

# MATH - 4360: Linear Statistical Models

## Chapter 3 - Results on Linear Algebra and Matrix Theory

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### Vectors

A vector  $Y$  is an ordered  $n$ -tuple of real numbers. A vector can be expressed as a row vector or a column vector as

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}_{n \times 1}$$

is a column vector of order  $n \times 1$  and

$$Y' = (y_1, y_2, \dots, y_n)_{1 \times n}$$

is a row vector of order  $1 \times n$ .

### Shortcut!

```
rm(list = ls())  
# I assumed that you have installed the following R packages. If not, please install  
# them using the R function: install.packages('package_name')  
library(ggplot2)  
library(Matrix)  
library(psych)  
library(pracma)  
z = c(10, 26, 9, 8, 4) # row vector  
Z = matrix(z) # convert into a column vector  
Z
```

```
##      [,1]  
## [1,] 10  
## [2,] 26  
## [3,] 9  
## [4,] 8  
## [5,] 4
```

```
as.vector(Z) # convert a column vector into a row vector
```

```
## [1] 10 26 9 8 4
```

If

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$Z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

Then,

$$X + Y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

$$kY = \begin{pmatrix} ky_1 \\ ky_2 \\ \vdots \\ ky_n \end{pmatrix}$$

```
x = c(7, 2, 7, 30, 5) # row vector
y = c(7, 3, 3, 4, 6) # row vector
X = matrix(c(7, 2, 7, 30, 5), ncol = 1) # column vector
X
```

```
##      [,1]
## [1,]    7
## [2,]    2
## [3,]    7
## [4,]   30
## [5,]    5
```

```
Y = matrix(c(7, 3, 3, 4, 6), ncol = 1) # column vector
Y
```

```
##      [,1]
## [1,]    7
## [2,]    3
## [3,]    3
## [4,]    4
## [5,]    6
```

```
# Sum X + Y
X + Y
```

```
##      [,1]
## [1,]   14
## [2,]    5
## [3,]   10
## [4,]   34
## [5,]   11
```

```
# Multiply X by a constant k = 3
3*X
```

```
##      [,1]
## [1,] 21
## [2,] 6
## [3,] 21
## [4,] 90
## [5,] 15
```

## Matrix

A matrix is a rectangular array of real numbers. For example

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

is a matrix of order  $m \times n$  with  $m$  rows and  $n$  columns.

```
a = c(7, 3, 3, 4, 6, 7, 2, 7, 30, 5, 16, 10)
# Let's create 4 by 3 matrix
A = matrix(c(7, 3, 3, 4, 6, 7, 2, 7, 30, 5, 16, 10), ncol = 4, nrow = 3)
A
```

```
##      [,1] [,2] [,3] [,4]
## [1,] 7    4    2    5
## [2,] 3    6    7   16
## [3,] 3    7   30   10
```

If  $m = n$ , then  $A$  is called a square matrix.

```
a = c(7, 3, 3, 4, 6, 7, 2, 7, 30, 5, 16, 10, 4, 6, 9, 10, 6, 7, 3, 17, 10, 26, 9, 8, 4)
# For example m = n = 5
A = matrix(a, ncol = 5, nrow = 5)
A
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,] 7    7   16   10   10
## [2,] 3    2   10    6   26
## [3,] 3    7    4    7    9
## [4,] 4   30    6    3    8
## [5,] 6    5    9   17    4
```

The diagonal elements of  $A$  can be obtained using the R function `diag()`

```
diag(A)
```

```
## [1] 7 2 4 3 4
```

If  $a_{ij} = 0$ ,  $i \neq j$ ,  $m = n$  then  $A$  is a diagonal matrix and is denoted as

$$A = \text{diag}(a_{11}, a_{22}, \dots, a_{mm})$$

```
diag(c(560, 220, 340, 80, 150))
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,] 560    0    0    0    0
## [2,] 0   220    0    0    0
## [3,] 0    0  340    0    0
## [4,] 0    0    0   80    0
## [5,] 0    0    0    0  150
```

**Null Matrix:** A matrix whose all elements are equal to zero is called a null matrix.

```
0 = matrix(0, nrow = 3, ncol = 3)
0
```

```
##      [,1] [,2] [,3]
## [1,]    0    0    0
## [2,]    0    0    0
## [3,]    0    0    0
```

**Identity Matrix:** The identity matrix of size  $n$  is the  $n \times n$  square matrix with ones on the main diagonal and zeros elsewhere. We can write

$$I_n = \text{diag}(1, 1, \dots, 1)$$

```
I=diag(5)
I
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    1    0    0    0    0
## [2,]    0    1    0    0    0
## [3,]    0    0    1    0    0
## [4,]    0    0    0    1    0
## [5,]    0    0    0    0    1
```

If  $m = n$  (square matrix) and  $a_{ij} = 0$  for  $i > j$ , then  $A$  is called an upper triangular matrix. On the other hand if  $m = n$  and  $a_{ij} = 0$  for  $i < j$  then  $A$  is called a lower triangular matrix.

```
a = c(7, 3, 3, 4, 6, 7, 2, 7, 30, 5, 16, 10, 4, 6, 9, 10, 6, 7, 3, 17, 10, 26, 9, 8, 4)
A = matrix(a, ncol = 5, nrow = 5)
A
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    7    7   16   10   10
## [2,]    3    2   10    6   26
## [3,]    3    7    4    7    9
## [4,]    4   30    6    3    8
## [5,]    6    5    9   17    4
```

```
triu(A) # upper triangular
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    7    7   16   10   10
## [2,]    0    2   10    6   26
## [3,]    0    0    4    7    9
## [4,]    0    0    0    3    8
## [5,]    0    0    0    0    4
```

```
tril(A) # lower triangular
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    7    0    0    0    0
## [2,]    3    2    0    0    0
## [3,]    3    7    4    0    0
## [4,]    4   30    6    3    0
## [5,]    6    5    9   17    4
```

**Symmetric Matrix:** If  $A = A'$  then  $A$  is a symmetric matrix.

```
a = c(1, 7, 3, 7, 4, 5, 3, 5, 6)
A = matrix(a, ncol = 3, nrow = 3)
A
```

```
##      [,1] [,2] [,3]
## [1,]    1    7    3
## [2,]    7    4    5
## [3,]    3    5    6
```

```
# Let's find transpose of A
t(A)
```

```
##      [,1] [,2] [,3]
## [1,]    1    7    3
## [2,]    7    4    5
## [3,]    3    5    6
```

**Skew-Symmetric Matrix** If  $A = -A'$  then  $A$  is skew-symmetric matrix.

```
a = c(0, -3, 2, 3, 0, -1, -2, 1, 0)
A = matrix(a, ncol = 3, nrow = 3, byrow = TRUE)
A
```

```
##      [,1] [,2] [,3]
## [1,]    0   -3    2
## [2,]    3    0   -1
## [3,]   -2    1    0
```

```
# Let's find transpose of A
t(A) # Not symmetric
```

```
##      [,1] [,2] [,3]
## [1,]    0    3   -2
## [2,]   -3    0    1
## [3,]    2   -1    0
```

```
# Let's find (-A)'
A_neg = -A
t(A_neg) # (-A)' = A
```

```
##      [,1] [,2] [,3]
## [1,]    0   -3    2
## [2,]    3    0   -1
## [3,]   -2    1    0
```

If  $A$  and  $B$  are matrices of order  $m \times n$  then  $(A + B)' = A' + B'$

```
a = c(7, 3, 3, 4, 6, 7, 2, 7, 30)
A = matrix(a, 3, 3)
A
```

```
##      [,1] [,2] [,3]
## [1,]    7    4    2
## [2,]    3    6    7
## [3,]    3    7   30
```

```
At = t(A)
At
```

```
##      [,1] [,2] [,3]
## [1,]    7    3    3
## [2,]    4    6    7
## [3,]    2    7   30
```

```
b = c(5, 16, 10, 4, 6, 9, 10, 6, 7)
B = matrix(b, 3, 3)
B
```

```
##      [,1] [,2] [,3]
## [1,]    5    4   10
## [2,]   16    6    6
## [3,]   10    9    7
```

```
Bt = t(B)
Bt
```

```
##      [,1] [,2] [,3]
## [1,]    5   16   10
## [2,]    4    6    9
## [3,]   10    6    7
```

```
# Sum of A + B
C = A + B
C
```

```
##      [,1] [,2] [,3]
## [1,]   12    8   12
## [2,]   19   12   13
## [3,]   13   16   37
```

```
# transpose of a matrix C
Ct = t(C)
Ct
```

```
##      [,1] [,2] [,3]
## [1,]   12   19   13
## [2,]    8   12   16
## [3,]   12   13   37
```

```
# Sum of At + Bt
At + Bt
```

```
##      [,1] [,2] [,3]
## [1,]   12   19   13
## [2,]    8   12   16
## [3,]   12   13   37
```

If  $A$  and  $B$  are the matrices of order  $m \times n$  and  $n \times p$  respectively and  $k$  is any scalar, then

$$(AB)' = B'A'$$

$$(kA)B = A(kB) = k(AB) = kAB$$

```
# Multiply A*B
```

```
D = A %*% B
```

```
D
```

```
##      [,1] [,2] [,3]
## [1,]  119   70  108
## [2,]  181  111  115
## [3,]  427  324  282
```

```
Dt = t(D) # transpose of D
```

```
Dt
```

```
##      [,1] [,2] [,3]
## [1,]  119  181  427
## [2,]   70  111  324
## [3,]  108  115  282
```

```
At_Bt = At %*% Bt # transpose(A) * transpose(B)
```

```
At_Bt
```

```
##      [,1] [,2] [,3]
## [1,]   77  148  118
## [2,]  114  142  143
## [3,]  338  254  293
```

If the orders of matrices  $A$  is  $m \times n$ ,  $B$  is  $n \times p$  and  $C$  is  $n \times p$  then  $A(B + C) = AB + AC$

```
# A is a 4 by 3 matrix
```

```
a = c(7, 3, 4, 6, 7, 2, 7, 30, 5, 16, 10)
```

```
A = matrix(a, 4, 3) # Matrix A
```

```
A
```

```
##      [,1] [,2] [,3]
## [1,]    7    6   30
## [2,]    3    7    5
## [3,]    3    2   16
## [4,]    4    7   10
```

```
# B is a 3 by 5 matrix
```

```
b = c(5, 16, 10, 4, 6, 9, 10, 6, 7, 3, 17, 10, 26, 9, 8)
```

```
B = matrix(b, 3, 5) # Matrix B
```

```
B
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    5    4   10    3   26
## [2,]   16    6    6   17    9
## [3,]   10    9    7   10    8
```

```
# C is a 3 by 5 matrix
```

```
c = c(3, 17, 10, 26, 9, 8, 4, 23, 16, 8, 17, 5.5, 19, 24, 2.5, 7.5)
```

```
C = matrix(c, 3, 5) # Matrix C
```

```
C
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    3   26    4  8.0 19.0
## [2,]   17    9   23 17.0 24.0
## [3,]   10    8   16  5.5  2.5
```

```
# Let's find  $A(B + C)$ 
```

```
A %*% (B + C)
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]  854  810  962 746.0 828.0
## [2,]  355  280  360 348.5 418.5
## [3,]  410  392  468 349.0 369.0
## [4,]  463  395  489 437.0 516.0
```

```
# Let's find  $AB + AC$ 
```

```
A %*% B + A %*% C
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]  854  810  962 746.0 828.0
## [2,]  355  280  360 348.5 418.5
## [3,]  410  392  468 349.0 369.0
## [4,]  463  395  489 437.0 516.0
```

If the orders of matrices  $A$  is  $m \times n$ ,  $B$  is  $n \times p$  and  $C$  is  $p \times q$  then  $(AB)C = A(BC)$

```
# A is a 4 by 3 matrix
```

```
a = c(7, 3, 4, 6, 7, 2, 7, 30, 5, 16, 10)
```

```
A = matrix(a, 4, 3) # Matrix A
```

```
A
```

```
##      [,1] [,2] [,3]
## [1,]    7    6   30
## [2,]    3    7    5
## [3,]    3    2   16
## [4,]    4    7   10
```

```
# B is a 3 by 5 matrix
```

```
b = c(5, 16, 10, 4, 6, 9, 10, 6, 7, 3, 17, 10, 26, 9, 8)
```

```
B = matrix(b, 3, 5) # Matrix B
```

```
B
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    5    4   10    3   26
## [2,]   16    6    6   17    9
## [3,]   10    9    7   10    8
```

```
# C is a 5 by 2 matrix
```

```
c = c(3, 17, 10, 26, 9, 8, 4, 23, 16, 8)
```

```
C = matrix(c, 5, 2) # Matrix C
```

```
C
```

```
##      [,1] [,2]
## [1,]    3    8
## [2,]   17    4
## [3,]   10   23
## [4,]   26   16
## [5,]    9    8
```

```
# Let's find  $(AB)C$ 
```

```
(A %*% B) %*% C
```



```
##      [,1] [,2]
## [1,] 25413 22628
## [2,]  9541  8569
## [3,] 12311 10910
## [4,] 12961 11616
```

```
# Let's find A(BC)
A %%% (B %%% C)
```

```
##      [,1] [,2]
## [1,] 25413 22628
## [2,]  9541  8569
## [3,] 12311 10910
## [4,] 12961 11616
```

If  $A$  is the matrix of order  $m \times n$  then  $I_m A = A I_n = A$

```
# A is a 4 by 3 matrix
a = c(7, 3, 3, 4, 6, 7, 2, 7, 30, 5, 16, 10)
A = matrix(a, 4, 3) # Matrix A
A
```

```
##      [,1] [,2] [,3]
## [1,]    7    6   30
## [2,]    3    7    5
## [3,]    3    2   16
## [4,]    4    7   10
```

```
I_4 = diag(rep(1,4)) # Identity matrix 4 by 4
I_4
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    0    0    0
## [2,]    0    1    0    0
## [3,]    0    0    1    0
## [4,]    0    0    0    1
```

```
I_3 = diag(rep(1,3)) # Identity matrix 3 by 3
I_3
```

```
##      [,1] [,2] [,3]
## [1,]    1    0    0
## [2,]    0    1    0
## [3,]    0    0    1
```

```
# Let's find I_4*A
I_4 %%% A
```

```
##      [,1] [,2] [,3]
## [1,]    7    6   30
## [2,]    3    7    5
## [3,]    3    2   16
## [4,]    4    7   10
```

```
# Let's find A*I_3
A %%% I_3
```

```
##      [,1] [,2] [,3]
## [1,]    7    6   30
## [2,]    3    7    5
## [3,]    3    2   16
## [4,]    4    7   10
```

## Trace of a Matrix

The trace of  $n \times n$  matrix  $A$ , denoted as  $tr(A)$  or  $tr(A)$  is defined to be the sum of all the diagonal elements of  $A$ , i.e.

$$tr(A) = \sum_{i=1}^n a_{ii}$$

```
# Create a 3 by 3 matrix
a = c(7, 3, 3, 4, 6, 7, 2, 7, 30)
A = matrix(a, 3, 3)
A
```

```
##      [,1] [,2] [,3]
## [1,]    7    4    2
## [2,]    3    6    7
## [3,]    3    7   30
```

```
# trace of A
tr(A)
```

```
## [1] 43
```

If  $A$  is of order  $m \times n$  and  $B$  is of order  $n \times m$ , then  $tr(AB) = tr(BA)$

```
# A is a 4 by 3 matrix
a = c(7, 3, 3, 4, 6, 7, 2, 7, 30, 5, 16, 10)
A = matrix(a, 4, 3) # Matrix A
A
```

```
##      [,1] [,2] [,3]
## [1,]    7    6   30
## [2,]    3    7    5
## [3,]    3    2   16
## [4,]    4    7   10
```

```
# B is a 3 by 4 matrix
b = c(5, 16, 10, 4, 6, 9, 10, 6, 7, 3, 17, 10)
B = matrix(b, 3, 4) # Matrix B
B
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    5    4   10    3
## [2,]   16    6    6   17
## [3,]   10    9    7   10
```

```
# Let's find tr(AB)
tr(A %*% B)
```

```
## [1] 915
```

```
# Let's find tr(BA)
tr(B %*% A)
```

```
## [1] 915
```

If  $A$  and  $B$  are  $n \times n$  matrices,  $a$  and  $b$  are scalars then  $tr(uA + vB) = utr(A) + vtr(B)$

```
# A is a 4 by 4 matrix
a = c(7, 3, 3, 4, 6, 7, 2, 7, 30, 5, 16, 10, 15.5, 23.75, 8, 17)
A = matrix(a, 4, 4) # Matrix A
A
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    7    6   30 15.50
## [2,]    3    7    5 23.75
## [3,]    3    2   16  8.00
## [4,]    4    7   10 17.00
```

```
# B is a 4 by 4 matrix
b = c(5, 16, 10, 4, 6, 9, 10, 6, 7, 3, 17, 10, 7.5, 11, 13, 3.75)
B = matrix(b, 4, 4) # Matrix B
B
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    5    6    7  7.50
## [2,]   16    9    3 11.00
## [3,]   10   10   17 13.00
## [4,]    4    6   10  3.75
```

```
# Let u = 2 and v = 4 be constants. Let's find tr(u*A + v*B)
u = 2
v = 4
tr(u*A + v*B)
```

```
## [1] 233
```

```
# Let's find u*tr(A) + v*tr(B)
u*tr(A) + v*tr(B)
```

```
## [1] 233
```

If  $A$  is an  $m \times n$  matrix, then

$$tr(A'A) = tr(AA') = \sum_{j=1}^n \sum_{i=1}^n a_{ij}^2$$

and  $tr(A'A) = tr(AA') = 0$  if and only if  $A = 0$

If  $A$  is  $n \times n$  matrix then  $tr(A')tr(A)$

```
# A is a 4 by 4 matrix
a = c(7, 3, 3, 4, 6, 7, 2, 7, 30, 5, 16, 10, 15.5, 23.75, 8, 17)
A = matrix(a, 4, 4) # Matrix A
A
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    7    6   30 15.50
## [2,]    3    7    5 23.75
## [3,]    3    2   16  8.00
## [4,]    4    7   10 17.00
```

```
# Let's find tr(A'A)
tr(t(A) %*% A)
```

```
## [1] 2659.312
```

```
# Let's find tr(AA')
tr(A %*% t(A))
```

```
## [1] 2659.312
```

```
sum(A^2)
```

```
## [1] 2659.312
```

## Rank of Matrices

The rank of a matrix  $A$  of  $m \times n$  is the number of linearly independent rows in  $A$ . Let  $B$  be any other matrix of order  $n \times q$ .

1. A square matrix of order  $m$  is called **non-singular** if it has full rank.  $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$
2.  $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$
3.  $\text{rank}(A)$  is equal to the maximum order of all nonsingular square sub-matrices of  $A$ .
4.  $\text{rank}(AA') = \text{rank}(A'A) = \text{rank}(A) = \text{rank}(A')$ .
5.  $A$  is of full row rank if  $\text{rank}(A) = m < n$ .
6.  $A$  is of full column rank if  $\text{rank}(A) = n < m$

We will use the `r` function `qr()`

```
# A is a 4 by 3 matrix
a = c(7, 3, 3, 4, 6, 7, 2, 7, 30, 5, 16, 10)
A = matrix(a, 4, 3, byrow = T) # Matrix A
A
```

```
##      [,1] [,2] [,3]
## [1,]    7    3    3
## [2,]    4    6    7
## [3,]    2    7   30
## [4,]    5   16   10
```

```
rref(A) # Reduced Row Echelon Form ( r package: pracma)
```

```
##      [,1] [,2] [,3]
## [1,]    1    0    0
## [2,]    0    1    0
## [3,]    0    0    1
## [4,]    0    0    0
```

```
qr(A)$rank
```

```
## [1] 3
```

```
# B is a 3 by 5 matrix
b = c(5, 16, 10, 4, 6, 9, 10, 6, 7, 3, 17, 10, 23.75, 8, 17)
B = matrix(b, 3, 5) # Matrix B
B
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    5    4   10    3 23.75
## [2,]   16    6    6   17  8.00
## [3,]   10    9    7   10 17.00
```

```
qr(B)$rank
```

```
## [1] 3
```

```
# C is a 4 by 3 matrix
```

```
c = c(3, 17, 10, 26, 9, 8, 9, 51, 30)
```

```
C = matrix(c, 3, 3, byrow = T) # Matrix C
```

```
C
```

```
##      [,1] [,2] [,3]
```

```
## [1,]    3    17    10
```

```
## [2,]   26     9     8
```

```
## [3,]    9    51    30
```

```
qr(C)$rank
```

```
## [1] 2
```

```
d = c(1,2,1,3,6,3,2,4,2)
```

```
D = matrix(d, nrow=3, ncol = 3, byrow=F)
```

```
D
```

```
##      [,1] [,2] [,3]
```

```
## [1,]    1     3     2
```

```
## [2,]    2     6     4
```

```
## [3,]    1     3     2
```

```
qr(D)$rank
```

```
## [1] 1
```

## The determinant of a matrix

The determinant of a matrix is a scalar value that is a function of the entries of a square matrix. The determinant of a matrix  $A$  is denoted  $\det(A)$  or  $|A|$ .

1. For a  $2 \times 2$  matrix

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11}a_{22} - a_{12}a_{21})$$

2. For a  $3 \times 3$  matrix

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

```
# A is a 2 by 2 matrix
```

```
a = c(7, 3, 3, 4)
```

```
A = matrix(a, 2, 2, byrow = TRUE) # Matrix A
```

```
A
```

```
##      [,1] [,2]
```

```
## [1,]    7     3
```

```
## [2,]    3     4
```

```
# Let's find det(A)
det(A)
```

```
## [1] 19
```

```
# B is a 3 by 3 matrix
b = c(5, 16, 10, 4, 6, 9, 10, 6, 7)
B = matrix(b, 3, 3, byrow = TRUE) # Matrix B
B
```

```
##      [,1] [,2] [,3]
## [1,]    5   16   10
## [2,]    4    6    9
## [3,]   10    6    7
```

```
# Let's find det(B)
det(B)
```

```
## [1] 572
```

## Inverse of a Matrix

The inverse of a square matrix  $A$  of order  $m$ , is a square matrix of order  $m$ , denoted as  $A^{-1}$ , such that  $A^{-1}A = AA^{-1} = I_m$ . The inverse of  $A$  exists if and only if  $A$  is non-singular. For a  $2 \times 2$  matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

the matrix inverse is

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

where  $\det(A) = (a_{11}a_{22} - a_{12}a_{21})$

```
# A is a 4 by 4 matrix
a = c(7, 3, 3, 4, 6, 7, 2, 7, 30, 5, 16, 10, 15.5, 23.75, 8, 17)
A = matrix(a, 4, 4) # Matrix A
A
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    7    6   30 15.50
## [2,]    3    7    5 23.75
## [3,]    3    2   16  8.00
## [4,]    4    7   10 17.00
```

```
# Inverse of a Matrix A
A_inv = solve(A)
A_inv
```

```
##      [,1]      [,2]      [,3]      [,4]
## [1,] 2.57394366 1.3591549 -3.67957746 -2.5140845
## [2,] -1.14964789 -0.8697183 1.43485915 1.5880282
## [3,] -0.38644366 -0.2341549 0.61707746 0.3890845
## [4,] 0.09507042 0.1760563 -0.08802817 -0.2323944
```

```
# Let's find A_inv*A
A_invA = A_inv %*% A
round(A_invA, 4) # Round to four decimal places
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    0    0    0
## [2,]    0    1    0    0
## [3,]    0    0    1    0
## [4,]    0    0    0    1
```

```
# Let's find A*A_inv
AA_inv = A %*% A_inv
round(AA_inv, 4) # Round to four decimal places
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    0    0    0
## [2,]    0    1    0    0
## [3,]    0    0    1    0
## [4,]    0    0    0    1
```

1.  $(A^{-1})^{-1} = A$
2. If  $A$  is non-singular, then  $(A')^{-1} = (A^{-1})'$
3. If  $A$  and  $B$  are non-singular matrices of the same order, then their product, if defined, is also nonsingular and

$$(AB)^{-1} = B^{-1}A^{-1}$$

```
# A is a 4 by 4 matrix
a = c(7, 3, 3, 4, 6, 7, 2, 7, 30, 5, 16, 10, 15.5, 23.75, 8, 17)
A = matrix(a, 4, 4) # Matrix A
A
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    7    6   30 15.50
## [2,]    3    7    5 23.75
## [3,]    3    2   16  8.00
## [4,]    4    7   10 17.00
```

```
# Inverse of a Matrix A
A_inv = solve(A)
A_inv
```

```
##      [,1]      [,2]      [,3]      [,4]
## [1,] 2.57394366 1.3591549 -3.67957746 -2.5140845
## [2,] -1.14964789 -0.8697183 1.43485915 1.5880282
## [3,] -0.38644366 -0.2341549 0.61707746 0.3890845
## [4,] 0.09507042 0.1760563 -0.08802817 -0.2323944
```

```
# 1) Inverse of an Inverse
solve(A_inv)
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    7    6   30 15.50
## [2,]    3    7    5 23.75
## [3,]    3    2   16  8.00
## [4,]    4    7   10 17.00
```

```
# Transpose of matrix A
```

```
At = t(A)
```

```
At
```

```
##      [,1] [,2] [,3] [,4]
## [1,]  7.0  3.00   3   4
## [2,]  6.0  7.00   2   7
## [3,] 30.0  5.00  16  10
## [4,] 15.5 23.75   8  17
```

```
# 2) Inverse of At
```

```
At_inv = solve(At)
```

```
At_inv
```

```
##      [,1]      [,2]      [,3]      [,4]
## [1,]  2.573944 -1.1496479 -0.3864437  0.09507042
## [2,]  1.359155 -0.8697183 -0.2341549  0.17605634
## [3,] -3.679577  1.4348592  0.6170775 -0.08802817
## [4,] -2.514085  1.5880282  0.3890845 -0.23239437
```

```
A_invt = t(A_inv)
```

```
A_invt
```

```
##      [,1]      [,2]      [,3]      [,4]
## [1,]  2.573944 -1.1496479 -0.3864437  0.09507042
## [2,]  1.359155 -0.8697183 -0.2341549  0.17605634
## [3,] -3.679577  1.4348592  0.6170775 -0.08802817
## [4,] -2.514085  1.5880282  0.3890845 -0.23239437
```

```
# B is a 4 by 4 matrix
```

```
b = c(5, 16, 10, 4, 6, 9, 10, 6, 7, 3, 17, 10, 7.5, 11, 13, 3.75)
```

```
B = matrix(b, 4, 4) # Matrix B
```

```
B
```

```
##      [,1] [,2] [,3] [,4]
## [1,]   5   6   7  7.50
## [2,]  16   9   3 11.00
## [3,]  10  10  17 13.00
## [4,]   4   6  10  3.75
```

```
# Let's find inverse of AB
```

```
AB_inv = solve(A %*% B)
```

```
AB_inv
```

```
##      [,1]      [,2]      [,3]      [,4]
## [1,] -1.0888083 -0.6021291  1.5455475  1.1072524
## [2,]  1.4083154  0.7884752 -2.0250054 -1.4159614
## [3,] -0.5820977 -0.3024236  0.8474547  0.5556390
## [4,]  0.4857036  0.2341219 -0.6919287 -0.4592068
```

```
# Let's find inverse of matrix A and inverse of matrix B separately
```

```
A_inv = solve(A)
```

```
A_inv
```

```
##      [,1]      [,2]      [,3]      [,4]
## [1,]  2.57394366  1.3591549 -3.67957746 -2.5140845
## [2,] -1.14964789 -0.8697183  1.43485915  1.5880282
## [3,] -0.38644366 -0.2341549  0.61707746  0.3890845
## [4,]  0.09507042  0.1760563 -0.08802817 -0.2323944
```



```
B_inv = solve(B)
B_inv
```

```
##           [,1]      [,2]      [,3]      [,4]
## [1,] -0.3623929  0.094109064  0.12639066  0.01057815
## [2,]  0.4847711  0.007477658 -0.36786431  0.28378625
## [3,] -0.2170345 -0.025168703  0.14061645  0.02042677
## [4,]  0.1896772 -0.045230713  0.07878898 -0.25314609
```

```
# Lets find the product of (inverse of matrix B)*(inverse of matrix A)
B_inv %*% A_inv
```

```
##           [,1]      [,2]      [,3]      [,4]
## [1,] -1.0888083 -0.6021291  1.5455475  1.1072524
## [2,]  1.4083154  0.7884752 -2.0250054 -1.4159614
## [3,] -0.5820977 -0.3024236  0.8474547  0.5556390
## [4,]  0.4857036  0.2341219 -0.6919287 -0.4592068
```

## Idempotent Matrix

A square matrix  $A$  is called idempotent if  $A^2 = AA = A$ . If  $A$  is an  $n \times n$  idempotent matrix with  $\text{rank}(A) = r < n$ . Then

1. eigenvalues of  $A$  are 1 or 0.
2.  $\text{tr}(A) = \text{rank}(A) = r$
3. If  $A$  is of full rank  $n$ , then  $A = I_n$ .
4. If  $A$  and  $B$  are idempotent and  $AB = BA$ , then  $AB$  is also idempotent.
5. If  $A$  is idempotent then  $(I - A)$  is also idempotent and  $A(I - A) = (I - A)A = 0$

```
# A is a 3 by 3 matrix
a = c(2, -2, -4, -1, 3, 4, 1, -2, -3)
A = matrix(a, nrow = 3, byrow = TRUE)
A
```

```
##           [,1] [,2] [,3]
## [1,]      2  -2  -4
## [2,]     -1   3   4
## [3,]      1  -2  -3
```

```
# Let's find the rank of matrix A
qr(A)$rank # rank(A) = 2 < 3
```

```
## [1] 2
```

```
# Let's find the trace of matrix A
tr(A) # So tr(A) = rank(A)
```

```
## [1] 2
```

```
# Let's check A*A = A ?
A %*% A
```

```
##           [,1] [,2] [,3]
## [1,]      2  -2  -4
## [2,]     -1   3   4
## [3,]      1  -2  -3
```

```
# Identity matrix 3 by 3
```

```
I_3 = diag(rep(1,3))
```

```
I_3
```

```
##      [,1] [,2] [,3]
## [1,]    1    0    0
## [2,]    0    1    0
## [3,]    0    0    1
```

```
# Let's find (I-A)
```

```
I_minus_A = I_3 - A
```

```
# Let's find A(I-A)
```

```
A %*% I_minus_A # this is a 3 by 3 null matrix
```

```
##      [,1] [,2] [,3]
## [1,]    0    0    0
## [2,]    0    0    0
## [3,]    0    0    0
```

## Quadratic Forms

If  $A$  is a given matrix of order  $m \times n$  and  $X$  and  $Y$  are two given vectors of order  $m \times 1$  and  $n \times 1$  respectively

$$X'AY = \sum_{i=1}^m \sum_{j=1}^n a_{ij}x_iy_j$$

where  $a_{ij}$  are the nonstochastic elements of  $A$

1. If  $A$  is a square matrix of order  $m$  and  $X = Y$ , then

$$X'AX = a_{11}x_1^2 + \cdots + a_{mm}x_m^2 + (a_{12} + a_{21})x_1x_2 + \cdots + (a_{m-1,m} + a_{m,m-1})x_{m-1}x_m$$

2. If  $A$  is symmetric also, then

$$X'AX = a_{11}x_1^2 + \cdots + a_{mm}x_m^2 + 2a_{12}x_1x_2 + \cdots + 2a_{m-1,m}x_{m-1}x_m = \sum_{i=1}^m \sum_{j=1}^n a_{ij}x_ix_j$$

is called a quadratic form in  $m$  variables  $x_1, x_2, \dots, x_m$  or a quadratic form in  $X$ .

```
x = c(10, 26, 9, 8, 4) # row vector
```

```
X = matrix(x) # column vector
```

```
X
```

```
##      [,1]
## [1,]   10
## [2,]   26
## [3,]    9
## [4,]    8
## [5,]    4
```

```
a = c(7, 3, 3, 4, 6, 7, 2, 7, 30, 5, 16, 10, 4, 6, 9, 10, 6, 7, 3, 17, 10, 26, 9, 8, 4)
```

```
A = matrix(a, ncol = 5, nrow = 5) # Matrix A
```

```
A
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    7    7   16   10   10
## [2,]    3    2   10    6   26
## [3,]    3    7    4    7    9
## [4,]    4   30    6    3    8
## [5,]    6    5    9   17    4
```

```
Quadratic = t(X) %*% A %*% X # X'AX
Quadratic
```

```
##      [,1]
## [1,] 25776
```

## Note

1. Positive definite if  $A'AX > 0$  for all  $X \neq 0$
2. Positive semidefinite if  $A'AX \geq 0$  for all  $X \neq 0$ .
3. Negative definite if  $A'AX < 0$  for all  $X \neq 0$ .
4. Negative semidefinite if  $A'AX \leq 0$  for all  $X \neq 0$ .
5. If  $P$  is any nonsingular matrix and  $A$  is any positive definite matrix (or positive semi-definite matrix) then  $P'AP$  is also a positive definite matrix (or positive semi-definite matrix).
6. A matrix  $A$  is positive definite if and only if there exists a non-singular matrix  $P$  such that  $A = P'P$ .
7. A positive definite matrix is a nonsingular matrix.
8. If  $A$  is  $m \times n$  matrix and  $\text{rank}(A) = m < n$  then  $AA'$  is positive definite and  $A'A$  is positive semidefinite.
9. If  $A$  is  $m \times n$  matrix and  $\text{rank}(A) = k < m < n$ , then both  $A'A$  and  $AA'$  are positive semidefinite.

## Orthogonal Matrix

A square matrix  $A$  is called an orthogonal matrix if  $A'A = AA' = I$  or equivalently if  $A^{-1} = A'$

1. An orthogonal matrix is non-singular.
2. If  $A$  is orthogonal, then  $AA'$  is also orthogonal.
3. If  $A$  is an  $n \times n$  matrix and let  $P$  is an  $n \times n$  orthogonal matrix, then the determinants of  $A$  and  $P'AP$  are the same.

## Random Vectors

Let  $Y_1, Y_2, \dots, Y_n$  be  $n$  random variables then  $Y = (Y_1, Y_2, \dots, Y_n)'$  is called a random vector.

1. The mean vector  $Y$  is

$$E(Y) = (E(Y_1), E(Y_2), \dots, E(Y_n))'$$

2. The covariance matrix or dispersion matrix of  $Y$  is

$$\text{Var}(Y) = \begin{pmatrix} \text{Var}(Y_1) & \text{Cov}(Y_1, Y_2) & \cdots & \text{Cov}(Y_1, Y_n) \\ \text{Cov}(Y_2, Y_1) & \text{Var}(Y_2) & \cdots & \text{Cov}(Y_2, Y_n) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Cov}(Y_n, Y_1) & \text{Cov}(Y_n, Y_2) & \cdots & \text{Var}(Y_n) \end{pmatrix}$$

which is a symmetric matrix.

1. If  $Y_1, Y_2, \dots, Y_n$  are independently distributed, then the covariance matrix is a diagonal matrix
2. If  $\text{Var}(Y_i) = \sigma^2$  for all  $i = 1, 2, \dots, n$  then  $\text{Var}(Y) = \sigma^2 I_n$

## Matrix Derivatives

Let  $A$  be a  $k \times k$  matrix of constants,  $a$  be a  $k \times 1$  vector of constants, and  $y$  be a  $k \times 1$  vector of variables.

- If  $z = a'y$ , then

$$\frac{\partial z}{\partial y} = \frac{\partial a'y}{\partial y} = a$$

- If  $z = y'y$ , then

$$\frac{\partial z}{\partial y} = \frac{\partial y'y}{\partial y} = 2y$$

- If  $z = a' Ay$ , then

$$\frac{\partial z}{\partial y} = \frac{\partial a' Ay}{\partial y} = A'a$$

- If  $z = y' Ay$ , then

$$\frac{\partial z}{\partial y} = \frac{\partial y' Ay}{\partial y} = Ay + A'y$$

If  $A$  is symmetric, then

$$\frac{\partial z}{\partial y} = \frac{\partial y' Ay}{\partial y} = 2Ay$$

## Expectations

Let  $A$  be a  $k \times k$  matrix of constants,  $a$  be a  $k \times 1$  vector of constants, and  $y$  be a  $k \times 1$  random vector with mean  $\mu$  and nonsingular variance - covariance matrix  $V$ .

- $E(a'y) = a'\mu$ .
- $E(Ay) = A\mu$ .
- $Var(a'y) = a'Va$ .
- $Var(Ay) = AVA'$ .

Note: If  $V = \sigma^2 I$ , then  $Var(Ay) = \sigma^2 AA'$ .

- $E(y' Ay) = trace(AV) + \mu' A\mu$ .

Note: If  $V = \sigma^2 I$ , then  $E(y' Ay) = \sigma^2 trace(A) + \mu' A\mu$ .

## References

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