Propositional and Predicate Logics
Solutions (2017)

1 Models and Entailment in Propositional Logic

1. (a) $\neg A \land \neg B \models \neg B$

   (b) $\neg A \lor \neg B \models \neg B$

   (c) $\neg A \land B \models A \lor B$

   (d) $A \Rightarrow B \models A \Leftrightarrow B$

   (e) $(A \Rightarrow B) \Leftrightarrow C \models A \lor \neg B \lor C$

   (f) $(\neg A \Rightarrow \neg B) \land (A \land \neg B)$ is satisfiable

   (g) $(\neg A \Leftrightarrow \neg B) \land (A \land \neg B)$ is satisfiable

2. In the following, let $Q = 2^{100}$.

   (a) The expression $A_{31} \land \neg A_{76}$ is satisfied by 1 models out of 4 possible for the variables A_{31}, A_{76}. For all 100 variables A_1, A_2, ..., A_{100}, the answer is thus $\boxed{\frac{1}{4}\boldsymbol{Q}}$.

   (b) The expression $A_{44} \land A_{49} \land A_{78}$ is satisfied by 1 models out of 8 possible for the variables A_{44}, A_{49}, A_{78}. For all 100 variables A_1, A_2, ..., A_{100}, the answer is thus $\boxed{\frac{1}{8}\boldsymbol{Q}}$.

   (c) The expression $A_{44} \lor A_{49} \lor A_{78}$ is satisfied by 7 models out of 8 possible for the variables A_{44}, A_{49}, A_{78}. For all 100 variables A_1, A_2, ..., A_{100}, the answer is thus $\boxed{\frac{7}{8}\boldsymbol{Q}}$.

   (d) The expression $A_{70} \Rightarrow \neg A_{92}$ is satisfied by 3 models out of 4 possible for the variables A_{70}, A_{92}. For all 100 variables A_1, A_2, ..., A_{100}, the answer is thus $\boxed{\frac{3}{4}\boldsymbol{Q}}$.

   (e) The expression $(A_7 \Leftrightarrow A_{72}) \land (A_{83} \Leftrightarrow A_{84})$ is satisfied by 4 models out of 16 possible for the variables A_7, A_{72}, A_{83}, A_{84}. For all 100 variables A_1, A_2, ..., A_{100}, the answer is thus $\boxed{\frac{4}{16}\boldsymbol{Q}}$.

   (f) The expression $\neg A_9 \land \neg A_{19} \land A_{37} \land A_{50} \land A_{68} \land A_{73} \land A_{79} \land A_{81}$ is satisfied by 1 models out of 256 possible for the variables A_{19}, A_{37}, A_{50}, A_{68}, A_{73}, A_{79}, A_{81}, A_9. For all 100 variables A_1, A_2, ..., A_{100}, the answer is thus $\boxed{\frac{1}{256}\boldsymbol{Q}}$.

   Another way to look at this is to realize that the 8 variables A_9, A_{19}, A_{37}, A_{50}, A_{68}, A_{73}, A_{79}, A_{81} all have their values "fixed" by the expression, so that the number of possible models is reduced from 2^{100} to $\boxed{\mathbf{2^{100-8} = 2^{92}} = \frac{1}{256}\boldsymbol{Q}}$.

3. Table 1 shows the 16 possible models. There are $16 = 2^4$ possibilities because we ignore the Wumpus and only consider whether there are pits in the four adjacent rooms $[3, 1]$, $[3, 2]$, $[3, 3]$ and $[4, 4]$.

   The 6th column of the table shows the models that are consistent with the knowledge base (KB), where the state of the KB is given in the assignment text. The 7th, 8th, 9th and 10th columns show the truth values of respectively α_1, α_2, α_3 and α_4.

Propositional and Predicate Logics
Solutions (2017)

1 Models and Entailment in Propositional Logic

1. (a) Truth table for $\neg A \wedge \neg B \models \neg B$:

A	B	$\neg A \wedge \neg B$	$\neg B$
0	0	1	1
0	1	0	0
1	0	0	1
1	1	0	0

The entailment is **true**.

(b) Truth table for $\neg A \vee \neg B \models \neg B$:

A	B	$\neg A \vee \neg B$	$\neg B$
0	0	1	1
0	1	1	0
1	0	1	1
1	1	0	0

The entailment is **false**.

(c) Truth table for $\neg A \wedge B \models A \vee B$:

A	B	$\neg A \wedge B$	$A \vee B$
0	0	0	0
0	1	1	1
1	0	0	1
1	1	0	1

The entailment is **true**.

(d) Truth table for $A \Rightarrow B \models A \Leftrightarrow B$:

A	B	$A \Rightarrow B$	$A \Leftrightarrow B$
0	0	1	1
0	1	1	0
1	0	0	0
1	1	1	1

The entailment is **false**.

The KB is only true in three models: 10, 11 and 12. α_1 and α_4 are both true in all three of these models, thus both sentences are entailed by the KB. α_2 is true in model 10 and 12, but not in model 11. α_3 is true in model 11, but not in model 10 and 12. These sentences are therefore *not* entailed by the KB.

Table 1: 16 models for a restricted view of the Wumpus World, where KB is the current state of the knowledge base after visiting $[4,1]$, $[4,2]$ and $[4,3]$. α_1 = "There is no pit in $[3,2]$". α_2 = "There is a pit in $[4,4]$". α_3 = "There is no a pit in $[4,4]$". α_4 = "There is a pit in $[3,3]$ or $[4,4]$".

	Pits								
Index	P_{31}	P_{32}	P_{33}	P_{44}	KB	α_1	α_2	α_3	α_4
1	0	0	0	0	0	1	0	1	0
2	0	0	0	1	0	1	1	0	1
3	0	0	1	0	0	1	0	1	1
4	0	0	1	1	0	1	1	0	1
5	0	1	0	0	0	0	0	1	0
6	0	1	0	1	0	0	1	0	1
7	0	1	1	0	0	0	0	1	1
8	0	1	1	1	0	0	1	0	1
9	1	0	0	0	0	1	0	1	0
10	1	0	0	1	1	1	1	0	1
11	1	0	1	0	1	1	0	1	1
12	1	0	1	1	1	1	1	0	1
13	1	1	0	0	0	0	0	1	0
14	1	1	0	1	0	0	1	0	1
15	1	1	1	0	0	0	0	1	1
16	1	1	1	1	0	0	1	0	1

2 Resolution in Propositional Logic

1. (a) $A \vee (B \wedge C \wedge \neg D) \equiv (B \vee A) \wedge (C \vee A) \wedge (\neg D \vee A)$

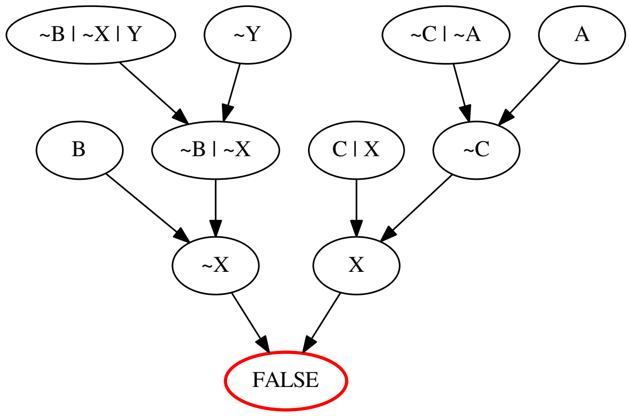
   (b) $\neg(A \Rightarrow \neg B) \wedge \neg(C \Rightarrow \neg D) \equiv B \wedge A \wedge D \wedge C$

   (c) $\neg((A \Rightarrow B) \wedge (C \Rightarrow D)) \equiv (\neg D \vee \neg B) \wedge (C \vee \neg B) \wedge (\neg D \vee A) \wedge (C \vee A)$

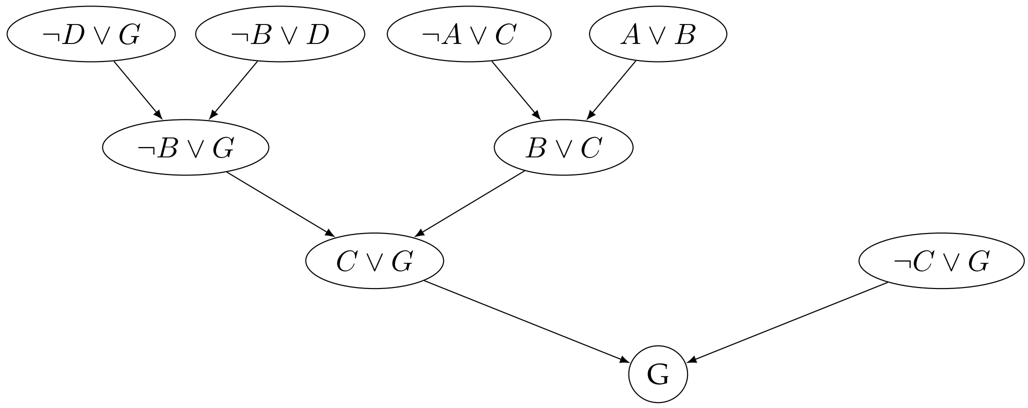
   (d) $(A \wedge B) \vee (C \Rightarrow D) \equiv (A \vee D \vee \neg C) \wedge (B \vee D \vee \neg C)$

   (e) $A \Leftrightarrow (B \Rightarrow \neg C) \equiv (C \vee A) \wedge (B \vee A) \wedge (\neg C \vee \neg B \vee \neg A)$

2. The following figure shows one possible example of a resolution.



3. (a) Resolution:



(b) You have $2n$ distinct literals, then pick two $\binom{2n}{2} = \frac{2n \times (2n-1)}{2} = 2n^2 - n$. The binomial doesn't account for $A \vee A$ though, so we have to add 2n clauses here, as these are all semantically distinct from the ones we have so far. We now have to remove some clauses, since we have n clauses that are like $A \vee \neg A$. These are all always true, so we have to remove all but one of them (keep one clause that is always true, since it will be distinct from the other clauses we have). Summing these up gives

$$2n^2 - n + 2n - (n-1) = 2n^2 + 1 \tag{1}$$

(c) Two 2-CNF clauses resolving always produce a 2-CNF clause. According to **b**, the total number of distrinct clauses is $2n^2 + 1$. The number of possible resolutions is then $(2n^2 + 1) \times 2n^2$. Any resolution step is one of these, so any process terminates in time polynomial in n given a 2-CNF clause containing n distinct symbols.

(d) Two 3-CNF clauses does not always produce a 3-CNF clause, i.e. they are not closed under resolution. For example $(A \vee B \vee C) \wedge (\neg A \vee D \vee E)$ resolves to $(B \vee C \vee D \vee E)$.

3 Representations in First-Order Logic

1. (a) $Occupation(Emily, Surgeon) \vee Occupation(Emily, Lawyer)$

(b) $Occupation(Joe, Actor) \wedge \exists x : x \neq Actor \wedge Occupation(Joe, x)$

(c) $\forall x : Occupation(x, Surgeon) \rightarrow Occupation(x, Doctor)$

(d) $\neg \exists x : Occupation(x, Lawyer) \wedge Customer(x, Joe)$

(e) $\exists x : Boss(x, Emily) \wedge Occupation(x, Lawyer)$

(f) $\exists x : Occupation(x, Lawyer) \wedge \forall y : Customer(y, x) \rightarrow Occupation(y, Doctor)$

(g) $\forall x : Occupation(x, Surgeon) \rightarrow \exists y : Occupation(y, Lawyer) \wedge Customer(y, x)$

2. (a) PlayedCharacter(ChristianBale,Batman) ∧ PlayedCharacter(GeorgeClooney,Batman) ∧ PlayedCharacter(ValKilmer,Batman)

   (b) ∀ a: PlayedCharacter(a,Batman) → Male(a)

   (c) ∀ a: PlayedCharacter(a,Batwoman) → Female(a)

   (d) ∀ c: ¬ PlayedCharacter(Bale,c) ∨¬ PlayedCharacter(Ledger,c)

   (e) ∀ m: CharacterInMovie(Batman,m) ∧ Directed(Nolan,m) → PlayedInMovie(ChristianBale,m)

   (f) ∃ m: CharacterInMovie(TheJoker,m) ∧ CharacterInMovie(Batman,m)

   (g) ∃ m: PlayedInMovie(KevinCostner,m) ∧ Directed(KevinCostner,m)

   (h) ∀ m: PlayedInMovie(GeorgeClooney,m) → ¬ (PlayedInMovie(Tarantino,m) ∨ Directed(Tarantino,m))

   This is equivalent to:

   ∀ m: ¬ (PlayedInMovie(GeorgeClooney,m) ∧ PlayedInMovie(Tarantino,m))∧¬ (PlayedInMovie(GeorgeCl
   ∧ Directed(Tarantino,m))

   And also

   ∀ m: (PlayedInMovie(Tarantino,m) ∨ Directed(Tarantino,m)) → ¬PlayedInMovie(GeorgeClooney,m)

   (i) Female(UmaThurman) ∧ ∃ m: PlayedInMovie(UmaThurman,m) ∧ Directed(Tarantino,m)

3. (a) $\forall x, y$: $Divisible(x, y) \leftrightarrow \exists z : (z \leq x) \wedge (x = z \times y)$

   (b) $\forall x$: $Even(x) \leftrightarrow Divisible(x, 2)$

   (c) $\forall x$: $Odd(x) \leftrightarrow \neg Divisible(x, 2)$

   (d) $\forall x$: $Odd(x) \leftrightarrow \exists y$:$Even(y) \wedge (x = y + 1)$

   (e) $\forall x$: $Prime(x) \leftrightarrow \forall y$:$\neg(x = y) \rightarrow \neg Divisible(x, y)$

   (f) $\exists x$: $\forall y : Prime(y) \wedge Even(y) \leftrightarrow x = y$

   (g) $\forall x$: $\exists p_1, p_2, \ldots, p_n : (\forall k : (1 \leq k \leq n) \wedge Prime(p_k)) \wedge x = \prod_{k=1}^{k=n} p_k$

4 Resolution in First-Order Logic

1. (a) $\theta = \{x/Rocky\}$

   (b) $\theta = \{x/Leo, y/Rocky\}$

   (c) $\theta = \{x/Rocky, y/Leo\}$

   (d) Not possible – cannot unify due to non-matching predicates

   (e) Not possible – cannot unify function FastestHorse with constant Rocky

   (f) $\theta = \{x/Leo, y/FastestHorse(Leo)\}$

   (g) $\theta = \{x/Marvin, y/Leo\}$

2. (a) Philosopher(c_x) ∧ StudentOf(c_y,c_x)

   Where c_y and c_x are Skolem constants substituting variables x and y.

(b) $\forall$ y,x: Philosopher(x) $\wedge$ StudentOf(y,x) $\rightarrow$ [Book(S_z(x,y)) $\wedge$ Write(x,S_z(x,y)) $\wedge$ Read(y,S_z(x,y))]

Where S_z(x,y) is a Skolem function substituting variable z.

3. (a) We start with the CNF form of the SuperActor applying inference rule and Skolemization

- $\forall$x: SuperActor(x) $\leftrightarrow$ [$\exists$ m: PlayedInMovie(x,m) $\wedge$ Directed(x,m)]

Break the double-implication into 2 conjoined implications

$$\begin{aligned}
&\forall x :(SuperActor(x) \rightarrow [\exists m : PlayedInMovie(x,m) \wedge Directed(x,m)]) \\
&\qquad \wedge ([\exists m : PlayedInMovie(x,m) \wedge Directed(x,m)] \rightarrow SuperActor(x)) && (2)\\
\Leftrightarrow &\forall x :(\neg SuperActor(x) \vee [\exists m : PlayedInMovie(x,m) \wedge Directed(x,m)]) \\
&\qquad \wedge (\neg[\exists m : PlayedInMovie(x,m) \wedge Directed(x,m)] \vee SuperActor(x)) && (3)\\
\Leftrightarrow &\forall x :(\neg SuperActor(x) \vee [\exists m : PlayedInMovie(x,m) \wedge Directed(x,m)]) \\
&\qquad \wedge ([\forall m : \neg PlayedInMovie(x,m) \vee \neg Directed(x,m)] \vee SuperActor(x)) && (4)\\
\Rightarrow &\forall x :(\neg SuperActor(x) \vee [PlayedInMovie(x,F[x]) \wedge Directed(x,F[x])]) \\
&\qquad \wedge ([\forall m : \neg PlayedInMovie(x,m) \vee \neg Directed(x,m)] \vee SuperActor(x)) && (5)\\
&\Rightarrow(\neg SuperActor(x) \vee [PlayedInMovie(x,F[x]) \wedge Directed(x,F[x])]) \\
&\qquad \wedge ([\neg PlayedInMovie(x,m) \vee \neg Directed(x,m)] \vee SuperActor(x)) && (6)\\
&\Rightarrow(\neg SuperActor(x) \vee PlayedInMovie(x,F[x])) \wedge (\neg SuperActor(x) \vee Directed(x,F[x])) \\
&\qquad \wedge (\neg PlayedInMovie(x,m) \vee \neg Directed(x,m) \vee SuperActor(x)) && (7)
\end{aligned}$$

Now let us look at the second and third formula

- $\forall$ m: Directed(Tarantino,m) $\leftrightarrow$ PlayedInMovie(UmaThurman,m)

Break the double-implication into 2 conjoined implications

$$\begin{aligned}
&\forall m : (Directed(Tarantino,m) \rightarrow PlayedInMovie(UmaThurman,m)) \\
&\quad \wedge (PlayedInMovie(UmaThurman,m) \rightarrow Directed(Tarantino,m)) && (8)\\
\Leftrightarrow &\forall m : (\neg Directed(Tarantino,m) \vee PlayedInMovie(UmaThurman,m)) \\
&\quad \wedge (\neg PlayedInMovie(UmaThurman,m) \vee Directed(Tarantino,m)) && (9)\\
\Rightarrow &(\neg Directed(Tarantino,m) \vee PlayedInMovie(UmaThurman,m)) \\
&\quad \wedge (\neg PlayedInMovie(UmaThurman,m) \vee Directed(Tarantino,m)) && (10)
\end{aligned}$$

- $\exists$ m: PlayedInMovie(UmaThurman,m) $\wedge$ PlayedInMovie(Tarantino,m)

$$\begin{aligned}
&\exists m : PlayedInMovie(UmaThurman,m) \wedge PlayedInMovie(Tarantino,m) \\
\Rightarrow &PlayedInMovie(UmaThurman,c) \wedge PlayedInMovie(Tarantino,c) && (11)
\end{aligned}$$

Now we add the hypothesis $\neg SuperActor(Tarantino)$ and apply resolution rule until we achieve a contradiction.

$$\neg SuperActor(Tarantino) \tag{12}$$
$$PlayedInMovie(UmaThurman, c) \tag{13}$$
$$PlayedInMovie(Tarantino, c) \tag{14}$$
$$\neg PlayedInMovie(UmaThurman, m) \vee Directed(Tarantino, m) \tag{15}$$
$$\neg Directed(Tarantino, m) \vee PlayedInMovie(UmaThurman, m) \tag{16}$$

Unification of 12, 14 and 15

$$\neg PlayedInMovie(UmaThurman, c) \vee Directed(Tarantino, c) \tag{17}$$
$$\neg Directed(Tarantino, c) \vee PlayedInMovie(UmaThurman, c) \tag{18}$$

Conjunction of 12 and 16

$$\Rightarrow PlayedInMovie(UmaThurman, c) \wedge Directed(Tarantino, c) \tag{19}$$
$$\Rightarrow Directed(Tarantino, c) \tag{20}$$

Conjunction of 19 and 13

$$\Rightarrow PlayedInMovie(Tarantino, c) \wedge Directed(Tarantino, c) \tag{21}$$

Given from 6

$$\neg PlayedInMovie(x, m) \vee \neg Directed(x, m) \vee SuperActor(x) \tag{22}$$

Conjunction of 20 and 21

$$\Rightarrow PlayedInMovie(Tarantino, c) \wedge Directed(Tarantino, c) \wedge SuperActor(x) \tag{23}$$
$$\Rightarrow SuperActor(x) \tag{24}$$
$$\Rightarrow \mathbf{F} \tag{25}$$

(b) Translate the information given in FOL into English (or Norwegian) and describe in high level the reasoning you could apply in English to have the same result (in other words, describe a proof of the result in natural language).

A SuperActor is someone that is a director and an actor in the same film. Uma Thurman is performing in a movie if and only if Tarantino is the director. There is a movie that has Uma Thurman and Tarantino as actors in it. We know that everytime Uma Thurman is playing a character in a movie Tarantino is directing. So in this specific movie Tarantino is director and actor. So we can conclude that he is an SuperActor.