Advanced Algorithms and Data Structures
Assignment 1: Minimum-cost Flow

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1 Exercise 1: b-flow

For Figure 1(a) in the assignment, one possible b-flow is shown in Figure 1.

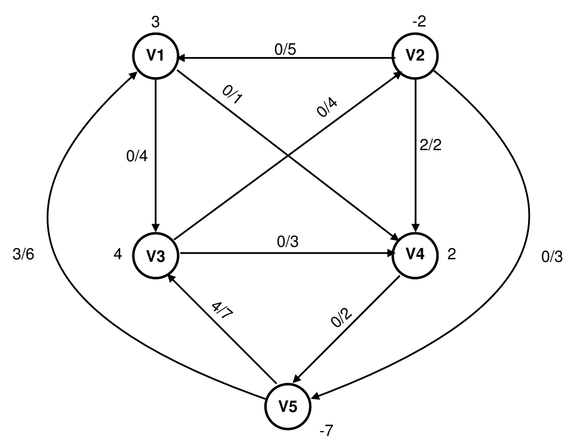

Figure 1: A b-flow in Figure 1(a) from the assignment text.

Figure 1(b) in the assignment text has no b-flow since vertex v_4 has a negative demand, but no outgoing edges. This is equivalent to producing 2 units of flow and since v_4 has no outgoing edges, the b-flow equation (flow conservation equation from the assignment text) cannot be satisfied. Because of the capacity constraint, the flow into the vertex will be positive and cannot possibly satisfy the negative demand when there are no outgoing edges.

2 Exercise 2: An application of MCFP: rectilinear planar embedding

2.1 Exercise 2.1

All the values of the variables z_{fg} and x_{vf}, for the rectilinear layout in Figure 3 in the the assignment text, are shown below.

$$
\begin{array}{llll}
z_{ab} = 0 & z_{ac} = 0 & z_{ad} = 0 & z_{ae} = 0 \\
z_{ba} = 2 & z_{bc} = 1 & z_{bd} = 1 & z_{be} = 0 \\
z_{ca} = 1 & z_{cb} = 1 & z_{cd} = 0 & z_{ce} = 0 \\
z_{da} = 0 & z_{db} = 1 & z_{dc} = 0 & z_{de} = 2 \\
z_{ea} = 4 & z_{eb} = 0 & z_{ec} = 0 & z_{ed} = 0
\end{array}
$$

x	v_1	v_2	v_3	v_4	v_5	v_6	v_7
a	0	0	1	0	1	1	0
b	1	0	0	0	0	1	0
c	1	1	1	0	0	0	0
d	0	1	1	-1	0	1	0
e	0	0	1	1	-1	1	0

The total number of breakpoints is 13.

A rectilinear layout of the graph from Figure 2(a) in the assignment, given the planar embedding in Figure 2(b) in the assignment, is shown in Figure 2.

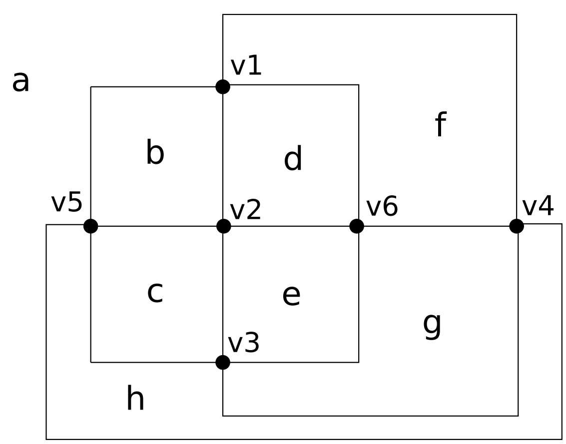

Figure 2: A rectilinear layout of the graph from Figure 2(a) in the assignment, given the planar embedding in Figure 2(b) in the assignment.

2.2 Exercise 2.2

Given $G = (V, E)$, a set of boundary cycles C. Then the following linear constraints can be stated. Let f be a boundary cycle in a rectilinear layout.

If f corresponds to an internal face, then

$$\sum_{v \in V} x_{vf} + \sum_{g \in C - \{f\}} (z_{fg} - z_{gf}) = 4 \ , \tag{1}$$

and if f corresponds to an external face, then

$$\sum_{v \in V} x_{vf} + \sum_{g \in C - \{f\}} (z_{fg} - z_{gf}) = -4 \ . \tag{2}$$

For boundary cycle a in Figure 3 in the assignment text, equation (2) evaluates to

$$(1 + 1 + 1) + (0 - 7) = -4$$

and for boundary cycle e it evaluates to

$$(1 + 1 - 1 + 1) + (4 - 2) = 4 \ .$$

2.3 Exercise 2.3

No vertex in G can have degree greater than 4 because rectilinear embedding tries to draw the graph in the plane using just horizontal and vertical lines and there are only four such possible directions from each vertex.

Furthermore, it will be shown that the following holds for each vertex v:

$$\sum_{f} x_{vf} = \begin{cases} 0 & \text{if } v \text{ has degree } 2 \\ 2 & \text{if } v \text{ has degree } 3 \\ 4 & \text{if } v \text{ has degree } 4 \end{cases} \tag{3}$$

Case 1 (v has degree 2):
In the case where v has degree 2, there are two possible situations that can occur:

1. The two incident edges are either on the same line (opposite of each other), i.e. two angles of 180° each, and x_{vf} is equal to 0 for both boundary cycles.
2. The two edges form a 90° angle and a 270° angle. In this case $\sum_f x_{vf} = 1 + (-1) = 0$.

This is illustrated in Figure 3.

Figure 3: Illustration of vertex with degree 2.

Case 2 (v has degree 3):
In this case, the only possibility is that the three edges form one 180° angle and two 90° angles. In this case $\sum_f x_{vf} = 1 + 1 + 0 = 2$. This is illustrated in Figure 4.

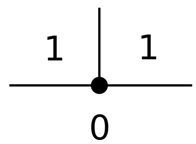

Figure 4: Illustration of vertex with degree 3.

Case 3 (v has degree 4):
In this case, the only possibility is that the four edges form four 90° angles. In this case $\sum_f x_{vf} = 1 + 1 + 1 + 1 = 4$. This is illustrated in Figure 5.

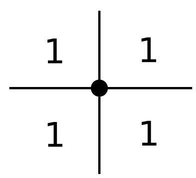

Figure 5: Illustration of vertex with degree 4.

2.4 Exercise 2.4

Minimizing the following objective function minimizes the number of breakpoints. Let C be the set of boundary cycles.

$$\text{Minimize} \sum_{\substack{f,g \in C \\ f \neq g}} z_{fg}$$

subject to

$$\sum_{v \in V} x_{vf} + \sum_{g \in C-\{f\}} (z_{fg} - z_{gf}) = 4 \qquad \text{if } f \text{ corresponds to an internal face}$$

$$\sum_{v \in V} x_{vf} + \sum_{g \in C-\{f\}} (z_{fg} - z_{gf}) = -4 \qquad \text{if } f \text{ corresponds to an external face}$$

$$v_1.deg, v_2.deg, \ldots, v_n.deg \leq 4$$

$$v_1.deg, v_2.deg, \ldots, v_n.deg \geq 2$$

$$\sum_f x_{vf} = 0 \qquad \text{if } v \text{ has degree 2}$$

$$\sum_f x_{vf} = 2 \qquad \text{if } v \text{ has degree 3}$$

$$\sum_f x_{vf} = 4 \qquad \text{if } v \text{ has degree 4}$$

$$z_{fg} \geq 0 \qquad \forall f, g \in C, f \neq g$$

2.5 Exercise 2.5

To make the rectilinear planar graph an instance of MCFP, each vertex and each face from the rectilinear planer graph transforms to a node in the MCFP graph. The demand of nodes corresponding to the external face is -4 and that of the nodes corresponding to the internal faces is 4 (as in equations (2) and (1) in exercise 2.2).

The demand of nodes corresponding to the vertices (from the rectilinear planar graph) is equivalent to the negative of equation (3).

An edge exist between each adjacent elements (faces/vertices from the rectilinear planar graph). We define a flow from a vertex u to v if v corresponds to a face in which the inner turn is made. If face a has the outer turn and face b has the inner turn at a vertex v, then there exists edges (a,v) and (v,b).

We associate a cost for the edge between 2 faces, since the flow along the edge corresponds to the no. of breakpoints. We associate no cost with the other edges. To reduce the no. of breakpoints, the cost must be minimum and thus this is an instance of the MCFP.

For each pair of adjacent faces, there is an edge from of infinite capacity having cost 1 with the amount of flow corresponding to the number of breakpoints. Give each node corresponding to an internal face in the rectilinear planar graph a demand of 4 and give the node corresponding to the external face a demand of -4. For every vertex that lies between two faces, add an edge between the vertex and the face of capacity 1.

$$V' = V \cup C$$

$$E' = \{(f, g) \in C \times C \mid f \text{ and } g \text{ share a vertex or an edge}\}$$

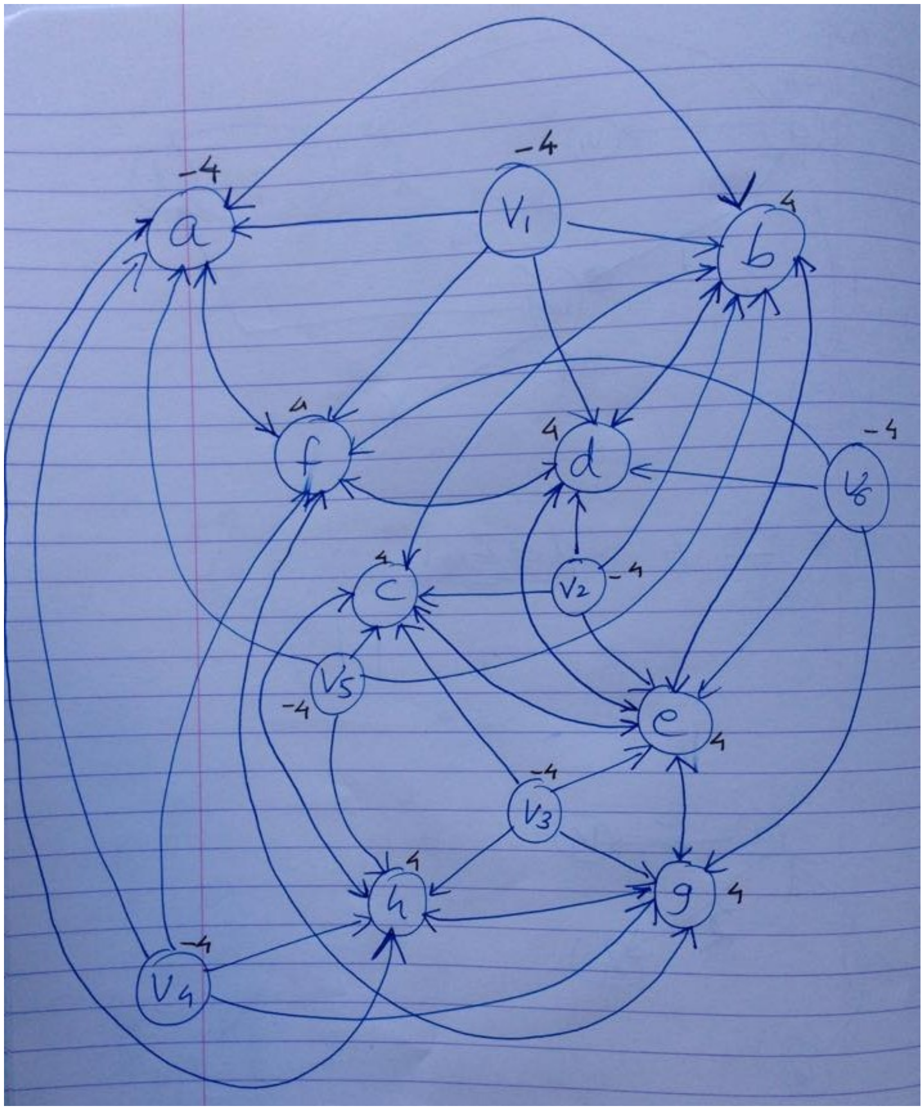

Figure 6: