Mathematic Analysis with Matlab

Lecture 7: Infinite Series and Differential Equations

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Outline

Commands for Operations on Series (1)

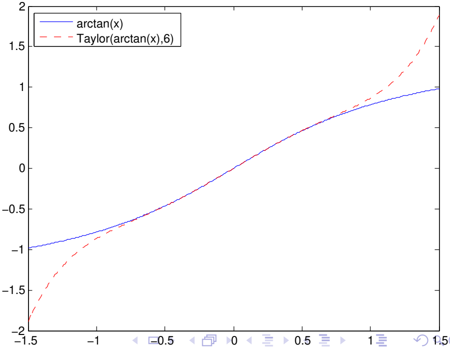
- Summation on series with variable
  1. symsum(S(k)): return summation from 0 to k-1
  2. symsum(S(k),v): substitute k by v and sum series from 0 to v-1
  3. symsum(S(k), a, b): sum S(k) that k changes from a to b
  4. symsum(S(k), v, a, b): substitute k by v, and sum series from a to b
- Taylor expansion and Maclaurin expansion
  1. taylor(f, x): 5 order Maclaurin expansion
  2. taylor(f, x, 'Order', n): n-1 order Taylor expansion
  3. taylor(f, x, 'ExpansionPoint',a, 'Order', n): n-1 order Taylor expansion at a
- Given f(x) is n order differentiable in [a,b] and n+1 order differentiable in (a, b), $x \in [a, b]$

$$f(x) = \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^n(a)}{n!}(x-a)^n + R_n(x)$$

Taylor Expansion

- Calculate 6 order Taylor expansion for cos(x)
- Step:
  1. syms x;
  2. ser1=taylor(cos(x), 'ExpansionPoint', 0 , 'Order', 7);
- Output: -x^6/720 + x^4/24 - x^2/2 + 1;

- Calculate 5 order Taylor expansion for arctan(x):
  1. syms x;
  2. ser2=taylor(atan(x), x, 6);
- Output: x^5/5 - x^3/3 + x
- plot the figure:
  1. x=-1.5:0.01:1.5;
  2. y1=atan(x);
  3. y2=x.^5/5 - x.^3/3 + x
  4. plot(x, y1,'b', x, y2, 'r--');



Fourier Series (1)

- Fourier Transform

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cos\frac{n\pi x}{L} + b_n sin\frac{n\pi x}{L}),$$

$$\text{where } a_n = \frac{1}{L}\int_{-L}^{L} f(x) cos\frac{n\pi x}{L} dx,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) sin\frac{n\pi x}{L} dx,$$

- Let's have a simple experiment

```
x  = 0:0.05:2*pi;
a  = cos(x);
b  = sin(x);
c  = sin(2*x);
r1 = a*b'
r2 = a*c'
```

Fourier Series (2)

- Let's have a simple experiment

```
x  = 0:0.05:2*pi;
a  = cos(x);
b  = sin(x);
c  = sin(2*x);
r1 = a*b'
r2 = a*c'
```

- if inner product between a and b is close to 0, it means what???

Fourier Series: exercise (1)-1

- Given f(x):

$$f(x) = \begin{cases} 1, & 0 \le x < 1 \\ -x, & -1 \le x < 0 \end{cases} \tag{1}$$

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cos\frac{n\pi x}{L} + b_n sin\frac{n\pi x}{L}),$$

$$\text{where } a_0 = \int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)cos\frac{n\pi x}{L}dx,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)sin\frac{n\pi x}{L}dx,$$

- Work out its Fourier series

Fourier Series: exercise (1)-2

- Given f(x):

$$f(x) = \begin{cases} x, & 0 \leq x < 1 \\ -x, & -1 \leq x < 0 \end{cases} \quad (2)$$

1. Work out a_0, a_n and b_n

$$a_0 = \int_{-L}^{L} f(x)dx, a_n = \frac{1}{L}\int_{-L}^{L} f(x)cos\frac{n\pi x}{L}dx, b_n = \frac{1}{L}\int_{-L}^{L} f(x)sin\frac{n\pi x}{L}dx$$

1. Plot f(x) in the range [-1, 3]
2. Plot g(x) in the range [-1, 3]

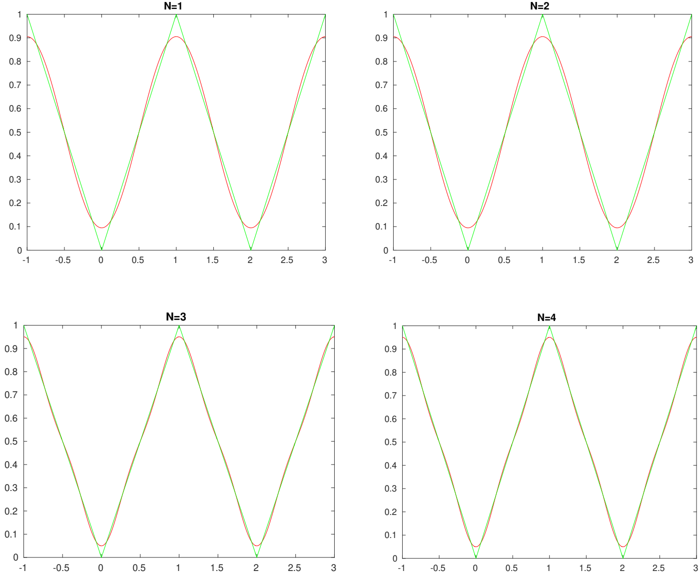
Fourier Series: exercise (1)-3

```matlab
function y=four1exp(lb,
     ub)
 y = []
 for x = lb:0.01:ub
     %filling your codes
 end
 x=lb:0.01:ub;
 %plot(x, y,'r');
end
```

```matlab
four2exp()
 syms x;
 a0  = int(-x,x,-1,0) + int(x,x,0,1);
 for k=1:N
   ak = ?;
   ak = ?;
   bk = ?;
   bk = ?;
   sk = ?;
 end

%filling your code here
plot(x1, y1, 'r',x1, y2,'g');
end
```

Fourier Series: exercise (1)-4



Outline

Commands for Solving Differential Equations: closed (analytic) form

- Solving **Odinary** Differential Equation
  1. dsolve('eq1,eq2,...','cond1,cond2,...','v')
  2. dsolve('Dx=-a*x');
  3. x=dsolve('Dx=-a*x','x(0)=1','s');
  4. w=dsolve('D3w=-w','w(0)=1,Dw(0)=0,D2w(0)=0');
  5. [f,g]=dsolve('df=f+g','Dg=-f+g','f(0)=1', 'g(0)=2');
- Solving Differential Equations: numerical form
  1. [T,Y]=ode23(odefun,tspan,y0);
  2. [T,Y]=ode45(odefun,tspan,y0);

Solving Differential Equation (1)

- Given differential equation:

$$y'' + y' - 2y = 0 \tag{3}$$

- Solving Differential Equation
  1. y=dsolve('D2y+D1y-2*y', 'x');
- Output: y=C1*exp(x)+C2*exp(-2*x);
- Given differential equation:

$$y' + 2xy = xe^{-x^2} \tag{4}$$

- Solving Differential Equation
  1. y=dsolve('Dy+2*x*y=x*exp(-x^2)', 'x');
- Output: y=(1/2*x^2+C1)*exp(-x^2);

Solving Differential Equation (2)

- Given differential following equation and $y\mid_{x=1}= 2e$:

$$xy' + y - e^{-x} = 0 \tag{5}$$

- Solving Differential Equation
  1. clear;
  2. f=dsolve('x*Dy+y-exp(-x)=0', 'y(1)=2*exp(1)','x');
- Output: (exp(-1) - exp(-x) + 2*exp(1))/x;
- Given differential equation:

$$y'' - 2y' + 5y = xe^{x}cos(2x) \tag{6}$$

- Solving Differential Equation
  1. clear;
  2. f=dsolve('D2y-2*Dy+5*y=x*exp(x)*cos(2*x)', 'x');
  3. f=simplify(f);
- Output: f=(exp(x)*(cos(2*x) + 2*x*sin(2*x) + 8*C2*cos(2*x) + 8*C3*sin(2*x)))/8;

Solving Differential Equations

- Given differential following equations:

$$\begin{cases} \frac{dx}{dt} + x + 2y = e^t \\ \frac{dy}{dt} - x - y = 0, \end{cases} \quad (7)$$

- under initial condition:

$$\begin{cases} x\big|_{t=0} = 1 \\ y\big|_{t=0} = 0 \end{cases}$$

- Steps:
  1. clear;
  2. [x,y]=dsolve('Dx=-x-2*y+exp(t)', 'Dy=x+y', 'x(0)=1','y(0)=0');
- Output: x=cos(t)
- y=exp(t)/2 - cos(t)/2 + sin(t)/2

Numerical Solution: Euler Method (1)

- Given differential following equation:

$$y' = f(x, y), \quad f(x_0) = y_0 \tag{8}$$

- Solve the differential equation by Euler Method
- f(x) is approximated as:

$$L(x) = f(x_0) + f'(x_0)(x - x_0) \tag{9}$$

- Think about Taylor expansion:

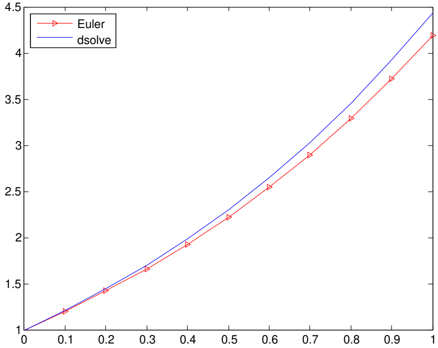
$$f(x) = \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^n(a)}{n!}(x-a)^n + R_n(x)$$

Numerical Solution: Euler Method (2)

- Given y'=f(x,y), the general steps of Euler Method:
  1. $x_1 = x_0 + dx$, $y_1 = y_0 + f(x_0, y_0)dx$
  2. $x_2 = x_1 + dx$, $y_2 = y_1 + f(x_1, y_1)dx$
  3. ...
  4. $x_n = x_{n-1} + dx$, $y_n = y_{n-1} + f(x_{n-1}, y_{n-1})dx$

Numerical Solution: Euler Method (3)

```
clear;clf;
f1 = [];
 y = 1;
f1 = [f1;y];
for x = 0.1:0.1:1
  y  = y+(1+y)*0.1;
  f1 =[f1; y];
end
f2 = dsolve('Dy=1+y','y(0)=1','x');
y2 = [];
for x1 = 0:0.1:1
   x = x1;
  y2 = [y2; eval(f2)];
end
x=[0:0.1:1];
plot(x, f1,'b-',x, y2, 'r->');
```



Numerical Solution: Runge-Kutta Method (1)

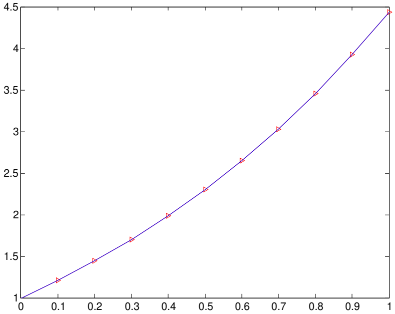
Carl Runge (1856-1927)

Martin W. Kutta (1867-1944)

- Given y'=f(x,y), $y(x_0) = y_0$ the general steps of Runge-Kutta method:
  1. h=dx
  2. $y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
  3. where $k_1 = f(x_n, y_n)$
  4. $k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1)$
  5. $k_3 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2)$
  6. $k_4 = f(x_n + h, y_n + hk_3)$
- Comments: the idea is to take average on different tangent values

Numerical Solution: Runge-Kutta Method (2)

```
clear; clf;
  x = 0:0.1:1;
  h = 0.1;
  y = 1;
f1 = [];
f1 = [f1; y];
for i = 1:1:(length(x)-1)
  k1 = ?; k2 = ?;
  k3 = ?;
  k4 = ?;
  y  = ?;
  f1 = [f1; y];
end
f2=dsolve('Dy=1+y', 'y(0)=1', 'x');
y2=[];
for x = 0:0.1:1
  y2 = [y2; eval(f2)];
end
x=[0:0.1:1];
plot(x, f1,'r->',x, y2, 'b-');
```



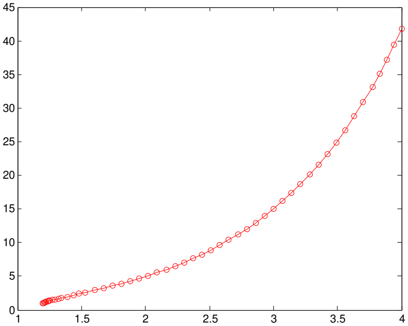
Numerical Solution: Runge-Kutta Method (4)

- Given differential equation:

$$(1 + xy)y + (1 - xy)y' = 0, y|_{x=1.2} = 1 \quad (10)$$

- Work out the approximate solution in range [1.2, 4].

- Steps:
  1. fun=inline('(1+x*y)*y/(x*y-1)','x','y')
  2. [x, y]=ode45(fun, [1.2,4],1);
  3. plot(x,y);



Numerical Solution: Runge-Kutta Method (5)

- Solve following differential equation in range [0, 20]

$$y'' - (1 - y^2)y' + y = 0, y\mid_{x=0} = 20, y'|_{x=0} = -0.5 \qquad (11)$$

- Given $y_1 = y$, $y_2 = \frac{dy}{dx}$, plug into Eqn. 11, we have

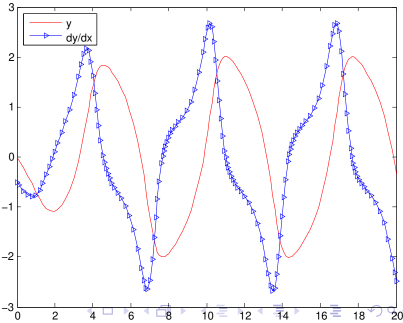
$$\begin{cases} \frac{dy_1}{dx} = y_2 \\ \frac{dy_2}{dx} = (1 - y_1^2)y_2 - y_1 \end{cases}$$

- Define a function:

```
1 function dydt = rungekutta(t, dy)
2    dydt = [dy(2); (1 - dy(1)^2)*dy(2) - dy(1)];
3 end
```

- Run following commands:

```
1 [t, y] = ode23(@rungekutta, [0 20], [0 -0.5]);
2 plot(t,y(:,1),'r-',t,y(:,2),'b->');
```



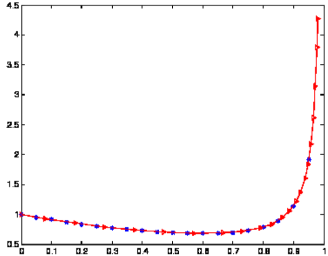
Exercise (2)-1

- Solve differential equation
- Under initial condition $y(0) = 1$

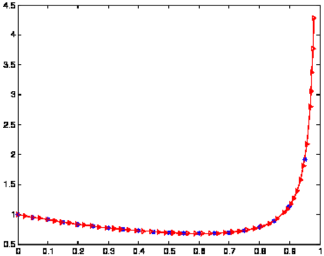
$$(x^2 - 1)y' + 2xy - cos(x) = 0 \tag{12}$$

- Work out the analytic solution and approximate solution $(0 \leq x < 1)$
- Plot out the answers on the same figure

Exercise (2)-3



(a) ode23



(b) ode45

Exercise (2)-4

```
clear;clf;
h     =  0.01; y  = 1;
f1    = [];    f1 = [f1;y];
dfstr = 'dy = 1+y+x';
for x = h:h:0.99
  k1 = df1(x, y);
  k2 = df1(x+h/2, y+h*k1/2);
  k3 = df1(x+h/2, y+h*k2/2);
  k4 = df1(x+h,   y+h*k3);
  y  = y + (k1 + 2*k2 + 2*k3 + k4)*h/6;
  f1 = [f1; y];
end
f2 = dsolve('Dy=1+y+x', 'y(0)=1', 'x');
y2 = [];
for x = 0:h:0.99
  y2 = [y2; eval(f2)];
end
x=[0:h:0.99];
plot(x, f1,'r->',x, y2, 'bx-');
lgd = legend('Rug-Kutta','True');
lgd.Location = 'northwest';
ttl=strcat(dfstr, ', h = ', num2str(h));
xlabel('x'); ylabel('y'); title(ttl);
```

```
function v = df2(x, y)
  v = (cos(x)-2*x*y)/(x^2-1);
end

function v = df1(x, y)
  v = 1+y+x;
end
```

Tangent field for Differential Equation (1)

- Display the Tangent field for differential equation in range [0, 4]

$$\frac{dy}{dx} = \frac{x}{y} \tag{13}$$

```
function slopefield(x, y)
   Fx = cos(atan(x./y));
   Fy = sqrt(1-Fx.^2);
   quiver(x,y,Fx,Fy,0.5);

   hold on;
   axis([0, 4, 0, 4]);
end
```

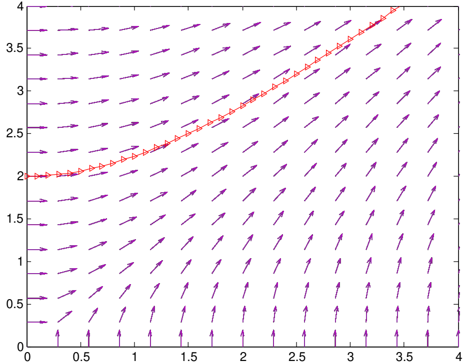
```
function Dy=tangfield(x,y)
  Dy=x./y;
end
```

```
[x, y]=meshgrid(linspace(0.1,4,15), linspace(0.1,4,15));
slopefield(x,y);
[x1,y1] = ode45(@tangfield, [0.1, 4], 2);
plot(x1, y1, 'r->');
```

Tangent field for Differential Equation (2)

- Display the Tangent field for differential equation in range [0, 4]
- r=dsolve('Dy-x/y=0','y(0)=2','x')

$$\frac{dy}{dx} = \frac{x}{y} \tag{14}$$

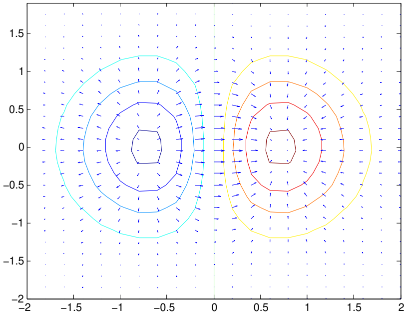


Slope field for function

- Display the slope field for differential equation in range [0, 4]

$$z = x * e^{(-x^2-y^2)} \tag{15}$$

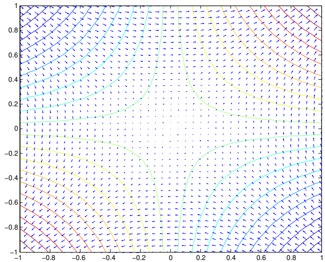
```
clf;
[x,y] = meshgrid(-2:0.15:2);
z = x.*exp(-x.^2-y.^2);
[px,py] = gradient(z,0.15,0.15);
contour(x, y, z); hold on;}
quiver(x, y, px, py, 0.5);
```



Exercise (1)

- Display the slope field for differential equation in range [-1, 1]

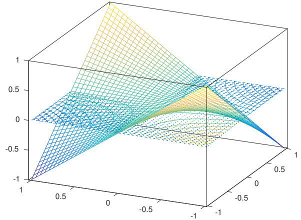
$$z = xy \tag{16}$$



Exercise (2)

- Display the slope field for differential equation in range [-1, 1]

```
clf;
clear;
[X,Y] = meshgrid([-1:0.05:1]);
Z = X.*Y;
[gx,gy] = gradient(Z,0.3,0.3);
contour(X,Y,Z,20); hold on;
quiver(X,Y,gx,gy,0.5); hold on;
mesh(X, Y, Z)
```



Q & A

Thanks for your attention!