§ Hochschild homology and THH

Def $k \in \mathrm{CRing}$. $A \in \mathrm{Alg}_k$

The complex of Hochschild homology is

$$HH(A/k) : A \xleftarrow{b} A \overset{\mathbb{L}}{\underset{k}{\otimes}} A \xleftarrow{b} A \overset{\mathbb{L}}{\underset{k}{\otimes}} A \overset{\mathbb{L}}{\underset{k}{\otimes}} A \xleftarrow{b} \cdots$$

$$b : A^{\otimes_k n+1} \longrightarrow A^{\otimes_k n}$$

$$\begin{aligned} a_0 \otimes \cdots \otimes a_n \longmapsto\ & \boxed{a_0 a_1 \otimes \cdots \otimes a_n} \\ & - a_0 \otimes a_1 a_2 \otimes \cdots \otimes a_n \\ & + (-1)^n a_0 \otimes \cdots \otimes a_{n-1} a_n \\ & \boxed{+ (-1)^{n+1} a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}} \end{aligned}$$

Bar construction + 2 terms

$HH_n(A/k)$ = n-th homology group of $HH(A/k)$

(generalization of de Rham in non-commutative setting)

Prop If A is commutative and smooth over k,

$$\varepsilon_n : \Omega^n_{A/k} \longrightarrow HH_n(A/k)$$

$$a\, db_1 \wedge \cdots \wedge db_n \longmapsto \sum_{\sigma \in C_n} (-1)^{\mathrm{sgn}(\sigma)}\, a \otimes b_{\sigma(1)} \otimes \cdots \otimes b_{\sigma(n)}$$

is an isomorphism

is an isomorphism

§ Connes's operation on HH.

Convention

$t_n \in \mathbb{Z}/(n+1)$ generator

$t_n : a_0 \otimes \cdots \otimes a_n := a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}$

"norm" $N := \sum_{i=0}^{n} \left((-1)^n t_n\right)^i : A^{\otimes_k n+1} \longrightarrow A^{\otimes_k n+1}$

"extra degeneracy" $s : A^{\otimes_k n} \longrightarrow A^{\otimes_k n+1}$

$a_0 \otimes \cdots \otimes a_{n-1} \longmapsto 1 \otimes a_0 \otimes \cdots \otimes a_{n-1}$

<u>Def</u> (Connes operation)

$$B := (1-(-1)^n t_n)\, sN : A^{\otimes_k n} \longrightarrow A^{\otimes_k n+1}$$

<u>Prop</u> (1) $B^2 = 0$

(2) $Bb = -bB$

<u>Def</u> (mixed complex) ① $A_\bullet : A_0 \xleftarrow{b} A_1 \xleftarrow{b} A_2$

$(A_\bullet, b, B)$ ② $B : A_n \to A_{n+1} \ \forall n.$

s.t. (1) $B^2 = 0$

(2) $Bb = -bB$.

$$
\begin{tikzcd}
& A_3 \arrow[d] & A_2 \arrow[l, "B"'] \arrow[d, "b"] & A_1 \arrow[l, "B"'] \\
{} & A_2 \arrow[l] \arrow[d] & A_1 \arrow[l, "B"'] \arrow[d, "b"] & A_0 \arrow[l, "B"'] \\
{} & A_1 \arrow[l] \arrow[d] & \underset{(0,0)}{A_0} \arrow[l, "B"'] & \\
{} & A_0 \arrow[l] & &
\end{tikzcd}
$$

periodic

$HP(A_\bullet, b, B)$: product totalization of the bicomplex.

$HC^-(A_\bullet, b, B)$ ———————————

——— $x \leq 0$ half plane

$HC(A_\bullet, b, B)$ ———————————

$x \geq 0$ half plane

§ Simplicial perspective

<u>Def</u> $B_n^{cyc}(A) = A^{\otimes_k n+1}$ $\left\} \begin{array}{l} d_i : X_n \to X_{n-1} \\ s_i : X_n \to X_{n+1} \end{array} \right.$

$$
d_i(a_0 \otimes \cdots \otimes a_n) = \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & i < n \\ a_n a_0 \otimes \cdots \otimes a_{n-1} & i = n \end{cases}
$$

$$s_i(a_0 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n$$

$$\pi_i : B^{cyc}_\bullet(A/k) = HH_i(A/k)$$

FACT 1 $B^{cyc}_\bullet(A)$ is endowed with an S^1-action

$$\begin{array}{c} B^{cyc}_\bullet(A) \\ \text{“}\|\text{”} \\ HH(A) \end{array}$$

$$HC := (HH)_{hS^1}$$

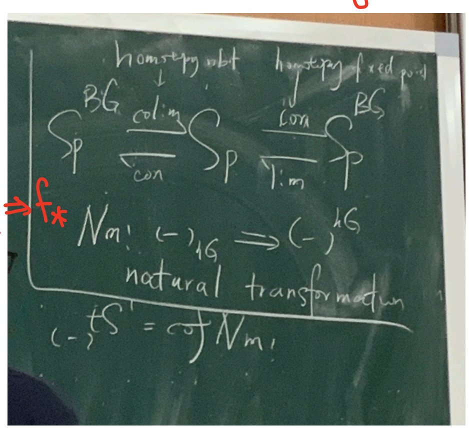
$$HC^- := HH^{hS^1}$$

$$HP := HH^{tS^1}$$

$\uparrow$

Tate construction

ambidexterity



$f_! \Rightarrow f_*$

$$HP = \mathrm{cofib}(\Sigma HC \xrightarrow{Nm} HC^-)$$

Intuitively, $\underset{\text{finite}}{G} \curvearrowright M$, $\quad M_G \to M^G$

$$[m] \mapsto \sum_{g \in G} g \cdot m$$

[Lurie, DAG III commutative algebra], [HA]

Important remark $\qquad k[B\mathbb{Z}] \quad B\mathbb{Z} \simeq S^1$

$S^1 \circlearrowright sSet \rightsquigarrow B \circlearrowright Ch_{\geq 0}(k)$

$\pi_0(TC^-(\mathbb{F}_p)) = \mathbb{Z}_p$

$\pi_0(THH(\mathbb{F}_p, \mathbb{Z}_p)^{hS^1_{triv}}) = \mathbb{F}_p[[t]]$

FACT Cyclic bar construction can be extended to any sym-monoidal ∞-category

Sp: sym monoidal infinity category

Def (THH). $A \in Alg_{E_1}(Sp) \qquad E_1 = A_\infty$

$$THH(A) = \underbrace{\operatorname{colim}(A \leftleftarrows A \otimes_{\mathbb{S}} A \leftleftarrows \cdots)}_{|B^{cyc}_\bullet(A)|}$$

$\otimes_k \rightsquigarrow \otimes_{\mathbb{S}}$

$TC^- := (THH)^{hS^1}$

$TP := (THH)^{tS^1}$

$TC \not:= (THH)_{hS^1}$

cyclotomic Frobenius

§ Cyclotomic structure

Def $A \in Sp^{BS^1}$ "S^1-equivar spectrum" Borel equivariant

$$\{\varphi_p : A \longrightarrow A^{tC_p}\}_{p \in \mathbb{P}}$$

$C_p \subseteq G \qquad (-)^{tC_p} : Sp^G \longrightarrow Sp^{G/C_p}$

S^1-equivariant $\quad (S^1 \simeq S^1/C_p)$

Prop $THH(A)$ admits a cyclotomic structure naturally.

Prop $A \to THH(A)$ is initial among all maps from A to an S^1-equivariant $\mathbb{E}_\infty$-ring spectrum.

Sketch of the construction of the cyclotomic structure on $THH(A)$

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\Delta_p"] & THH(A) \arrow[d, dashed, "\varphi_p"] \\
(A \otimes \cdots \otimes A)^{tC_p} \arrow[r] & THH(A)^{tC_p}
\end{tikzcd}$$

$$\underbrace{\qquad\qquad\qquad}_{p}$$

$$\Delta_p(\mathbb{S}): \quad \mathbb{S} \longrightarrow (\mathbb{S}^{\otimes p})^{tC_p}$$

$$S^1 \longrightarrow S^p \qquad x \longmapsto (x, \dots, x)$$

diagonal map

§ TC

$$\begin{tikzcd}
THH(A)^{hS^1} = TC^-(A) \arrow[d, "(\varphi_p)^{hS^1}"] \arrow[ddr, "\text{canonical}"] & \simeq \left(THH(A)^{hC_p}\right)^{hS^1/C_p} \arrow[d, phantom, "\cong"] \\
\left(THH(A)^{tC_p}\right)^{hS^1} & \left(THH(A)^{hC_p}\right)^{hS^1} \arrow[d] \\
 & \left(THH(A)^{tC_p}\right)^{hS^1}
\end{tikzcd}$$

$$TC = \mathrm{Eq}\left(THH(A)^{hS^1} \underset{\text{can}}{\overset{(\varphi_p)^{hS^1}_{p\in\mathbb{P}}}{\rightrightarrows}} \prod_{p\in\mathbb{P}} \left(THH(A)^{tC_p}\right)^{hS^1} \right)$$

Frobenius fixed points from Guo's talk

p-complete case

$TP(A;\mathbb{Z}_p) \simeq (THH^{tC_p})^{hS^1}$

$\quad \| $

$TP(A)^\wedge_p$ $\quad \boxed{A \text{ bounded below}}$

§ Compute ~~THH by~~ HH.

<u>Prop</u> A $\mathbb{E}_\infty$-ring spectrum $\left(\begin{array}{c} D(\mathbb{Z}) \to Sp \\ A \mapsto HA \end{array} \right)$

$$THH(A) \underset{THH(\mathbb{Z})}{\otimes} \mathbb{Z} \overset{S^1}{\simeq} HH(A)$$

<u>Prop</u> $THH_i(A) := \pi_i THH(A)$

The natural map

$$THH_i(A) \longrightarrow HH_i(A)$$

is an isomorphism when $i \leq 2$

$\mathbb{L}_{A/k} = \mathbb{L}\Omega$

§ HKR filtration

w/o smooth, need cotangent complex below

<u>Thm</u> k (perfect field of char p) comm ring. A smooth over k

Then $HC^-(R/k)$, HP, HC admit natural complete descending $\mathbb{Z}$-indexed filtrations

whose i-th graded pieces are

$$\Omega^{\geq i}_{R/k}[2i],\ \Omega^{i}_{A/k}[2i],\ \Omega^{\leq i}_{R/k}[2i]$$

comm ring

Thm A commutative k-algebra.

(1) $HH(A/k)$ ——— Shukla

graded pieces $\mathbb{L}^{i}_{A/k}[\ell]_{\geq 0}$.

(2) HC $\bigoplus_{n\geq 0} \mathbb{L}^{i}_{A/k}[\ell+2n]$ for $i \geq 0$.

k quasi-syntomic

$\{$Connes operation B $\circlearrowright$ complexes$\}$

$\downarrow\sim$ $k[B]/B^2$ - module

$H_*(S^1,k)$

$\cong$

$\{k[B]/B^2$ dg-module $\}$

$Bb = -bB$

$0 \leftarrow B \xleftarrow{0} B\cdot\varepsilon \leftarrow$

$$
\begin{tikzcd}
{} \arrow[d, "\text{quasi-iso}"] & & S^1 \arrow[d, phantom, "\wr"] \\
\{k[S^1]\text{-dg module}\} \arrow[rr, "DK"] & & \{B\mathbb{Z}\text{-simplicial set}\} \arrow[d, "\text{CC. Connes}"] \\
\{\text{cyclic objects}\} & & \{\text{CC. simplicial set}\} \arrow[ll, "\text{Quillen B}"']
\end{tikzcd}
$$