Mathematic Analysis with Matlab

Lecture 8: Vector, Matrix and Determinant

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Outline

Commands for Operations on vector and matrix (1)

- Operations on vector
  1. **linspace(a, b, n)**: returns vector of n elements in range [a, b]
  2. **cross(a,b)**: cross product between vectors a and b
  3. **dot(a, b)**: dot product between vectors a and b, same as 'sum(a.*b)'
- Operations on matrix
  1. Define a matrix: $M = [a_{11} a_{12} a_{13}; a_{21} a_{22} a_{23}]$
  2. Define $m \times n$ matrix whose entries are all zeros: **zeros(m, n)**;
  3. Define n order identity matrix: **eye(n)**
  4. Define diagonal matrix: **v=[1 2 3 4]; diag(v)**
  5. Define random matrix: **rand(m, n)**
  6. Define random matrix following N(0,1) distribution: **randn(n)**
  7. Define n order magic matrix: **magic(n)**
  8. Define Hilbert matrix: **hilb(n)**
  9. Define invered Hilbert matrix: **invhilb(n)**
  10. Define Vandermonde matrix: **v=[1 2 3 4]; vander(v)**

Commands for Operations on vector and matrix (2)

- Operations on matrix (continued)
  1. A+B: matrix A plus B
  2. A' or transpose(A): A^T
  3. k*A: matrix A is multiplied by k
  4. A*B: matrix multiplication
  5. A/B: $A * B^{-1}$
  6. inv(A): A^{-1}
  7. det(A): derterminant of A, i.e. $|A|$

Simple exercies on matrix basic operations

- Try following operations:
- Special matrices:
  1. h=hilb(4);
  2. h1=inv(h);
  3. m=magic(4);
  4. i=eye(4);
  5. v=[1 2 3 4 5]
  6. M=vander(v);
  7. M=M';

- Hilbert matrix: $h(i,j) = \frac{1}{i+j-1}$
- Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 & x_5^3 \\ x_1^4 & x_2^4 & x_3^4 & x_4^4 & x_5^4 \end{bmatrix} \tag{1}$$

Determinant of a matrix

- Given Vandermonde matrix:

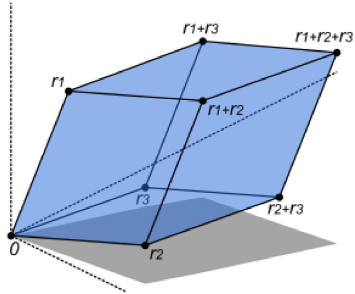
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 & x_5^3 \\ x_1^4 & x_2^4 & x_3^4 & x_4^4 & x_5^4 \end{bmatrix} \tag{2}$$

- work out its derminant.

```
clear;
syms x1 x2 x3 x4 x5 real; %Matlab 2012b or later
v=[x1 x2 x3 x4 x5];
A=vander(v)'
B=flipud(A) %flip up and down
det(B)
factor(det(B)) %factorize the result, Matlab 2012b or later

```

What's the determinant for?



- The volume of this parallelepiped is the absolute value of the determinant of the matrix formed by the rows constructed from the vectors r1, r2, and r3
- So it is reasonable that the determinant of non squared matrix is 0

Inverse operation on matrix

- Given matrices:

$$A = \begin{bmatrix} 3 & 0 & 4 & 4 \\ 2 & 1 & 3 & 3 \\ 1 & 5 & 3 & 4 \\ 1 & 2 & 1 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 3 & 2 \\ 7 & 1 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 2 \end{bmatrix}$$

- $A^{-1}B$=?
- Steps:
  1. clear;
  2. A=[3 0 4 4;2 1 3 3;1 5 3 4;1 2 1 5];
  3. B=[0 3 2;7 1 3;1 3 3;1 2 2];
  4. inv(A)*B

Solve linear equations

- Given linear equations:

$$A = \begin{cases} 3x + 2y + z = 7 \\ x - y + 3z = 6 \\ 2x + 4y - 4z = 2 \end{cases}$$

- solve the equations.

- Cases:
  1. det(A)$=0$: no solution
  2. det(A)$\neq 0$: unique solution

```
clear;
A=[3 2 1;1 -1 3;2 4 -4];
b=[7;6;-2];
det(A)  %check determinant
inv(A)*b

```

Two Interpretations about $Ax = b$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$\begin{aligned} \alpha_1 x &= b_1 \\ \alpha_2 x &= b_2 \\ \alpha_2 x &= b_3 \\ \alpha_2 x &= b_4 \end{aligned}$$

$$\beta_1^T x_1 + \beta_2^T x_2 + \beta_3^T x_3 + \beta_4^T x_4 = b$$

- Different interpretations have different physical meaning
- Should not make a mixture use

Outline

1. Basic Operations on Matrix

2. Rank and Reduced Row Echelon Form of a Matrix

3. Solve Linear Equations

Commands for rank and Reduced Row Echelon Form

- Calculate the rank for matrix
  1. rank(A): generates rank for A
  2. rref(A): produces reduced row echelon form for A;

$$A = \begin{bmatrix} 3 & 0 & 4 & 4 \\ 2 & 1 & 3 & 3 \\ 1 & 5 & 3 & 4 \\ 1 & 2 & 1 & 5 \end{bmatrix} \qquad rref(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

[Explained] $[A^{(1)} = \alpha_1 - 3\alpha_4; \alpha_2 - 2\alpha_4; \alpha_3 - \alpha_4; \alpha_4 - \alpha_4]$

$$A = \begin{bmatrix} \alpha_1 = & 3 & 0 & 4 & 4 \\ \alpha_2 = & 2 & 1 & 3 & 3 \\ \alpha_3 = & 1 & 5 & 3 & 4 \\ \alpha_4 = & 1 & 2 & 1 & 5 \end{bmatrix} \qquad A^{(1)} = \begin{bmatrix} 3 & 0 & 4 & 4 \\ 2 & 1 & 3 & 3 \\ 1 & 5 & 3 & 4 \\ 1 & 2 & 1 & 5 \end{bmatrix} \times T$$

Obtain rank for a matrix

- Given matrix:

$$A = \begin{bmatrix} 3 & 2 & -1 & -3 & -2 \\ 2 & -1 & 3 & 1 & -3 \\ 7 & 0 & 5 & -1 & -8 \end{bmatrix}$$

- Steps
  1. M=[3 2 -1 -3 -2;2 -1 3 1 -3;7 0 5 -1 -8]
  2. rank(M)

Obtain rank for a matrix (1)

- Given matrix:

$$A = \begin{bmatrix} 3 & 2 & -1 & -3 \\ 2 & -1 & 3 & 1 \\ 7 & 0 & t & -1 \end{bmatrix}$$

- its rank is 2, t=?

Obtain rank for a matrix (2)

- Given matrix:

$$A = \begin{bmatrix} 3 & 2 & -1 & -3 \\ 2 & -1 & 3 & 1 \\ 7 & 0 & t & -1 \end{bmatrix}$$

- its rank is 2, t=?

```
syms t;
M = [3 2 -1 -3;2 -1 3 1;7 0 t -1];
a = det(M(1:3,1:3));
solve(a);
```

Obtain rank for a matrix by ‘rref()’

- Given matrix:

$$A = \begin{bmatrix} 2 & -3 & 8 & 2 \\ 2 & 12 & -2 & 12 \\ 1 & 3 & 1 & 4 \end{bmatrix}$$

- Rank(A)=?
- Steps:
  1. M=[2 -3 8 2;2 12 -2 12;1 3 1 4]
  2. A=rref(M);

Inverse matrix by 'rref()'

- Given matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 4 & 3 \end{bmatrix}$$

- A^{-1}=?
- Steps
  1. M=[1 2 3;2 2 1;3 4 3];
  2. I=eye(3);
  3. AE=[M I];
  4. IA=rref(AE);
  5. IA=IA(:,[4,5,6]);
  6. or: IA(:,[1,2,3])=[];

Judge whether vectors are linearly dependent

- Given vectors: $\alpha_1 = (1, 1, 2, 3)$, $\alpha_2 = (1, -1, 1, 1)$,
- $\alpha_3 = (1, 3, 4, 5)$, $\alpha_4 = (3, 1, 5, 7)$
- whether they are linearly independent?

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & -1 & 1 & 1 \\ 1 & 3 & 4 & 5 \\ 3 & 1 & 5 & 7 \end{bmatrix}$$

- Steps
  1. M=[1 1 2 3;1 -1 1 1;1 3 4 5;3 1 5 7];
  2. A=rref(M);

Judge whether vectors are linearly dependent

- Given vectors: $\alpha_1 = (1, -1, 2, 4)$, $\alpha_2 = (0, 3, 1, 2)$,
- $\alpha_3 = (3, 0, 7, 14)$, $\alpha_4 = (1, -1, 2, 0)$, $\alpha_5 = (2, 1, 5, 0)$
- whether they are linearly independent?

$$A = \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 3 & 1 & 2 \\ 3 & 0 & 7 & 14 \\ 1 & -1 & 2 & 0 \\ 2 & 1 & 5 & 0 \end{bmatrix}$$

- Steps
  1. A=[1 -1 2 4;0 3 1 2;3 0 7 14;1 -1 2 0;2 1 5 0];
  2. B=A';
  3. rref(B)
  4. $\alpha_3 = 3\alpha_1 + \alpha_2$
  5. $\alpha_5 = -\frac{1}{2}\alpha_1 + \alpha_2 + \frac{5}{2}\alpha_4$

Judge whether two groups of vectors are equivalent

- Given vector group A: $\alpha_1 = (2, 1, -1, 3)$, $\alpha_2 = (3, -2, 1, -2)$,
- group B: $\beta_1 = (-5, 8, -5, 12)$, $\beta_2 = (4, -5, 3, -7)$
- whether they are equivalent?
- Steps:
  1. A=[2 1 -1 3; 3 -2 1 -2];
  2. B=[-5 8 -5 12; 4 -5 3 -7];
  3. rref(A)
  4. rref(B)
- A can be represented by linear combination of vectors from B
- B can be represented by linear combination of vectors from A

Outline

Commands for solving linear equations

- Solve Linear Equations:
  1. null(A): solve Ax=0
  2. $A\backslash b$: solve Ax=b
     - if A (square matrix) is invertible, $A\backslash b$ outputs the unique solution
     - if rank(A) is less than number of variables, $A\backslash b$ returns one particular solution
     - if Ax=b is not solvable, $A\backslash b$ returns an approximate solution (minimize square error)
  3. rref([A, b])
  4. solve('exp1','exp2',...,'x1','x2',...);

Solve Linear Equations (1)

- Solve linear equations:

$$\begin{cases} x_1 + x_2 - 2x_3 - x_4 = 0 \\ 3x_1 - 2x_2 - 3x_3 + 2x_4 = 0 \\ 5x_2 + 7x_3 + 3x_4 = 0 \\ 2x_1 - 3x_2 - 5x_3 - x_4 = 0 \end{cases}$$

- Cases:
  1. Det(A)=0: multiple solutions
  2. Det(A)≠0: 1 solution (zero vector)

- Steps
  1. clear;
  2. A=[1 1 -2 -1;3 -2 -3 2;0 5 7 3; 2 -3 -5 -1];
  3. det(A)
  4. x=null(A)
  5. Or: A=sym(A);
  6. x=null(A)

Solve Linear Equations: particular solution

- Solve linear equations:

$$\begin{cases} x_1 + x_2 - 2x_3 - x_4 = 4 \\ 3x_1 - 2x_2 - x_3 + 2x_4 = 2 \\ 5x_2 + 7x_3 + 3x_4 = -2 \\ 2x_1 - 3x_2 - 5x_3 - x_4 = 4 \end{cases}$$

- Steps
  1. clear;
  2. A=[1 1 -2 -1;3 -2 -1 2;0 5 7 3; 2 -3 -5 -1];
  3. D=det(A)
  4. b=transpose([4 2 -2 4]);
  5. rank([A b])
  6. format rat
  7. rref([A b])
  8. Or: linsolve(A,b);

Solve Linear Equations: general solution (1)

- Solve linear equations:

$$\begin{cases} x_1 - x_2 + 2x_3 + x_4 = 1 \\ 2x_1 - x_2 + x_3 + 2x_4 = 3 \\ x_1 - x_3 + x_4 = 2 \\ 3x_1 - x_2 + x_4 = 5 \end{cases}$$

- Steps
  1. clear;
  2. A=[1 -1 2 1;2 -1 1 2;1 0 -1 1; 3 -1 0 3];
  3. b=([1 3 2 5])'
  4. A=sym(A); b=sym(b);
  5. x0=$A\backslash b$;
  6. null(A)
- The general solution is: $(2, 1, 0, 0)^T + k_1(1, 3, 1, 0)^T + k_2(-1, 0, 0, 1)^T$

Solve Linear Equations: general solution (2)

- Given linear equations:

$$\begin{cases} ax_1 + x_2 + x_3 = 1 \\ x_1 + ax_2 + x_3 = 1 \\ x_1 + x_3 + ax_4 = 1 \end{cases}$$

- Discuss the equations lead to no solution, one solution, or multiple solution when a is set to different values
- Steps:
  1. clear; syms a;
  2. A=[a 1 1; 1 a 1;1 1 a];
  3. D=det(A)
  4. a=solve('a^3-3*a+2=0')
- Given $a \neq -2$ or $a \neq 1$,
  1. clear;
  2. [x y z]=solve('a*x+y+z=1','x+a*y+z=1','x+y+a*z=1');

Solve Linear Equations: general solution (3)

- Given linear equations:

$$\begin{cases} ax_1 + x_2 + x_3 = 1 \\ x_1 + ax_2 + x_3 = 1 \\ x_1 + x_3 + ax_4 = 1 \end{cases}$$

- Discuss the equations lead to no solution, one solution, or multiple solutions when a is set to different values

Solve Linear Equations: general solution (4)

- Given linear equations:

$$\begin{cases} ax_1 + x_2 + x_3 = 1 \\ x_1 + ax_2 + x_3 = 1 \\ x_1 + x_3 + ax_4 = 1 \end{cases}$$

- Discuss the equations lead to no solution, one solution, or multiple solutions when a is set to different values

set a=2;

```
clear;
[x y z]=solve('-2*x+y+z=1','x-2*y+z=1','x+y-2*z=1');
```

set a=1

```
    clear;
    [x y z]=solve('x+y+z=1','x+y+z=1','x+y+z=1');
```

Linear representation of one vector by others (1)

- Given $\beta = (2, -1, 3, 4)$,
- Can it be represented by $\alpha_1 = (1, 2, -3, 1)$, $\alpha_2 = (5, -5, 12, 11)$,
- $\alpha_3 = (1, -3, 6, 3)$

$$x_1\alpha_1^T + x_2\alpha_2^T + x_3\alpha_3^T = \beta^T \tag{3}$$

Linear representation of one vector by others (2)

- Given $\beta = (2, -1, 3, 4)$,
- Can it be represented by $\alpha_1 = (1, 2, -3, 1)$, $\alpha_2 = (5, -5, 12, 11)$,
- $\alpha_3 = (1, -3, 6, 3)$

$$x_1\alpha_1^T + x_2\alpha_2^T + x_3\alpha_3^T = \beta^T \tag{4}$$

```
clear;
A=transpose([1 2 -3 1;5 -5 12 11;1 -3 6 3]);
B=transpose([2 -1 3 4]);
format rat;
A\B
```

Linear orthogonal vector to a group of vectors (1)

- Given vectors $\alpha_1 = (1, 1, -1, 1)$, $\alpha_2 = (1, -1, -1, 1)$ and $\alpha_3 = (2, 1, 1, 3)$
- Find out a vector $\beta = (x_1, x_2, x_3, x_4)$ that is orthogonal to α_1, α_2 and α_3

$$\begin{cases} \alpha_1 \beta^t = 0 \\ \alpha_2 \beta^t = 0 \\ \alpha_3 \beta^t = 0 \end{cases}$$

Linear orthogonal vector to a group of vectors (2)

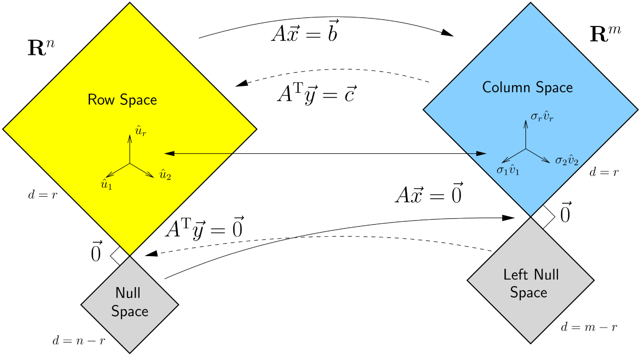
- Given vectors $\alpha_1 = (1, 1, -1, 1)^T$, $\alpha_2 = (1, -1, -1, 1)^T$ and $\alpha_3 = (2, 1, 1, 3)^T$
- Find out a vector $\beta = (x_1, x_2, x_3, x_4)$ that is orthogonal to α_1, α_2 and α_3

$$\begin{cases} \alpha_1 \beta^t = 0 \\ \alpha_2 \beta^t = 0 \\ \alpha_3 \beta^t = 0 \end{cases}$$

```
A = [1 1 -1 1;1 -1 -1 1;2 1 1 3];
format rat;
b = null(A);
```

A summary

- Four subspaces for a matrix $A^{m \times n}$
  1. Null space of A
  2. Left null space of A, viz. null(A^t)
  3. Spanning space of A
  4. Left span space of A, viz. span(A^t)



Q & A

Thanks for your attention!