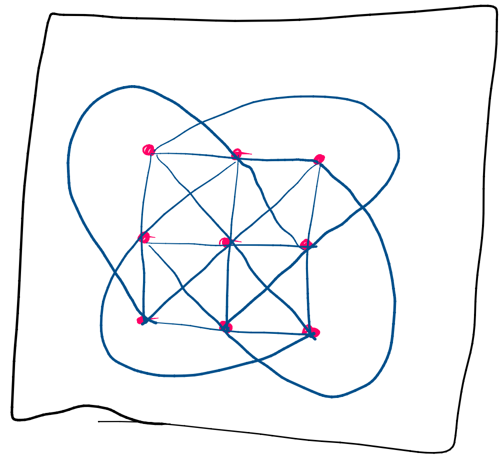
Choosing Points on Cubic Plane Curves



Weiyan Chen 陈伟彦

Tsinghua University

Joint work with Ishan Banerjee

arXiv link at weiyanc.com

Arithmetic and Topology, Dec. 16, 2022

Southern University of Science and Technology

Plan:

1. Motivation (algebraic geometry)
2. Theorems (topology)
3. Broader context
4. Proof ideas.

A cubic plane curve is

$$C_F = \{ [x:y:z] \in \mathbb{CP}^2 \mid F(x,y,z) = 0 \}$$

where F is homogeneous, $\deg = 3$.

If smooth, $C_F \cong$ 

e.g. $C_F: x^3 + y^3 + z^3 = 0$

Classical results in A.G.

Every cubic curve has:

Have we discovered all the structures?

the structures?

Every cubic curve has:

- 9 flex points (Maclaurin, 1700s)
- 27 sextatic points. (Cayley, 1865)
- 72 points of type nine (Gattazzo 1979)

⋮

Question (Farb, 2018):

Are these known algebraic structures the only ways to continuously choose n distinct points on cubic curves?

Reformulating the question.

$$X := \{\text{smooth cubic curves}\}$$

$$= \{F(x,y,z) \mid \text{homogeneous, deg 3, nonsingular}\} / \mathbb{C}^\times$$

$$\subseteq \mathbb{CP}^9$$

$$\begin{array}{ccc} \mathrm{UConf}^n(C_F) & \longrightarrow & E_n \\ & & \downarrow \uparrow \exists \quad \xi_n \\ & & F \in X \end{array}$$

$$\mathrm{UConf}^n C_F := \left\{ (x_1, \cdots, x_n) \;\middle|\; \begin{array}{l} x_i \in C_F \\ x_i \neq x_j \end{array} \right\} / S_n$$

$$\subseteq \mathrm{Sym}^n C_F := C_F^{\times n} / S_n$$

Farb's question reformulated

Is it true that the known algebraic constructions give all the ~~continuous~~ algebraic sections of ξ_n ~~up to homotopy~~?

A: Yes, when $9 \nmid n$, $n = 9$, $n = 18$,

No, when $n = 108k$, $k \geq 1$.

Thm 1 (C, 2018)

ξ_n has no section when $9 \nmid n$.

Rmk: ξ_1 has no section

$\Rightarrow$ it is not possible to continuously choose an elliptic curve structure

Consider sections of ξ_n for $9 \mid n$.

Thm 2 (Banerjee – C, 2021)

Sections of ξ_9 are unique up to homotopy.

Thm 3 (B – C, 2021)

ξ_{18} has no section.

choose an elliptic curve structure for each cubic curve.

ξ_{18} has no section.

Thm 4 (B–C, 2021)

When $108 | n$ and $n \geq 108$, ξ_n has a continuous section that is NOT homotopic to any existing algebraic ones.

McMullen's observation:

If we ask the question for algebraic sections, then the answer is always yes.

$$\left\{ \begin{matrix} \text{known algebraic} \\ \text{sections} \end{matrix} \right\} \underset{\substack{= \\ M}}{\subseteq} \left\{ \begin{matrix} \text{algebraic} \\ \text{sections} \end{matrix} \right\} \underset{\substack{\neq \\ BC.}}{\subseteq} \left\{ \begin{matrix} \text{topological} \\ \text{sections} \end{matrix} \right\}$$

Broader context:

"Rigidity of moduli spaces".

General phenomenon:

$\boxed{\text{algebro-geometric construction}} \rightsquigarrow$ a map of moduli spaces

$$\psi : \mathcal{M}_m \longrightarrow \mathcal{N}_n$$

Guiding principle: ψ is a miracle.

Topological rigidity:

$F : \mathcal{M}_{m'} \longrightarrow \mathcal{N}_{n'}$ continuous, "nontrivial"

$\Rightarrow (m', n') = (m, n)$. $F \sim \psi$

Holomorphic rigidity:

$F : \mathcal{M}_{m'} \longrightarrow \mathcal{N}_{n'}$ holomorphic

$\Rightarrow (m', n') = (m, n)$ and $F = \psi$.

Examples:

(1) Our works:

section problem for cubic curves is topologically rigid.

McMullen's observation:

$\longrightarrow$ holomorphically rigid.

(2) Resolving the quartic.

quartic polynomial $f(x)$ square-free $\rightsquigarrow$ cubic poly. $g(y)$ square-free.

$$\mathrm{UConf}^4\mathbb{C} \xrightarrow{R} \mathrm{UConf}^3\mathbb{C}$$

Thm. (Lin, 2004)

Suppose $F: \mathrm{UConf}^m\mathbb{C} \to \mathrm{UConf}^n\mathbb{C}$ is nonconstant holomorphic $(m > n > 2)$

Then 1. either F is "trivial" $\mathrm{UConf}^m\mathbb{C} \to \mathrm{UConf}^n\mathbb{C}$, $\Delta \to \mathbb{C}^\times \to$

2. or $(m,n) = (4,3)$, and $F = R$.

(3) Period mapping.

an algebraic curve C $\mapsto$ its Jacobian $J(C)$

$$\mathcal{M}_g \xrightarrow{J} \mathcal{A}_g$$

Thm. (Farb, 2021)

Let $F: \mathcal{M}_g \to \mathcal{A}_h$ be nonconstant and holomorphic, $g \geq 4$, $g \geq h \geq 1$.

Then $h = g$, $F = J$.