Complexity Analysis

By

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Which one is most efficient approach?

Approach 3

```python
def gcd(m: int, n: int):
    common_factors = 1
    for f in range(2, min(m, n)+1):
        if m % f == 0 and n % f == 0:
            common_factors = f

    return common_factors
```

Approach 4

```python
def gcd(m: int, n: int):
    cf = min(m, n)
    while cf > 0:
        if m % cf == 0 and n % cf == 0:
            return cf
        cf -= 1

```

Asymptotic analysis

- Estimation of CPU time and main memory space required to complete the execution of the algorithm
  - Time complexity:
    - Frequency count/Sum of frequency
  - Space complexity:
    - Extra space consumption for the execution of the algorithm
    - Not the input size

Asymptotic analysis: Time complexity

```
def gcd(m: int, n: int):
    factors_m = []------------------------→ 1
    factors_n = []------------------------→ 1
    for i in range(1, m+1):---------------→ m
        if m % i == 0:--------------------→ m
            factors_m.append(i)-----------→ m
    for i in range(1, n+1):---------------→ n
        if n % i == 0:--------------------→ n
            factors_n.append(i)-----------→ n
    common_factors = []-------------------→ 1
    for f in factors_m:-------------------→ m
        if f in factors_n:----------------→ mxn
            common_factors.append(f)------→ mxn
    return common_factors[-1]-------------→ 1
```

TC = 4m+3n+2(mn)+4

TC = Θ(mn)

Asymptotic analysis: Space complexity

```
def gcd(m: int, n: int):
    factors_m = []-------------------------→ m
    factors_n = []-------------------------→ n
    for i in range(1, m+1):----------------→ 1
        if m % i == 0:
            factors_m.append(i)
    for i in range(1, n+1):
        if n % i == 0:
            factors_n.append(i)
    common_factors = []--------------------→ m
    for f in factors_m:
        if f in factors_n:
            common_factors.append(f)
    return common_factors[-1]
```

SC = 2m + n + 1

SC = Θ(m)

Time complexity: Θ(n)

```
for (i=1; i<=n; i++)
     printf(“Text”);
```

```
for (i=n; i>=1; i--)
     printf(“Text”);
```

Time complexity?

```
for (i=1; i<=n; i=i+5)
     printf(“Text”)
```

Time complexity?

```
for (i=1; i<=n; i=i+5)
     printf(“Text”)
```

Exit when: i = 1+(k x 5) <= n

➔k <= (n-1)/5

TC: Θ(n)

1. i=1
2. i=1+5
3. i=1+(2x5)
4. i=1+(3x5)
5. i=1+(4x5)
6. ...
7. i=1+(k x 5)

Time complexity: Θ(n)

```
for (i=1; i<=n; i++)
     printf(“Text”);
```

```
for (i=n; i>=1; i--)
     printf(“Text”);
```

```
for (i=1; i<=n; i=i+5)
     printf(“Text”);
```

```
for (i=n; i>=1; i=i-5)
     printf(“Text”);
```

Time complexity?

```
for (i=1; i<=n; i=i*2)
     printf(“Text”);
```

Time complexity?

```
for (i=1; i<=n; i=i*2)
      printf("Text");

for (i=n; i>=1; i=i/2)
      printf("Text");
```

1. i=1
2. i=2
3. $i=2^2$
4. $i= 2^3$
5. $i= 2^4$
6. ...
7. $i= 2^k$

Exit when: $i = 2^k <= n$

➔ $k = \log n$

TC: $\Theta(\log n)$

Time complexity?

```
for (i=2; i<=n; i=i^2)
     printf(“Text”);

for (i=n; i>=1; i=i^1/2)
     printf(“Text”);
```

Time complexity?

```
for (i=2; i<=n; i=i^2)
      printf("Text");

for (i=n; i>=1; i=i^1/2)
      printf("Text");
```

1. i=2
2. $i=2^2$
3. $i=2^{2^2}$
4. $i= 2^{2^3}$
5. $i= 2^{2^4}$
6. ...
7. $i= 2^{2^k}$

Exit when: $i = 2^{(2^k)} <= n$

➔ $2^k = \log n$

➔ $k = \log \log n$

TC: $\Theta(\log \log n)$

Time complexity?

```
for (i=1; i<=n; i++)
     for (j=1; j<=n; j++)
          printf(“Text”);
```

```
for (i=1; i<=n; i++)
     for (j=1; j<=n; j=j*2)
          printf(“Text”);
```

Time complexity?

```
for (i=1; i<=n; i++) ------------------→n
        for (j=1; j<=n; j++) ----------→ n
                printf("Text");--------→ n^2
```

TC: $\Theta(n^2)$

```
for (i=1; i<=n; i++) ------------------→ n
        for (j=1; j<=n; j=j*2) -------→ log n
                printf("Text");-------→ n log n
```

TC: $\Theta(n \log n)$

Time complexity?

```
def fact(n: int) -> int:
    if n == 0 or n == 1:
        return 1
    else:
        return n * fact(n-1)
```

Space complexity

1. fact(5)
2. fact(4)
3. fact(3)
4. fact(2)
5. fact(1)

Maximum recursion depth (STACK)

$\Theta(n)$

Time complexity?

```
def fact(n: int) -> int:
    if n == 0 or n == 1:
        return 1
    else:
        return n * fact(n-1)
```

- $T(n) = T(n-1) + C$, $n>1$
- $T(n) = 1$, $n<=1$

(n-k) =1
K = (n-1)

Time complexity?

```
def fact(n: int) -> int:
    if n == 0 or n == 1:
        return 1
    else:
        return n * fact(n-1)
```

- T(n) = T(n-1) + C , n>1
- T(n) = 1, n<=1

T(n) = T(n-1)+C
= [T(n-2)+C]+C
= [T(n-3)+C]+C+C
= [T(n-4)+C]+C+C+C
…
= T(n-k) + k.C
= T(n-(n-1)) + (n-1) C
= T(1) + (n-1)C
= 1+ C (n-1) → Θ(n)

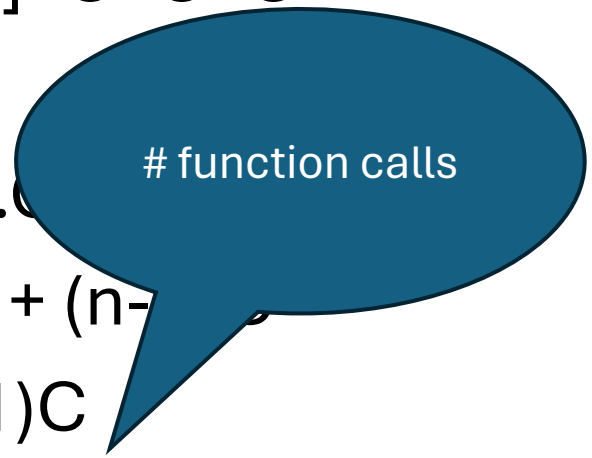
(n-k) =1
K = (n-1)

Time complexity?

```
def fact(n: int) -> int:
    if n == 0 or n == 1:
        return 1
    else:
        return n * fact(n-1)
```

- T(n) = T(n-1) + C , n>1
- T(n) = 1, n<=1

T(n) = T(n-1)+C
= [T(n-2)+C]+C
= [T(n-3)+C]+C+C
= [T(n-4)+C]+C+C+C
.
= T(n-k) + k.C
= T(n-(n-1)) + (n-
= T(1) + (n-1)C
= 1+ C (n-1) → Θ(n)



Time complexity?

```
1. def rec(n):
2.     if n <=1:
3.         return 1
4.     return n + rec(n-1) + rec(n-1)
```

Time complexity?

```
def rec(n):
    if n <=1:
        return 1
     return n + rec(n-1) + rec(n-1)
```

- $T(n) = 2T(n-1) + C$, $n>1$
- $T(n) = 1$, $n<=1$

Time complexity?

$T(n-1) = 2T(n-2)+C$

$T(n) = 2T(n-1)+C$

$= 2[2T(n-2)+C]+C$

$= 2^2T(n-2)+2C+C$

$= 2^2\,[2T(n-3)+C]+2C+C$

$= 2^3T(n-3)+2^2C+2C+C$

$= 2^kT(n-k) +(2^k+\ldots+2^2+2+1)C$

$= 2^{n-1}T(n-(n-1)) +(2^{n-1}+\ldots+2^2+2+1)C$

$= 2^{n-1}T(1) +(2^{n-1}+\ldots+2^2+2+1)C \rightarrow \Theta(2^n)$

$(n-k) = 1$

$k = (n-1)$

Time complexity?

```
def rec(n):
    if n <=1:
        return 1
     for i in range(n):
         //code//
     return n + rec(n-1)
```

Time complexity?

```
def rec(n):
    if n <=1:
        return 1
     for i in range(n):
         //code//
     return n + rec(n-1)
```

- T(n) = 1, n<=1
- T(n) = T(n-1)+n, n > 1

Time complexity?

$T(n-1) = T(n-2)+(n-1)$

$T(n) = T(n-1) + n$

$= T(n-2) + (n-1) + n$

$= T(n-3) +(n-2) + (n-1) + n$

$(n-k) = 1$

$k = (n-1)$

..

$= T(n-k) + (n-(k-1)) + (n-(k-2)) + \dots + (n-2) + (n-1) + n$

$= T(n-(n-1)) + (n-((n-1)-1)) + (n-((n-1)-2)) + \dots + (n-2) + (n-1) + n$

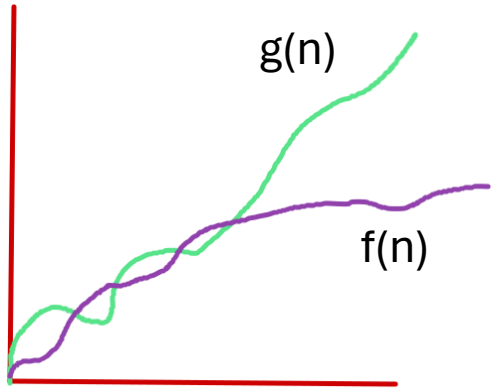
$= 1 + 2 + 3 + 4 + \dots + n$

$= n(n+1)/2 \rightarrow \Theta(n^2)$

Asymptotic notation

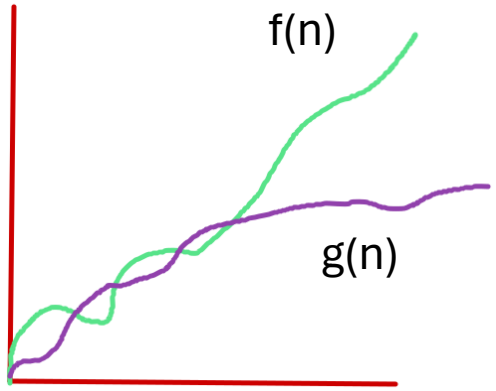
- Growth rate comparison of functions
- For large input values

Asymptotic notation



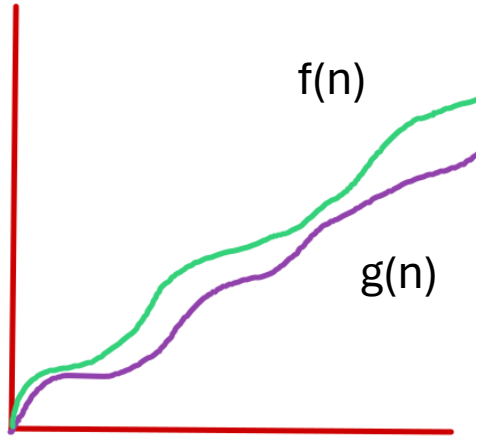
$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

f(n) asymptotically smaller than g(n)



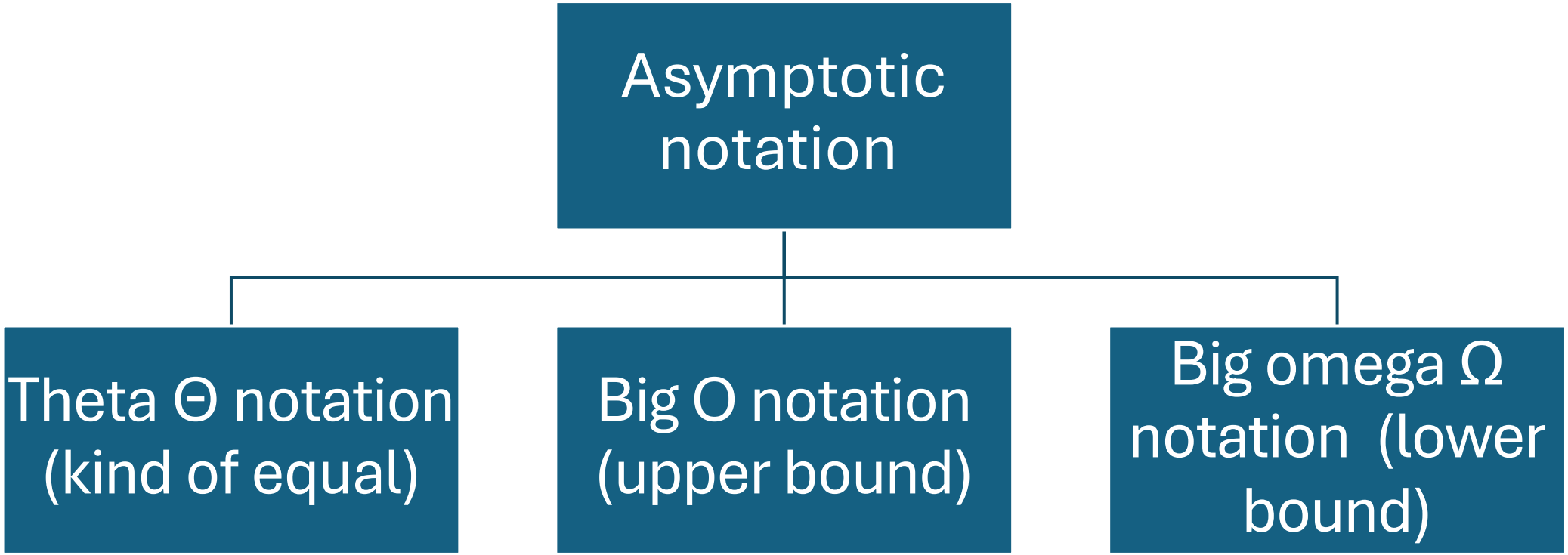
$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$$

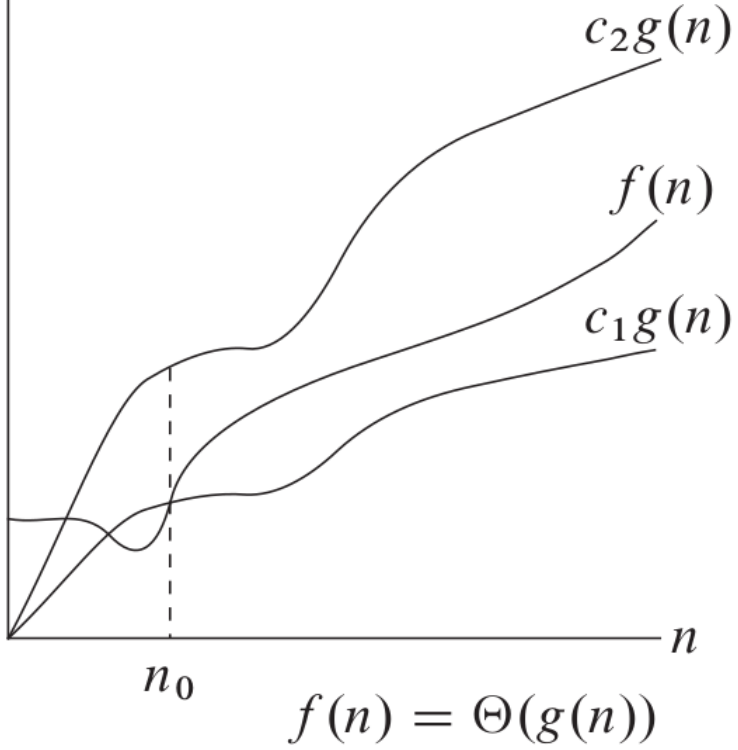
g(n) asymptotically smaller than f(n)

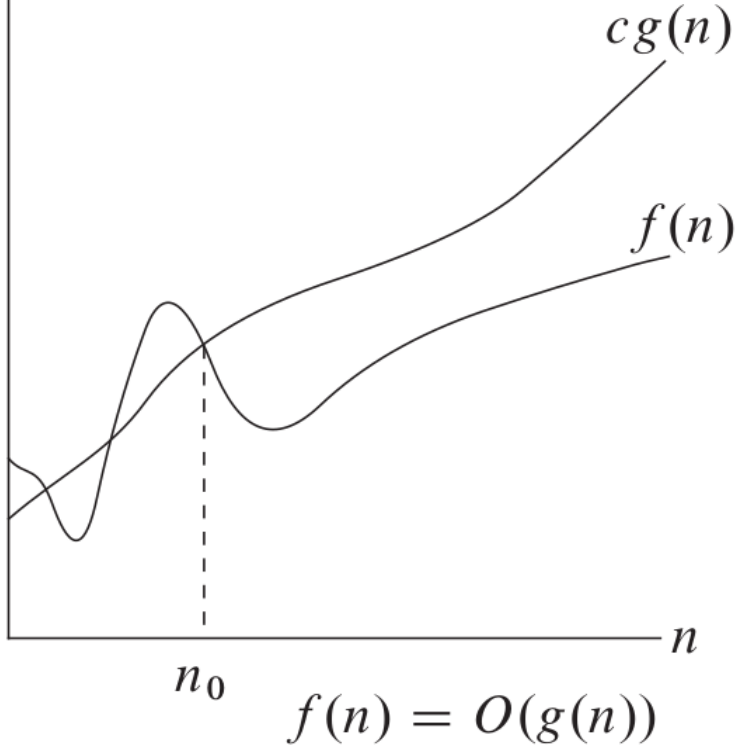


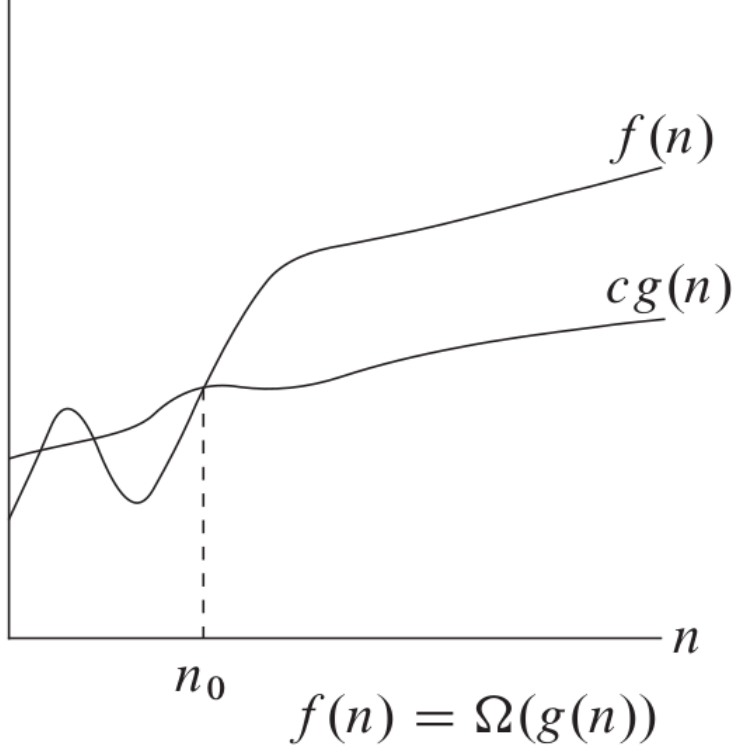
$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = C$$

f(n) & g(n) asymptotically equal


Asymptotic notation
Theta Θ notation (kind of equal)
Big O notation (upper bound)
Big omega Ω notation (lower bound)



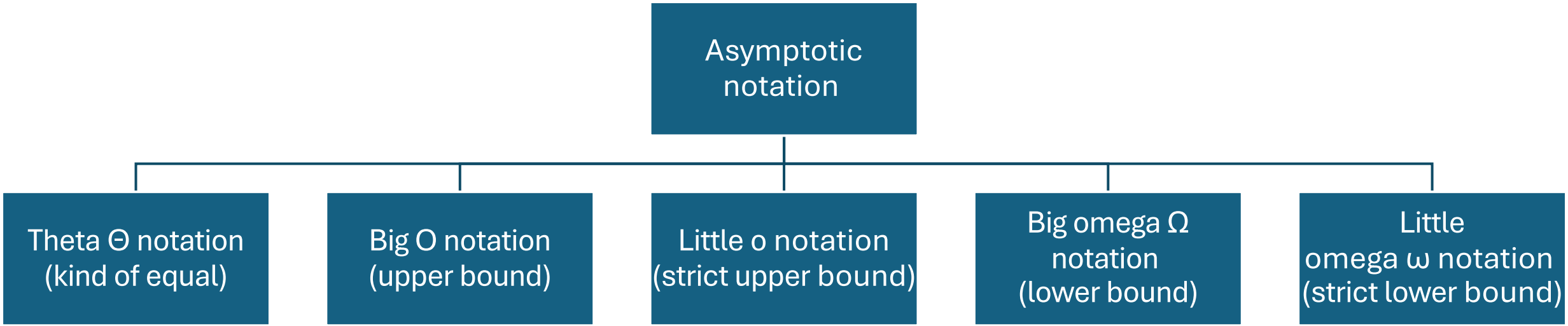




- **Θ notation**: Θ(g(n)) = { f(n): there exist positive constant c_1, c_2 and n_0 such that $0 <= c_1g(n) <= f(n) <= c_2g(n)$ for all $n >= n_0$ }

- **O notation**: O(g(n)) = { f(n): there exist positive constant c and n_0 such that $0 <= f(n) <= cg(n)$ for all $n >= n_0$ }

- **Ω notation**: Ω(g(n)) = { f(n)} there exist positive constant c and n_0 such that $0 <= cg(n) <= f(n)$ for all $n >= n_0$ }


Asymptotic notation
Theta Θ notation (kind of equal)
Big O notation (upper bound)
Little o notation (strict upper bound)
Big omega Ω notation (lower bound)
Little omega ω notation (strict lower bound)

Algorithm complexity: Approach 4

- Best case: Θ(1)
- Average case: O(n)
- Worst case: O(n)

Algorithm complexity: Approach 4

- Best case: Θ(1)
- Average case: O(n)
- Worst case: O(n)

Nothing to do with notations: O, o, Θ, Ω, and ω

Efficiency of algorithm

Polynomial time

- n
- $\log n$
- $n \log n$
- n^2
- n^3
- n^{10}
- n^k

Exponential time

- $n!$
- 2^n
- 3^n
- k^n
- n^n

Efficiency of algorithm

Polynomial time (**Theoretically efficient**)

- n
- $\log n$
- $n \log n$
- n^2
- n^3
- n^{10}
- n^k

Exponential time

- $n!$
- 2^n
- 3^n
- k^n
- n^n

Efficiency of algorithm

Polynomial time (**Practically efficient**)

- n
- log n
- n log n
- n^2
- n^3
- n^{10}
- n^k

Exponential time

- n!
- 2^n
- 3^n
- k^n
- n^n

References

- https://youtu.be/EHp4FPyajKQ?si=SGAiBtDPjsJ9lyEI
- https://www.claymath.org/millennium-problems/
- Cormen, Thomas H., et al. Introduction to algorithms. MIT press, 2022.