Mathematic Analysis with Matlab

Lecture 12: Linear Programming

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Outline

Linear Programming: the problem (1)

- Given following problem:
- An oil refinery produces two products: jet fuel and gasoline. The profit for the refinery is 0.10$ per barrel for jet fuel and 0.20$ per barrel for gasoline. The following conditions must be met.
  1. Only 10,000 barrels of crude oil are available for processing;
  2. Government contract requires to produce at least 1,000 barrels of jet fuel;
  3. Private contract requires to produce at least 2,000 barrels of gasoline;
  4. The delivery capacity of the truck fleet is 180,000 barrel-miles;
  5. The jet fuel is delivered to an airfield 10 miles from the refinery;
  6. The gasoline is transported 30 miles to the distributor;
- How to maximize the profit?
- x_1: quantity of jet fuel; x_2: quantity of gasoline
- Target: maximize $0.1 * x_1 + 0.2 * x_2$

Linear Programming: the problem (2)

- x_1: quantity of jet fuel; x_2: quantity of gasoline
- Target: maximize $0.1 * x_1 + 0.2 * x_2$
- Formularize the conditions:
  1. Only 10,000 barrels of crude oil are available for processing;
     - $x_1 + x_2 \leq 10000$
  2. Government contract requires to produce at least 1,000 barrels of jet fuel;
     - $x_1 \geqslant 1000$
  3. Private contract requires to produce at least 2,000 barrels of gasoline;
     - $x_2 \geqslant 2000$

Linear Programming: the problem (3)

- x_1: quantity of jet fuel; x_2: quantity of gasoline
- Target: maximize $0.1 * x_1 + 0.2 * x_2$
- Conditions:
  1. $x_1 + x_2 \leq 10000$
  2. $x_1 \geqslant 1000$
  3. $x_2 \geqslant 2000$
- Formularize the conditions:
  4. The delivery capacity of the truck fleet is 180,000 barrel-miles;
  5. The jet fuel is delivered to an airfield 10 miles from the refinery;
  6. The gasoline is transported 30 miles to the distributor;

  - $10 * x_1 + 30 * x_2 \leq 180000$

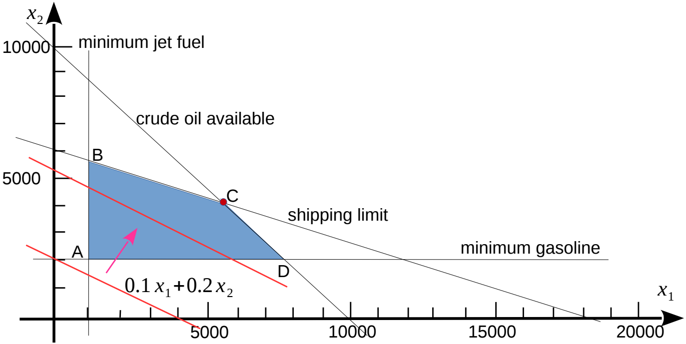
Linear Programming: the complete model

- x_1: quantity of jet fuel; x_2: quantity of gasoline
- Target: maximize $0.1 * x_1 + 0.2 * x_2$
- Conditions:
  1. $x_1 + x_2 \leq 10000$
  2. $x_1 \geqslant 1000$
  3. $x_2 \geqslant 2000$
  4. $10 * x_1 + 30 * x_2 \leq 180000$
- The formal linear programming form:

$$\text{maximize } 0.1 * x_1 + 0.2 * x_2$$
$$\text{subject to} \begin{cases} x_1 + x_2 \leq 10000 \\ x_1 \geqslant 1000 \\ x_2 \geqslant 2000 \\ 10 * x_1 + 30 * x_2 \leq 180000 \end{cases} \tag{1}$$

Linear Programming: solve the problem with graph

$$
\text{maximize } 0.1 * x_1 + 0.2 * x_2
$$
$$
\text{subject to} \begin{cases} x_1 + x_2 \leq 10000 \\ x_1 \geqslant 1000 \\ x_2 \geqslant 2000 \\ 10 * x_1 + 30 * x_2 \leq 180000 \end{cases} \tag{2}
$$



Outline

1. Linear Programming: the problem

2. Linear Programming: solve LP by Matlab

Linear Programming: the standard form

$$
\begin{aligned}
&\text{minimize } f(x), x \in R^n \\
&\text{s.t.} \quad \begin{aligned} A \cdot x &\preceq b \\ A_e \cdot x &= b_e \\ lb \preceq x &\preceq ub \end{aligned}
\end{aligned}
\tag{3}
$$

- 'Maximize problem' can be converted to 'minimize problem'
- $Ax \preceq b$ covers all inequalities
- $A_e x = b_e$ covers all equalities
- lb and ub are the lower and upper bounds for x respectively
- Observations:
  1. The target is a linear function
  2. All conditions are linear
  3. The region scoped by all conditions is **convex**
  4. Target function must be **convex** too!!

Linear Programming: matlab commands

$$\text{minimize } f(x), x \in R^n \\ \text{s.t.} \quad \begin{aligned} A \cdot x &\preceq b \\ A_e \cdot x &= b_e \\ lb \preceq x &\preceq ub \end{aligned} \tag{4}$$

- x=linprog(f, A, b, A_e, b_e, lb, ub);
- x=linprog(f, A, b, A_e, b_e, lb, ub, x0);
    - **x0**: initial value for x
- [x, fval]=linprog(f, A, b, A_e, b_e, lb, ub, x0);
    - **fval**: the optimal value f(x)

Linear Programming: solve "oil production problem" (1)

$$
\text{maximize } 0.1 * x_1 + 0.2 * x_2
$$
$$
\text{subject to } \begin{cases} x_1 + x_2 \leq 10000 \\ x_1 \geqslant 1000 \\ x_2 \geqslant 2000 \\ 10 * x_1 + 30 * x_2 \leq 180000 \end{cases} \tag{5}
$$

- Regularize above problem to:

$$
\text{minimize } -0.1 * x_1 - 0.2 * x_2
$$
$$
\text{subject to } \begin{cases} x_1 + x_2 \leq 10000 \\ 10 * x_1 + 30 * x_2 \leq 180000 \\ x_1 \geqslant 1000 \\ x_2 \geqslant 2000 \end{cases} \tag{6}
$$

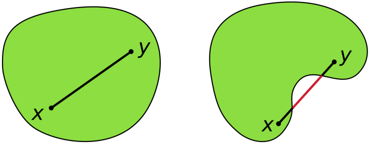
Linear Programming: solve "oil production problem" (2)

$$\text{minimize } -0.1*x_1-0.2*x_2$$
$$\text{subject to } \begin{cases} x_1+x_2\leq 10000 \\ 10*x_1+30*x_2\leq 180000 \\ x_1\geqslant 1000 \\ x_2\geqslant 2000 \end{cases} \tag{7}$$

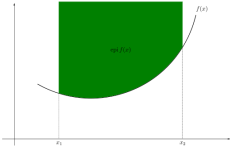
1. $f=-0.1*x_1-0.2*x_2$
2. A=[1 1;10 30];
   b=[10000;180000]
3. lb=[1000;2000]; ub=[]

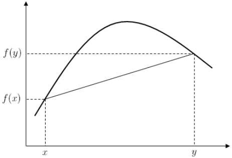
- Steps:
  1. syms x1 x2;
  2. c=[-0.1,-0.2];
  3. A=[1 1;10 30];
  4. b=[10000;180000]
  5. Ae=[]; be=[];
  6. lb=[1000;2000]; ub=[];
  7. x=linprog(c, A, b,Ae, be, lb, ub);
- Output: x=[6000; 4000]

- Convex set



- Convex function





1. defined on a convex set
2. function should be convex/concave on this set

$$\begin{bmatrix} \frac{\partial^2 F}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 F}{\partial x_1 \partial x_n} \\ \frac{\partial^2 F}{\cdots} & \cdots & \frac{\partial^2 F}{\cdots} \end{bmatrix}$$

Linear Programming: example 2

$$\text{maximize } 2*x_1+3*x_2-5*x_3$$
$$\text{s.t.} \begin{cases} x_1+x_2+x_3=7 \\ 2*x_1-5*x_2+x_3 \geqslant 10 \\ x_1,x_2,x_3 \geqslant 0 \end{cases} \tag{9}$$

1. $f=-2*x_1-3*x_2+5*x_3$
2. A=[-2 5 -1]; b=[-10]
3. Ae=[1 1 1]; be=[7]
4. lb=[0;0;0]; ub=[]

- Steps:
  1. syms x1 x2 x3;
  2. c=[-2 -3 5];
  3. A=[-2 5 -1];
  4. b=[-10]
  5. Ae=[1 1 1]; be=[7];
  6. lb=[0;0;0]; ub=[];
  7. x=linprog(c, A, b,Ae, be, lb, ub);
- Output: x=[6.4286; 0.5714;0]

Linear Programming: transportation problem (1)

1. Factories **A**, **B** and **C** are going to buy raw materials from X and Y
2. A: **17** tons; B: **18** tons; C: **15** tons
3. Factories **X** and **Y** are going to supply
4. Capacity of X: **23** tons; Capacity of Y: **27**
5. The transportation costs are shown in the table

Table: Transportation costs (yuan/ton)

	A	B	C
X	50	60	70
Y	60	110	160

1. Question: how to arrange the transaction to minimize the transportation cost?

Linear Programming: transportation problem (2)

- Target function

  $$\min 50 * x_A + 60 * x_B + 70 * x_C + 60 * y_A + 110 * y_B + 160 * y_C$$

- The constraints

$$\begin{cases} x_A + y_A = 17 \\ x_B + y_B = 18 \\ x_C + y_C = 15 \\ x_A + x_B + x_C = 23 \\ y_A + y_B + y_C = 27 \\ x_i \geq 0, y_i \geq 0 \end{cases} \tag{10}$$

- The answer: **[0, 8, 15, 17, 10, 0]**

Linear Programming: transportation problem (3)

```
function r=transport()
 f   = [50,60,70,60,110,160]
 a   = [];
 b   = [];
 aeq = [1 0 0 1 0 0;0 1 0 0 1 0;0 0 1 0 0 1;1 1 1 0 0 0;0 0 0 1 1 1];
 beq = [17 18 15 23 27]'
 lub = [];
 r   = linprog(f,a, b, aeq, beq, zeros(6,1)',lub);
end
```

Linear Programming: product profit problem (1)

- **Problem**: A company manufactures two products (A and B) and the profit per unit sold is 3$ and 5$ respectively. Each product has to be assembled on a particular machine, each unit of product A taking 12 minutes of assembly time and each unit of product B 25 minutes of assembly time. The company estimates that the machine used for assembly has an effective working week of only 30 hours
- **Constraint**: every five units of product A produced at least two units of product B must be produced
- Question: how many units of A and B should be produced to maximize the profit?

Linear Programming: product profit problem (2)

- Profit: 3 for A, 5 for B
- Time costs: 12m for every A, 25m for every B
- Constraint 1: 30 hours available
- Constraint 2: at least 2 units of B should be poroduced when 5 units of A are produced
- Any other **implicit** constraints??
- Intepretation of **Constraint 2:** $\frac{x_A}{x_B} \leq \frac{5}{2}$
- Try your best to organize them into standard linear programming form

Linear Programming: product profit problem (3)

- Target function

$$\max.\ 3x_A + 5x_B$$

- The constraints

$$\begin{cases} 12x_A + 25y_A \le 1800 \\ 2x_A - 5x_B \le 0 \\ x_A \ge 0, x_B \ge 0 \end{cases}$$

- Is this the standard form?

Linear Programming: product profit problem (4)

- Target function

$$\text{min.} \quad -3x_A - 5x_B$$

- The constraints

$$\begin{cases} 12x_A + 25y_A \leq 1800 \\ 2x_A - 5x_B \leq 0 \\ x_A \geq 0, x_B \geq 0 \end{cases}$$

Linear Programming: product profit problem (5)

```
function r=products()
f = [-3, -5]
a = [12 25;2 -5];
b = [1800 0];
aeq = [];
beq = [];
ub = [];
lb = zeros(1,2);
r=linprog(f,a, b, aeq, beq, lb, ub);
end
```

- Answers: [81.8182, 32.7273]

Scientists behind Linear Programming:

(a) George B. Dantzig (1914 -2005)

(b) John Von Neumann (1903 - 1957)

Q & A

Thanks for your attention!