AI 701 Bayesian machine learning, Fall 2021

Homework assignment 2

1. (b) We will use the law of large numbers to verify your code. For that, compute the mean and variance of X with Probability Density Function (PDF) $f_X(x) = \lambda x e^{-\lambda x} \mathbb{1}_{\{x \geq 0\}}$.

Solution:

$$\begin{aligned} \mathbb{E}[X] &= \int_0^\infty \lambda x e^{-\lambda x} \mathrm{d}x \\ &= \left[-x e^{-\lambda x}\right]_0^\infty + \int_0^\infty e^{-\lambda x} \mathrm{d}x \\ &= \left[-e^{-\lambda x}/\lambda\right]_0^\infty = 1/\lambda. \end{aligned} \tag{S.1}$$

$$\begin{aligned} \mathbb{E}[X^2] &= \int_0^\infty \lambda x^2 e^{-\lambda x} \mathrm{d}x \\ &= \left[-x^2 e^{-\lambda x}\right]_0^\infty + 2 \int_0^\infty x e^{-\lambda x} \mathrm{d}x \\ &= (2/\lambda)\mathbb{E}[X] \\ &= 2/\lambda^2 \qquad \text{(S.2)} \\ \mathrm{Var}(X) &= \mathbb{E}[X^2] - \mathbb{E}^2[X] = 1/\lambda^2. \qquad \text{(S.3)} \end{aligned}$$

(d) Let $Y_1 \sim \mathrm{Gamma}(a_1, b)$ be a gamma random variable with shape parameter $a_1 > 0$ and rate parameter $b > 0$ with PDF

$$f_{Y_1}(y) = \frac{b^{a_1} y^{a_1 - 1} e^{-by}}{\Gamma(a_1)}.$$

Similarly, let $Y_2 \sim \mathrm{Gamma}(a_2, b)$ be another gamma random variable with shape $a_2 > 0$ and rate $b > 0$. Assuming Y_1 and Y_2 are independent, show that $Y = Y_1 + Y_2 \sim \mathrm{Gamma}(a_1 + a_2, b)$.

Solution: The Cumulative Distribution Function (CDF) of Y is computed as

$$\begin{aligned} F_Y(z) &= \mathbb{P}(Y_1 + Y_2 \leq z) \\ &= \int_0^z \int_0^{z - y_2} f_{Y_1}(y_1) f_{Y_2}(y_2) \mathrm{d}y_1 \mathrm{d}y_2. \end{aligned} \tag{1}$$

Hence, we have

$$
\begin{aligned}
f_Y(z) &= \frac{\mathrm{d}F_Y(z)}{\mathrm{d}z} \\
&= \int_0^z f_{Y_1}(z-y_2)f_{Y_2}(y_2)\mathrm{d}y_2 \\
&= \frac{b^{a_1+a_2}e^{-bz}}{\Gamma(a_1)\Gamma(a_2)}\int_0^z (z-y_2)^{a_1-1}y_2^{a_2-1}\mathrm{d}y_2 \\
&= \frac{b^{a_1+a_2}z^{a_1+a_2-1}e^{-bz}}{\Gamma(a_1)\Gamma(a_2)}\int_0^1 (1-u)^{a_1-1}u^{a_2-1}\mathrm{d}u \quad (u := y_2/z) \\
&= \frac{b^{a_1+a_2}z^{a_1+a_2-1}e^{-bz}}{\Gamma(a_1)\Gamma(a_2)}\frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(a_1+a_2)} \\
&= \frac{b^{a_1+a_2}z^{a_1+a_2-1}e^{-bz}}{\Gamma(a_1+a_2)}, \qquad (2)
\end{aligned}
$$

which corresponds to the PDF of a gamma random variable with shape $a_1 + a_2$ and rate b.

(e) Explain how to draw samples from a gamma distributed random variable $Y \sim \mathrm{Gamma}(m, 1)$ with $m \in \mathbb{N}$ ($\mathbb{N}$ is a set of natural numbers) using `rand_exp`.

Solution: An exponential random variable X with parameter $\lambda = 1$ is also a gamma random variable with shape $a = 1$ and rate $b = 1$. The result we proved in (d) implies

$$X_1, \ldots, X_m \overset{\text{i.i.d.}}{\sim} \mathrm{Gamma}(1,1), \quad Y = \sum_{i=1}^m X_i \sim \mathrm{Gamma}(m,1). \qquad (S.4)$$

Hence we can draw m i.i.d. samples using `rand_exp` and sum them to get a sample from Y.

2. (a) Describe how to sample from $q(x)$ using `npr.rand`, and compute the mean and variance of the distribution.

Solution: We can sample from $q(x)$ via inverse CDF method. The CDF is given as

$$Q(x) := \mathbb{P}(X \leq x) = \int_{-2}^x q(y)\mathrm{d}y = \frac{a^3+8}{16}. \qquad (3)$$

Hence, given a sample u drawn from $\mathrm{Unif}(0,1)$, we can construct a sample from $q(x)$ as

$$x = Q^{-1}(u) = \sqrt[3]{16u-8} \qquad (4)$$

The mean and variances are easily computed as follows:

$$\mathbb{E}[X] = \int_{-2}^2 xq(x)\mathrm{d}x = 0 \quad (xq(x) \text{ is symmetric }) \qquad (5)$$

$$\mathrm{Var}(X) = \frac{3}{16}\int_{-2}^2 x^2q(x)\mathrm{d}x = \frac{12}{5}. \qquad (6)$$