TWISTED NEUMANN-ZAGIER MATRICES AND LOOP INVARIANTS

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OVERVIEW

K : KNOT IN S^3

(1980's) KASHAEV DEFINED A QUANTUM INVARIANT

$$\langle K \rangle_N \in \mathbb{C} \quad (N \geq 2)$$

AND CONJECTURED THAT

$$\log |\langle K \rangle_N| \underset{N\to\infty}{\sim} \frac{\mathrm{Vol}(S^3 \setminus K)}{2\pi i} N$$

THE VOLUME CONJECTURE IS GENERALIZED TO

$$\log \langle K \rangle_N \overset{N\to\infty}{\sim} \frac{\mathrm{Vol}_{\mathbb{C}}(S^3\setminus K)}{2\pi i} N + \frac{3}{2}\log N - \frac{1}{2}\log \phi_1 + \phi_2 \cdot \frac{2\pi i}{N} + \phi_3 \cdot \left(\frac{2\pi i}{N}\right)^2 + \cdots$$

$$\therefore\ S^3\setminus K \longrightarrow \underbrace{\{\phi_1, \phi_2, \phi_3, \ldots\ldots}_{\text{LOOP INVARIANTS}}\} \subset \left(\begin{array}{c}\text{THE TRACE}\\ \text{FIELD OF } K\end{array}\right)$$

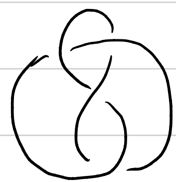
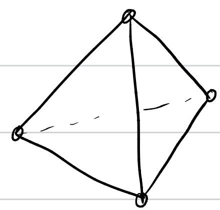
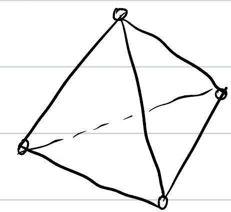
(QUESTION) HOW THE ℓ-LOOP INV. ϕ_ℓ BEHAVES UNDER CYCLIC COVERINGS?

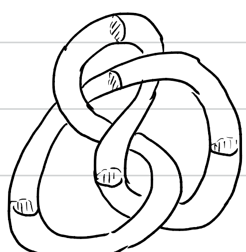
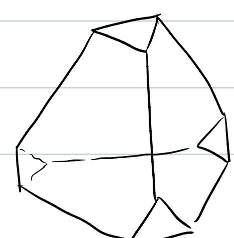
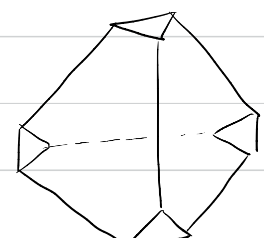
LOOP INVARIANTS

(1977) THURSTON SHOWED

$$S^3 \setminus K = \Delta_1 \cup \cdots \cup \Delta_N$$

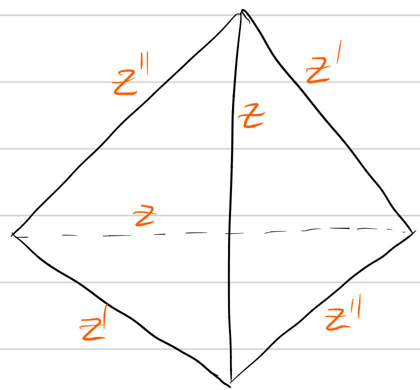
WHERE Δ_i ARE IDEAL TETRAHEDRA.

(Ex) $S^3 \setminus$  $=$  $\cup$ 

OR $S^3 \setminus$  $=$  $\cup$ 

THE SHAPE OF EACH Δ_i

$\rightsquigarrow$ ONE VARIABLE $z_i \in \mathbb{C} \setminus \{0, 1\}$



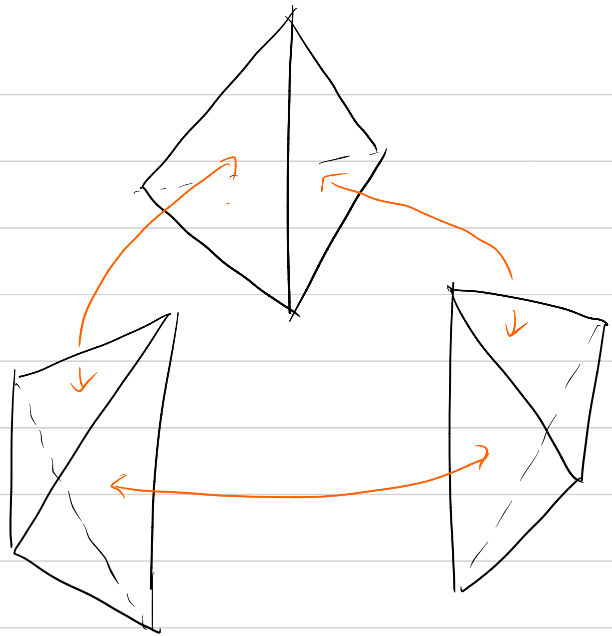
$$z$$

$$z' = \frac{1}{1-z}$$

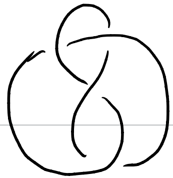
$$z'' = 1 - \frac{1}{z}$$

WE NEED SOME CONIDITION ON $z_1 \dots z_N$

TO GUARANTEE THE WELL-GLUEDNESS.



EACH EDGE $\rightsquigarrow \prod \left(\begin{array}{c} \text{SHAPES AROUND} \\ \text{THE EDGE} \end{array} \right) = 1.$

(Ex) $S^3 \setminus$  $= \Delta_1 \cup \Delta_2$

$e_1 : \quad z_1^2 \, z_2^2 \, z_1' \, z_2' = 1$

$e_2 : \quad z_1' \, z_2' \, (z_1'')^2 (z_2'')^2 = 1$

$$G = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}, \quad G' = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad G'' = \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix}$$

FOR SOME REASON WE SET

$$A := G - G', \qquad B := G'' - G'$$

A, B ARE CALLED NEUMANN – ZAGIER MATRICES

(1985) NEUMANN – ZAGIER PROVED THAT

A & B HAVE A SYMPLECTIC PROPERTY.

IN PARTICULAR, $A^T B = B^T A$.

(2012) DIMOFTE – GAROUFALIDIS SHOWED

$$\phi_1 = \frac{1}{2} \det \left[A \begin{pmatrix} z_1'' & & \\ & \ddots & \\ & & z_N'' \end{pmatrix} + B \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_N \end{pmatrix}^{-1} \right] \cdot \quad *$$

FOR $\ell \geq 2$ $\phi_\ell = \cdots$

TWISTED NZ MATRICES

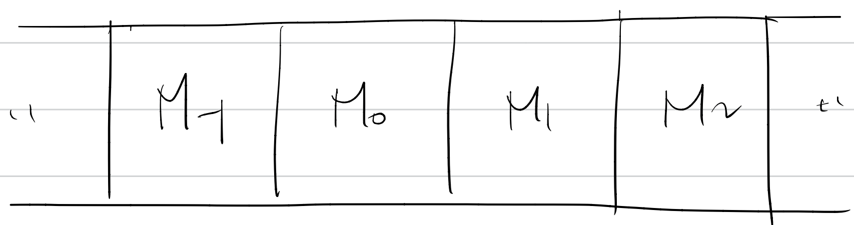
$\tilde{M}$: INFINITE CYCLIC COVER OF $M = S^3 \setminus K$

(DECK TRANSFORMATION GROUP $\cong \mathbb{Z}$)

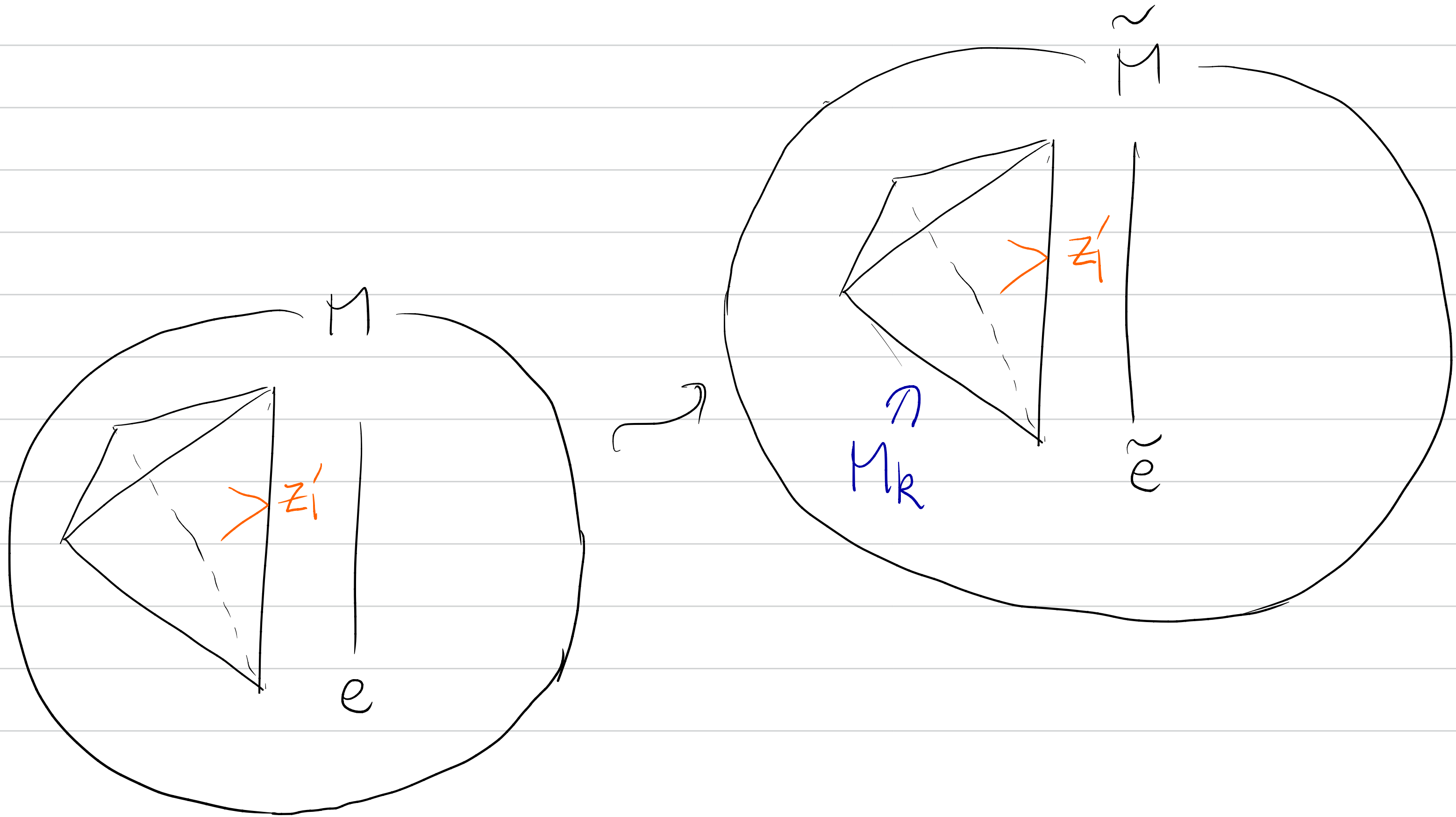
$$\begin{tikzcd} \tilde{M} \arrow[d] \\ M \end{tikzcd}$$

M_0 : FUNDAMENTAL DOMAIN

$\tilde{M} = \cdots$



$\tilde{M}$ AND M ARE LOCALLY SAME!



$$G = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}, \quad G' = \begin{pmatrix} \boxed{1} & 1 \\ 1 & 1 \end{pmatrix}, \quad G'' = \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix}.$$

Z_1' APPEARS ONCE AROUND e_1

USING A FORMAL VARIABLE t,

$$G(t) = \begin{pmatrix} 2t & 2t \\ 0 & 0 \end{pmatrix}, \quad G'(t) = \begin{pmatrix} \boxed{t^2} & 1 \\ t & t^2 \end{pmatrix}, \quad G''(t) = \begin{pmatrix} 0 & 0 \\ 2t^2 & 2t \end{pmatrix}$$

Z_1' APPEARS ONCE AROUND $\tilde{e}_1$

AND IT LIVES IN M_2

WE SET

$$A(t) := G(t) - G'(t), \quad B(t) := G''(t) - G'(t)$$

AND CALL THEM TWISTED NZ MATRICES

NOTE THAT $A(1) = A$, $B(1) = B$

<u>THEOREM</u> (GAROUFALIDIS-Y.)

$$A(t)^T B(\tfrac{1}{t}) = B(t)^T A(\tfrac{1}{t}).$$

RECALL

$$\begin{pmatrix} \text{1-LOOP} \\ \text{INVARIANT} \end{pmatrix} := \frac{1}{2} \det\left[A \begin{pmatrix} z_1'' & & \\ & \ddots & \\ & & z_N'' \end{pmatrix} + B \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_N \end{pmatrix} \right] \cdot *$$

$$\begin{pmatrix} \text{TWISTED} \\ \text{1-LOOP INV} \end{pmatrix} := \frac{1}{2} \det\left[A(t) \begin{pmatrix} z_1'' & & \\ & \ddots & \\ & & z_N'' \end{pmatrix} + B(t) \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_N \end{pmatrix} \right] \cdot *$$

THEOREM (GAROUFALIDIS-Y.)

THE TWISTED 1-LOOP INV. DETERMINES 1-LOOP INV. OF ALL FINITE CYCLIC COVERS

$\Phi_{1,M}(t)$: THE TWISTED 1-LOOP INV. OF M

$\Phi_{1,M^{(n)}}$: THE 1-LOOP INV OF $\underbrace{M^{(n)}}$

THE n-CYCLIC COVER OF M.

$$\rightsquigarrow \quad \Phi_{1,M^{(n)}} = \prod_{\omega^n=1} \Phi_{1,M}(\omega)$$

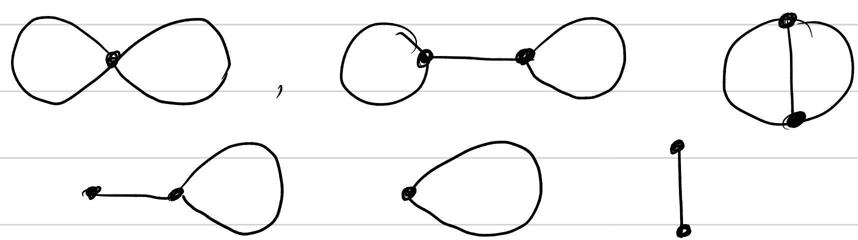
HIGHER LOOP INV

TO DEFINE ϕ_ℓ $(\ell \geq 2)$

WE NEED NZ MATRICES

+ GRAPHS w/ $\#(\text{LOOP}) \leq \ell$

FOR $\ell = 2$,



THE FEYNMAN RULE ASSOCIATES EACH GRAPH WITH A CONTRIBUTION, AND

$$\phi_\ell := \sum (\text{CONTRIBUTION})$$

THEOREM (GAROUFALIDIS-Y.)

FOR $\ell \geq 2$ ϕ_ℓ ARE ASYMPTOTICALLY MULTIPLICATIVE

NAMELY, $\exists\, I_{\ell,M}$ s.t.

$$\phi_{\ell,M^{(n)}} \underset{n\to\infty}{\sim} n \cdot I_{\ell,M} + \left(\begin{array}{c}\text{EXPONENTIALLY}\\ \text{SMALL TERM}\end{array}\right)$$

(EXAMPLE) $M = S^3 \setminus 4_1$

$$\phi_{2,M}(n) = \frac{55\sqrt{-3}}{1512} \cdot n - \frac{8\sqrt{-3} \cdot n}{63(1-\lambda^n)} + \frac{8\sqrt{-3} \cdot n}{63(1-\lambda^n)^2}$$

WHERE $\lambda = \frac{5+\sqrt{21}}{2} \cong 4.7912$ IS A ROOT OF $\underbrace{\Phi_{1,M}(t)}_{= t^2-5t+1}$

$$\therefore \quad \phi_{2,M}(n) \sim \frac{55\sqrt{-3}}{1512} n + \left(\text{EXPONENTIALLY SMALL} \right)$$

$\lambda_1 \ldots \lambda_r$: ROOTS OF $\Phi_{1,M}(t)$

<u>THEOREM</u> $\exists$ POLYNOMIAL $P_{\ell,M}(x_1, \ldots x_r, y)$

$$\text{s.t.} \quad \phi_{\ell,M}(n) = P_{\ell,M}\left(\frac{1}{1-\lambda_1^n}, \ldots, \frac{1}{1-\lambda_r^n}, n\right)$$

FOR ALL BUT FINITELY MANY n.

IN ADDITION, UNDER A MILD CONDITION,

$$\deg_{x_1} P + \cdots + \deg_{x_r} P \leq 2\ell - 2 \quad \& \quad 1 \leq \deg_y P \leq \ell - 1$$

$$\text{(ex)} \quad \phi_{2,M}(n) = n\left[\sum_{1 \leq i < j \leq r} \frac{C_{ij}}{(1-\lambda_i^n)(1-\lambda_j^n)} + \sum_{1 \leq i \leq r} \frac{C_i}{1-\lambda_i^n} + C_0\right]$$

(A KEY INGREDIENT OF PROOFS.)

$M \to N$ TETRAHEDRA

$M^{(n)} \to nN$ TETRAHEDRA

$A^{(n)}, B^{(n)}$: NZ-MATRICES FOR $M^{(n)}$.

$$(B^{(n)})^{-1} A^{(n)} = \underbrace{\boxed{}}_{nN}$$

ADMITS A BLOCK DIAGONALIZATION IN TERM OF $B(t)^{-1} A(t)$

THANKS FOR

YOUR ATTENTION.