Dimensionality Reduction Based on Local Homeomorphism and Global Second Order Nonlinearity

$$x \in R^D,\ y = \begin{bmatrix} y^{(1)} \\ \vdots \\ y^{(d)} \end{bmatrix} \in R^d \text{, where } d << D$$

Dimensionality Reduction:

$$X = \begin{bmatrix} x_1 & \cdots & x_N \end{bmatrix} \in R^{D \times N} \Rightarrow Y = \begin{bmatrix} y_1 & \cdots & y_N \end{bmatrix} \in R^{d \times N}$$

Dimensionality Reduction Based Local Homeomorphism

$$tr\left(YLY^T\right) \xrightarrow[choose\,Y]{} min \ \ \& \ subject\ to\ YY^T = I_d$$

Global Second Order Nonlinearity

$$y = \begin{bmatrix} y^{(1)} \\ \vdots \\ y^{(d)} \end{bmatrix} = f\left(x\right) = \begin{bmatrix} f_1\left(x\right) \\ \vdots \\ f_d\left(x\right) \end{bmatrix}$$

$$y^{(i)} = f_i\left(x\right) = a_i + b_i^T x + x^T H_i x = a_i + b_i^T x + h_i^T \eta$$

where

$$a_i \in R,\ b_i \in R^D,\ H_i \in R^{D \times D},\ H_i = H_i^T$$

$$\eta \in R^{\frac{D(D+1)}{2}},\ h_i \in R^{\frac{D(D+1)}{2}} \text{ (See Appendix)}$$

$$i = 1, \cdots, d$$

$$y=\begin{bmatrix} y^{(1)} \\ \vdots \\ y^{(d)} \end{bmatrix}=\begin{bmatrix} a_1+b_1^T x+h_1^T\eta \\ \vdots \\ a_d+b_d^T x+h_d^T\eta \end{bmatrix}=\begin{bmatrix} a_1 \\ \vdots \\ a_d \end{bmatrix}+\begin{bmatrix} b_1^T \\ \vdots \\ b_d^T \end{bmatrix}x+\begin{bmatrix} h_1^T \\ \vdots \\ h_d^T \end{bmatrix}\eta$$

$$=a+Bx+\Phi\eta=\begin{bmatrix} a & B & \Omega \end{bmatrix}\begin{bmatrix} 1 \\ x \\ \eta \end{bmatrix}=\Pi z$$

where

$$a=\begin{bmatrix} a_1 \\ \vdots \\ a_d \end{bmatrix}\in R^d,\ B=\begin{bmatrix} b_1^T \\ \vdots \\ b_d^T \end{bmatrix}\in R^{d\times D},\ \Phi=\begin{bmatrix} h_1^T \\ \vdots \\ h_d^T \end{bmatrix}\in R^{d\times\frac{D(D+1)}{2}}$$

$$\Pi=\begin{bmatrix} a & B & \Omega \end{bmatrix}\in R^{d\times\left(1+D+\frac{D(D+1)}{2}\right)},\ z\in\begin{bmatrix} 1 \\ x \\ \eta \end{bmatrix}\in R^{1+D+\frac{D(D+1)}{2}}$$

$$Y=\begin{bmatrix} y_1 & \cdots & y_N \end{bmatrix}=\begin{bmatrix} f(x_1) & \cdots & f(x_N) \end{bmatrix}$$

$$=\begin{bmatrix} \Pi z_1 & \cdots & \Pi z_N \end{bmatrix}=\Pi Z$$

where

$$Z=\begin{bmatrix} z_1 & \cdots & z_N \end{bmatrix}\in R^{\left(1+D+\frac{D(D+1)}{2}\right)\times N}$$

Dimensionality Reduction Based on Local Homeomorphism and Global Second Order Nonlinearity

$$tr\left(YLY^T\right)=tr\left(\Pi ZLZ^T\Pi^T\right)\xrightarrow[choose\ \Pi]{}min$$

Appendix Quadratic Form

For all vectors $x \in R^D$ and symmetric matrices $H \in R^{D\times D}$, $H^T = H$

$$x^T Hx = \sum_{i=1}^{D}\sum_{j=1}^{D} h_{ij} x^{(i)} x^{(j)} = \sum_{i=1}^{D} h_{ii} \left(x^{(i)}\right)^2 + 2\sum_{i=1}^{D-1}\sum_{j=1}^{d} h_{ij} x^{(i)} x^{(j)}$$

$$= \begin{bmatrix} h_{11} & \cdots & h_{DD} \end{bmatrix} \begin{bmatrix} \left(x^{(1)}\right)^2 \\ \vdots \\ \left(x^{(D)}\right)^2 \end{bmatrix} + \begin{bmatrix} 2h_{12} & \cdots & 2h_{(D-1)D} \end{bmatrix} \begin{bmatrix} x^{(1)} x^{(2)} \\ \vdots \\ x^{(D-1)} x^{(D)} \end{bmatrix}$$

$$= \begin{bmatrix} h_{11} & \cdots & h_{DD} & 2h_{12} & \cdots & 2h_{(D-1)D} \end{bmatrix} \begin{bmatrix} \left(x^{(1)}\right)^2 \\ \\ \left(x^{(D)}\right)^2 \\ x^{(1)} x^{(2)} \\ \vdots \\ x^{(D-1)} x^{(D)} \end{bmatrix} h^T \eta$$

where

$$\eta = \begin{bmatrix} \left(x^{(1)}\right)^2 \\ \\ \left(x^{(D)}\right)^2 \\ x^{(1)} x^{(2)} \\ \vdots \\ x^{(D-1)} x^{(D)} \end{bmatrix} \in R^{\frac{D(D+1)}{2}}, \ h = \begin{bmatrix} h_{11} \\ \\ h_{DD} \\ 2h_{12} \\ \vdots \\ 2h_{(D-1)D} \end{bmatrix} \in R^{\frac{D(D+1)}{2}}$$