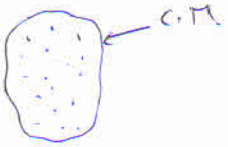
Lecture - 2 (8 - Jan - 2020)

Review of Fluid Mechanics

1. Control volume (CV) and Control Mass (system) analysis

Control Mass or System Approach



The the mass particles remains the ~~same~~ same as the system moves.

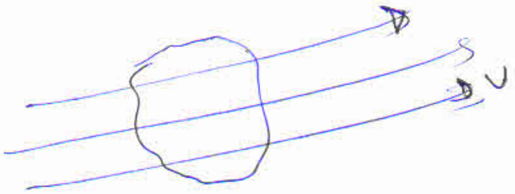
→ Since the fluid is squashy, ~~the~~ to contain the same mass particles, the system boundaries will change in time,

→ The analysis becomes complex, since the system boundary changes with time.

→ However all Newtonian Laws are written for a system.

→ Mass conservation is naturally satisfied $\frac{dM}{dt} = 0$ for the system

Control Volume (C.V)



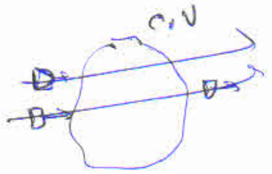
→ Fixed volume in space through which fluid crosses.

→ Easy for fluid flow analysis since C.V shape is fixed in space.

→ However Newtonian laws are defined for control mass (system) and not for C.V

→ For eg: Mass conservation for system is $\frac{dM}{dt} = 0$ ②

However for C.V $\frac{dM}{dt} \neq 0$



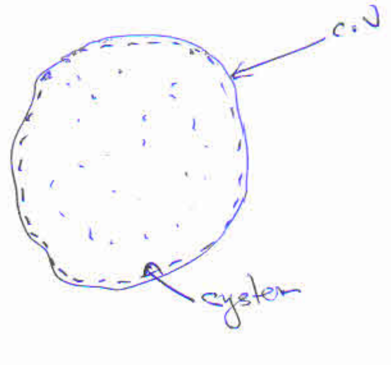
→ There can be fluid crossing the C.V boundary with out any uniform flux, leading to mass accumulation.

→ Therefore the conservation laws for C.V approach takes different forms than that of the system approach.

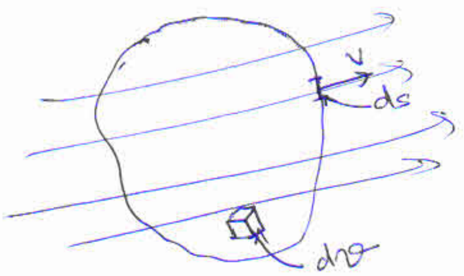
Reynolds Transport Theorem

→ Relates the conservation laws for system to C.V

→ Consider a C.V fixed in space. At $t=0$, ~~assume a~~ consider a system which exactly matches with the C.V



→ Consider any property N whose variation is sought with in the C.V, in a time interval of Δt.



→ Since there is inflow and outflow of mass with in the C.V, the property will be transported through the C.V boundary with the net ~~flux~~ flux of fluid ~~matter~~ mass crossing the boundary

Let η be the specific quantity of N

ie $\eta = \frac{N}{m}$

→ The net flux of property η crossing the boundary

$$\int_{cs} \rho \, v \cdot n \, ds \, \eta.$$

→ Similarly, ~~there~~ the time rate of change of property η with in the C.V is,

$$\frac{\partial}{\partial t} \int_{cv} \rho \eta \, dv.$$

The ~~net~~ total change of property within C.V in a time interval Δt

$$\Rightarrow \frac{\partial}{\partial t} \int_{cv} \rho \eta \, dv + \int_{cs} \rho \eta \, v \cdot ds$$

Similarly the time rate of change of property 'N' in the system as it moves is $\frac{dN}{dt}$

→ If we consider a very small time interval, $\Delta t \to 0$, then the movement of the system is infinitesimally small and the system and c.v can be assumed to occupy the same region.

→ In this limit $\Delta t \to 0$, the changes happening in system can be equated to the changes happening in c.V ie Reynold's Transport theorem.

$$\boxed{\lim_{\Delta t \to 0} \left.\frac{dN}{dt}\right)_{system} = \frac{d}{dt}\int_{CV} \rho \eta \, d\forall + \int_{CS} \rho \eta \, V \cdot n \, ds.}$$

The equivalent form in differential approach is called the Material derivative | substantial derivative or Total derivative.

$$\frac{d}{dt} \quad \boxed{\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \vec{V} \cdot \nabla \phi}$$

Mass Conservation.

$N = M \quad ; \quad \eta = 1.$

$$\left.\frac{DM}{Dt}\right)_{system} = 0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \, V \cdot n \, ds.$$

ie $$\boxed{\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \, V \cdot n \, ds. = 0}$$

$$\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CV} \nabla \cdot (\rho \vec{V}) \, d\forall = 0$$

$$\int_{CV} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V}\right] d\forall = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V} = 0.}$$ ⇒ Mass conservation in differential form / ~~Cont~~ Continuity equation

$$\frac{\partial \rho}{\partial t} + \rho (\nabla \cdot \vec{V}) + \vec{V} \cdot \nabla \rho \Rightarrow \boxed{\frac{D\rho}{Dt} + \rho (\nabla \cdot \vec{V}) = 0}$$ Another form.

$$\boxed{\nabla \cdot \vec{V} = \frac{1}{\forall}\frac{D\forall}{Dt}}$$

for Incompressible flows $\frac{D\forall}{Dt} = 0$ or $\nabla \cdot \vec{V} = 0$

Incompressible C.E ⇒ $\frac{D\rho}{Dt} = 0$ or $\nabla \cdot \vec{V} = 0$

Conservation of Momentum

$N = m\vec{V}$

$\eta = \vec{V}$

$\Sigma F =$ rate of change of momentum.

~~$\frac{d\, m \cdot \vec{V}}{dt} = \Sigma F$~~

$$\left[\frac{d(m\vec{V})}{dt}\right]_{system} = \Sigma F.$$

$$\frac{\partial}{\partial t}\int_{cv} \rho \vec{V}\, dv + \int_{cs} \rho \vec{V}\, \vec{V} \cdot n\, ds = \cancel{\Sigma F} = -\int_{cs} P\, ds + \int_{cv} \rho f\, dv$$

$\Rightarrow \int_{cv} (\nabla \cdot \rho \vec{V}\vec{V})\, dv \qquad \Rightarrow -\int_{cv} \nabla P\, dv$

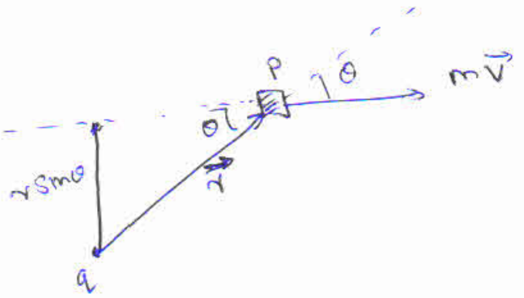
$$\boxed{\frac{\partial(\rho\vec{V})}{\partial t} + \nabla \cdot \rho\vec{V}\vec{V} = -\nabla P + \rho f}$$

or

$$\boxed{\frac{\partial V}{\partial t} + \vec{V} \cdot \nabla \vec{V} = -\frac{\nabla P}{\rho} + \vec{f}}$$

Conservation of Angular Momentum

Rate of change of angular momentum $= \Sigma$ Torque.



Angular momentum is defined as the cross product of the position vector of a mass particle w.r.t to a datum point q, to the linear momentum of the mass particle.

$$\boxed{\vec{L} = (\vec{r} \times m\vec{V}) \Rightarrow (\vec{r} \times \vec{V})\, m}$$

$$\frac{D(\vec{L})}{Dt} = \Sigma \tau$$

$N = \vec{L} = (\vec{r} \times \vec{V})\, m$

$\eta = (\vec{r} \times \vec{V})$

$$\boxed{\frac{D(\vec{r} \times \vec{V})m}{Dt} = \frac{\partial}{\partial t}\int_{cv} \rho(\vec{r} \times \vec{V})\, dv + \int_{cs} \rho(\vec{r} \times \vec{V})\, \vec{V} \cdot n\, ds = \Sigma \tau}$$

Conservation of Energy

$$\boxed{\delta Q = \delta E + \delta W} \Rightarrow \text{first law of thermodynamics}$$

$\delta Q \Rightarrow$ heat supplied into the system

$\delta W \Rightarrow$ Work done by the system.

$$\boxed{\frac{d(\delta E)}{dt} = \frac{\delta Q}{\delta t} - \frac{\delta W}{\delta t}.}$$

$$\frac{dE}{dt} = \dot{Q} - \dot{W}$$

$E \rightarrow$ ~~internal~~ energy.

$\dot{Q} \rightarrow$ rate of heat given to system

$\dot{W} \rightarrow$ rate of work done by the system.

$E \rightarrow \left(e + gz + \frac{V^2}{2}\right) m$

$$\left.\frac{dE}{dt}\right)_{system} = \frac{\partial}{\partial t}\int_{cv} \rho\left(e + gz + \frac{V^2}{2}\right) dv + \int_{cs} \rho\left(e + gz + \frac{V^2}{2}\right)\vec{V}\cdot n\,ds$$

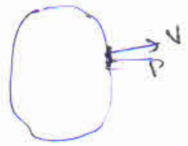
$$= \dot{Q} - \dot{W}$$

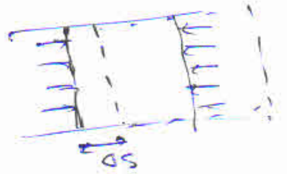
$\dot{W} \rightarrow$ Work done due to pressure force, $(\dot{W}_P)$ shaft work $(\dot{W}_S)$ etc.

$W_P = (\Delta P)\, A\, (\Delta S)$

$\dot{W}_P = \Delta P\, A \left(\frac{\Delta S}{\Delta t}\right)$

$= \Delta P\, A\, \vec{V}$





In vector form $\dot{W}_P = \int_{cs} P\,dA \cdot \vec{V}$

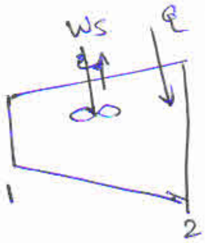
$\dot{W} = \dot{W}_P + \dot{W}_S$

$$\frac{\partial}{\partial t}\int_{cv} \rho\left(e + gz + \frac{V^2}{2}\right) dv + \int_{cs} \rho(\vec{V}\cdot ds)\left[e + \frac{P}{\rho} + gz + \frac{V^2}{2}\right] = \dot{Q} - \dot{W}_S$$

$$\boxed{h = e + Pv = \text{specific enthalpy}}$$

$$\therefore \boxed{\frac{\partial}{\partial t}\int_{cv} \rho\left(e + gz + \frac{V^2}{2}\right) dv + \int_{cs} \rho\vec{V}\cdot ds\left[h + gz + \frac{V^2}{2}\right] = \dot{Q} - \dot{W}_S}$$

One dimensional steady flow energy equation.



$$\dot{Q} - \dot{W}_s = \rho_2 V_2 A_2 \left[h + gz + \frac{V^2}{2} \right]_2 - \rho_1 V_1 A_1 \left[h + gz + \frac{V^2}{2} \right]_1$$

From C.E $\rho_1 A_1 V_1 = \rho_2 A_2 V_2 = \dot{m}$

$$\dot{Q} - \dot{W}_s = \dot{m}\left(\left[h_2 + gz_2 + \frac{V_2^2}{2} \right] - \left[h_1 + gz_1 + \frac{V_1^2}{2} \right] \right)$$

Let $q = \frac{\dot{Q}}{\dot{m}}$ and $w = \frac{\dot{W}_s}{\dot{m}}$

$$\therefore \boxed{q - w = \left[h_2 + gz_2 + \frac{V_2^2}{2} \right] - \left[h_1 + gz_1 + \frac{V_1^2}{2} \right]}$$

The term ~~$h_2 + V$~~ $h + \frac{V^2}{2} + gz$ is equal to the total energy and the term $\boxed{h + \frac{V^2}{2}}$ is called the total or stagnation enthalpy

$$\therefore \boxed{h_0 = h + \frac{V^2}{2}}$$

$$\therefore \boxed{h_{01} - h_{02} = W} \Rightarrow$$ If there is no heat transfer.

From thermodynamics. $h = C_p T$ and $u = C_v T$

$$\therefore h_0 = h + \frac{V^2}{2}$$

$$C_p T_0 = C_p T + \frac{V^2}{2} \Rightarrow \boxed{T_0 = T + \frac{V^2}{2c_p}}$$

→ In the absence of heat and work transfer, ($\dot{Q} = \dot{W} = 0$), the internal energy change across the inlet and outlet of a constant density fluid ($\rho = \frac{1}{v} = \text{Const}$) is $\underline{\underline{u_1 - u_2 = 0}}$

∴ The energy equation reduces to.

$$\boxed{\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2}$$

→ which is the Bernoulli's eqn:

→ The Bernoulli's eqn can also be derived from Euler equation. (Inviscid Momentum)

$$m\frac{d\vec{V}}{dt} = \Sigma F$$

$$\frac{\partial u}{\partial t} + \vec{V}\cdot\nabla\vec{V} = -\frac{\nabla P}{\rho} + \vec{f}$$

Assumptions

① Steady.

$\vec{f} \rightarrow$ body force / unit mass

$\vec{f} = -g\hat{k}$

$$\vec{V}\cdot\nabla\vec{V} = \nabla\left(\frac{V^2}{2}\right) - V \times \text{curl}\, V$$

$$\nabla\left(\frac{V^2}{2}\right) - V \times \text{curl}\, V = -\frac{\nabla P}{\rho} + -g\hat{k}$$

$$\nabla\left(\frac{V^2}{2}\right) + \frac{\nabla P}{\rho} + g\hat{k} = \vec{V} \times \text{curl}\,\vec{V}$$

$$d\vec{s} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

Take dot product with respect to a position vector $d\vec{s}$

$$\nabla\left(\frac{V^2}{2}\right)\cdot ds + \frac{\nabla P}{\rho}\cdot ds + g\hat{k}\cdot ds$$

$$\frac{\partial}{\partial x}\left(\frac{v^2}{2}\right) \cdot dx + \frac{\partial}{\partial y}\left(\frac{v^2}{2}\right) \cdot dy + \frac{\partial}{\partial z}\left(\frac{v^2}{2}\right) \cdot dz$$

$$\Rightarrow \quad d\left(\frac{v^2}{2}\right)$$

$$\therefore \quad \int d\frac{v^2}{2} + \int d\frac{p}{\rho} + \int g\, dz = \int \vec{v} \times \text{curl}\,\vec{v} \cdot ds \rightarrow (1)$$

~~This~~ This is a differential form of eq: which can be integrated b/w two points. However evaluating $\int v \times \text{curl}\,\vec{v} \cdot ds$ is a problem

The integral $\int \vec{v} \times \text{curl}\,\vec{v} \cdot ds$ vanishes under two cases

→ (1) if ~~curl~~ $\text{curl}\,\vec{v} = 0$ ie flow is irrotational

→ (2) if ds is a streamline.

$$\therefore \quad \int d\left(\frac{v^2}{2}\right) + \int \frac{dp}{\rho} + \int g\, dz = 0$$

If flow is irrotational or if integration is taken along a streamline.

$$\therefore \quad \boxed{\frac{v^2}{2} + \frac{p}{\rho} + gz = C} \rightarrow (A)$$

→ The equation is valid between any two points in the flow, if the flow is irrotational and incompressible

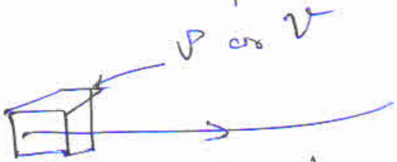
→ If the flow is rotational, the equation is valid only along a streamline

Compressibility & Compressible Flows

$$\text{Compressibility} = \frac{1}{\text{Bulk Modulus}} \approx \frac{1}{\rho}\frac{\Delta\rho}{\Delta P}$$

Large Compressibility $\Rightarrow$ very easy to get compressed eg. air.

$\rightarrow$ Hower large compressibility does not ensure that flow is compressible or not.



If the density of the fluid element doesnot change as it moves, flow is called incompressible. $\frac{D\rho}{Dt} = 0$

If density changes, flow is called compressible. $\frac{D\rho}{Dt} \neq 0$

$\rightarrow$ We try to estimate the percentage change in density w.r.t to the flow speed (Inertial forces)

$$\Delta\rho \Rightarrow \frac{\Delta\rho}{\Delta P} \times \Delta P \Rightarrow \frac{\Delta\rho}{\left(\frac{\Delta P}{\Delta\rho}\right)\Delta\rho} \Delta P$$

$$\Delta\rho \Rightarrow \frac{\rho V^2}{\left(\frac{\Delta P}{\Delta\rho}\right)} \Rightarrow \frac{\Delta\rho}{\rho} = \frac{V^2}{a^2} \quad ; \quad \frac{\Delta P}{\Delta\rho} = a^2$$

ie $\boxed{\frac{\Delta\rho}{\rho} \approx M^2}$

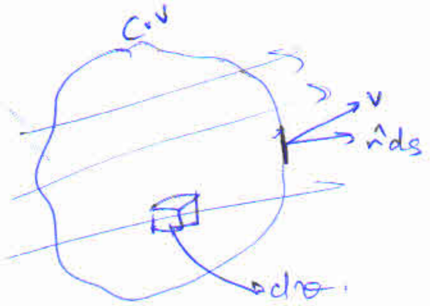
If we limit the Incompressible flows ~~as~~ as those with density change within 10%.

the $\frac{\Delta\rho}{\rho} = 0.1 \simeq M^2$

$\therefore \boxed{M \simeq 0.33}$ $\rightarrow$ This corresponds to roughly 100 m/s

Linear Momentum Equation

Consider a c.v as shown in fig.



$N = m\vec{V}$

$\eta = \vec{V}$

$\vec{f}$ = body force per unit mass

$\frac{D m\vec{V}}{Dt} = \Sigma f$, No shear stress.

$$\frac{DN}{Dt} = \frac{Dm\vec{V}}{Dt} = \int_{cv} \frac{\partial}{\partial t} \rho \vec{V} \, dv + \int_{cs} \rho \vec{V} \vec{V} \cdot n \, ds = -\int_{cs} p \, ds + \int_{cv} \rho \vec{f} \, dv$$

$$\int_{cv} \left[\frac{\partial}{\partial t} \rho \vec{V}\right] + \underbrace{\int_{cs} \rho \vec{V} \vec{V} \cdot n \, ds}_{\Rightarrow \nabla \cdot \rho \vec{V} \vec{V}} + \underbrace{\int_{cs} p \, ds}_{\Rightarrow \nabla p \, dv} - \int_{cv} \left[\rho \vec{f}\right] dv = 0$$

$$\Rightarrow \int_{cv} \left[\frac{\partial}{\partial t}(\rho \vec{V}) + \nabla \cdot \rho \vec{V} \vec{V} + \nabla p - \rho f\right] dv = 0$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}(\rho \vec{V}) + \nabla \cdot \rho \vec{V} \vec{V} = -\nabla p + \rho \vec{f}}$$

Euler Equation

$$\rho \frac{\partial \vec{V}}{\partial t} + \underbrace{\vec{V} \frac{\partial \rho}{\partial t} + \vec{V}(\nabla \cdot \rho \vec{V})}_{C.E \,\Rightarrow 0} + (\rho \vec{V} \cdot \nabla \vec{V}) = -\nabla p + \rho f$$

$$\therefore \quad \rho \frac{\partial \vec{V}}{\partial t} + \rho \vec{V} \cdot \nabla \vec{V} = -\nabla p + \rho f$$

or

$$\boxed{\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} = -\frac{\nabla p}{\rho} + f.}$$

Another form of Euler equation.