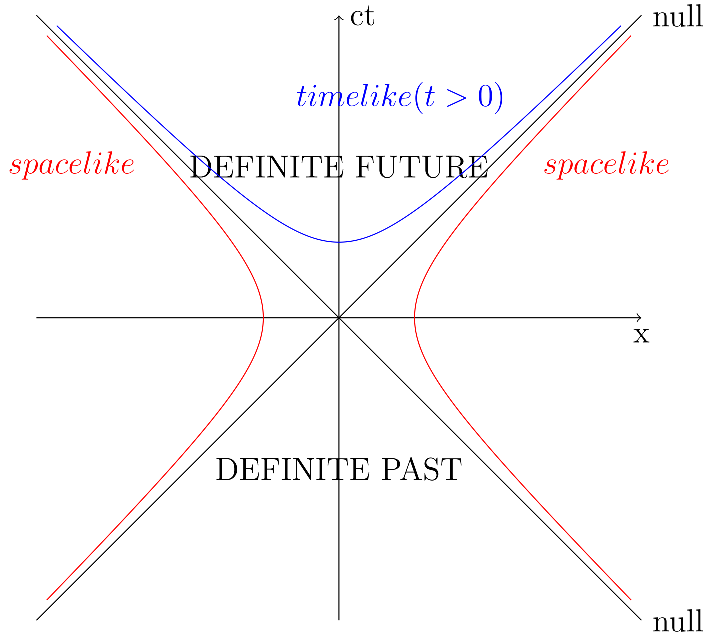


FIGURE 5.1. Different Orbits

We have on the graph orbits drawn for a timelike point with $x^0 > 0$ (the blue curve), a lightlike point (the black cross), and a spacelike point (the two red curves). Clearly, the cone of the null points is critical, and we call it the lightcone because light emitted from the origin will travel on the cone. Inside the cone, any points can only generate orbits that have the same sign of the time component. Outside the cone, the orbit's time components spread everywhere. That pretty much explains why I marked the $t+$ half of the inner lightcone "DEFINITE FUTURE" because events in that part are future events on the origin no matter how you change the referencing frames. It's the same for the "DEFINITE PAST" part. This fact has a lot to do with causalities. Basically, the "DEFINITE FUTURE" partition is where the influence of the event at the origin, traveling at the speed of light, can reach at most. Such events must not happen before the event at the origin, otherwise we violate the causality! We don't want someone shot to death to get killed in some other referencing frames even before the bullet reaches his head. Similarly, the "DEFINITE PAST" is the point that which their influences can reach the origin. They must not happen before the original event! In contrast, points outside the lightcone can never influence the origin, so they can happen at any time by changing the referencing frame.

5.2. **The Lagrangian.** A physics system can be defined by the Lagrangian. To put it simple, the Lagrangian $\mathcal{L}$ is what used to define the *action* $\mathcal{S}$:

$$\mathcal{S} = \int \mathcal{L}\left(q_i, \dot{q}_i, t\right) dt$$

The system evolves according to the *least action principle* $\delta\mathcal{S} = 0$. That is, the variables q_i follow the Euler-Lagrange equation:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i} = \frac{\partial\mathcal{L}}{\partial q_i} \tag{5.1}$$

In which, we define the *general momentum* p_i to be $p_i = \dfrac{\partial \mathcal{L}}{\partial \dot{q}_i}$

Now we pose the action for a massive relativistic particle:

$$\mathcal{S} = -mc \int ds$$

Where $ds = \sqrt{dx^\mu dx_\mu}$ is the spacetime interval of the trajectory of the particle. Massive particles are not allowed to even reach the speed of light because then the action will be singular. By taking $c\,dt$ out of the square root, we get the Lagrangian:

$$\mathcal{L} = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} \tag{5.2}$$

However, we don't favor this form of the Lagrangian in relativity. Instead, we'd love to put the time and space components on an equal footing, because the time and the space are not very different from the perspective of relativity as they are allowed to mix in Lorentz transforms. To achieve that, we make λ the dynamic variable, and rewrite the action as:

$$\mathcal{S} = -mc \int \sqrt{\frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda}} d\lambda$$

Therefore, the Lagrangian becomes:

$$\mathcal{L}\left(x^\mu, \frac{dx^\mu}{d\lambda}, \lambda\right) = \sqrt{\frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda}} \tag{5.3}$$

We say the trajectory of the particle in the spacetime coordinates the *worldline* of the particle, and λ is a *parameterization* of the worldline, just as how you parameterize a line in Euclidean geometry. When we use (5.2), we're actually parameterizing the worldline using the *proper time*, the time felt by the particle itself.

Note that, we hope our Lagrangian is possessed with what-so-called *reparameterization invariance*. That is, we can parameterize the worldline in whatever way we want. This is important because since we strip off the key character of time from the Lagrangian, there should be no special parameterization influencing our Lagrangian, otherwise our previous discussion will become nonsense. You can check that if you reparameterize the worldline by $\lambda \mapsto \tilde{\lambda}(\lambda)$, you will get exactly the same Lagrangian, in the sense that

$$-mc \int \sqrt{\frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda}} d\lambda \mapsto -mc \int \sqrt{\frac{dx^\mu}{d\tilde{\lambda}}\frac{dx_\mu}{d\tilde{\lambda}}} d\tilde{\lambda}$$

Observing the trajectory of the particle in another referencing frame is nothing more than a reparameterization plus a Lorentz transform.

5.3. **The Equation of Motion.** Let's see how free relativistic particles move in the spacetime by deriving their equation of motion. The result could be dull because we know they must run straight. However, they help us introduce some notions that are useful in general relativity.

By plugging (5.3) into (5.1), we get:

$$\ddot{x}^\mu = f(\lambda)\, \dot{x^\mu} \quad \dot{x^u} = \frac{dx^\mu}{d\lambda}$$

That's quite ugly and it depends on some random function of λ! What went wrong? Well, the problem is in the parameter λ. The worldline is "straight" in the sense of proper time parameterization. One can prove that, under reparameterization in the form $\lambda = a\tau + b$, the equation of motion takes the expected form [3]

$$\ddot{x}^\mu = 0 \quad \tau = ct$$

In the above formulae, t is the time measured by an observer staying stationary with the particle, what we call the proper time earlier. However, we prefer to call τ the proper time because it has the same dimension as space components.

We can further utilize Lorentz invariance to strip off the referencing frame involved in the definition. In the referencing frame where we originally defined the proper time, we know $dx^i = 0$ because the particle is static. Therefore, we have $ds^2 = dx^\mu dx_\mu = c^2\, dt^2 = d\tau^2$. Great, the spacetime interval of the trajectory of the particle is exactly the proper time! And since ds^2 is manifestly Lorentz invariant, ds naturally defines the proper time. This also provides a new perspective to think of arbitrary particle motions in special relativity: particles are moving along their worldlines with a constant speed of 1, no matter in what frames of reference. Note, however, that the "speed" here is not the regular speed, but the *four-velocity* defined by

$$u^\mu = \frac{dx^\mu}{ds}$$

and then it satisfies $u^\mu u_\mu = 1$, a constant amplitude. If you find those arguments too abstract to you, just recall your knowledge from Calculus: if you animate the motion of a point on the curve $\gamma(t)$ with $vt = ds = \sqrt{dx^2 + dy^2 + dz^2}$, it moves on the curve with a constant speed v.

5.4. **Lorentz Boost.** You may have noticed that the group $1 \oplus SO(3)$ is a subgroup of the Lorentz group because we're free to choose to reference frames of any pose in the space components. But the Lorentz transforms involving the time component are less trivial, called *boost*, and are what we're actually interested in. Let's work out one special example of it!

Consider an observer flying away from you in direction $\hat{x}$ with speed v. The spacetime trajectory of that observer to you is $(t, vt, 0, 0)^T$. The infinitesimal spacetime interval is $ds = \sqrt{c^2 - v^2}dt$. The four-speed is

$$u = \left(\frac{c}{\sqrt{c^2 - v^2}}, \frac{v}{\sqrt{c^2 - v^2}}, 0, 0\right)^T$$

But the four-speed of the moving observer in the frame of himself is

$$u' = (1, 0, 0, 0)^T$$

They should be related by a Lorentz transform Λ

$$(u')^\mu = \Lambda^\mu{}_\nu u^\nu$$

We can find such Λ by applying a modified version of the Gram-Schmidt process to ensure Λ is an isometry. I will put the result here:

[3] Such parameterizations are called affine parameterizations. They only differ from the Lorentzian arc length of the worldline by affine transformations.

$$\Lambda = \begin{pmatrix} \gamma & -\frac{v\gamma}{c} & 0 & 0 \\ -\frac{v\gamma}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{5.4}$$

I eliminated the degree of freedom of $SO(3)$ by pinning $\hat{y}$ and $\hat{z}$. And this is the simplest Lorentz boost. Also note that γ is defined to be $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ if not otherwise explicitly stated.

5.5. **Time Dilation and Length Contraction.** Although time dilation and length contraction are rather specific results from Lorentz transforms, I want to explain them here because they're too famous.

Let's consider two observers moving with relative speed v, at some a moment overlaps, simultaneously set their clocks to 0. From the next moment, their trajectory deviates, what happens if one continues to calculate the clock speed of the other observer? [4] From one of the observers, say the reference, the infinitesimal spacetime interval of the trajectory of the target observer is

$$ds = c\sqrt{1-\frac{v^2}{c^2}}dt$$

When the clock in the reference runs for Δt, we get the integrated interval

$$\Delta s = c\sqrt{1-\frac{v^2}{c^2}}\Delta t$$

Considering the same span of the worldline of the target observer, the integrated interval Δs is just the proper time $c\,\Delta t'$. But ds is a Lorentz invariant quantity, so must be its sum! This then implies $c\sqrt{1-\frac{v^2}{c^2}}\Delta t = c\,\Delta t'$, or $\Delta t' = \sqrt{1-\frac{v^2}{c^2}}\Delta t$.

Wow, the clock in the target observer runs slower than the clock in the reference, just as what we often hear from those science videos! But wait, you might think, the same argument can also be made from the target observer to the reference. Can we have two referencing frames both thinking the other has a slower clock than themselves? Well, it's fine as long as the two observers never meet again. Why? Because if they don't meet, they are just asserting something on a very distant clock, and that's all fine. But if they meet, they are then enabled to check what actually happened! Then it becomes a disaster if there could be two possible results! If that happens, they could have experienced and further developed into completely different futures, and our predictability completely breaks down! Luckily, the special relativity won't allow that. The caveats are that if they can meet again, v must be a function of time, and the integral's result is no longer trivial. This is the well-known Twin Paradox.

Length contraction is less tricky, we just need to make clear what length we're talking about. Again, we're talking about the length at the same time coordinates. Because you just don't call the quantity not measured at the same time a length. Imagine a car, which you know it's $3m$ when stopped, just ran past you at $1m/s$. You marked the position of

[4] You need to compensate for the extra time the light needs to travel to you. If you merely observe the clock, the time will have another extra factor.

the front tire and $1s$ later, you marked the position of the back tire. And you measured the distance between the two marks. You read $2m$. Will you call that the length of the car? No! Because you didn't mark the two points at the same time. The same thing happens in special relativity.

If you try to measure a rod of length L_0 moving in and aligned with $\hat{x}$ with speed v. We employ the Lorentz boost

$$x' = \gamma\left(x - vt\right)$$

on both ends of the rod

$$\begin{cases} x_1' = \gamma\left(x_1 - vt\right) \\ x_2' = \gamma\left(x_2 - vt\right) \end{cases}$$

and subtract them

$$\Delta x' = \gamma \Delta x$$

The t terms cancel out because you're conducting the measure at an equal time. You need to note that $\Delta x'$ is always L_0 because they are static in the moving frame so their difference is independent of the time. And yes, you get the formulae for length contraction, $L = \frac{1}{\gamma} L_0$.

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INTRODUCTORY LECTURES ON EINSTEIN FIELD EQUATION

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Abstract. These lectures are for the salon series *Principles in Relativity Theories* held by the Club of Mathematics and Physics, Southern University of Science and Technology. These lectures present the developments of the field equation of Einstein's gravity theory. We start with a review of Newtonian gravity and aim to explain how it implies the form of Einstein's equation.

1. Introduction

The special theory of relativity asks the physical formulas to satisfy the principle of covariance. Before Einstein's gravity theory, *Newton's law of universal gravitation* was the most widely accepted theory of various gravitation phenomena. In Newton's theory, the matter fields are taken as the sources of the gravitational fields, and mass as the charge of gravity. The awful thing is that it failed when physicists tried to write Newton's equation or some simple modification of the equation into a covariant form. Some interesting point of view of Newtonian gravity can refer to [14].

As Newton's theory, there are two parts to Einstein's theory: the first is how the distribution of matter determines the geometry of spacetime and the second part is how the spacetime geometry determines the motion of matter. Einstein made the assumption of the equivalence principle through a genius observation of the gravitation phenomenon, which gives the geodesic equation as the equation of motion of the particles in some specific spacetime geometry

$$\nabla_v v = 0 \tag{1.1}$$

and

$$v = \frac{dx}{d\tau} \tag{1.2}$$

where τ is the proper time. The thing left is the first part of Einstein's theory. We need a formula for how the distribution of matter determines the geometry of spacetime. The formula somehow is an assumption but it should be covariant and correspondent to Newton's theory. More resources about general relativity background preliminary refers to [15] [9]. This interesting article [16] gives more resources.

We will compare the description of particles' deviation in Newton's theory and geometry. In Newton's theory, it is caused by tidal force. Such we can drive a relation of matter field distribution and the acceleration of deviation caused by tidal force. On the other hand, we find the relation between the spacetime geometry and the acceleration of deviation. Such we can find the relation between matter field and spacetime geometry.

A brief review of Newton's theory is in section 2. The tidal force is introduced in section 3. The deviation theory of particles in spacetime is described in section 4. Finally, in section

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5 we put all these together and get the Einstein field equation. The symbol and definition of Riemann geometry we refer to [5].

2. Newton's Law of Universal Gravitation

One of the greatest and most famous physics laws in history is Newton's law of universal gravitation. It can be described by the formula:

$$\mathbf{F} = -\frac{GMm\hat{r}}{r^2} \tag{2.1}$$

which describes the force acting on a point particle that masses m, which is initiated by some particle masses M. Where r is the distance of these two particles, $\hat{r}$ is the unite vector from the mass M particle to the mass m particle, and G is the *gravitational constant*, which also appears in Einstein's gravity. The value of G is approximately $6.6743 \times 10^{-11} N \cdot m^2/kg^2$, and G was first measured implicitly by Henry Cavendish in a 1798 experiment.

We can rewrite equation (2.1) in a field form. We should introduce the gravitational potential $\phi(r)$, which is defined by:

$$\nabla\phi = -\mathbf{F} \tag{2.2}$$

which is well-defined because the gravitational force is a conservative force by equation (2.1). Such that the gravitational potential initiated by a particle mass M is :

$$\phi = -\frac{GM}{r^2} \tag{2.3}$$

For rewriting (2.1) as an equation of field, we should also replace particles as the sources of gravity by the matter field. If gravity is initiated by a matter field, that is, the mass distribution is continuous and replaced by a density of mass. The equation (2.1) turns to

$$\phi = -\int_M d\mu \frac{G\rho}{r^2} \tag{2.4}$$

By taking the gradient and divergence of (2.4), we obtain

$$\nabla^2\phi = 4\pi G\rho \tag{2.5}$$

The complete discretion of Newtonian gravity is to take this formula together with Newton's famous laws of motion:

(1) A body remains at rest or in motion at a constant speed in a straight line, except insofar as it is acted upon by a force.
(2) At any instant of time, the net force on a body is equal to the body's acceleration multiplied by its mass or, equivalently, the rate at which the body's momentum changes with time.
(3) If two bodies exert forces on each other, these forces have the same magnitude but opposite directions.

The logic of Newton's gravity is that the mass initiates force by equation (2.1), and the force determines the motion of the particles by the three laws. The second law can be described by the formula

$$\mathbf{F} = m\mathbf{a} \tag{2.6}$$

where $\mathbf{a}$ is the acceleration of the particles.

3. Tidal Force

The explanation of tidal phenomena is an important achievement of Newton's gravity. The tidal forces, which are the effective part of Newtonian gravitational force, are the cause of tidal phenomena. The gravity force of the sun acts on the seawater. However, because seawater is distributed in various places on the earth, the gravitational force the seawater experiences will also be different. Considering that the difference of the distribution of seawater is much smaller than the distance between the sun and the earth, we can describe the distribution of the seawater by $\mathbf{R}+\mathbf{r}$, where $\mathbf{R}$ is the distance vector from the mass center of the sun to the mass center of the earth, and $\mathbf{r}$ is the distance vector from the mass center of the earth to the seawater we describe. Therefore, the gravity force of the sun acting on the seawater can be written by $\mathbf{F}(\mathbf{R}+\mathbf{r})$ or the gravitational potential by $\phi(\mathbf{R}+\mathbf{r})$. Notice that $|\mathbf{r}| << |\mathbf{R}|$, we can expand the gravitational potential by the power series of r. Such that the tidal forces are actually the different parts of the gravity of the sun, which is just the second term of the expanding of Newton's gravity force.

Now, we are going to drive the form of the tidal force. Notice that:

$$F(\mathbf{R}+\mathbf{r}) = -\frac{GM}{R^2} + \frac{2GM}{R^2}\frac{r}{R} + \cdots \tag{3.1}$$

The first term is the gravitational force acting on the center of mass of the earth, and the second term is the tidal force,e which is what we want. We define the tidal force by:

$$T(\mathbf{r}) \equiv \frac{2GM}{R^2}\frac{r}{R} \tag{3.2}$$

Another way here is to expand potential but not the gravitational force. For convent, we use the coordinate index to replace the vector sign.

$$\phi(x+\Delta x) = \phi(x) + \Delta x^j \partial_j \phi(x) + o(\Delta x) \tag{3.3}$$

such that the gravitational force:

$$a^i = -\partial_i \phi(x+\Delta x) = -\partial_i \phi(x) - \Delta x^j \partial_i \partial_j \phi(x) + o(\Delta x) \tag{3.4}$$

and

$$a_i(t) + \Delta r^j(t) \partial_i \partial_j \phi(x,t) = 0 \tag{3.5}$$

For discussing the physical meaning of a_i, we consider the case that some particles in the gravitational field are close to each other. Imagine that one of these particles is an observer who is located at x. In a stationary reference frame, it is obvious that a_i is the acceleration of the other particles.

4. Geodesic Deviation

Imagine you have some apples in an elevator, and you find that the apples are suspended in the elevator. The question is, does the elevator free-fall in some gravitational field or wander somewhere without gravity? You have learned that the Newton gravity of the apples at different locations in the elevator is slightly different. The difference can be described by tidal force. With Newton's laws of motion, the tidal force will cause an acceleration between the apples, which can be calculated by measuring the apples' deviation.

On the other hand, spacetime can be viewed as a pseudo-Riemann manifold, and the apples moving along geodesics on the manifold, which is based on the equivalent principle. Therefore, the deviation of the apples can be described by the deviation of the geodesics.

In Riemannian geometry, one can associate the Riemann curvature with the deviation of geodesic curves. The formula that makes the association is called *Jacobi equation*, named after Carl Jacobi.

Let spacetime manifold $\mathcal{M}$ be a C^∞ connected maximally unextendable pseudo-Romanian manifold equipped with a C^∞ pseudo-Riemann metric g of Lorentzian signature $(-+++)$.

To begin with, we are going to consider a parameter surface $S(\mathcal{U}) \in \mathcal{M}$ with parameter (x, y) in some open set $U \subset \mathbb{R}^2$. Take two vector fields on the surface $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$. Since (x, y) is a local coordinate chart of S, the vector field X describes the distance between the integral curves of Y and vice versa. Notice that

$$(\nabla_X Y)^\mu = \left(\frac{\partial^2}{\partial x \partial y}\right)^\mu + \Gamma^\mu{}_{\nu\sigma} X^\nu Y^\sigma = \left(\frac{\partial^2}{\partial y \partial x}\right)^\mu + \Gamma^\mu{}_{\nu\sigma} Y^\nu X^\sigma = (\nabla_Y X)^\mu \tag{4.1}$$

and the commutator $[X, Y]$ between the two files vanished

$$[X, Y] \equiv \nabla_X Y - \nabla_Y X = 0 \tag{4.2}$$

In other words, for two fields with a commutation relation $[X, Y] = 0$ the deviation of their integral curves can be "measured" by each other. Recall that the curvature R of a Riemannian manifold $\mathcal{M}$ is a correspondence that associates to every pair $X, Y \in \mathfrak{X}(\mathcal{M})$ a mapping $R(X, Y) : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ given by

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z \tag{4.3}$$

for any $Z \in \mathfrak{X}(\mathcal{M})$. Take the coordinate tangent fields $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$ into the curvature $R(X, Y) : \mathfrak{X}(\mathcal{U}) \to \mathfrak{X}(\mathcal{U})$ definite locally

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z \tag{4.4}$$

for any $Z \in \mathfrak{X}(\mathcal{U})$.

Take $Y = \frac{\partial}{\partial y}$ as a vector field such that the integral curves of Y are affine geodesic curves. That is to say, the curves $\gamma_x(y) \equiv f(x, y)$ are geodesic curves and y is the affine parameter. As the tangent vector of geodesic curves

$$\nabla_Y Y = 0 \tag{4.5}$$

hence

$$0 = \nabla_X \nabla_Y Y = R\left(Y, X\right) Y + \nabla_Y \nabla_X Y \tag{4.6}$$

which implies that along each $\gamma_x(y)$ we obtain

$$\frac{D^2}{dy^2} X + R\left(Y, X\right) Y = 0 \tag{4.7}$$

or in the form of coordinate components

$$\left(\frac{D^2}{dy^2} X\right)^\sigma + R^\sigma{}_{\rho\mu\nu} X^\mu Y^\nu Y^\rho = 0 \tag{4.8}$$

which is the Jacobi equation that we want to drive.

Consider a Cauchy surface $\Sigma \in \mathcal{M}$ and each $\gamma_x(y)$ be future time-like such that $\gamma_x(0) \in \Sigma$ for any $(x, 0) \in \mathcal{U}$ and $Y|_{y=0}$ be the unit normal vector of Σ. Unit means that $\langle Y(p), Y(p) \rangle = -1$ for any $p \in f(\mathcal{U})$ (this is because the magnitude of vector who parallel along geodesic do not change). Then

$$0 = \nabla_X \left\langle Y, Y \right\rangle = 2Y \nabla_X Y \tag{4.9}$$

and

$$\nabla_Y \langle X, Y \rangle = Y \nabla_Y X = Y \nabla_X Y = 0 \tag{4.10}$$

Therefore, we have $\langle X(p), Y(p) \rangle = 0$ for any $p \in f(\mathcal{U})$. This is also a result given by *Gauss lemma* and the coordinate satisfied the condition above we call it a *Gaussian coordinate* [4].

The trajectory of two apples who are close to each other in the elevator can be described in a Gaussian coordinate by two geodesic curves $\gamma(t) \equiv \gamma_0(t)$ and $\gamma_\epsilon(t)$ for ϵ small enough, here we use $t \equiv y$ to remind us the parameter is the proper time in the elevator. The vector field along $\gamma(t)$ satisfies the Jacobi equation

$$\frac{D^2}{dt^2} J + R\left(\gamma'(t), J\right)\gamma'(t) = 0 \tag{4.11}$$

is said to be a *Jacobi field*. Denote the distance of the two apples at time t by

$$l(t) = \int_0^\epsilon ds \sqrt{\langle \beta_t'(s), \beta_t'(s) \rangle} \tag{4.12}$$

along the curve $\beta_t(x) = f(t, x)$. Since ϵ is small enough, we can make the approximation:

$$l(t) \simeq \epsilon |J(t)| \tag{4.13}$$

where we use $J(t)$ to denote the field $\beta_t'(0)$ who satisfy the Jacobi equation along $\gamma(t)$. Therefore, the vector $\epsilon J(t)$ for some time t is the relative position vector between two apples.

Take the apple moving along $\gamma(t)$ as an observer with a moving coordinate bases $\{e_i(t)\}$ $(i = 1, 2, 3)$ such that $e_1(t), e_2(t), e_3(t)$ be parallel, orthonormal fields along γ and for all $e_i(t)$ orthogonal to $\gamma'(t)$. We shall write:

$$J(t) = J^i(t) e_i(t) \tag{4.14}$$

and

$$c_{ij} = \langle R(\gamma'(t), e_i(t))\gamma'(t), e_j(t) \rangle \tag{4.15}$$

then the Jacobi equation is equivalent to the system

$$\frac{d^2}{dt^2} J_i(t) + c_{ji} J^j(t) = 0 \tag{4.16}$$

the position of the apple moving along $\gamma_\epsilon(t)$ in the coordinate is $\Delta \mathbf{r}(t) = \epsilon(J^1, J^2, J^3)$. We obtain

$$a_i(t) + c_{ji}(t) \Delta r^j(t) = 0 \tag{4.17}$$

where $\mathbf{a}(t) = \Delta \mathbf{r}''(t)$ is the accelerate of the apple moving along $\gamma_\epsilon(t)$. With the moving bases $\{e_i\}$, the equation 4.17 can take another form called the Raychaudhuri equation with a direct result from the equation called focusing theorem, which is impotent in Penrose–Hawking singularity theorems [8] [17]. Another way to describe geodesic deviation can refer to [1] [6].

5. Einstein Field Equations

In this section, we will drive the Einstein field equation, which describes the association of the distribution of matter fields with the geometry of spacetime. There are two conditions that a gravitational field equation must satisfy:

(1) The field equation satisfies the general covariant principle.

(2) When the gravity is weak and the speed of the matter field is low, the field equation will "back" to Newton's law.

What is "back" means? It does not imply the value of the scalar field in Newtonian gravity is equal to some field induced by the metric field. We should consider some observable phenomena to be similar to each other. The simplest phenomenon is the kinetics of particles under the gravitational field. Recall that the apples in the elevator will deviate because of Newtonian gravity's tidal force and spacetime's curvature in Einstein's theory. The acceleration of geodesic deviation should be approximately the same as that caused by tidal force. Therefore, compare the equation (3.5) and equation (4.17), we obtain

$$c_{ij} = \partial_i \partial_j \phi \tag{5.1}$$

Recall that the *Ricci curvature* in the direction x at $p \in \mathcal{M}$ is defined by

$$Ric_p(u) = \frac{1}{3} \sum_i \langle R(u, e_i)u, e_i \rangle \,, \; i = 1, 2, 3 \tag{5.2}$$

where $x \in T_p\mathcal{M}$ be a unit vector and $\{e_i\}$ is a an orthonormal basis of the hyperplane in $T_p\mathcal{M}$. Let $u \in T_p\mathcal{M}$ denote $\gamma'(t)$ take the value at p. Take the trace of equation (5.1) at any point p

$$\frac{1}{3} Ric_p(u) = \partial^2 \phi = 4\pi G \rho \tag{5.3}$$

where $\partial^2\phi$ and ρ take the value at p. The left-hand side of (5.3) can be written as $3Ric(u, u) = Ric_p(u)$, where *Ricci tensor Ric* is a symmetric bilinear form on $T_p\mathcal{M}$ given by

$$Ric(u, v) = \sum_i \langle R(u, e_i)v, e_i \rangle \tag{5.4}$$

for $u, v \in T_p\mathcal{M}$. Therefore, equation (5.3) is equivalent to

$$R_{\mu\nu} u^\mu u^\nu = R_{00} = 4\pi\rho \tag{5.5}$$

where we choose the Gaussian coordinate and $R_{\mu\nu}$ is the coordinate components of *Ric*. As listed above, the field equation should satisfy the general covariant principle. That is to say, the field equation should not depend on the choice of coordinate, which implies that equation (5.5) is a "00" component equation of some tensor equation. When the velocity of the matter field is much smaller than the speed of light, the mass density ρ at the right-hand side of (5.5) will equal energy density, which is the "00" component of energy-momentum tensor T. Thus, the simplest form of gravitational fields equation is a tensor equation:

$$Ric = 4\pi G T \tag{5.6}$$

At first, Einstein published this equation as the field equation of the gravitational field. But he quickly realized the problem and made changes.

The Energy-Momentum tensor should be conserved in flat spacetime, and the conservation should also be satisfied in a free-falling elevator, that is to say

$$\nabla_\mu T^\mu{}_\nu = 0 \tag{5.7}$$

For convenience, we use the component form of tensor equations from now on. We take the derivative of both sides of (5.6) we find that

$$\nabla^\mu R_{\mu\nu} = 0 \tag{5.8}$$

with the second (differential) Bianchi identity:

$$\nabla_\rho R^\sigma{}_{\lambda\mu\nu} + \nabla_\nu R^\sigma{}_{\lambda\rho\mu} + \nabla_\mu R^\sigma{}_{\lambda\nu\rho} = 0 \tag{5.9}$$

and take the contract

$$\nabla_\rho R^\rho{}_{\lambda\mu\nu} + \nabla_\nu R_{\lambda\mu} - \nabla_\mu R_{\lambda\nu} = 0 \tag{5.10}$$

with

$$\nabla_\rho R^\rho{}_{\lambda\mu\nu} = \nabla^\rho R_{\rho_{\lambda\mu\nu}} = \nabla^\rho R_{\lambda\rho\nu\mu} \tag{5.11}$$

we obtain

$$2\nabla_\rho R^\rho{}_\nu - \nabla_\mu R = 0 \tag{5.12}$$

hence

$$\nabla_\mu R = 0 \tag{5.13}$$

where $R = trRic$ is the scalar curvature. Equation (5.13) implies that the spacetime is a manifold of constant curvature. By *Cartan theorem* [5], the spacetime should be *de Sitter* [13], *anti-de Sitter* [7], or Minkowski. The problem is that in this case the geometry of spacetime and the matter field distribution is uniform and has been determined already.

To solve the problem Einstein constructed a tensor to replace $R_{\mu\nu}$ in (14), which is said to be *Einstein's tensor*:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \tag{5.14}$$

And the field equation replaces (5.6) Einstein constructed the famous Einstein field equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{5.15}$$

Einstein tensor satisfied $\nabla_\rho G_{\mu\nu} = 0$ automatically because of (5.12). But it should be compatible with (5.5) under weak gravity and low-speed approximation. Take the tare of both sides of the Einstein field equation

$$R = -8\pi G trT \tag{5.16}$$

then

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}trT) \tag{5.17}$$

and for the "00" component in the Gaussian coordinate

$$R_{00} = 8\pi G(\rho + \frac{1}{2}trT) \tag{5.18}$$

Now we are going to discuss the Newtonian limit. Then the trace of energy-momentum tensor is $trT = \Sigma_i p_i - \rho$ and p_i representing the pressure density of matter. The pressure will cause acceleration of the matter field or cancel with gravity. Under weak gravity and low-speed approximation, the acceleration and the gravity should be small such that the pressure density should be much smaller than the energy density. Therefore, equation (5.18) will return to the form of (5.5).

Equation (5.15) is not the only form of field equation. We can take some higher-order terms of geometry quantities or some constants small enough. The latter can be expressed as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{5.19}$$

which is said to be the Einstein field equation with a *cosmological constant* Λ [2]. One of the other ways to drive the Einstein field equation, which develops from the action form of Einstein gravity[10], is given by *Lovelock theorem* [9] [11]. It contains gravity action that makes the Einstein field equation with a cosmological constant the only form of the Euler–Lagrange expression of this action. The alternative form of field equation can refer to [3] [12].

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S-MATRIX

QIONGYU QI

Abstract. In quantum theory, interactions between particles are the things that interest physicists most. However, experiments generally do not track the details of events in particle interactions. Rather, the paradigm of scattering experiments is several particles approaching each other from a macroscopically large distance, interacting in a microscopically small region, after which the products of the interaction travel out again to a macroscopically large distance. The physical states before and after the collision consist of particles so far apart that they are effectively noninteracting, allowing their description as direct products of the one-particle states. In such experiments, all observables are the probability distribution, or 'cross sections', for transitions between the initial and final states. This note will clarify the definition of the S-matrix and tackle many details.

1. In and Out states

Now we want to figure out what we mean by "several particles approaching each other from a macroscopically large distance, interacting in a microscopically small region, after which the products of the interaction travel out again to a macroscopically large distance". It is a difficult mission because we must consider the Hamiltonian with interaction terms. In such a case, we must ask what the meaning of the free particles in the far past and future is; the interaction terms may allow particles to self-interact, and should we redefine our one-particle state? And how could we define a well-defined physical amplitude to describe this process? To eventually reach the answer to these questions, we shall define the "in" and "out" states as our first step.

1.1. **Multiparticles states.**

As the process that physicists are interested in usually contains more than one particle, we should first consider the interaction-free multiparticle state.

Because the Hamiltonian we now consider is without interaction, a state consisting of several particles may be regarded as one that transforms under the inhomogeneous Lorentz group as a direct product of the representation (tensor product of states). To label the one-particle states, we use their four-momenta p^μ, spin z-component (or, for massless particles, helicity) σ, and, since we now may be dealing with more than one species of particle, an additional discrete label n for the particle type, which includes a specification of its mass, spin charge, etc. Following the group representation theory, the general transformation rule reads

$$\begin{aligned}\mathcal{U}(\Lambda, a)\Psi_{p_1,\sigma_1,n_1;p_2,\sigma_2,n_2} &= \\ &exp(-ia_\mu((\Lambda p_1)^\mu + (\Lambda p_2)^\mu + \cdots)) \\ &\times \sqrt{\frac{(\Lambda p_1)^0(\Lambda p_2)^0\cdots}{p_1^0 p_2^0\cdots}} \sum_{\sigma_1'\sigma_2'\cdots} D^{(j_1)}_{\sigma_1'\sigma_1}(W(\Lambda, p_1))D^{(j_2)}_{\sigma_2'\sigma_2}(W(\Lambda, p_2))\cdots \\ &\times \Psi_{\Lambda p_1,\sigma_1',n_1;\Lambda p_2,\sigma_2',n_2\cdots}\end{aligned}$$

where $W(\Lambda, p_1)$ is the Wigner rotation, and $D^{(j)}_{\sigma'\sigma}$ are the conventional $(2j+1)$-dimensional unitary matrices representing $SO(3)$. The states are normalized as

$$(\Psi_{p_1,\sigma_1,n_1;p_2,\sigma_2,n_2\cdots}, \Psi_{p'_1,\sigma'_1,n'_1;p'_2,\sigma'_2,n'_2\cdots})$$
$$= \delta^3(p_1 - p'_1)\delta_{\sigma_1\sigma'_1}\delta_{n_1n'_1}\delta^3(p_2 - p'_2)\delta_{\sigma_2\sigma'_2}\delta_{n_2n'_2}\cdots \pm \textit{permutations}$$

with the term $\pm$ *permutations* included to take account of the possibility that there are some permutation of the particle types $n_1, n_2 \cdots$ of the same species as $n'_1, n'_2 \cdots$.

We often use an abbreviated notation, letting one Greek letter, say α, stand for the whole collection. In this notation, the normalization condition is written simply

$$(\Psi_{\alpha'}, \Psi_\alpha) = \delta(\alpha' - \alpha)$$

In particular, the completeness relation for states normalized as in reads

$$\Psi = \int d\alpha \Psi_\alpha(\Phi_\alpha, \Psi)$$

The transformation rule written above is only possible for particles that are not interacting. Obviously, Ψ_α is a eigenstate of Hamiltonian operator

$$H\Psi_\alpha = E_\alpha\Psi_\alpha$$

with an energy equal to the sum of the one-particle energies

$$E_\alpha = p_1^0 + p_2^0 + \cdots$$

1.2. **In and Out State.** Suppose the particles we now discussing are all stable. What we want to do is to find out what happens in the far future about a state which "looks like" a tensor product of free particles in the far past after interaction at a small region. Therefore, we should first define a state which "looks like" a free multiparticle state.

Suppose in the far past and far future we make two observations, and we find in the far past the particles collection is α, in the far future the particles collection is β. Because the states "look like" free multiparticle states, the transformation rule should apply (we have not proved it yet, we shall prove it when we discuss the symmetry of the S-matrix) in scattering processes at times $t \to \pm\infty$. As explained at the beginning of this paragraph, in the typical scattering experiment, we start with particles at time $t \to -\infty$ so far apart that they are not yet interacting, and end with particles at $t \to +\infty$ so far apart that they have ceased interacting. We therefore have not one but two sets of states that transform as free multiparticle states: *the 'in' and 'out' states Ψ^+_α and Ψ^-_β will be found to contain the particles described by the label a if observations are made at $t \to -\infty$ or $t \to +\infty$, respectively.*

Note how this definition is framed. To maintain the manifest Lorentz invariance, in the formalism we are using here, state-vectors do not change with time — a state-vector Ψ describes the whole spacetime history of a system of particles. (This is known as the Heisenberg picture, in distinction with the Schrodinger picture, where the operators are constant and the states change with time.) The last two sentences are excerpted from the third chapter of Weinberg's Quantum Field Theory Volume *1*, which are very abstract to understanding. Now we will explain these two sentences clearly so that everyone can understand them.

Because the physicists (at least when we do the experiments) do not care about the details of the interaction process most time, we only care about the far past and far future. Therefore, to describe the process, we only need to know the probability amplitude that a "in" state evolves to an "out" state, prescribed by an inner product of "in" state and "out" state. But the proper probability amplitude of two states (Schrodinger) must be simultaneous. To do so, one seemingly has to evolve the "in" state to the far future. But considering that we will discuss the scattering theory of quantum field theory, describing the state in the Heisenberg picture shall be more convenient. Since the state in the Heisenberg picture never evolves, we need the initial value of the states, as the states in the Heisenberg picture are just the Schrodinger states at some reference time (take it $t = 0$), and we could evolve the states to the far past or far future. In the far past and the far future, the state must resemble the free multiparticle state. In this way, we could get a precise description of in and out states.

There is still a problem, we could not just let in and out states be the eigenstates of full Hamiltonian (because of the interaction, without which the scattering process can not happen), because it cannot be localized in time — the operator $exp(-iH\tau)$ yields an inconsequential phase factor $exp(-iE_\alpha\tau)$. Therefore, we must consider wave-packets, which are a superpositions $\int d\alpha g(\alpha)\Psi_\alpha$ of states, with an amplitude $g(\alpha)$ that is non-zero and smoothly varying over some finite range ΔE of energies. The 'in' and 'out' states are defined so that the superposition

$$exp(-iH\tau)\int d\alpha\ g(\alpha)\Psi_\alpha = \int d\alpha\ exp(-iE_\alpha\tau)g(\alpha)\Psi_\alpha$$

has the appearance of a corresponding superposition of free-particle states for $\tau \ll -1/\Delta E$ or $\tau\Delta E$, respectively.

To make this concrete, suppose we can divide the time-translation generator H into two terms, a free-particle Hamiltonian H_0 and an interaction V, $H = H_0 + V$ in such a way that H_0 has eigenstates Φ_α that have the same appearance as the eigenstates Ψ_α^+ and Ψ_α^- of the complete Hamiltonian

$$H_0\Phi_\alpha = E_\alpha\Phi_\alpha$$
$$(\Phi_{\alpha'}, \Phi_\alpha) = \delta(\alpha' - \alpha)$$

Note that H_0 is assumed here to have the same spectrum as the full Hamiltonian H. But why? Because this requires that the masses appearing in H_0 be the physical masses that are physically measured, which are not necessarily the same as the 'bare' mass terms appearing in H; the difference, if there is any, must be included in the interaction V, not just H_0. A simple example is the self-energy of an electron, present even in classical electrodynamics, that requires mass renormalization. That is, the electron two-point propagator is equal to the sum of diagrams.

$$\langle\Omega|T\psi(x)\bar{\psi}(y)|\Omega\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x-y)}}{\not{p} - m_0 - \Sigma(\not{p})} = \int \frac{d^4p}{(2\pi)^4}\frac{iZ_2e^{-ip\cdot(x-y)}}{\not{p} - m}$$

where m is the physical mass that we could observe. In this way, we avoid some troubles, because this is just mass renormalization.

The 'in' and 'out' states can now be defined as eigenstates of H, not H_0, $H\Psi_\alpha^\pm = E_\alpha\Psi_\alpha^\pm$ which satisfy the condition

$$\int d\alpha\ e^{-iE_\alpha\tau}g(\alpha)\Psi_\alpha^\pm \to \int d\alpha\ e^{-iE_\alpha\tau}g(\alpha)\Phi_\alpha$$

for $\tau \to -\infty$ or $\tau \to +\infty$, respectively. The equation can be rewritten as

$$e^{-iH\tau} \int d\alpha\ g(\alpha)\Psi_\alpha^\pm \to e^{-iH_0\tau} \int d\alpha\ g(\alpha)\Phi_\alpha$$

This is sometimes rewritten as a formula for the 'in' and 'out' states:

$$\Psi_\alpha^\pm = \Omega(\mp\infty)\Phi_\alpha$$

where

$$\Omega(\mp\tau) = exp(iH\tau)exp(-iH_0\tau).$$

However, it keep in mind that $\Omega(\mp\infty)$ in the equation gives meaningful results only when acting on a smooth superposition of energy eigenstates.

One immediate consequence of the definition is that the 'in' and 'out' states are normalized just like the free-particle states.

$$(\Psi_\alpha^\pm, \Psi_{\alpha'}^\pm) = \delta(\alpha - \alpha')$$

It is useful for some purposes (at least when we want to know how to calculate the 'in' and 'out' state) to have an explicit, though formal, solution of the energy eigenvalue equation satisfying the conditions. For this purpose, we write

$$(E_\alpha - H_0)\Psi_\alpha^\pm = V\Psi_\alpha^\pm.$$

The operator $E_\alpha - H_0$ is not invertible; it annihilates not only the free particle state Φ_α, but also the continuum of other free particle states Φ_β of the same energy. Since the 'in' and 'out' states become just Φ_α for $V \to 0$, we tentatively write the formal solutions as Φ_α plus a term proportional to V:

$$\Psi_\alpha^\pm = \phi_\alpha + (E_\alpha - H_0 \pm i\epsilon)^{-1}V\Psi_\alpha^\pm$$

or, expanding in a complete set of free-particle states,

$$\Psi_\alpha^\pm = \Phi_\alpha + \int d\beta \frac{T_{\beta\alpha}^\pm \Phi_\alpha}{E_\alpha - E_\beta \pm i\epsilon}$$

$$T_{\beta\alpha}^\pm = (\Phi_\alpha, V\Psi_\alpha^\pm)$$

With ϵ a positive infinitesimal quantity, inserted to give meaning to the reciprocal of $E_\alpha - H_0$. These are known as the *Lippmann-Schwinger equations.*

2. The S-matrix

2.1. **Definition.** An experimentalist generally prepares a state to have a definite particle content at $t \to -\infty$, and then measures what this state looks like at $t \to +\infty$. If the state is prepared to have a particle content α for $t \to -\infty$, then it is the 'in' state Ψ_α^+, and if it is found to have the particle content β at $t \to +\infty$, then it is the 'out' state Ψ_α^+. The probability amplitude for the transition $\alpha \to \beta$ is thus the scalar product

$$S_{\beta\alpha} = (\Psi_\beta^-, \Psi_\alpha^+)$$

This array of complex amplitudes is known as the *S-matrix.* If there were no interactions, then 'in' and 'out' states would be the same, and then $S_{\beta\alpha}$ would just be $\delta(\beta - \alpha)$. The rate for a reaction $\alpha \to \beta$ is thus proportional to $|S_{\beta\alpha} - \delta(\alpha - \beta)|^2$.

Perhaps it should be stressed that 'in' and 'out' states do not inhabit two different Hilbert spaces. They differ only in how they are labelled: by their appearance either at $t \to +\infty$ or

$t \to -\infty$. Any 'in' state can be expanded as a sum of 'out' states, with expansion coefficients given by the S-matrix.

Since $S_{\beta\alpha}$ is the matrix connecting two complete sets of orthonormal states, it must be unitary. To see this in greater detail, apply the completeness relation to the 'out' states, and write

$$\int d\beta\ S^*_{\beta\gamma} S_{\beta\alpha} = \int d\beta\ (\Psi^+_\gamma, \Psi^-_\beta)(\Psi^-_\beta, \Psi^+_\alpha) = (\Psi^+_\gamma, \Psi^+_\alpha) = \delta(\gamma - \alpha)$$

or in brief, $S^\dagger S = 1$. In the same way, completeness for the 'in' states gives $SS^\dagger = 1$.

It is often convenient instead of dealing with the S-matrix to work with an operator S, defined to have matrix elements between free-particle states equal to the corresponding elements of the S-matrix

$$(\Phi_\beta, S\Phi_\alpha) = S_{\beta\alpha}$$

The explicit, though highly formal, expression for the 'in' and 'out' states yields a formula for the S-operator:

$$S = \Omega(\infty)\Omega^\dagger(-\infty) = U(\infty, -\infty)$$

where

$$U(\tau, \tau_0) = \Omega(\tau)\Omega(\tau_0)^\dagger = exp(iH_0\tau)exp(-iH(\tau - \tau_0))exp(-iH_0\tau_0).$$

Seemingly, we should use the *Lippmann-Schwinger equation* to derive the explicit formula of S-matrix. But we will not do that, as this will lead to the *old-fashioned perturbation theory*, which is notoriously ugly and is completely outdated. For the most part in this note, we will rely on a rewritten version of S-matrix, known as *time-dependent perturbation theory*, which has the virtue of making Lorentz invariance much more transparent, while somewhat obscuring the contribution of individual intermediate states.

2.2. **Time-dependent perturbation theory.** The easiest way to derive the time-ordered perturbation expansion is to use S operator

$$S = U(\infty, -\infty).$$

Differentiating the formula for $U(\tau, \tau_0)$ with respect to τ gives the differential equation

$$i\frac{d}{d\tau}U(\tau, \tau_0) = V(\tau)U(\tau, \tau_0)$$

where

$$V(t) = exp(iH_0t)V exp(-iH_0t).$$

(Operators with this sort of time-dependence are said to be defined in the *interaction picture*, to distinguish their time-dependence from the time-dependence $O_H(t) = exp(iHt)O exp(-iHt)$ required in the Heisenberg picture of quantum mechanics.) The differential equation as well as the initial condition $U(\tau_0, \tau_0) = 1$ is obviously satisfied by the solution of the integral equation

$$U(\tau, \tau_0) = 1 - i\int_{\tau_0}^{\tau} dt\ V(t)U(t, \tau_0).$$

By iteration of this integral equation, we obtain an expansion for $U(\tau, \tau_0)$ in powers of V

$$U(\tau, \tau_0) = 1 - i\int_{\tau_0}^{\tau} dt\ V(t_1) + (-i)^2\int_{\tau_0}^{\tau} dt_1 \int_{\tau_0}^{t_1} dt_2 V(t_1)V(t_2) + \cdots .$$

Setting $\tau \to \infty$ and $\tau_0 \to -\infty$ then gives the perturbation expansion for the S-operator. There is a way of rewriting the equation that proves very useful in carrying out manifestly

Lorentz-invariant calculations. Define the time-ordered product of any time-dependent operators as the product with factors arranged so that the one with the latest time-argument is placed leftmost, the next-latest next to the leftmost, and so on. For instance,

$$T\{V(t)\} = V(t) = V(t)$$

$$T\{V(t_1)V(t_2)\} = \theta(t_1 - t_2)V(t_1)V(t_2) + \theta(t_2 - t_1)V(t_2)V(t_1).$$

The time-ordered product of n Vs is a sum over all $n!$ permutations of the Vs, each of which gives the same integral over all $t_1 \cdots_n t$, so the equation may be written

$$S = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 dt_2 \dots dt_n T\{V(t_1)V(t_2)\cdots V(t_n)\}.$$

This is sometimes known as the *Dyson series.* This series can be summed if the $V(t)$ at different times all commute; the sum is then

$$S = exp(-i \int_{-\infty}^{\infty} dt V(t))$$

Of course, this is not usually the case; in general, the series does not even converge, and is at best an asymptotic expansion in whatever coupling constant factors appear in V. However, the equation is sometimes written in the general case as

$$S = T\ exp(-i \int_{-\infty}^{\infty} dt V(t))$$

With T indicating here that the expression is to be evaluated by time-ordering each term in the series expansion for the exponential. Now it's time to introduce the interaction picture.

3. The interaction picture

The origin of the interaction picture is the purpose that describes the scattering process by states and operators that are evolved by the free Hamiltonian. But in the last section, we already used the free states and free operators to derive the S-matrix. Therefore, we will use such states and operators to define the interaction picture (abbreviated as *IP* below).

In the IP, both states and operators are time-dependent but in a way that is well adapted to perturbation theory, especially in quantum field theory. The operators have a time dependence generated by the free Hamiltonian H_0, say, and so a 'free-particle' mode expansion like survives intact. The states have a time dependence generated by the interaction V. Thus, as $V \to 0$ we return to the free-particle HP.

The way this works formally is as follows. In terms of the time-independent SP operator V, we define the corresponding IP operator $V_I(t)$ by

$$V_I(t) = exp(iH_0 t)V exp(-iH_0 t)$$

It follows that the time dependence of $V_I(t)$ is given by:

$$\frac{d}{dt} V_I(t) = -i[V_I(t), H_0]$$

Thus, as mentioned already, the time dependence of $V_I(t)$ is generated by the free part of the Hamiltonian, by construction.

Now, for convenience, we just consider the interacting scalar field, as other type of fields resembles its behavior. The free Hamiltonian of the scalar field has been well studied. But there is a remaining question: when we consider the interaction picture, will the canonical commutation relation change? If it changes, we will fail to apply the linear expansion of ϕ_I. Then our formula will be very ugly. But if the canonical commutation relation still holds in IP, we could use creation and annihilation operators to linearly expand the ϕ_I (we already use the creation and annihilation operators as a complete basis of operators). Therefore, we now verify that the canonical commutation relation remain unchanged in IP.

For example, suppose $\mathcal{L} = \frac{1}{2}\phi(\Box - m^2)\phi - V(\phi)$. We still let ϕ be the 'position' operator. Since $V(\phi)$ is not a function of derivatives of ϕ, the 'momentum' operator of the scalar field does not change. Therefore, we arrive at the canonical commutation relation

$$[\phi(x,t), \pi(y,t)] = i\delta^3(x-y).$$

Now we get into the interaction picture, where we find that $\phi_I = e^{iH_0 t}e^{-iHt}\phi(t)e^{iHt}e^{-iH_0 t}$. Therefore, we get:

$$[\phi_I(x,t), \pi_I(y,t)] = i\delta^3(x-y).$$

Indeed, the canonical commutation relation still holds in IP.

Now we reconstruct the S-matrix in the interaction picture. Now we define the states in the interaction picture, and define

$$\psi_I(t) = exp(iH_0 t)\psi_t = exp(iH_0 t)exp(-iHt)\psi.$$

We see that when $t \to \infty$ $\psi_I(\infty) = \Phi$ which we already defined before. Then we define the time evolution operator $U(t, t_1)$ by

$$U(t, t_1)\psi_I(t_1) = \psi_I(t).$$

Differentiate it, we get

$$i\frac{d}{dt}U(t, t_1) = V_I(\phi)U(t, t_1).$$

Which is the same as before, we could get

$$U(\tau, \tau_0) = 1 - i\int_{\tau_0}^{\tau} dt\ V(t_1) + (-i)^2\int_{\tau_0}^{\tau} dt_1 \int_{\tau_0}^{t_1} dt_2 V(t_1)V(t_2) + \cdots .$$

We find that $S = U(+\infty, -\infty)$. Therefore, it is natural to define the S-matrix by:

$$S_{\alpha\beta} = (\psi_{I,\alpha}, U(+\infty, -\infty)\psi_{I,\beta}).$$

which is the origin of the interaction picture!

4. Constructing in and out states

Now we have already defined the in and out states, and the proper way to calculate the S-operator. But there is still one problem: the in and out states are difficult to calculate in field theory. Therefore, if we could construct in and out states with interacting field operators, then the calculation would be simpler. Another thing is that, by using interaction field operators, we could calculate the S-matrix non-perturbatively.

4.1. **Renormalized field.** The vacuum is an eigenstate of the energy and momentum operators, with eigenvalues zero, and it is normalized to one. We assume we have physical one-particle states $|p\rangle$ in our theory. (If the particle is unstable, there's no point in trying to compute particle–particle scattering matrix elements.) We will relativistically normalize them:

$$\langle p|p'\rangle = (2\pi)^2 2\omega_p \delta^3(p - p')$$

These states are eigenstates of the momentum operator:

$$P^\mu|p\rangle = p^\mu|p\rangle$$

where $p^0 = \omega_p = \sqrt{p^2 + \mu^2}$. where μ is the physical mass of a real meson (or other particles). Those are just notational conventions. We will not write down the normalization for a two-meson state now, because a two-meson state could be an in state or an out state, and they are not the same thing; a state that looks like two mesons in the far past may look like a nucleon and an antinucleon in the far future. One of the problems we're going to confront is how to construct those states. We'll have troubles enough with just the vacuum and the one-particle states.

Since our field operators are interacting, they're not going to make only one-particle states when they hit the vacuum. They'll make one-particle states, two-particle states, three-particle states, and 72-particle states; they're capable of doing a lot. We're going to make several definitions to construct a limit in time that will enable us to get, from the field operator hitting the vacuum, only the one-particle part. If we can do that, we will have crafted something like a creation operator for a single particle. And then I will be able to use these "creation operators" next time to create states that look like two-particle states, either in the far past or the far future, by making a time limit $-\infty$ or ∞, respectively. All that will be shown in detail. Our first job is to find a time limit that makes exclusively a one-particle state.

We will need some conventions about the scale of our field. We're going to work with these Heisenberg fields, without using any details of the equations of motion, just the fact that there are equations of motion; $\phi(x)$ is a local scalar field. We're not even going to say this field obeys the canonical commutators. We will require two normalization conditions.

The first condition concerns the (physical) vacuum expectation value of the Heisenberg field. By translational invariance, this will be independent of x:

$$\langle 0|\phi(x)|0\rangle = \langle 0|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}|0\rangle = \langle 0|\phi(0)|0\rangle$$

We require our field to have a vacuum expectation value of zero. If it is not zero, I will redefine the field, subtracting from it the constant $\langle 0|\phi(0)|0\rangle$:

$$\phi(x) \to \phi'(x) = \phi(x) - \langle 0|\phi(0)|0\rangle$$

Second, we need to specify the normalization of the one-particle matrix element. Because these one-particle states are momentum eigenstates,

$$\langle k|\phi'(x)|0\rangle = \langle k|e^{iP\cdot x}\phi'(0)e^{-iP\cdot x}|0\rangle = e^{ik\cdot x}\langle k|\phi'(0)|0\rangle$$

Since Lorentz transformations don't change $\phi'(0)$, or change any one-meson state to any other one-meson state, the coefficient $\langle k|\phi'(0)|0\rangle$ of $e^{ik\cdot x}$ must be Lorentz invariant, and so can depend only on k^2. Presumably, the one-particle state is on its mass shell. Then $k^2 = \mu^2$, and $\langle k|\phi'(0)|0\rangle$ is a constant. By convention this constant is denoted by $\sqrt{Z_3}$;

$$\langle k|\phi'(0)|0\rangle = \sqrt{Z_3}$$

Now redefine $\phi'(x)$ by

$$\phi'(x) = Z_3^{-1/2}(\phi(x) - \langle 0|\phi(0)|0\rangle)$$

We will assume Z_3 is not zero, so this definition makes sense. Then $\phi'(x)$ has the property that it has the same matrix element between the physical vacuum and the renormalized one-particle state as a free field has between the bare vacuum and the bare one-particle state:

$$\langle k|\phi'(0)|0\rangle = \sqrt{Z_3} = e^{ik\cdot x}$$

These two conditions, (13.49) and (13.52), are just matters of definition. $\phi'(x)$ is called the *renormalized field*, if $\phi(x)$ is the canonical field, obeying canonical commutators. It's called "renormalized" for an obvious reason: we have changed the normalization. Z_3 is called, for reasons so obscure and so embedded in the early history of quantum electrodynamics that I don't want to describe them, "the wave function renormalization constant". It is nowadays referred to as "the field renormalization constant".

The renormalized fields have been scaled in such a way that if all they did was to create and annihilate single-particle states when hitting the vacuum, they would do so in exactly the same way as a free field. They do more than that, however, and therefore we've got to define a limiting procedure. It's not so bad, though. Most of our work will consist of writing down a bunch of definitions and then investigating their implications.

4.2. **Constructing stable one particle state.** Unfortunately, we would get into a lot of trouble if we were to try to do limits involving plane wave states, so we would like to develop some notation for normalizable wave packet states:

$$|f\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k} F(k)|k\rangle$$

Associated with each of these wave packets is a function $f(x)$,

$$f(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} F(k)e^{-ik\cdot x}$$

which is obtained with exactly the same integral as the ket $|f\rangle$, but whose integrand has $e^{-ik\cdot x}$ instead of the ket $|k\rangle$. For reasons that will become clear in a moment, we don't want to denote $F(k)$ by $\tilde{f}(k)$. This function $f(x)$ is a positive frequency solution to the free Klein-Gordon equation:

$$(\Box^2 + \mu^2)f(x) = 0$$

We also have

$$\langle k'|f\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k} F(k)\langle k'|k\rangle = F(k')$$

Furthermore, if the one-particle state $|f\rangle$ goes to a plane wave state $|k\rangle$, $F(k')$ goes to $(2\pi)^3 2\omega_k \delta^3(k-k')$, and $f(x)$ goes to the plane wave solution $e^{-ik\cdot x}$. We've arranged a one-to-one mapping such that our relativistically normalized states correspond to plane waves with no factors in front of them.

We are now going to define an operator that, at first glance, looks disgusting:

$$\phi'^f(t) = i\int d^3x[\phi'(x)\partial_0 f(x) - f(x)\partial_0\phi'(x)]$$

Remember, $\phi'(x)$ is a Heisenberg field, a function of x and t; this produces a function of t only. We can say some things about this object. In particular, we know its vacuum-to-vacuum matrix element:

$$\langle k|\phi'^f(t)|0\rangle = i\int d^3x \int \frac{d^3k'}{(2\pi)^3 2\omega_{k'}} F(k')[-i\omega_{k'}e^{-ik'\cdot x} - e^{-ik'\cdot x}\partial_0]\langle k|\phi'^f(t)|0\rangle$$
$$= i[\frac{-2i\omega_k}{2\omega_k}]F(k) = F(k) = \langle k|f\rangle$$

so that, as part of an inner product with a one-particle bra, we can say

$$\phi'^f(t)|0\rangle = |f\rangle.$$

A calculation analogous to the above, but differing in one crucial minus sign, gives

$$\langle 0|\phi'^f(t)|k\rangle = 0.$$

Thus, this operator $\phi'^f(t)$ has *time-independent* matrix elements from vacuum to vacuum, and from vacuum to any one-particle state; the time-dependent phases cancel in the calculation. If we just restrict ourselves to the one-particle subspace at any given time, $\phi'^f(t)$ is like a creation operator for the normalized state $|f\rangle$.

What about a multiparticle state? Suppose we take a state $|n\rangle$ with two or more particles, and total momentum p_n^μ;

$$P^\mu|n\rangle = p_n^\mu|n\rangle$$

The matrix element of the state $|n\rangle$ with our new creation operator $\phi'^f(t)$ can be worked out in exactly the same way. There is a small complication in that we don't know the normalization of $\langle n|\phi'(x)|0\rangle$:

$$\langle n|\phi'(x)|0\rangle = e^{ip_n\cdot x}\langle n|\phi'(0)|0\rangle$$

and we don't know what $\langle n|\phi'(0)|0\rangle$ is, yet. In terms of this quantity,

$$\langle n|\phi'^f(t)|0\rangle = [\frac{\omega_{p_n} + E_n}{2\omega_{p_n}}]F(p_n)\langle n|\phi'(0)|0\rangle e^{-i(\omega_{p_n}-E_n)t}.$$

The real killer is in the exponential, $e^{-i(\omega_{p_n}-E_n)t}$. A multiparticle state always has energy $E_n > \omega_{p_n}$, more energy than a single particle state with momentum p_n. For example, a two-meson state with $p_n = 0$ can have any energy from $E = 2\mu$ to infinity. The one-meson state with $p = 0$ has energy $E = \mu$. So the exponential will provide the same sort of oscillatory factor as we saw in the calculation of interaction vacuum excitation. Thus we can use the same argument with the operator $\phi'^f(t)$ as we did with the U_I matrix:

$$\lim_{t\to\pm\infty}\langle n|\phi'^f\,(t)\,|0\rangle = 0$$

by the Riemann–Lebesgue lemma, provided $|n\rangle$ is a multiparticle state.

Let $\langle\psi|$ be a fixed, normalizable state, and consider the limit as $t\to\pm\infty$ of the matrix element $\langle\psi|\phi'^f(t)|0\rangle$:

$$\lim_{t\to\pm\infty}\langle\psi|\phi'^f(t)|0\rangle = \lim_{t\to\pm\infty}\sum_n\langle\psi|n\rangle\langle n|\phi'^f(t)|0\rangle$$

$$= \lim_{t\to\pm\infty}[\langle\psi|0\rangle\langle 0|\phi'^f(t)|0\rangle + \sum_{n,\ single\ particle}\langle\psi|n\rangle\langle n|\phi'^f(t)|0\rangle + \sum_{n,\ muti-particle}\langle\psi|n\rangle\langle n|\phi'^f(t)|0\rangle]$$

For any fixed state $|\psi$ sitting on the left of the operator, the matrix element with the vacuum state will give us nothing; the matrix elements with the one-particle states will give us $F(k)$, independent of time; and everything else in the whole wide world will give us oscillations which vanish, by the calculation above. Thus

$$\lim_{t\to\pm\infty}\langle\psi|\phi'^f(t)|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\langle\psi|k\rangle\langle k|\phi'^f(t)|0\rangle = \langle\psi|f\rangle.$$

So this is exactly analogous to the formula we found with the one-particle state $|k\rangle$ sitting on the left. The operator just projects out the part $F(k)$ and gives you $\langle\psi|f\rangle$. That is, we have something that can act either in the far past or the far future as a creation operator for a normalizable state $|f\rangle$.

An analogous calculation gives

$$\lim_{t\to\pm\infty}\langle 0|\phi'^f(t)|\psi\rangle = 0$$

because the arguments of the exponentials add and never cancel, for every single particle or multiparticle momentum eigenstate.

Now this procedure looks very tempting as a prescription for constructing two-particle in states and two-particle out states, and to find S-matrix elements. We will yield to that temptation at the beginning of the next section.

4.3. **Constructing multi-particles states.** We can, of course, get some related formulas from the result we obtained before. For example, if we put the vacuum on the other side,

$$\lim_{t\to\pm\infty}\langle 0|\phi'^f(t)|\psi\rangle = 0$$

Then, as we found above, even for the one-particle states we have a phase mismatch: in this matrix element, all of the phases have a positive frequency and never cancel. So this limit is zero. All the phases mismatch, which is again what you would expect if this asymptotic limit is producing something like a creation operator. A creation operator does indeed annihilate the vacuum on the left. Of course, we have certain trivial equations that follow from above just by taking the adjoint;

$$\lim_{t\to\pm\infty}\langle 0|\phi'^{f\dagger}(t)|\psi\rangle = \langle f|\psi\rangle$$

And the adjoint equation has a limit of zero.

Again, this is what you would expect if ϕ'^f is a creation operator, because $\phi'^{f\dagger}$ should then be an annihilation operator. This is just what an annihilation operator does: it makes a one-particle state from the vacuum on the left, and kills the vacuum on the right.

Now we come to the great leap of faith. I assume I have two functions $F_1(k)$ and $F_2(k)$, which are associated with nice, normalized, non-interacting wave packet states $|f_1\rangle$ and $|f_2\rangle$, respectively, in the sense of (14.1). We require that the functions $F_1(k)$ and $F_2(k)$ have no common support in momentum space. That is,

$$F_1(k)F_2(k) = 0, \ \textit{for each } k.$$

By making this statement, we are leaving out only a negligible region of phase space. When we eventually let the kets $|f_1\rangle$ and $|f_2\rangle$ go to plane wave states, this restriction will exclude just the configurations with two collinear momenta, which correspond to scattering at threshold in the center-of-mass frame, a case we excluded in our other analysis also. Thus, one of these kets is associated with a one-particle state which is going off in some direction, and the other is associated with a one-particle state going off in another direction. I'll call the functions and states associated with $F_1(k)$ and $F_2(k)$, $f_1(x)$, $f_2(x)$ and $|f_1\rangle$, $|f_2\rangle$, respectively.

I now want to consider what happens if I take the limit

$$\lim_{t\to+\infty}\langle\psi|\phi'^{f_2}(t)|f_1\rangle$$

the operator $\phi'^{f_2}(t)$ acting on not the vacuum now, but on the state $|f_1\rangle$. Well, (14.11) is a matrix element, which we can think about in either the Schrodinger or the Heisenberg picture: matrix elements are matrix elements, even though these are all Heisenberg fields. Let's think about this operation in the Schrodinger picture. I have a state $|f_1\rangle$, described by some wave packet, say with the center of the wave packet traveling in some direction. I wait for some very large future time, say, a billion years. If I wait long enough, that wave packet has gotten very very far away, maybe several galaxies over in the original direction. Now I come into this room. I have an operator which if I applied it to the vacuum would make a state $|f_2\rangle$. If I were now to go a billion light years in the opposite direction, carrying this operator, and hit the vacuum with it there, it would make a single-particle state with distribution f_2. That's the physics of what is going on.

So let me ask a question. What happens if I apply it not to the vacuum, but to the state that has that other particle over there, way beyond the Andromeda galaxy, two million light years away? Well, if there's any sense in the world whatsoever, the fact that that other particle is on the other side of the Andromeda galaxy should be completely irrelevant. I'd have to travel to the other side of Andromeda to see it's there. As far as I'm concerned, I don't know in the whole region of spacetime in which I'm working that I haven't got the vacuum state. The particle that is really there, that is secretly there, I can hardly expect to see in any experiment I can do, because it is all the way over on the other side of Andromeda. It can't affect what I'm doing in this room, or what I'm doing two million light years away in the other direction. I am making a state by this operation that is effectively a two-particle state, with the two particles in the far future moving away from each other, one going in one direction and the other going in another direction. Therefore, I assert this limit should exist and should give the definition of a two-particle out state, a state that in the far future looks like two particles moving away from each other:

$$\lim_{t\to+\infty}\langle\psi|\phi'^{f_2}(t)|f_1\rangle = \langle\psi|f_1,f_2\rangle^{out}$$

That's an argument, not a proof. If you want a mathematical proof you have to read a long paper by Klaus Hepp; but it is physically very reasonable. The only thing I am incorporating is that there is some rough idea of localization in this theory, some sort of approximation to position. And if there's a particle on the other side of the Andromeda galaxy traveling away from me, I'll never know it.

In fact the analysis can be extended to collinear momenta, but it requires much more complicated reasoning, and the result is not even on the rigorous level of Hepp's argument. The physics is clear, even if the momenta are collinear, because wave packets tend to spread out. If I wait long enough, I'll have a negligible probability for the first particle to be anywhere near the second particle even though the centers of the wave packets are moving in the same direction. So it turns out it's also true for collinear momenta. The limit will be a little slower, because in spreading out, they're not moving away from each other as fast as they would if their motions were pointing in different directions.

Of course, if our limit were for the time to approach minus infinity, all the arguments would be exactly the same, but time reversed. Instead of an "out", I would have an "in":

$$\lim_{t\to-\infty}\langle\psi|\phi'^{f_2}(t)|f_1\rangle = \langle\psi|f_1,f_2\rangle^{in}$$

Thus we have the prescription for constructing in states and out states, states that look like two-particle states in the far past, and states that look like two-particle states in the far future. I use two-particle states only for simplicity. After I go through all the agonies I will go through for two particles scattering into two particles, if you wish you can extend the arguments to two into three or two into four or seven into eighteen. We can construct states that are indeed asymptotic states.