Chapter 7

Uncertain Differential Equation

Uncertain differential equation, proposed by Liu [121] in 2008, is a type of differential equation driven by canonical process. Uncertain differential equation was then introduced into finance by Liu [123] in 2009. After that, an existence and uniqueness theorem of solution of uncertain differential equation was proved by Chen and Liu [17], and a stability theorem was showed by Chen [20].

This chapter will discuss the existence, uniqueness and stability of solutions of uncertain differential equations. This chapter will also provide a 99-method to solve uncertain differential equations numerically. Finally, some applications of uncertain differential equation in finance are documented.

7.1 Uncertain Differential Equation

Definition 7.1 *(Liu [121]). Suppose C_t is a canonical process, and f and g are some given functions. Then*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{7.1}$$

is called an uncertain differential equation. A solution is an uncertain process X_t that satisfies (7.1) identically in t.

Remark 7.1: Note that there is no precise definition for the terms $\mathrm{d}X_t$, $\mathrm{d}t$ and $\mathrm{d}C_t$ in the uncertain differential equation (7.1). The mathematically meaningful form is the uncertain integral equation

$$X_s = X_0 + \int_0^s f(t, X_t)\mathrm{d}t + \int_0^s g(t, X_t)\mathrm{d}C_t. \tag{7.2}$$

However, the differential form is more convenient for us. This is the main reason why we accept the differential form.

Example 7.1: Let C_t be a canonical process. Then the uncertain differential equation

$$\mathrm{d}X_t = a\mathrm{d}t + b\mathrm{d}C_t \tag{7.3}$$

B. Liu: Uncertainty Theory: A Branch of Mathematics, SCI 300, pp. 147–161.
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has a solution

$$X_t = at + bC_t \tag{7.4}$$

which is just an arithmetic canonical process.

Example 7.2: Let C_t be a canonical process. Then the uncertain differential equation

$$\mathrm{d}X_t = aX_t\mathrm{d}t + bX_t\mathrm{d}C_t \tag{7.5}$$

has a solution

$$X_t = \exp\left(at + bC_t\right) \tag{7.6}$$

which is just a geometric canonical process.

Example 7.3: Let C_t be a canonical process. Then the uncertain differential equation

$$\mathrm{d}X_t = (m - aX_t)\mathrm{d}t + \sigma\mathrm{d}C_t \tag{7.7}$$

has a solution

$$X_t = \frac{m}{a} + \exp(-at)\left(X_0 - \frac{m}{a}\right) + \sigma\exp(-at)\int_0^t \exp(as)\mathrm{d}C_s \tag{7.8}$$

provided that $a \neq 0$. It follows from Theorem 6.4 that X_t is a normal uncertain variable, i.e.,

$$X_t \sim \mathcal{N}\left(\frac{m}{a} + \exp(-at)\left(X_0 - \frac{m}{a}\right),\ \frac{\sigma}{a} - \exp(-at)\frac{\sigma}{a}\right). \tag{7.9}$$

Example 7.4: Let u_t and v_t be some continuous functions with respect to t. Consider the *homogeneous linear uncertain differential equation*

$$\mathrm{d}X_t = u_tX_t\mathrm{d}t + v_tX_t\mathrm{d}C_t. \tag{7.10}$$

It follows from the chain rule that

$$\mathrm{d}\ln X_t = \frac{\mathrm{d}X_t}{X_t} = u_t\mathrm{d}t + v_t\mathrm{d}C_t.$$

Integration of both sides yields

$$\ln X_t - \ln X_0 = \int_0^t u_s\mathrm{d}s + \int_0^t v_s\mathrm{d}C_s.$$

Therefore the solution of (7.10) is

$$X_t = X_0\exp\left(\int_0^t u_s\mathrm{d}s + \int_0^t v_s\mathrm{d}C_s\right). \tag{7.11}$$

Example 7.5: Suppose $u_{1t}, u_{2t}, v_{1t}, v_{2t}$ are continuous functions with respect to t. Consider the *linear uncertain differential equation*

$$\mathrm{d}X_t = (u_{1t}X_t + u_{2t})\mathrm{d}t + (v_{1t}X_t + v_{2t})\mathrm{d}C_t. \tag{7.12}$$

At first, we define two uncertain processes U_t and V_t via

$$\mathrm{d}U_t = u_{1t}U_t\mathrm{d}t + v_{1t}U_t\mathrm{d}C_t, \quad \mathrm{d}V_t = \frac{u_{2t}}{U_t}\mathrm{d}t + \frac{v_{2t}}{U_t}\mathrm{d}C_t.$$

Then we have $X_t = U_tV_t$ because

$$\begin{aligned}\mathrm{d}X_t &= V_t\mathrm{d}U_t + U_t\mathrm{d}V_t \\ &= (u_{1t}U_tV_t + u_{2t})\mathrm{d}t + (v_{1t}U_tV_t + v_{2t})\mathrm{d}C_t \\ &= (u_{1t}X_t + u_{2t})\mathrm{d}t + (v_{1t}X_t + v_{2t})\mathrm{d}C_t.\end{aligned}$$

Note that

$$U_t = U_0 \exp\left(\int_0^t u_{1s}\mathrm{d}s + \int_0^t v_{1s}\mathrm{d}C_s\right),$$

$$V_t = V_0 + \int_0^t \frac{u_{2s}}{U_s}\mathrm{d}s + \int_0^t \frac{v_{2s}}{U_s}\mathrm{d}C_s.$$

Taking $U_0 = 1$ and $V_0 = X_0$, we get a solution of the linear uncertain differential equation as follows,

$$X_t = U_t\left(X_0 + \int_0^t \frac{u_{2s}}{U_s}\mathrm{d}s + \int_0^t \frac{v_{2s}}{U_s}\mathrm{d}C_s\right) \tag{7.13}$$

where

$$U_t = \exp\left(\int_0^t u_{1s}\mathrm{d}s + \int_0^t v_{1s}\mathrm{d}C_s\right). \tag{7.14}$$

7.2 Existence and Uniqueness Theorem

Theorem 7.1 *(Chen and Liu [17], Existence and Uniqueness Theorem). The uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{7.15}$$

has a unique solution if the coefficients $f(x,t)$ and $g(x,t)$ satisfy the Lipschitz condition

$$|f(x,t) - f(y,t)| + |g(x,t) - g(y,t)| \le L|x - y|, \quad \forall x, y \in \Re, t \ge 0 \tag{7.16}$$

and linear growth condition

$$|f(x,t)| + |g(x,t)| \le L(1 + |x|), \quad \forall x \in \Re, t \ge 0 \tag{7.17}$$

for some constant L. Moreover, the solution is sample-continuous.

Proof: We first prove the existence of solution by a successive approximation method. Define $X_t^{(0)} = X_0$, and

$$X_t^{(n)} = X_0 + \int_0^t f\left(X_s^{(n-1)}, s\right)\mathrm{d}s + \int_0^t g\left(X_s^{(n-1)}, s\right)\mathrm{d}C_s$$

for $n = 1, 2, \cdots$ and write

$$D_t^{(n)}(\gamma) = \max_{0\le s\le t}\left|X_s^{(n+1)}(\gamma) - X_s^{(n)}(\gamma)\right|$$

for each $\gamma \in \Gamma$. It follows from the Lipschitz condition and linear growth condition that

$$\begin{aligned}
D_t^{(0)}(\gamma) &= \max_{0\le s\le t}\left|\int_0^s f(X_0, v)\mathrm{d}v + \int_0^s g(X_0, v)\mathrm{d}C_v(\gamma)\right| \\
&\le \int_0^t |f(X_0, v)|\,\mathrm{d}v + K_\gamma \int_0^t |g(X_0, v)|\,\mathrm{d}v \\
&\le (1 + |X_0|)L(1 + K_\gamma)t
\end{aligned}$$

where K_γ is the Lipschitz constant to the sample path $C_t(\gamma)$. In fact, by using the induction method, we may verify

$$D_t^{(n)}(\gamma) \le (1 + |X_0|)\frac{L^{n+1}(1 + K_\gamma)^{n+1}}{(n+1)!}t^{n+1}$$

for each n. This means that, for each sample γ, the paths $X_t^{(k)}(\gamma)$ converges uniformly on any given interval $[0, T]$. Write the limit by $X_t(\gamma)$ that is just a solution of the uncertain differential equation because

$$X_t = X_0 + \int_0^t f(X_s, s)\mathrm{d}s + \int_0^t g(X_s, s)\mathrm{d}s.$$

Next we prove that the solution is unique. Assume that both X_t and X_t^* are solutions of the uncertain differential equation. Then for each $\gamma \in \Gamma$, it follows from the Lipschitz condition and linear growth condition that

$$|X_t(\gamma) - X_t^*(\gamma)| \le L(1 + K_\gamma)\int_0^t |X_v(\gamma) - X_v^*(\gamma)|\mathrm{d}v.$$

By using Gronwall inequality, we obtain

$$|X_t(\gamma) - X_t^*(\gamma)| \le 0 \cdot \exp(L(1 + K_\gamma)t) = 0.$$

Hence $X_t = X_t^*$. The uniqueness is proved. Finally, let us prove the sample-continuity of X_t. The Lipschitz condition and linear growth condition may produce

$$\begin{aligned}
|X_t(\gamma) - X_s(\gamma)| &= \left|\int_s^t f(X_v(\gamma), v)\mathrm{d}v + \int_s^t g(X_v(\gamma), v)\mathrm{d}C_v(\gamma)\right| \\
&\le (1 + K_\gamma)(1 + |X_0|)\exp(L(1 + K_\gamma)t)(t - s) \\
&\to 0 \text{ as } s \to t.
\end{aligned}$$

Thus X_t is sample-continuous and the theorem is proved.

7.3 Stability Theorem

Definition 7.2 *(Liu [123]). An uncertain differential equation is said to be stable if for any given numbers $\kappa > 0$ and $\varepsilon > 0$, there exists a number $\delta > 0$ such that for any solutions X_t and Y_t, we have*

$$\mathcal{M}\{|X_t - Y_t| > \kappa\} < \varepsilon, \quad \forall t > 0 \tag{7.18}$$

whenever $|X_0 - Y_0| < \delta$.

In other words, an uncertain differential equation is stable if for any given number $\kappa > 0$, we have

$$\lim_{|X_0 - Y_0| \to 0} \mathcal{M}\{|X_t - Y_t| > \kappa\} = 0, \quad \forall t > 0. \tag{7.19}$$

Example 7.6: The uncertain differential equation $\mathrm{d}X_t = a\mathrm{d}t + b\mathrm{d}C_t$ is stable since for any given numbers $\kappa > 0$ and $\varepsilon > 0$, we may take $\delta = \kappa$ and have

$$\mathcal{M}\{|X_t - Y_t| > \kappa\} = \mathcal{M}\{|X_0 - Y_0| > \kappa\} = \mathcal{M}\{\emptyset\} = 0 < \varepsilon$$

for any time $t > 0$ whenever $|X_0 - Y_0| < \delta$.

Example 7.7: The uncertain differential equation $\mathrm{d}X_t = X_t\mathrm{d}t + b\mathrm{d}C_t$ is unstable since for any given number $\kappa > 0$ and any different initial solutions X_0 and Y_0, we have

$$\mathcal{M}\{|X_t - Y_t| > \kappa\} = \mathcal{M}\{\exp(t)|X_0 - Y_0| > \kappa\} = 1$$

provided that t is sufficiently large.

Theorem 7.2 *(Chen [20], Stability Theorem). Suppose u_t and v_t are continuous functions such that*

$$\sup_{s \geq 0} \int_0^s u_t\mathrm{d}t < +\infty, \quad \int_0^{+\infty} |v_t|\mathrm{d}t < +\infty. \tag{7.20}$$

Then the uncertain differential equation

$$\mathrm{d}X_t = u_t X_t \mathrm{d}t + v_t X_t \mathrm{d}C_t \tag{7.21}$$

is stable.

Proof: It has been proved that the unique solution of the uncertain differential equation $\mathrm{d}X_t = u_t X_t \mathrm{d}t + v_t X_t \mathrm{d}C_t$ is

$$X_t = X_0 \exp\left(\int_0^t u_s\mathrm{d}s + \int_0^t v_s\mathrm{d}C_s\right).$$

Thus for any given number $\kappa > 0$, we have

$$\begin{aligned}\mathcal{M}\{|X_t - Y_t| > \kappa\} &= \mathcal{M}\left\{|X_0 - Y_0| \exp\left(\int_0^t u_s \mathrm{d}s + \int_0^t v_s \mathrm{d}C_s\right) > \kappa\right\} \\ &= \mathcal{M}\left\{\int_0^t v_s \mathrm{d}C_s > \ln \frac{\kappa}{|X_0 - Y_0|} - \int_0^t u_s \mathrm{d}s\right\} \to 0\end{aligned}$$

as $|X_0 - Y_0| \to 0$ because

$$\int_0^t v_s \mathrm{d}C_s \sim \mathcal{N}\left(0, \int_0^t |v_s| \mathrm{d}s\right)$$

is a normal uncertain variable with expected value 0 and finite variance, and

$$\ln \frac{\kappa}{|X_0 - Y_0|} - \int_0^t u_s \mathrm{d}s \to +\infty.$$

The theorem is proved.

7.4 Numerical Method

It is almost impossible to find analytic solutions for general uncertain differential equations. This fact provides a motivation to design numerical methods to solve uncertain differential equations.

Definition 7.3. *Let α be a number with $0 < \alpha < 1$. An uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{7.22}$$

is said to have an α-path X_t^α if it solves the corresponding ordinary differential equation

$$\mathrm{d}X_t^\alpha = f(t, X_t^\alpha)\mathrm{d}t + g(t, X_t^\alpha)\Phi^{-1}(\alpha)\mathrm{d}t \tag{7.23}$$

where $\Phi^{-1}(\alpha)$ is the inverse uncertainty distribution of standard normal uncertain variable, i.e.,

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}. \tag{7.24}$$

Example 7.8: The uncertain differential equation $\mathrm{d}X_t = a\mathrm{d}t + b\mathrm{d}C_t$ with $X_0 = 0$ has an α-path

$$X_t^\alpha = at + b\Phi^{-1}(\alpha)t. \tag{7.25}$$

Example 7.9: The uncertain differential equation $\mathrm{d}X_t = aX_t\mathrm{d}t + bX_t\mathrm{d}C_t$ with $X_0 = 1$ has an α-path

$$X_t^\alpha = \exp\left(at + b\Phi^{-1}(\alpha)t\right). \tag{7.26}$$

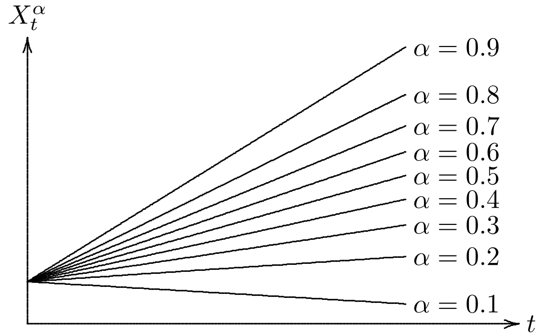

Figure 7.1: A Spectrum of α-Paths of $\mathrm{d}X_t = a\mathrm{d}t + b\mathrm{d}C_t$

Definition 7.4 . *The uncertain differential equation (7.22) is said to be monotone increasing if for any $\alpha \in (0,1)$ and any $t \geq 0$, we have*

$$\mathcal{M}\{X_t \leq X_t^\alpha\} = \alpha \tag{7.27}$$

where X_t and X_t^α are the solution and α-path of (7.22), respectively.

Example 7.10: The homogeneous linear uncertain differential equation

$$\mathrm{d}X_t = aX_t\mathrm{d}t + bX_t\mathrm{d}C_t \tag{7.28}$$

is monotone increasing whenever $b > 0$.

Example 7.11: The special linear uncertain differential equation

$$\mathrm{d}X_t = (m - aX_t)\mathrm{d}t + \sigma\mathrm{d}C_t \tag{7.29}$$

is monotone increasing whenever $\sigma > 0$.

Theorem 7.3. *If an uncertain differential equation is monotone increasing, then its α-path X_t^α is increasing with respect to α at each time t. That is,*

$$X_t^\alpha \leq X_t^\beta \tag{7.30}$$

at each time t whenever $\alpha < \beta$.

Proof: Since the uncertain differential equation is monotone increasing, we immediately have

$$\mathcal{M}\{X_t \leq X_t^\alpha\} = \alpha < \beta = \mathcal{M}\{X_t \leq X_t^\beta\}.$$

It follows from the monotonicity of uncertain measure that $X_t^\alpha \leq X_t^\beta$.

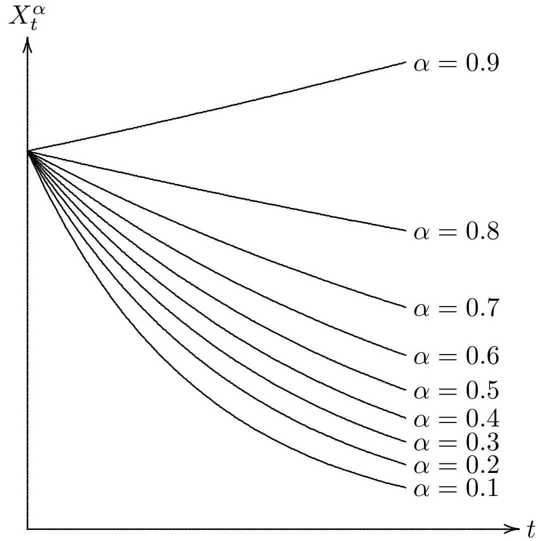

Figure 7.2: A Spectrum of α-Paths of $\mathrm{d}X_t = aX_t\mathrm{d}t + bX_t\mathrm{d}C_t$

Definition 7.5. *The uncertain differential equation (7.22) is said to be monotone decreasing if for any $\alpha \in (0,1)$ and any $t \geq 0$, we have*

$$\mathcal{M}\{X_t \leq X_t^\alpha\} = 1 - \alpha \tag{7.31}$$

where X_t and X_t^α are the solution and α-path of (7.22), respectively.

Theorem 7.4. *If an uncertain differential equation is monotone decreasing, then its α-path X_t^α is decreasing with respect to α at each time t. That is,*

$$X_t^\alpha \geq X_t^\beta \tag{7.32}$$

at each time t whenever $\alpha < \beta$.

Proof: Since the uncertain differential equation is monotone decreasing, we immediately have

$$\mathcal{M}\{X_t \leq X_t^\alpha\} = 1 - \alpha > 1 - \beta = \mathcal{M}\{X_t \leq X_t^\beta\}.$$

It follows from the monotonicity of uncertain measure that $X_t^\alpha \geq X_t^\beta$.

99-Method for Solving $\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t$

For solving a monotone uncertain differential equation, a key point is to obtain a 99-table of its solution X_s. In order to do so, a 99-method is designed as follows:

Step 1. Fix a time s and set $\alpha = 0$.

Step 2. Set $\alpha \leftarrow \alpha + 0.01$.

Step 3. Employ a classical numerical method to solve the corresponding ordinary differential equation $\mathrm{d}X_t^\alpha = f(t, X_t^\alpha)\mathrm{d}t + g(t, X_t^\alpha)\Phi^{-1}(\alpha)\mathrm{d}t$ and obtain X_s^α.

Step 4. Repeat the second and third steps until $\alpha = 0.99$.

Step 5. For a monotone increasing equation, the solution X_s has a 99-table,

0.01	0.02	$\cdots$	0.99
$X_s^{0.01}$	$X_s^{0.02}$	$\cdots$	$X_s^{0.99}$

(7.33)

Step 6. For a monotone decreasing equation, the solution X_s has a 99-table,

0.01	0.02	$\cdots$	0.99
$X_s^{0.99}$	$X_s^{0.98}$	$\cdots$	$X_s^{0.01}$

(7.34)

Note that the 99-method works only when the uncertain differential equation is almost monotone. In addition, the 99-method may be extended to the 999-method if a more precise result is needed. It is suggested that the ordinary differential equations in Step 3 are approximated by the recursion formula

$$X_{i+1}^\alpha = X_i^\alpha + f(t_i, X_i^\alpha)\Delta + g(t_i, X_i^\alpha)\Phi^{-1}(\alpha)\Delta \tag{7.35}$$

where Δ is the step length.

Example 7.12: Consider a monotone increasing uncertain differential equation

$$\mathrm{d}X_t = X_t\mathrm{d}t + X_t\mathrm{d}C_t, \quad X_0 = 1 \tag{7.36}$$

whose solution is $X_t = \exp(t + C_t)$. The 99-method may solve this equation successfully and obtain a 99-table of X_t at time $t = 1$ shown in Figure 7.3. The computer program is available at http://orsc.edu.cn/liu/resources.htm.

Example 7.13: Consider a monotone increasing uncertain differential equation

$$\mathrm{d}X_t = (1 - X_t)\mathrm{d}t + \mathrm{d}C_t, \quad X_0 = 1 \tag{7.37}$$

whose solution is

$$X_t = 1 + \int_0^t \exp(s - t)\mathrm{d}C_s. \tag{7.38}$$

The 99-method obtains a 99-table of X_t at time $t = 1$ shown in Figure 7.4.

Example 7.14: Consider a nonlinear uncertain differential equation

$$\mathrm{d}X_t = (t + X_t)\mathrm{d}t + \sqrt{1 + X_t}\mathrm{d}C_t, \quad X_0 = 2. \tag{7.39}$$

This equation is not completely monotone, even is not well defined because $1 + X_t$ may take negative values on some extreme sample paths. However,

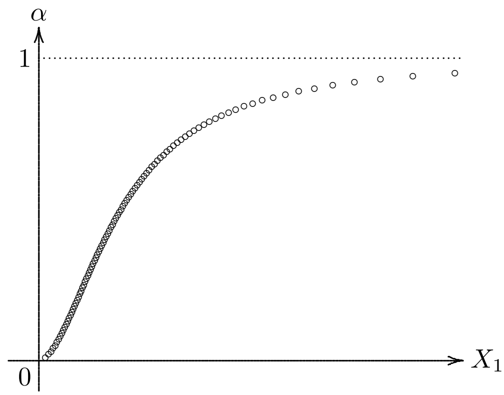

Figure 7.3: The 99-Table of $\mathrm{d}X_t = X_t\mathrm{d}t + X_t\mathrm{d}C_t$ with $X_0 = 1$

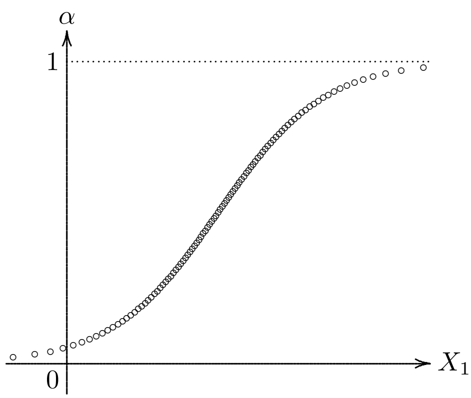

Figure 7.4: The 99-Table of $\mathrm{d}X_t = (1 - X_t)\mathrm{d}t + \mathrm{d}C_t$ with $X_0 = 1$

this blemish may be ignored and the 99-method is still valid. The 99-method obtains a 99-table of X_t at time $t = 1$ shown in Figure 7.5.

Open Problem: A necessary condition of monotone uncertain differential equation is that its α-path X_t^α is monotone with respect to α. What is a sufficient condition?

7.5 Uncertain Differential Equation with Jumps

In many cases the stock price is not continuous because of economic crisis or war. In order to incorporate those into stock model, we should develop an uncertain calculus with jump process. For many applications, a renewal

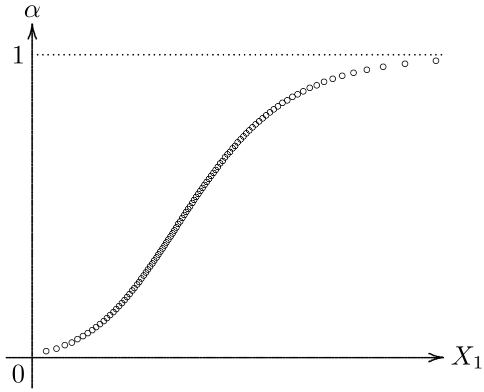

Figure 7.5: The 99-Table of $\mathrm{d}X_t = (t + X_t)\mathrm{d}t + \sqrt{1 + X_t}\mathrm{d}C_t$ with $X_0 = 2$

process N_t is sufficient. The uncertain integral of uncertain process X_t with respect to N_t is

$$\int_a^b X_t \mathrm{d}N_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i} \cdot (N_{t_{i+1}} - N_{t_i}) = \sum_{a \le t \le b} X_t \cdot (N_t - N_{t-}). \quad (7.40)$$

Definition 7.6. *Suppose C_t is a canonical process, N_t is a renewal process, and f, g, h are some given functions. Then*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t + h(t, X_t)\mathrm{d}N_t \quad (7.41)$$

is called an uncertain differential equation with jumps. A solution is an uncertain process X_t that satisfies (7.41) identically in t.

Example 7.15: Let C_t be a canonical process and N_t a renewal process. Then the uncertain differential equation with jumps

$$\mathrm{d}X_t = a\mathrm{d}t + b\mathrm{d}C_t + c\mathrm{d}N_t$$

has a solution $X_t = at + bC_t + cN_t$ which is just a jump process.

Example 7.16: Let C_t be a canonical process and N_t a renewal process. Then the uncertain differential equation with jumps

$$\mathrm{d}X_t = aX_t\mathrm{d}t + bX_t\mathrm{d}C_t + cX_t\mathrm{d}N_t$$

has a solution $X_t = \exp(at + bC_t + cN_t)$ which may be employed to model stock price with jumps.

7.6 Uncertain Finance

If we assume that the stock price follows some uncertain differential equation, then we may produce a new topic of uncertain finance. As an example, Liu [123] supposed that the stock price follows geometric canonical process and presented a *stock model* in which the bond price X_t and the stock price Y_t are determined by

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t \\ \mathrm{d}Y_t = eY_t\mathrm{d}t + \sigma Y_t\mathrm{d}C_t \end{cases} \tag{7.42}$$

where r is the riskless interest rate, e is the stock drift, σ is the stock diffusion, and C_t is a canonical process.

European Call Option Price

A European call option gives the holder the right to buy a stock at a specified time for specified price. Assume that the option has strike price K and expiration time s. Then the payoff from such an option is $(Y_s - K)^+$. Considering the time value of money resulted from the bond, the present value of this payoff is $\exp(-rs)(Y_s - K)^+$. Hence the European call option price should be the expected present value of the payoff,

$$f_c = \exp(-rs)E[(Y_s - K)^+]. \tag{7.43}$$

It is clear that the option price is a decreasing function of interest rate r. That is, the European call option will devaluate if the interest rate is raised; and the European call option will appreciate in value if the interest rate is reduced. In addition, the option price is also a decreasing function of strike price K.

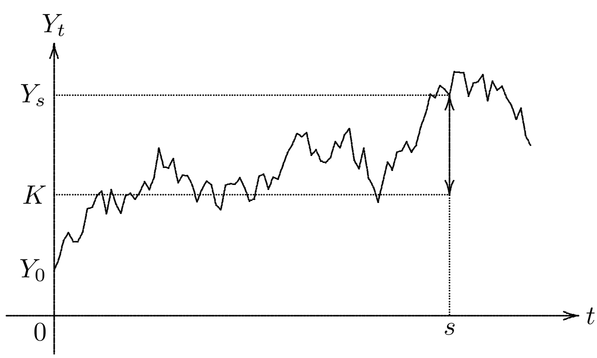

Figure 7.6: Payoff $(Y_s - K)^+$ from European Call Option

Let us consider the financial market described by the stock model (7.42). The European call option price is

$$\begin{aligned}
f_c &= \exp(-rs)E[(Y_0\exp(es+\sigma C_s)-K)^+]\\
&= \exp(-rs)\int_0^{+\infty}\mathcal{M}\{Y_0\exp(es+\sigma C_s)-K\ge x\}\mathrm{d}x\\
&= \exp(-rs)Y_0\int_{K/Y_0}^{+\infty}\mathcal{M}\{\exp(es+\sigma C_s)\ge y\}\mathrm{d}y\\
&= \exp(-rs)Y_0\int_{K/Y_0}^{+\infty}\mathcal{M}\{es+\sigma C_s\ge \ln y\}\mathrm{d}y\\
&= \exp(-rs)Y_0\int_{K/Y_0}^{+\infty}\left(1+\exp\left(\frac{\pi(\ln y-es)}{\sqrt{3}\sigma s}\right)\right)^{-1}\mathrm{d}y.
\end{aligned}$$

Thus Liu [123] derived the following European call option price formula,

$$f_c = \exp(-rs)Y_0\int_{K/Y_0}^{+\infty}\left(1+\exp\left(\frac{\pi(\ln y-es)}{\sqrt{3}\sigma s}\right)\right)^{-1}\mathrm{d}y. \tag{7.44}$$

European Put Option Price

A European put option gives the holder the right to sell a stock at a specified time for specified price. Assume that the option has strike price K and expiration time s. Then the payoff from such an option is $(K-Y_s)^+$. Considering the time value of money resulted from the bond, the present value of this payoff is $\exp(-rs)(K-Y_s)^+$. Hence the European put option price should be the expected present value of the payoff,

$$f_p = \exp(-rs)E[(K-Y_s)^+]. \tag{7.45}$$

It is easy to verify that the option price is a decreasing function of interest rate r, and is an increasing function of strike price K.

Let us consider the financial market described by the stock model (7.42). The European put option price is

$$\begin{aligned}
f_p &= \exp(-rs)E[(K-Y_0\exp(es+\sigma C_s))^+]\\
&= \exp(-rs)\int_0^{+\infty}\mathcal{M}\{K-Y_0\exp(es+\sigma C_s)\ge x\}\mathrm{d}x\\
&= \exp(-rs)Y_0\int_{K/Y_0}^{+\infty}\mathcal{M}\{\exp(es+\sigma C_s)\le y\}\mathrm{d}y\\
&= \exp(-rs)Y_0\int_0^{K/Y_0}\mathcal{M}\{es+\sigma C_s\le \ln y\}\mathrm{d}y\\
&= \exp(-rs)Y_0\int_0^{K/Y_0}\left(1+\exp\left(\frac{\pi(es-\ln y)}{\sqrt{3}\sigma s}\right)\right)^{-1}\mathrm{d}y.
\end{aligned}$$

Thus Liu [123] derived the following European put option price formula,

$$f_p = \exp(-rs)Y_0 \int_0^{K/Y_0} \left(1 + \exp\left(\frac{\pi(es - \ln y)}{\sqrt{3}\sigma s}\right)\right)^{-1} \mathrm{d}y. \tag{7.46}$$

Multi-factor Stock Model

Now we assume that there are multiple stocks whose prices are determined by multiple canonical processes. For this case, we have a *multi-factor stock model* in which the bond price X_t and the stock prices Y_{it} are determined by

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t \\ \mathrm{d}Y_{it} = e_iY_{it}\mathrm{d}t + \displaystyle\sum_{j=1}^{n} \sigma_{ij}Y_{it}\mathrm{d}C_{jt}, \ i = 1, 2, \cdots, m \end{cases} \tag{7.47}$$

where r is the riskless interest rate, e_i are the stock drift coefficients, σ_{ij} are the stock diffusion coefficients, C_{it} are independent canonical processes, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$.

Portfolio Selection

For the stock model (7.47), we have the choice of $m+1$ different investments. At each instant t we may choose a portfolio $(\beta_t, \beta_{1t}, \cdots, \beta_{mt})$ (i.e., the investment fractions meeting $\beta_t + \beta_{1t} + \cdots + \beta_{mt} = 1$). Then the wealth Z_t at time t should follow the uncertain differential equation

$$\mathrm{d}Z_t = r\beta_tZ_t\mathrm{d}t + \sum_{i=1}^{m} e_i\beta_{it}Z_t\mathrm{d}t + \sum_{i=1}^{m}\sum_{j=1}^{n} \sigma_{ij}\beta_{it}Z_t\mathrm{d}C_{jt}. \tag{7.48}$$

Portfolio selection problem is to find an optimal portfolio $(\beta_t, \beta_{1t}, \cdots, \beta_{mt})$ such that the expected wealth $E[Z_s]$ is maximized.

No-Arbitrage

The stock model (7.47) is said to be *no-arbitrage* if there is no portfolio $(\beta_t, \beta_{1t}, \cdots, \beta_{mt})$ such that for some time $s > 0$, we have

$$\mathcal{M}\{\exp(-rs)Z_s \geq Z_0\} = 1 \tag{7.49}$$

and

$$\mathcal{M}\{\exp(-rs)Z_s > Z_0\} > 0 \tag{7.50}$$

where Z_t is determined by (7.48) and represents the wealth at time t. We may prove that the stock model (7.47) is no-arbitrage if and only if its diffusion matrix

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{m1} & \sigma_{m2} & \cdots & \sigma_{mn} \end{pmatrix}$$

has rank m, i.e., the row vectors are linearly independent.

Stock Model with Mean-Reverting Process

Peng [175] assumed that the stock price follows a type of mean-reverting uncertain process and proposed the following stock model,

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t \\ \mathrm{d}Y_t = (m - aY_t)\mathrm{d}t + \sigma\mathrm{d}C_t \end{cases} \tag{7.51}$$

where r, m, a, σ are given constants.

Stock Model with Periodic Dividends

Liu [134] assumed that a dividend of a fraction δ of the stock price is paid at deterministic times $T_1, T_2, \cdots$ and presented a stock model with periodic dividends,

$$\begin{cases} X_t = X_0 \exp(rt) \\ Y_t = Y_0(1-\delta)^{n_t} \exp(et + \sigma C_t) \end{cases} \tag{7.52}$$

where $n_t = \max\{i : T_i \leq t\}$ is the number of dividend payments made by time t.

Currency Models

Liu [133] assumed that the exchange rate follows a geometric canonical process and proposed a currency model with uncertain exchange rate,

$$\begin{cases} \mathrm{d}X_t = eX_t\mathrm{d}t + \sigma X_t\mathrm{d}C_t \quad \text{(Exchange rate)} \\ \mathrm{d}Y_t = uY_t\mathrm{d}t \quad \text{(Yuan Bond)} \\ \mathrm{d}Z_t = vZ_t\mathrm{d}t \quad \text{(Dollar Bond)} \end{cases} \tag{7.53}$$

where e, σ, u, v are constants. If the exchange rate follows a mean-reverting uncertain process, then the currency model with uncertain exchange rate is

$$\begin{cases} \mathrm{d}X_t = (m - \alpha X_t)\mathrm{d}t + \sigma\mathrm{d}C_t \quad \text{(Exchange rate)} \\ \mathrm{d}Y_t = uY_t\mathrm{d}t \quad \text{(Yuan Bond)} \\ \mathrm{d}Z_t = vZ_t\mathrm{d}t \quad \text{(Dollar Bond)} \end{cases} \tag{7.54}$$

where m, α, σ, u, v are constants.