MATH - 4360: Linear Statistical Models

Chapter 3 - Results on Linear Algebra and Matrix Theory

Suthakaran Ratnasingam

Vectors

A vector Y is an ordered n-tuple of real numbers. A vector can be expressed as a row vector or a column vector as

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}_{n\times 1}$$

is a column vector of order $n \times 1$ and

$$Y' = (y_1, y_2, \ldots, y_n)_{1\times n}$$

is a row vector of order $1 \times n$.

Shortcut!

```
rm(list = ls())
# I assumed that you have installed the following R packages. If not, please install
# them using the R function: install.packages('package_name')
library(ggplot2)
library(Matrix)
library(psych)
library(pracma)
z = c(10, 26,  9,  8,  4) # row vector
Z = matrix(z) # convert into a column vector
Z
```

```
##      [,1]
## [1,]   10
## [2,]   26
## [3,]    9
## [4,]    8
## [5,]    4
```

```
as.vector(Z) # convert a column vector into a row vector
```

```
## [1] 10 26  9  8  4
```

If

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$Z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

Then,

$$X + Y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

$$kY = \begin{pmatrix} ky_1 \\ ky_2 \\ \vdots \\ ky_n \end{pmatrix}$$

```
x = c(7,  2,  7, 30,  5) # row vector
y = c(7,  3,  3,  4,  6) # row vector
X = matrix(c(7,  2,  7, 30,  5), ncol = 1) # column vector
X
```

```
##      [,1]
## [1,]    7
## [2,]    2
## [3,]    7
## [4,]   30
## [5,]    5
```

```
Y = matrix(c(7,  3,  3,  4,  6), ncol = 1) # column vector
Y
```

```
##      [,1]
## [1,]    7
## [2,]    3
## [3,]    3
## [4,]    4
## [5,]    6
```

```
# Sum X + Y
X + Y
```

```
##      [,1]
## [1,]   14
## [2,]    5
## [3,]   10
## [4,]   34
## [5,]   11
```

```
# Multiply X by a constant k = 3
3*X
```

```
##      [,1]
## [1,]   21
## [2,]    6
## [3,]   21
## [4,]   90
## [5,]   15
```

Matrix

A matrix is a rectangular array of real numbers. For example

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

is a matrix of order $m \times n$ with m rows and n columns.

```
a = c(7,  3,  3,  4,  6,  7,  2,  7, 30,  5, 16, 10)
# Let's create 4 by 3 matrix
A = matrix(c(7,  3,  3,  4,  6,  7,  2,  7, 30,  5, 16, 10), ncol = 4, nrow = 3)
A
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    7    4    2    5
## [2,]    3    6    7   16
## [3,]    3    7   30   10
```

If $m = n$, then A is called a square matrix.

```
a = c(7,  3 , 3 , 4,  6,  7,  2,  7, 30,  5, 16, 10,  4,  6,  9, 10,  6,  7,  3, 17, 10, 26,  9,  8,  4)
# For example m = n = 5
A = matrix(a, ncol = 5, nrow = 5)
A
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    7    7   16   10   10
## [2,]    3    2   10    6   26
## [3,]    3    7    4    7    9
## [4,]    4   30    6    3    8
## [5,]    6    5    9   17    4
```

The diagonal elements of A can be obtained using the `R` function `diag()`

```
diag(A)
```

```
## [1] 7 2 4 3 4
```

If $a_{ij} = 0,\ i \neq j,\ m = n$ then A is a diagonal matrix and is denoted as

$$A = \text{diag}(a_{11}, a_{22}, \dots, a_{mm})$$

```
diag(c(560,  220,  340,   80,  150 ))
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]  560    0    0    0    0
## [2,]    0  220    0    0    0
## [3,]    0    0  340    0    0
## [4,]    0    0    0   80    0
## [5,]    0    0    0    0  150
```

Null Matrix: A matrix whose all elements are equal to zero is called a null matrix.

```
O = matrix(0, nrow = 3, ncol = 3)
O
```

```
##      [,1] [,2] [,3]
## [1,]    0    0    0
## [2,]    0    0    0
## [3,]    0    0    0
```

Identity Matrix: The identity matrix of size n is the $n \times n$ square matrix with ones on the main diagonal and zeros elsewhere. We can write

$$I_n = \text{diag}(1, 1, \ldots, 1)$$

```
I=diag(5)
I
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    1    0    0    0    0
## [2,]    0    1    0    0    0
## [3,]    0    0    1    0    0
## [4,]    0    0    0    1    0
## [5,]    0    0    0    0    1
```

If $m = n$ (square matrix) and $a_{ij} = 0$ for $i > j$, then A is called an upper triangular matrix. On the other hand if $m = n$ and $a_{ij} = 0$ for $i < j$ then A is called a lower triangular matrix.

```
a = c(7,  3 , 3 , 4,  6,  7,  2,  7, 30,  5, 16, 10,  4,  6,  9, 10,  6,  7,  3, 17, 10, 26,  9,  8,  4)
A = matrix(a, ncol = 5, nrow = 5)
A
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    7    7   16   10   10
## [2,]    3    2   10    6   26
## [3,]    3    7    4    7    9
## [4,]    4   30    6    3    8
## [5,]    6    5    9   17    4
```

```
triu(A) # upper triangular
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    7    7   16   10   10
## [2,]    0    2   10    6   26
## [3,]    0    0    4    7    9
## [4,]    0    0    0    3    8
## [5,]    0    0    0    0    4
```

```
tril(A) # lower triangular
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    7    0    0    0    0
## [2,]    3    2    0    0    0
## [3,]    3    7    4    0    0
## [4,]    4   30    6    3    0
## [5,]    6    5    9   17    4
```

Symmetric Matrix: If $A = A'$ then A is a symmetric matrix.

```r
a = c(1, 7, 3, 7, 4, 5, 3, 5, 6)
A = matrix(a, ncol = 3, nrow = 3)
A
```

```
##      [,1] [,2] [,3]
## [1,]    1    7    3
## [2,]    7    4    5
## [3,]    3    5    6
```

```r
# Let's find transpose of A
t(A)
```

```
##      [,1] [,2] [,3]
## [1,]    1    7    3
## [2,]    7    4    5
## [3,]    3    5    6
```

Skew-Symmetric Matrix If $A = -A'$ then A is skew-symmetric matrix.

```r
a =  c(0, -3, 2, 3, 0, -1, -2, 1, 0)
A = matrix(a, ncol = 3, nrow = 3, byrow = TRUE)
A
```

```
##      [,1] [,2] [,3]
## [1,]    0   -3    2
## [2,]    3    0   -1
## [3,]   -2    1    0
```

```r
# Let's find transpose of A
t(A) # Not symmetric
```

```
##      [,1] [,2] [,3]
## [1,]    0    3   -2
## [2,]   -3    0    1
## [3,]    2   -1    0
```

```r
# Let's find (-A)'
A_neg = -A
t(A_neg) # (-A)' = A
```

```
##      [,1] [,2] [,3]
## [1,]    0   -3    2
## [2,]    3    0   -1
## [3,]   -2    1    0
```

If A and B are matrices of order $m \times n$ then $(A + B)' = A' + B'$

```r
a = c(7,  3 , 3 , 4,  6,  7,  2,  7, 30)
A = matrix(a, 3, 3)
A
```

```
##      [,1] [,2] [,3]
## [1,]    7    4    2
## [2,]    3    6    7
## [3,]    3    7   30
```

```r
At = t(A)
At
```

```
##      [,1] [,2] [,3]
## [1,]    7    3    3
## [2,]    4    6    7
## [3,]    2    7   30
```

```r
b = c(5, 16, 10,  4,  6,  9, 10,  6,  7)
B = matrix(b, 3, 3)
B
```

```
##      [,1] [,2] [,3]
## [1,]    5    4   10
## [2,]   16    6    6
## [3,]   10    9    7
```

```r
Bt = t(B)
Bt
```

```
##      [,1] [,2] [,3]
## [1,]    5   16   10
## [2,]    4    6    9
## [3,]   10    6    7
```

```r
# Sum of A + B
C = A + B
C
```

```
##      [,1] [,2] [,3]
## [1,]   12    8   12
## [2,]   19   12   13
## [3,]   13   16   37
```

```r
# transpose of a matrix C
Ct = t(C)
Ct
```

```
##      [,1] [,2] [,3]
## [1,]   12   19   13
## [2,]    8   12   16
## [3,]   12   13   37
```

```r
# Sum of At + Bt
At + Bt
```

```
##      [,1] [,2] [,3]
## [1,]   12   19   13
## [2,]    8   12   16
## [3,]   12   13   37
```

If A and B are the matrices of order $m \times n$ and $n \times p$ respectively and k is any scalar, then

$$(AB)' = B'A'$$
$$(kA)B = A(kB) = k(AB) = kAB$$

```r
# Multiply A*B
D = A %*% B
D
```

```
##      [,1] [,2] [,3]
## [1,]  119   70  108
## [2,]  181  111  115
## [3,]  427  324  282
```

```r
Dt = t(D) # transpose of D
Dt
```

```
##      [,1] [,2] [,3]
## [1,]  119  181  427
## [2,]   70  111  324
## [3,]  108  115  282
```

```r
At_Bt = At %*% Bt # transpose(A) * transpose(B)
At_Bt
```

```
##      [,1] [,2] [,3]
## [1,]   77  148  118
## [2,]  114  142  143
## [3,]  338  254  293
```

If the orders of matrices A is $m \times n$, B is $n \times p$ and C is $n \times p$ then $A(B+C) = AB + AC$

```r
# A is a 4 by 3 matrix
a = c(7,  3 , 3 , 4,  6,  7,  2,  7, 30, 5, 16, 10)
A = matrix(a, 4, 3) # Matrix A
A
```

```
##      [,1] [,2] [,3]
## [1,]    7    6   30
## [2,]    3    7    5
## [3,]    3    2   16
## [4,]    4    7   10
```

```r
# B is a 3 by 5 matrix
b = c(5, 16, 10,  4,  6,  9, 10,  6,  7, 3, 17, 10, 26,  9,  8)
B = matrix(b, 3, 5) # Matrix B
B
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    5    4   10    3   26
## [2,]   16    6    6   17    9
## [3,]   10    9    7   10    8
```

```r
# C is a 3 by 5 matrix
c = c(3, 17, 10, 26,  9,  8,  4, 23, 16, 8, 17, 5.5, 19, 24, 2.5, 7.5)
C = matrix(c, 3, 5) # Matrix C
C
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    3   26    4  8.0 19.0
## [2,]   17    9   23 17.0 24.0
## [3,]   10    8   16  5.5  2.5
```

```
# Let's find A(B + C)
A %*% (B + C)
```

```
##      [,1] [,2] [,3]  [,4]  [,5]
## [1,]  854  810  962 746.0 828.0
## [2,]  355  280  360 348.5 418.5
## [3,]  410  392  468 349.0 369.0
## [4,]  463  395  489 437.0 516.0
```

```
# Let's find AB + AC
A %*% B + A %*% C
```

```
##      [,1] [,2] [,3]  [,4]  [,5]
## [1,]  854  810  962 746.0 828.0
## [2,]  355  280  360 348.5 418.5
## [3,]  410  392  468 349.0 369.0
## [4,]  463  395  489 437.0 516.0
```

If the orders of matrices A is $m \times n$, B is $n \times p$ and C is $p \times q$ then $(AB)C = A(BC)$

```
# A is a 4 by 3 matrix
a = c(7,  3 , 3 , 4,  6,  7,  2,  7, 30, 5, 16, 10)
A = matrix(a, 4, 3) # Matrix A
A
```

```
##      [,1] [,2] [,3]
## [1,]    7    6   30
## [2,]    3    7    5
## [3,]    3    2   16
## [4,]    4    7   10
```

```
# B is a 3 by 5 matrix
b = c(5, 16, 10,  4,  6,  9, 10,  6,  7, 3, 17, 10, 26,  9,  8)
B = matrix(b, 3, 5) # Matrix B
B
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    5    4   10    3   26
## [2,]   16    6    6   17    9
## [3,]   10    9    7   10    8
```

```
# C is a 5 by 2 matrix
c = c(3, 17, 10, 26,  9,  8,  4, 23, 16, 8)
C = matrix(c, 5, 2) # Matrix C
C
```

```
##      [,1] [,2]
## [1,]    3    8
## [2,]   17    4
## [3,]   10   23
## [4,]   26   16
## [5,]    9    8
```

```
# Let's find (AB)C
(A %*% B) %*% C
```

```
##       [,1]  [,2]
## [1,] 25413 22628
## [2,]  9541  8569
## [3,] 12311 10910
## [4,] 12961 11616
```

```r
# Let's find A(BC)
A %*% (B %*% C)
```

```
##       [,1]  [,2]
## [1,] 25413 22628
## [2,]  9541  8569
## [3,] 12311 10910
## [4,] 12961 11616
```

If A is the matrix of order $m \times n$ then $I_m A = A I_n = A$

```r
# A is a 4 by 3 matrix
a = c(7,  3 , 3 , 4,  6,  7,  2,  7, 30, 5, 16, 10)
A = matrix(a, 4, 3) # Matrix A
A
```

```
##      [,1] [,2] [,3]
## [1,]    7    6   30
## [2,]    3    7    5
## [3,]    3    2   16
## [4,]    4    7   10
```

```r
I_4  = diag(rep(1,4)) # Identity matrix 4 by 4
I_4
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    0    0    0
## [2,]    0    1    0    0
## [3,]    0    0    1    0
## [4,]    0    0    0    1
```

```r
I_3  = diag(rep(1,3)) # Identity matrix 3 by 3
I_3
```

```
##      [,1] [,2] [,3]
## [1,]    1    0    0
## [2,]    0    1    0
## [3,]    0    0    1
```

```r
# Let's find I_4*A
I_4 %*% A
```

```
##      [,1] [,2] [,3]
## [1,]    7    6   30
## [2,]    3    7    5
## [3,]    3    2   16
## [4,]    4    7   10
```

```r
# Let's find A*I_3
A %*% I_3
```

```
##      [,1] [,2] [,3]
## [1,]    7    6   30
## [2,]    3    7    5
## [3,]    3    2   16
## [4,]    4    7   10
```

Trace of a Matrix

The trace of $n \times n$ matrix A , denoted as $tr(A)$ or $tr(A)$ is defined to be the sum of all the diagonal elements of A, i.e.

$$tr(A) = \sum_{i=1}^{n} a_{ii}$$

```
# Create a 3 by 3 matrix
a = c(7,  3 , 3 , 4,  6,  7,  2,  7, 30)
A = matrix(a, 3, 3)
A
```

```
##      [,1] [,2] [,3]
## [1,]    7    4    2
## [2,]    3    6    7
## [3,]    3    7   30
```

```
# trace of A
tr(A)
```

```
## [1] 43
```

If A is of order $m \times n$ and B is of order $n \times m$, then $tr(AB) = tr(BA)$

```
# A is a 4 by 3 matrix
a = c(7,  3 , 3 , 4,  6,  7,  2,  7, 30, 5, 16, 10)
A = matrix(a, 4, 3) # Matrix A
A
```

```
##      [,1] [,2] [,3]
## [1,]    7    6   30
## [2,]    3    7    5
## [3,]    3    2   16
## [4,]    4    7   10
```

```
# B is a 3 by 4 matrix
b = c(5, 16, 10,  4,  6,  9, 10,  6,  7, 3, 17, 10)
B = matrix(b, 3, 4) # Matrix B
B
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    5    4   10    3
## [2,]   16    6    6   17
## [3,]   10    9    7   10
```

```
# Let's find tr(AB)
tr(A %*% B)
```

```
## [1] 915
```

```
# Let's find tr(BA)
tr(B %*% A)
```

```
## [1] 915
```

If A and B are $n \times n$ matrices, a and b are scalars then $tr(uA + vB) = utr(A) + vtr(B)$

```
# A is a 4 by 4 matrix
a = c(7,  3 , 3 , 4,  6,  7,  2,  7, 30, 5, 16, 10, 15.5, 23.75,  8, 17)
A = matrix(a, 4, 4) # Matrix A
A
```

```
##      [,1] [,2] [,3]  [,4]
## [1,]    7    6   30 15.50
## [2,]    3    7    5 23.75
## [3,]    3    2   16  8.00
## [4,]    4    7   10 17.00
```

```
# B is a 4 by 4 matrix
b = c(5, 16, 10,  4,  6,  9, 10,  6,  7, 3, 17, 10, 7.5, 11, 13,  3.75)
B = matrix(b, 4, 4) # Matrix B
B
```

```
##      [,1] [,2] [,3]  [,4]
## [1,]    5    6    7  7.50
## [2,]   16    9    3 11.00
## [3,]   10   10   17 13.00
## [4,]    4    6   10  3.75
```

```
# Let u = 2 and v = 4 be constants. Let's find tr(u*A + v*B)
u = 2
v = 4
tr(u*A + v*B)
```

```
## [1] 233
```

```
# Let's find u*tr(A) + v*tr(B)
u*tr(A) + v*tr(B)
```

```
## [1] 233
```

If A is an $m \times n$ matrix, then

$$tr(A'A) = tr(AA') = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}^2$$

and $tr(A'A) = tr(AA') = 0$ if and only if $A = 0$

If A is $n \times n$ matrix then $tr(A')tr(A)$

```
# A is a 4 by 4 matrix
a = c(7,  3 , 3 , 4,  6,  7,  2,  7, 30, 5, 16, 10, 15.5, 23.75,  8, 17)
A = matrix(a, 4, 4) # Matrix A
A
```

```
##      [,1] [,2] [,3]  [,4]
## [1,]    7    6   30 15.50
## [2,]    3    7    5 23.75
## [3,]    3    2   16  8.00
## [4,]    4    7   10 17.00
```

```
# Let's find tr(A'A)
tr(t(A) %*% A)
```

```
## [1] 2659.312
```

```r
# Let's find tr(AA')
tr(A %*% t(A))
```

```
## [1] 2659.312
```

```r
sum(A^2)
```

```
## [1] 2659.312
```

Rank of Matrices

The rank of a matrix A of $m \times n$ is the number of linearly independent rows in A. Let B be any other matrix of order $n \times q$.

1. A square matrix of order m is called **non-singular** if it has full rank. $rank(AB) \leq \min\{rank(A), rank(B)\}$
2. $rank(A+B) \leq rank(A) + rank(B)$
3. $rank(A)$ is equal to the maximum order of all nonsingular square sub-matrices of A.
4. $rank(AA') = rank(A'A) = rank(A) = rank(A')$.
5. A is of full row rank if $rank(A) = m < n$.
6. A is of full column rank if $rank(A) = n < m$

We will use the `r` function `qr()`

```r
# A is a 4 by 3 matrix
a = c(7,  3 , 3 , 4,  6,  7,  2,  7, 30, 5, 16, 10)
A = matrix(a, 4, 3, byrow = T) # Matrix A
A
```

```
##      [,1] [,2] [,3]
## [1,]    7    3    3
## [2,]    4    6    7
## [3,]    2    7   30
## [4,]    5   16   10
```

```r
rref(A) # Reduced Row Echelon Form ( r package: pracma)
```

```
##      [,1] [,2] [,3]
## [1,]    1    0    0
## [2,]    0    1    0
## [3,]    0    0    1
## [4,]    0    0    0
```

```r
qr(A)$rank
```

```
## [1] 3
```

```r
# B is a 3 by 5 matrix
b = c(5, 16, 10,  4,  6,  9, 10,  6,  7, 3, 17, 10, 23.75,  8, 17)
B = matrix(b, 3, 5) # Matrix B
B
```

```
##      [,1] [,2] [,3] [,4]  [,5]
## [1,]    5    4   10    3 23.75
## [2,]   16    6    6   17  8.00
## [3,]   10    9    7   10 17.00
```

```r
qr(B)$rank
```

```
## [1] 3
```

```r
# C is a 4 by 3 matrix
c = c(3, 17, 10, 26,  9,  8, 9, 51, 30)
C = matrix(c, 3, 3, byrow = T) # Matrix C
C
```

```
##      [,1] [,2] [,3]
## [1,]    3   17   10
## [2,]   26    9    8
## [3,]    9   51   30
```

```r
qr(C)$rank
```

```
## [1] 2
```

```r
d = c(1,2,1,3,6,3,2,4,2)
D = matrix(d, nrow=3, ncol = 3, byrow=F)
D
```

```
##      [,1] [,2] [,3]
## [1,]    1    3    2
## [2,]    2    6    4
## [3,]    1    3    2
```

```r
qr(D)$rank
```

```
## [1] 1
```

The determinant of a matrix

The determinant of a matrix is a scalar value that is a function of the entries of a square matrix. The determinant of a matrix A is denoted $det(A)$ or $|A|$.

1. For a 2×2 matrix

$$det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11}a_{22} - a_{12}a_{21})$$

2. For a 3×3 matrix

$$det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

```r
# A is a 2 by 2 matrix
a = c(7,  3 , 3 , 4)
A = matrix(a, 2, 2, byrow = TRUE) # Matrix A
A
```

```
##      [,1] [,2]
## [1,]    7    3
## [2,]    3    4
```

```r
# Let's find det(A)
det(A)
```

```
## [1] 19
```

```r
# B is a 3 by 3 matrix
b = c(5, 16, 10,  4,  6,  9, 10,  6,  7)
B = matrix(b, 3, 3, byrow = TRUE) # Matrix B
B
```

```
##      [,1] [,2] [,3]
## [1,]    5   16   10
## [2,]    4    6    9
## [3,]   10    6    7
```

```r
# Let's find det(B)
det(B)
```

```
## [1] 572
```

Inverse of a Matrix

The inverse of a square matrix A of order m, is a square matrix of order m, denoted as A^{-1}, such that $A^{-1}A = AA^{-1} = I_m$. The inverse of A exists if and only if A is non-singular. For a 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

the matrix inverse is

$$A^{-1} = \frac{1}{det(A)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

where $det(A) = (a_{11}a_{22} - a_{12}a_{21})$

```r
# A is a 4 by 4 matrix
a = c(7,  3 , 3 , 4,  6,  7,  2,  7, 30, 5, 16, 10, 15.5, 23.75,  8, 17)
A = matrix(a, 4, 4) # Matrix A
A
```

```
##      [,1] [,2] [,3]  [,4]
## [1,]    7    6   30 15.50
## [2,]    3    7    5 23.75
## [3,]    3    2   16  8.00
## [4,]    4    7   10 17.00
```

```r
# Inverse of a Matrix A
A_inv = solve(A)
A_inv
```

```
##             [,1]       [,2]        [,3]       [,4]
## [1,]  2.57394366  1.3591549 -3.67957746 -2.5140845
## [2,] -1.14964789 -0.8697183  1.43485915  1.5880282
## [3,] -0.38644366 -0.2341549  0.61707746  0.3890845
## [4,]  0.09507042  0.1760563 -0.08802817 -0.2323944
```

```r
# Let's find A_inv*A
A_invA = A_inv %*% A
round(A_invA, 4) # Round to four decimal places
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    0    0    0
## [2,]    0    1    0    0
## [3,]    0    0    1    0
## [4,]    0    0    0    1
```

```r
# Let's find A*A_inv
AA_inv = A %*% A_inv
round(AA_inv, 4) # Round to four decimal places
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    0    0    0
## [2,]    0    1    0    0
## [3,]    0    0    1    0
## [4,]    0    0    0    1
```

1. $(A^{-1})^{-1} = A$
2. If A is non-singular, then $(A')^{-1} = (A^{-1})'$
3. If A and B are non-singular matrices of the same order, then their product, if defined, is also nonsingular and
$$(AB)^{-1} = B^{-1}A^{-1}$$
.

```r
# A is a 4 by 4 matrix
a = c(7,  3 , 3 , 4,  6,  7,  2,  7, 30, 5, 16, 10, 15.5, 23.75,  8, 17)
A = matrix(a, 4, 4) # Matrix A
A
```

```
##      [,1] [,2] [,3]  [,4]
## [1,]    7    6   30 15.50
## [2,]    3    7    5 23.75
## [3,]    3    2   16  8.00
## [4,]    4    7   10 17.00
```

```r
# Inverse of a Matrix A
A_inv = solve(A)
A_inv
```

```
##             [,1]       [,2]        [,3]       [,4]
## [1,]  2.57394366  1.3591549 -3.67957746 -2.5140845
## [2,] -1.14964789 -0.8697183  1.43485915  1.5880282
## [3,] -0.38644366 -0.2341549  0.61707746  0.3890845
## [4,]  0.09507042  0.1760563 -0.08802817 -0.2323944
```

```r
# 1) Inverse of an Inverse
solve(A_inv)
```

```
##      [,1] [,2] [,3]  [,4]
## [1,]    7    6   30 15.50
## [2,]    3    7    5 23.75
## [3,]    3    2   16  8.00
## [4,]    4    7   10 17.00
```

```r
# Transpose of matrix A
At = t(A)
At
```

```
##      [,1]  [,2] [,3] [,4]
## [1,]  7.0  3.00    3    4
## [2,]  6.0  7.00    2    7
## [3,] 30.0  5.00   16   10
## [4,] 15.5 23.75    8   17
```

```r
# 2) Inverse of At
At_inv = solve(At)
At_inv
```

```
##           [,1]       [,2]       [,3]        [,4]
## [1,]  2.573944 -1.1496479 -0.3864437  0.09507042
## [2,]  1.359155 -0.8697183 -0.2341549  0.17605634
## [3,] -3.679577  1.4348592  0.6170775 -0.08802817
## [4,] -2.514085  1.5880282  0.3890845 -0.23239437
```

```r
A_invt = t(A_inv)
A_invt
```

```
##           [,1]       [,2]       [,3]        [,4]
## [1,]  2.573944 -1.1496479 -0.3864437  0.09507042
## [2,]  1.359155 -0.8697183 -0.2341549  0.17605634
## [3,] -3.679577  1.4348592  0.6170775 -0.08802817
## [4,] -2.514085  1.5880282  0.3890845 -0.23239437
```

```r
# B is a 4 by 4 matrix
b = c(5, 16, 10,  4,  6,  9, 10,  6,  7, 3, 17, 10, 7.5, 11, 13,  3.75)
B = matrix(b, 4, 4) # Matrix B
B
```

```
##      [,1] [,2] [,3]  [,4]
## [1,]    5    6    7  7.50
## [2,]   16    9    3 11.00
## [3,]   10   10   17 13.00
## [4,]    4    6   10  3.75
```

```r
# Let's find inverse of AB
AB_inv = solve(A %*% B)
AB_inv
```

```
##            [,1]       [,2]       [,3]       [,4]
## [1,] -1.0888083 -0.6021291  1.5455475  1.1072524
## [2,]  1.4083154  0.7884752 -2.0250054 -1.4159614
## [3,] -0.5820977 -0.3024236  0.8474547  0.5556390
## [4,]  0.4857036  0.2341219 -0.6919287 -0.4592068
```

```r
# Let's find inverse of matrix A and inverse of matrix B separately
A_inv = solve(A)
A_inv
```

```
##             [,1]       [,2]        [,3]       [,4]
## [1,]  2.57394366  1.3591549 -3.67957746 -2.5140845
## [2,] -1.14964789 -0.8697183  1.43485915  1.5880282
## [3,] -0.38644366 -0.2341549  0.61707746  0.3890845
## [4,]  0.09507042  0.1760563 -0.08802817 -0.2323944
```

```r
B_inv = solve(B)
B_inv
```

```
##            [,1]         [,2]        [,3]        [,4]
## [1,] -0.3623929  0.094109064  0.12639066  0.01057815
## [2,]  0.4847711  0.007477658 -0.36786431  0.28378625
## [3,] -0.2170345 -0.025168703  0.14061645  0.02042677
## [4,]  0.1896772 -0.045230713  0.07878898 -0.25314609
```

```r
# Lets find the product of (inverse of matrix B)*(inverse of matrix A)
B_inv %*% A_inv
```

```
##            [,1]       [,2]       [,3]       [,4]
## [1,] -1.0888083 -0.6021291  1.5455475  1.1072524
## [2,]  1.4083154  0.7884752 -2.0250054 -1.4159614
## [3,] -0.5820977 -0.3024236  0.8474547  0.5556390
## [4,]  0.4857036  0.2341219 -0.6919287 -0.4592068
```

Idempotent Matrix

A square matrix A is called idempotent if $A^2 = AA = A$. If A is an $n \times n$ idempotent matrix with $rank(A) = r < n$. Then

1. eigenvalues of A are 1 or 0.
2. $tr(A) = rank(A) = r$
3. If A is of full rank n, then $A = I_n$.
4. If A and B are idempotent and $AB = BA$, then AB is also idempotent.
5. If A is idempotent then $(I - A)$ is also idempotent and $A(I - A) = (I - A)A = 0$

```r
#  A is a 3 by 3 matrix
a = c(2, -2, -4, -1, 3, 4, 1, -2, -3)
A = matrix(a, nrow = 3, byrow = TRUE)
A
```

```
##      [,1] [,2] [,3]
## [1,]    2   -2   -4
## [2,]   -1    3    4
## [3,]    1   -2   -3
```

```r
# Let's find the rank of matrix A
qr(A)$rank # rank(A) = 2 < 3
```

```
## [1] 2
```

```r
# Let's find the trace of matrix A
tr(A) # So tr(A) = rank(A)
```

```
## [1] 2
```

```r
# Let's check A*A = A ?
A %*% A
```

```
##      [,1] [,2] [,3]
## [1,]    2   -2   -4
## [2,]   -1    3    4
## [3,]    1   -2   -3
```

```r
# Identity matrix 3 by 3
I_3  = diag(rep(1,3))
I_3
```

```
##      [,1] [,2] [,3]
## [1,]    1    0    0
## [2,]    0    1    0
## [3,]    0    0    1
```

```r
# Let's find (I-A)
I_minus_A = I_3 - A
# Let's find A(I-A)
A %*% I_minus_A # this is a 3 by 3 null matrix
```

```
##      [,1] [,2] [,3]
## [1,]    0    0    0
## [2,]    0    0    0
## [3,]    0    0    0
```

Quadratic Forms

If A is a given matrix of order $m \times n$ and X and Y are two given vectors of order $m \times 1$ and $n \times 1$ respectively

$$X'AY = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_i y_j$$

where a_{ij} are the nonstochastic elements of A

1. If A is a square matrix of order m and $X = Y$, then

$$X'AX = a_{11}x_1^2 + \cdots + a_{mm}x_m^2 + (a_{12} + a_{21})x_1x_2 + \cdots + (a_{m-1,m} + a_{m,m-1})x_{m-1}x_m$$

2. If A is symmetric also, then

$$X'AX = a_{11}x_1^2 + \cdots + a_{mm}x_m^2 + 2a_{12}x_1x_2 + \cdots + 2a_{m-1,m}x_{m-1}x_m = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_i x_j$$

is called a quadratic form in m variables $x_1, x_2, \ldots, x_m$ or a quadratic form in X.

```r
x = c(10, 26,  9,  8,  4) # row vector
X = matrix(x) # column vector
X
```

```
##      [,1]
## [1,]   10
## [2,]   26
## [3,]    9
## [4,]    8
## [5,]    4
```

```r
a = c(7,  3 , 3 , 4,  6,  7,  2,  7, 30,  5, 16, 10,  4,  6,  9, 10,  6,  7,  3, 17, 10, 26,  9,  8,  4)
A = matrix(a, ncol = 5, nrow = 5) # Matrix A
A
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    7    7   16   10   10
## [2,]    3    2   10    6   26
## [3,]    3    7    4    7    9
## [4,]    4   30    6    3    8
## [5,]    6    5    9   17    4
```

```
Quadratic = t(X) %*% A %*% X  # X'AX
Quadratic
```

```
##       [,1]
## [1,] 25776
```

Note

1. Positive definite if $A'AX > 0$ for all $X \neq 0$
2. Positive semidefinite if $A'AX \geq 0$ for all $X \neq 0$.
3. Negative definite if $A'AX < 0$ for all $X \neq 0$.
4. Negative semidefinite if $A'AX \leq 0$ for all $X \neq 0$.
5. If P is any nonsingular matrix and A is any positive definite matrix (or positive semi-definite matrix) then $P'AP$ is also a positive definite matrix (or positive semi-definite matrix).
6. A matrix A is positive definite if and only if there exists a non-singular matrix P such that $A = P'P$.
7. A positive definite matrix is a nonsingular matrix.
8. If A is $m \times n$ matrix and $rank(A) = m < n$ then AA' is positive definite and $A'A$ is positive semidefinite.
9. If A is $m \times n$ matrix and $rank(A) = k < m < n$, then both $A'A$ and AA' are positive semidefinite.

Orthogonal Matrix

A square matrix A is called an orthogonal matrix if $A'A = AA' = I$ or equivalently if $A^{-1} = A'$

1. An orthogonal matrix is non-singular.
2. If A is orthogonal, then AA' is also orthogonal.
3. If A is an $n\times$ matrix and let P is an $n \times n$ orthogonal matrix, then the determinants of A and $P'AP$ are the same.

Random Vectors

Let $Y_1, Y_2, \ldots, Y_n$ be n random variables then $Y = (Y_1, Y_2, \ldots, Y_n)'$ is called a random vector.

1. The mean vector Y is

$$E(Y) = (E(Y_1), E(Y_2), \ldots, E(Y_n))'$$

2. The covariance matrix or dispersion matrix of Y is

$$Var(Y) = \begin{pmatrix} Var(Y_1) & Cov(Y_1, Y_2) & \cdots & Cov(Y_1, Y_n) \\ Cov(Y_2, Y_1) & Var(Y_2) & \cdots & Cov(Y_2, Y_n) \\ \vdots & \vdots & \ddots & \vdots \\ Cov(Y_n, Y_1) & Cov(Y_n, Y_2) & \cdots & Var(Y_n) \end{pmatrix}$$

which is a symmetric matrix.

1. If $Y_1, Y_2, \ldots, Y_n$ are independently distributed, then the covariance matrix is a diagonal matrix
2. If $Var(Y_i) = \sigma^2$ for all $i = 1, 2, \ldots, n$ then $Var(Y) = \sigma^2 I_n$

Matrix Derivatives

Let A be a $k \times k$ matrix of constants, a be a $k \times 1$ vector of constants, and y be a $k \times 1$ vector of variables.

- If $z = a'y$, then

$$\frac{\partial z}{\partial y} = \frac{\partial a'y}{\partial y} = a$$

- If $z = y'y$, then

$$\frac{\partial z}{\partial y} = \frac{\partial y'y}{\partial y} = 2y$$

- If $z = a'Ay$, then

$$\frac{\partial z}{\partial y} = \frac{\partial a'Ay}{\partial y} = A'a$$

- If $z = y'Ay$, then

$$\frac{\partial z}{\partial y} = \frac{\partial y'Ay}{\partial y} = Ay + A'y$$

If A is symmetric, then

$$\frac{\partial z}{\partial y} = \frac{\partial y'Ay}{\partial y} = 2Ay$$

Expectations

Let A be a $k \times k$ matrix of constants, a be a $k \times 1$ vector of constants, and y be a $k \times 1$ random vector with mean μ and nonsingular variance - covariance matrix V.

- $E(a'y) = a'\mu$.
- $E(Ay) = A\mu$.
- $Var(a'y) = a'Va$.
- $Var(Ay) = AVA'$.

Note: If $V = \sigma^2 I$, then $Var(Ay) = \sigma^2 AA'$.

- $E(y'Ay) = trace(AV) + \mu' A\mu$.

Note: If $V = \sigma^2 I$, then $E(y'Ay) = \sigma^2 trace(A) + \mu' A\mu$.

References

- Introduction to Linear Regression Analysis, 5th Edition, by Douglas C. Montgomery, Elizabeth A. Peck, G. Geoffrey Vining (Wiley), ISBN: 978-0-470-54281-1.
- R Core Team (2020). R: A Language and Environment for Statistical Computing. Vienna, Austria: R Foundation for Statistical Computing.
- RStudio Team (2020). RStudio: Integrated Development Environment for R. Boston, MA: RStudio, PBC.