§1 Introduction

$K/\mathbb{Q}_p$ finite ext. $n=2$., mod p Langlands corresp.

$$\left\{ \bar\rho : \mathrm{Gal}_K \to \mathrm{Gl}_2(\bar{\mathbb{F}}_p) \right\} \overset{?}{\longleftrightarrow} \left\{ \underbrace{\text{adm smooth rep of } \mathrm{Gl}_2(K)}_{\pi(\bar\rho)} \right\}.$$

$\updownarrow$

$\mathbb{F}/\mathbb{F}_p$ finite

$\mathrm{ind}_B^{\mathrm{Gl}_2} \chi_1 \otimes \chi_2$

principal series (irred)
1-dim
Steinberg
Supersingular ($=$ supercuspidal)

classified by Breuil for $\mathrm{Gl}_2(\mathbb{Q}_p)$.

Known: $\mathrm{Gl}_2(\mathbb{Q}_p)$: well understood (Breuil, Colmez)

$K \neq \mathbb{Q}_p$: (even for $K = \mathbb{Q}_{p^2}$) little is known. $\leftarrow$ no classification

recall: two main results in $\mathrm{Gl}_2(\mathbb{Q}_p)$ case:

(i) Colmez's functor: an exact functor

local

$$\left\{ \text{adm } \mathrm{Gl}_2(\mathbb{Q}_p)\text{-rep. of finite length} \right\}_{/\mathbb{F}} \longrightarrow \left\{ \text{étale } (\varphi,\Gamma)\text{-mod} \right\}_{/\mathbb{F}((T))}$$

$\Big\downarrow \mathcal{S}$ Fontaine

$$\pi(\bar\rho) \longmapsto \bar\rho \qquad \left\{ \text{rep of } \mathrm{Gal}_{\mathbb{Q}_p} \right\}_{/\mathbb{F}}$$

(ii) Emerton: local-global compatibility : $\bar r : \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{Gl}_2(\mathbb{F})$., $\bar r|_{\mathrm{Gal}_{\mathbb{Q}_p}} \simeq \bar\rho$

global.

$$\varinjlim_{\substack{U_p \subseteq \mathrm{Gl}_2(\mathbb{Q}_p) \\ \text{open compact}}} \underbrace{H^1_{et}(X_{U_pU^p,\bar{\mathbb{Q}}}, \mathbb{F})[\bar r]}_{\text{eigenspace}} \simeq \pi(\bar\rho)^{\oplus d}, \quad \text{for some } d \geq 1.$$

$\circlearrowright$ $\mathrm{Gal}_{\mathbb{Q}} \times \mathrm{Gl}_2(\mathbb{Q}_p)$.

$\uparrow$ depends on U^p.

§2. generalized Colmez's functor.

① Colmez's construction. (φ,Γ)-mod. : M over $\mathbb{F}((T))$.

$p \leftrightarrow \varphi, \circlearrowright \Gamma \simeq \mathbb{Z}_p^\times$.

$$P^+ := \begin{pmatrix} \mathbb{Z}_p - \{0\} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix} \circlearrowright M. ; \qquad \mathbb{F}[[T]] \simeq \mathbb{F}\left[\left[\begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}\right]\right].$$

monoid.

$$\mathbb{Z}_p - \{0\} = \mathbb{Z}_p^\times \times p^{\mathbb{N}}.$$

$\pi \longmapsto \pi|_{P^+} \longmapsto$ Pontryagin dual. ;

$\pi \longmapsto D(\pi) \xrightarrow{\text{Fontaine}} V(\pi)$: exact functor.

étale (φ,Γ)-mod. : Galois rep. finite dim (use classification theorem of $GL_2(\mathbb{Q}_p)$)

$K \neq \mathbb{Q}_p$, $\mathcal{O}_K$; unramified, $k = \mathbb{F}_{p^f}$ for some $f \geq 1$.

Breuil (2015) : $\mathcal{O}_K \xrightarrow{tr} \mathbb{Z}_p$, $N_0 = \begin{pmatrix} 1 & \mathcal{O}_K \\ 0 & 1 \end{pmatrix} \rhd N_1 = \begin{pmatrix} 1 & \ker(tr) \\ 0 & 1 \end{pmatrix}$

$N_0/N_1 \simeq \mathbb{Z}_p$.

π = adm smooth rep of $GL_2(K)$.

Consider $\pi^{N_1} \circlearrowleft N_0/N_1 \circlearrowleft \begin{pmatrix} \mathbb{Z}_p\setminus\{0\} & 0 \\ 0 & 1 \end{pmatrix}$

$\rightsquigarrow$ generalized Colmez's functor. $D_{Breuil}(\pi)$ = pro-étale (φ,Γ)-module / $\mathbb{F}((T))$

= proj limite of étale (φ,Γ)-mod.

(no finiteness, no exactness) $\Gamma = \mathbb{Z}_p^\times$

Breuil-Herzig-H.-Morra-Schraen. $(\varphi, \mathcal{O}_K^\times)$-module over A.

ring A: $N_0 = \begin{pmatrix} 1 & \mathcal{O}_K \\ 0 & 1 \end{pmatrix}$, $\mathbb{F}[[N_0]] \supseteq m_{N_0}$ max ideal. $S := \{(Y_{\sigma_0} \cdots Y_{\sigma_{f-1}})^n \mid n \geq 0\}$ multip. subset

where $Y_{\sigma_i} := \sum_{\lambda \in \mathbb{F}_{p^f}^\times} \sigma_i(\lambda)^{-1} \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} \in \mathbb{F}[[N_0]]$, $\sigma_i : \mathbb{F}_{p^f} \hookrightarrow \mathbb{F}$

$A := \widehat{\mathbb{F}[[N_0]]_S}$ completion wrt the filtration. $\varprojlim_n M/F^nM$

(If $f=1$, $\mathbb{F}[[N_0]] = \mathbb{F}[[Y]]$.
$\mathbb{F}[[N_0]]_S = \mathbb{F}((Y))$)

given π = adm rep of $GL_2(K)$.

$\pi^\vee = \mathrm{Hom}_{\mathbb{F}}(\pi, \mathbb{F})$, f.g. $\mathbb{F}[[I_1]]$-module. $I_1 = \begin{pmatrix} 1+p\mathcal{O}_K & \mathcal{O}_K \\ p\mathcal{O}_K & 1+p\mathcal{O}_K \end{pmatrix}$ open.
$\mathbb{F}[[I_1]] \supseteq m_{I_1}$

define $D_A(\pi) := A \widehat{\otimes}_{\mathbb{F}[[N_0]]} \pi^\vee$ m_{I_1}-adic filtration, not m_{N_0}-adic one!

(back to $GL_2(\mathbb{Q}_p)$ case: $\mathbb{F}((Y)) \otimes_{\mathbb{F}[[Y]]} \pi^\vee$: example: π = Steinberg. $\pi \simeq C_c(\mathbb{Q}_p, \mathbb{F})$
$\pi^\vee \simeq \mathbb{F}[[\mathbb{Z}_p]] \otimes_{\mathbb{F}[\mathbb{Z}_p]} \mathbb{F}[\mathbb{Q}_p]$
$\pi^\vee$ as $\mathbb{F}[[\mathbb{Z}_p]]$-mod, is free, ∞ rank

Thm: $D_{\mathrm{Breuil}}(\pi)$:

$\mathcal{O}_K \overset{tr.}{\twoheadrightarrow} \mathbb{Z}_p$

- $\boxed{D_A^{\text{ét}}(\pi)/(\mathrm{Ker}) \simeq D_{\mathrm{Breuil}}(\pi)}$

$\mathbb{F}[[N_0]] \twoheadrightarrow \mathbb{F}[[\mathbb{Z}_p]]$

$0 \to \mathrm{Ker} \to A \twoheadrightarrow \mathbb{F}((T)) \to 0.$

- $D_A^{\text{ét}}(\cdot)$ is exact; $\Rightarrow$ $D_{\mathrm{Breuil}}(\cdot)$ is exact.

(ii) global method.

$F/\mathbb{Q}$. tot real field, $F_v \simeq K$.

replace modular curve $X_{U^p U_p}$, $\rightsquigarrow$ Shimura curve $Sh_{U^v U_v}$, $v | p$.

D/F

$\bar{\rho}: \mathrm{Gal}_K \to GL_2(\mathbb{F})$. $\underset{\text{globalize}}{\rightsquigarrow}$ $\bar{r}: \mathrm{Gal}_F \to GL_2(\mathbb{F})$

$$\varinjlim_{U_v \subseteq GL_2(F_v)} H^1_{\text{ét}}(Sh_{U^v U_v}, \bar{\mathbb{Q}}, \mathbb{F}) = \pi_v(\bar{\rho})^{\oplus d}$$

$\circlearrowleft GL_2(F_v) = GL_2(K)$.

Choose U^v s.t. $d=1$

problem: Structure of $\pi_v(\bar{\rho})$ is not clear.

Is $\pi_v(\bar{\rho})$ local ?? (only depends on $\bar{\rho}$)?

Thm (BHHMS). $\bar{\rho}$: semi-simple

$D_A(\pi_v(\bar{\rho}))$ is local.

$$\bar{\rho} \overset{\text{global}}{\rightsquigarrow} \underset{GL_2(K)}{\pi_v(\bar{\rho})} \overset{\text{local}}{\rightsquigarrow} D_A(\pi_v(\bar{\rho})).$$

$D_{\mathrm{Breuil}}(\pi_v(\bar{\rho})) =$ tensor induction. $\mathrm{ind}_{\mathrm{Gal}_K}^{\otimes \mathrm{Gal}_{\mathbb{Q}_p}} \bar{\rho}$. $\left(\dim = (\dim \bar{\rho})^{[\mathrm{Gal}_{\mathbb{Q}_p} : \mathrm{Gal}_K]}\right)$

$= 2^f$

relation:

$\bar{\rho}/G_K \overset{\text{Fontaine}}{\rightsquigarrow}$ $(\varphi, \mathbb{Z}_p^\times)$-mod over $\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_q((T))$ $\quad q = p^f$

linear $\quad$ semi-linear.

$\simeq D_{\sigma_0}(\bar{\rho}) \times \cdots \times D_{\sigma_{f-1}}(\bar{\rho})$

each over $\mathbb{F}((T))$, $(\varphi_q, \mathbb{Z}_p^\times)$-module. $\quad \varphi_q = \varphi^f$

$(\varphi, \mathbb{Z}_p^\times)$-mod of $\mathrm{ind}^{\otimes}\bar{\rho} = D_{\sigma_0}(\bar\rho) \underset{\mathbb{F}((T))}{\otimes} \cdots\cdots \otimes D_{\sigma_{f-1}}(\bar\rho)$

$(\varphi, \mathbb{Z}_p^\times)$: diag.

$$\begin{tikzcd}
D_A(\pi_v(\bar\rho)) \arrow[d] & \overset{?}{=} & D_{A,\sigma_0}(\bar\rho) \arrow[d, dashed] & \underset{A}{\otimes} \cdots\cdots \otimes & D_{A,\sigma_{f-1}}(\bar\rho) \arrow[d, dashed] \\
D_{\mathrm{Breuil}}(\pi_v(\bar\rho)) & = & D_{\sigma_0}(\bar\rho) & \underset{\mathbb{F}((T))}{\otimes} \cdots\cdots \otimes & D_{\sigma_{f-1}}(\bar\rho)
\end{tikzcd}$$

find $D_{A,\sigma_0}(\bar\rho)$: $(\varphi_q, \mathcal{O}_K^\times)$-module over A,

need Lubin-Tate $(\varphi, \mathcal{O}_K^\times)$-module.

$$\begin{array}{c} K_\infty \\ | \;\}\; \mathcal{O}_K^\times \\ K \end{array}$$

$\bar\rho \rightsquigarrow D_K(\bar\rho) = (\varphi, \mathcal{O}_K^\times)$-mod over $(\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_q)((T_K))$

$\quad\quad\; \|$

$D_{K,\sigma_0}(\bar\rho) \times \cdots\cdots \times D_{K,\sigma_{f-1}}(\bar\rho)$

each $(\varphi_q, \mathcal{O}_K^\times)$-mod over $\mathbb{F}((T_{K,\sigma_i}))$

$\exists$ embedding
$$\mathbb{F}((T_{K,\sigma_i}^{q-1})) \hookrightarrow A,$$
$$T_{K,\sigma_i}^{q-1} \mapsto \frac{\varphi(Y_{\sigma_i})}{Y_{\sigma_i}}$$

not canonical.

then take $\underline{D_{A,\sigma_i}(\bar\rho) = A \underset{\mathbb{F}((T_{K,\sigma_i}))}{\otimes} D_{K,\sigma_i}(\bar\rho)}$

and $D_A^{\otimes}(\bar\rho) = D_{A,\sigma_0}(\bar\rho) \otimes \cdots\cdots \otimes D_{A,\sigma_{f-1}}(\bar\rho)$

$$\begin{tikzcd} D_A^{\otimes}(\bar\rho) \arrow[d, dashed] \\ D_{\mathrm{Breuil}}(\pi_v(\bar\rho)) \end{tikzcd}$$

local.

Thm: $\boxed{\underline{D_A(\pi_v(\bar\rho))} = D_A^{\otimes}(\bar\rho)}$ $\longleftarrow$ geometric way to construct it.

(under assumption $\bar\rho$ semi-simple, generic)