Optimization and Optimal Control

Lecture 5: Optimal Control: Linear MPC

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Outline of the Course

Part I: Constrained Optimization

Part II: Optimal Control of Linear Systems

Part III: Optimal Control of Hybrid Systems

Outline of Part II

Part II: Optimal Control of Linear Systems
- II.1. Continuous Time Optimal Control
  - II.1.1. Motivation
  - II.1.2. Optimal Control using Variational Approach
- II.2. Discrete-Time Linear Quadratic Regulator
  - II.2.1. Problem Formulation
  - II.2.2. LQR via least-squares
  - II.2.3. LQR via Dynamic Programming
  - II.2.4. LQR for Nonlinear Systems
  - II.2.5. Infinite Horizon LQR problem
- II.3. Model Predictive Control
  - II.3.1. Linear MPC
  - II.3.2. Properties of MPC
  - II.3.3. MPC and LQR
  - II.3.4. Linear MPC based on LP
  - II.3.5. Explicit MPC

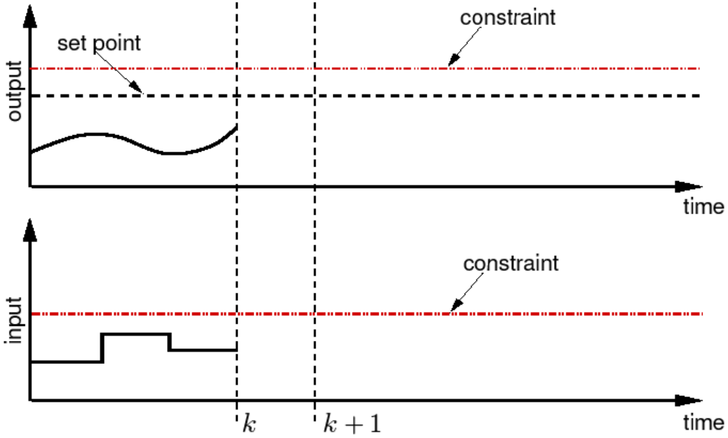
Outline

1. Model Predictive Control
1.1 Linear MPC
1.2 Properties of MPC
1.3 MPC and LQR
1.4 Linear MPC based on LP
1.5 Explicit MPC

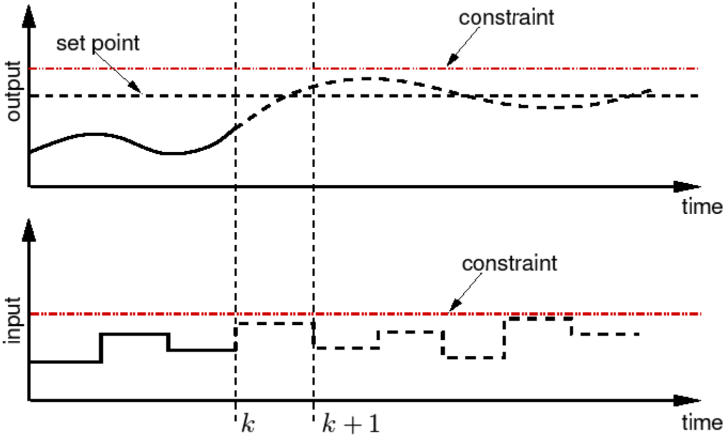
Constraints in Control

- All physical systems have constraints:
  - physical constraints, e.g., actuator limits
  - performance constraints, e.g., overshoot
  - safety constraints, e.g., temperature/presure limits
- Optimal operating points are often near constraints
- Most control methods address constraints a posteriori
  - anti-windup methods, etc.

Receding Horizon Principle



Receding Horizon Principle



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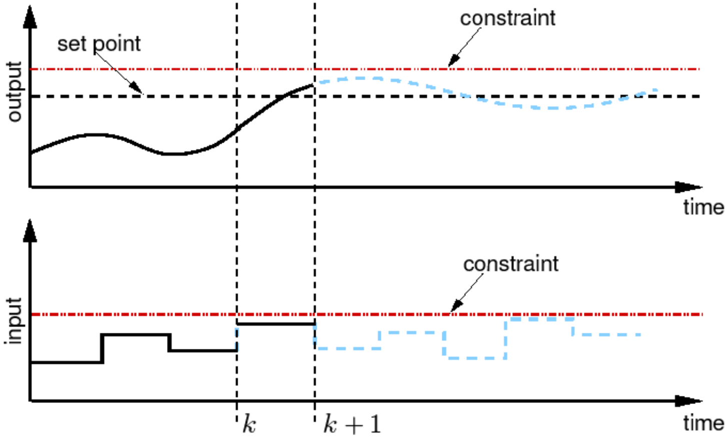

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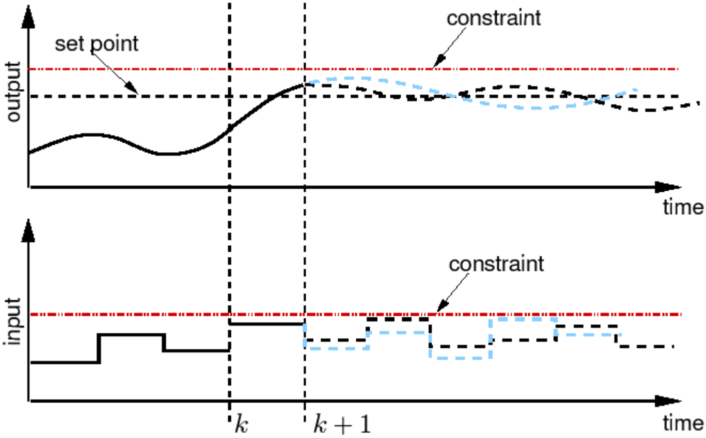

Principal Ideas

- At each time instant, a MPC
  1. Takes a measurement of the system
  2. Computes a finite horizon control sequence that
     - Uses an internal model to predict system behavior
     - Minimizes some cost functions
     - Doesn't violate any constraints
  3. Implements the first part of the optimal sequence
- This is a feedback control law

Properties of MPC technique

- Is this a new idea?
  - No - Standard finite horizon LQR
  - Yes - Optimization in the loop
- The main problems:
  - Computation should be fast enough
  - Infinite horizon constraints satisfaction is difficult
  - Stability of the control scheme
- The main advantages:
  - Systematic method for handling constraints
  - Flexible performance specifications

Constraint Optimal Control

Discrete-time system

Linear Model: $\begin{cases} x(t+1) &= Ax(t)+Bu(t) \\ y(t) &= Cx(t) \end{cases}$

Constraints: $\begin{cases} u_{min} \le u(t) \le u_{max} \\ y_{min} \le y(t) \le y_{max} \end{cases}$

Constraint Optimal Control:

$$\begin{aligned} \min_{u(t),\ldots,u(t+N-1)} \quad & \sum_{k=0}^{N-1} (x(t+k)'Qx(t+k)+ \\ & + u(t+k)'Ru(t+k)) + x'(t+N)Px(t+N) \\ \text{s.t.} \quad & x(t+k+1) = Ax(t+k)+Bu(t+k),\ \ k=1,\ldots,N-1 \\ & u_{min} \le u(t+k) \le u_{max},\ \ k=0,\ldots,N-1 \\ & y_{min} \le y(t+k) \le y_{max},\ \ k=1,\ldots N \end{aligned}$$

Constraint Optimal Control

- Using the same technique as for unconstrained optimal control (LQR) we have:

Optimization Problem

$$\begin{aligned} V(x_0) &= \tfrac{1}{2}x_0'Yx_0 + \min_U \tfrac{1}{2}U'HU + x_0'FU \\ \text{s.t.} &\quad GU \leq W + Sx_0 \end{aligned}$$

- Properties of the optimal solution:
  - It is a global minimum when $H \geq 0$
  - it is unique if $H > 0$

Example - Double Integrator System

- System:

$$
\begin{aligned}
x(t+1) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) \\
y(t) &= (1\ 0)\ x(t)
\end{aligned}
$$

- Constraints:

$$|u(t)| \leq 1$$

- Timing horizon: $N = 2$
- Quadratic Cost:

$$Q = P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; R = 0.1$$

Example - Double Integrator System

Step 1: Put the Problem in the standard form

$$\begin{aligned} V(x_0) &= \tfrac{1}{2}x_0' Y x_0 + \min_U \tfrac{1}{2} U'HU + x_0' FU \\ \text{s.t.} &\quad GU \leq W + Sx_0 \end{aligned}$$

- $N = 2 \Longrightarrow U = (u(t)\ u(t+1))'$ and $X = (x(t+1)\ x(t+2))'$
- $|u(t)| \leq 1 \Longleftrightarrow (u(t) \leq 1)$ and $(u(t) \geq -1 \Longleftrightarrow -u(t) \leq 1)$, therefore:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u(t) \\ u(t+1) \end{pmatrix} \leq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Example - Double Integrator System

Step 1: Put the Problem in the standard form

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Example - Double Integrator System

- $V(x_0) = \frac{1}{2}x_0' Y x_0 + \min_{U} \frac{1}{2}U' H U + x_0' F U$, where

$$H = \begin{pmatrix} 4.2 & 2 \\ 2 & 2.2 \end{pmatrix}, F = \begin{pmatrix} 2 & 0 \\ 6 & 2 \end{pmatrix}, Y = \begin{pmatrix} 6 & 6 \\ 6 & 12 \end{pmatrix}$$

- subject to $GU \leq W + Sx_0$, where

$$G = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}, W = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, S = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

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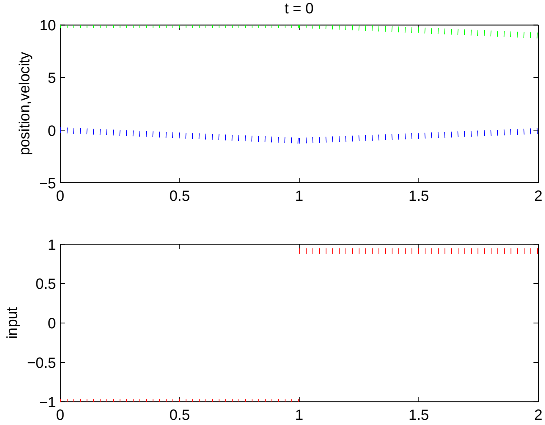

- Receding Horizon Control
- $t = 0$, $x_0 = (10\ 0)'$
- Optimal Input:

$$U = (u(t{+}0)\ u(t{+}1))' = (-1\ 0.9091)'$$

- Predicted States:

$$\begin{aligned}\hat{x}(t+1) &= Ax_0 + Bu(t+0)\\ &= (10\ \ -1)'\\ \hat{x}(t+2) &= A\hat{x}(t+1) + Bu(t+1)\\ &= (9\ \ -0.0909)'\end{aligned}$$

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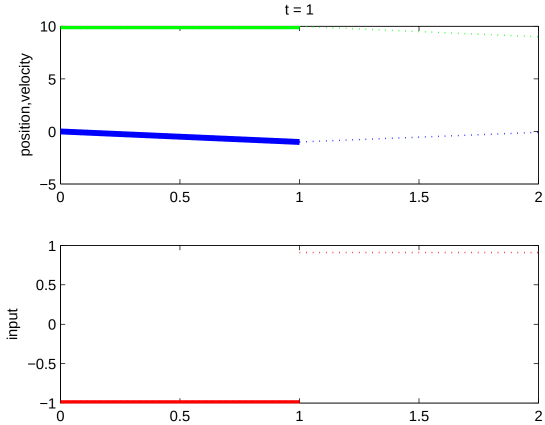

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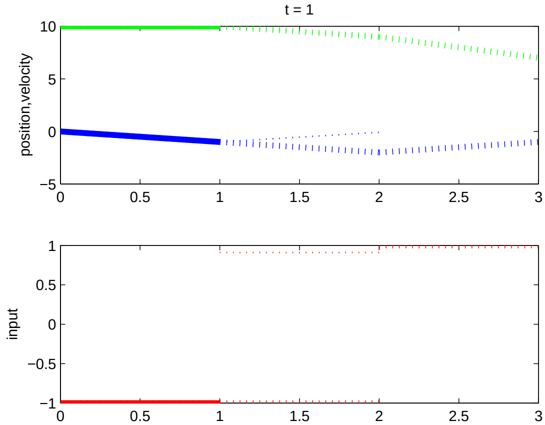

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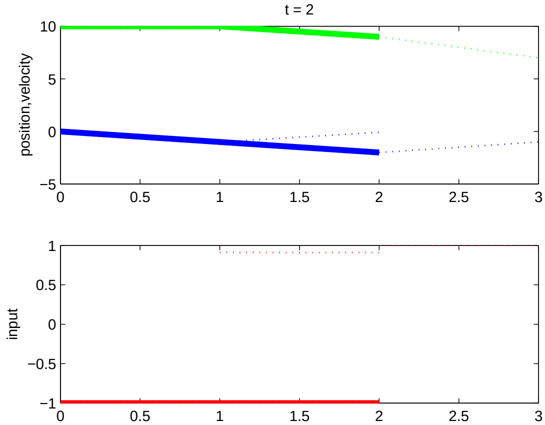

- Receding Horizon Control
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- Optimal Input:

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- $t = 2$, $x_0 = (9 \;\; -2)'$
- Optimal Input:

$$U = (u(t+0) \;\; u(t+1))' = (-1 \;\; 1)'$$

- Predicted States:

$$\begin{aligned} \hat{x}(t+1) &= Ax_0 + Bu(t+0) \\ &= (7 \;\; -3)' \\ \hat{x}(t+2) &= A\hat{x}(t+1) + Bu(t+1) \\ &= (4 \;\; -2)' \end{aligned}$$

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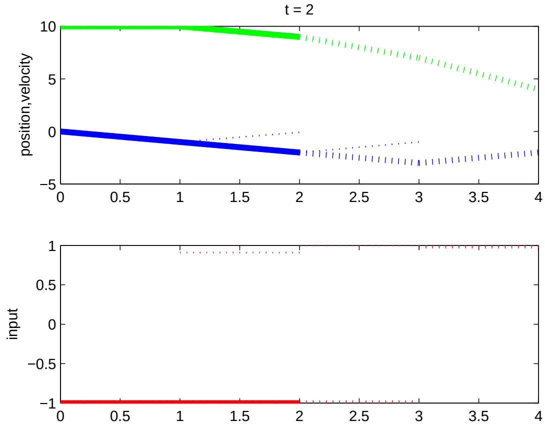

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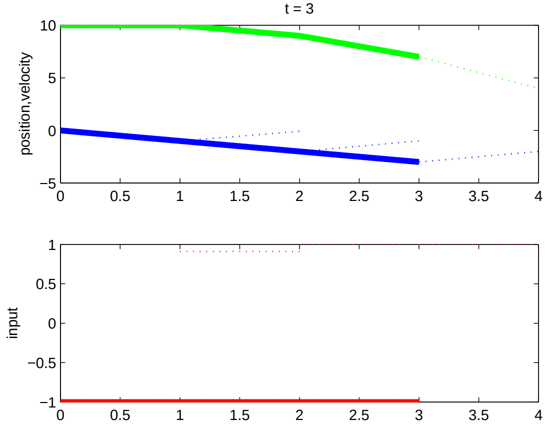
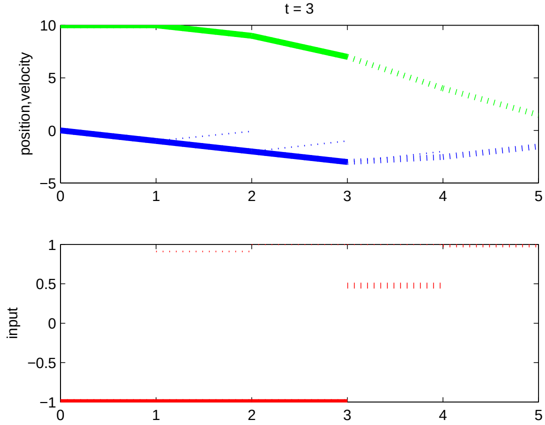


- Receding Horizon Control

Example - Double Integrator System



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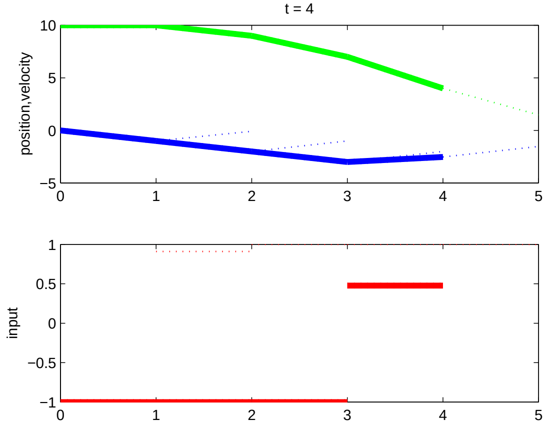

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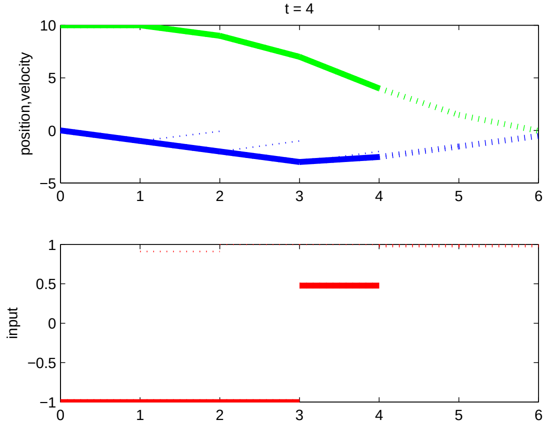

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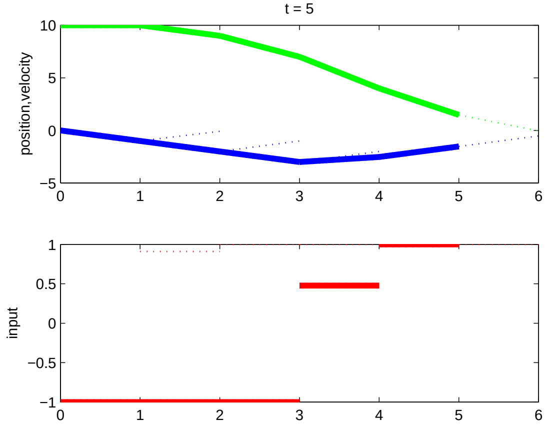

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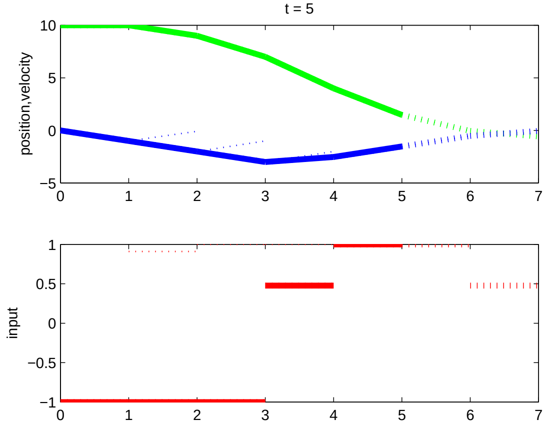

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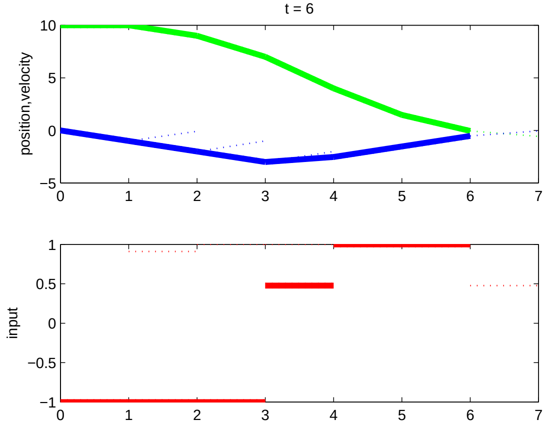

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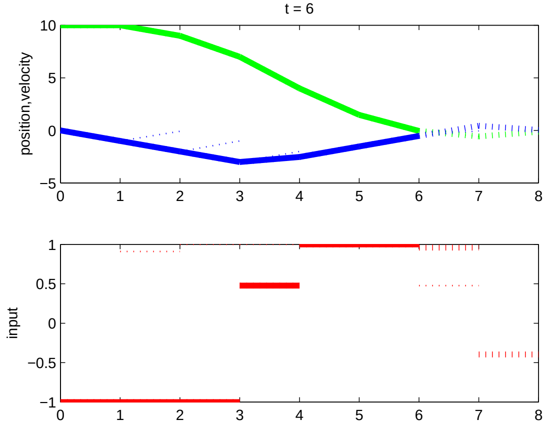

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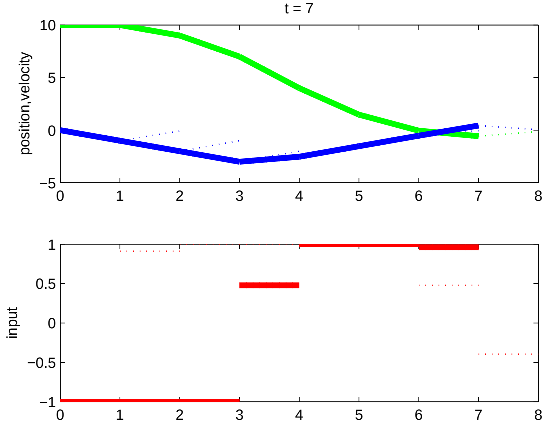

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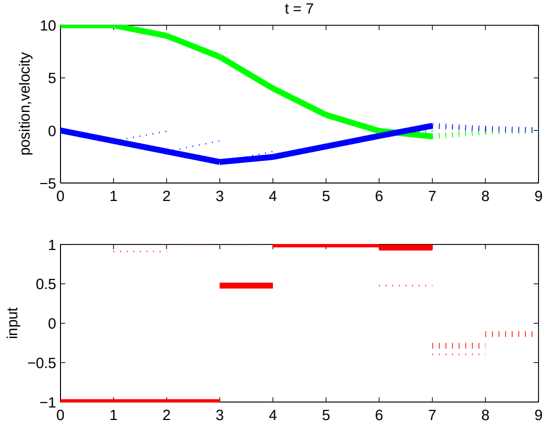

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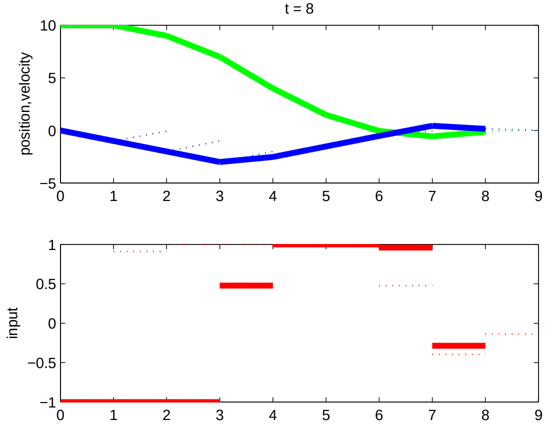

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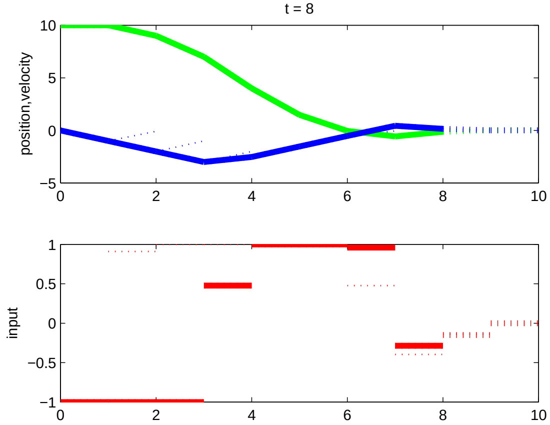

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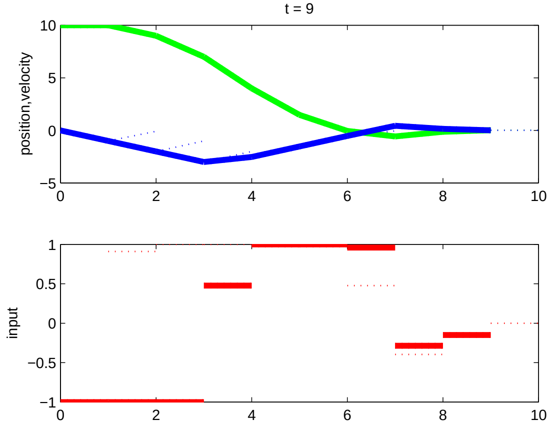

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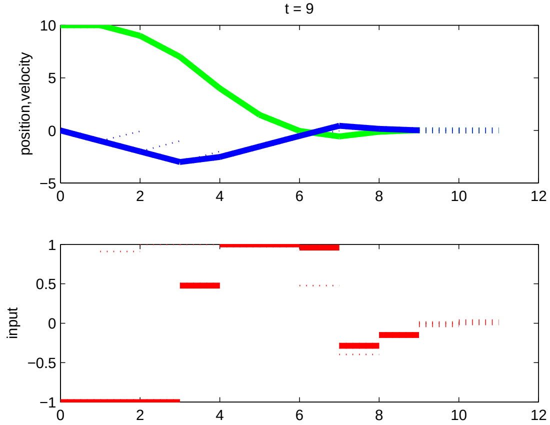

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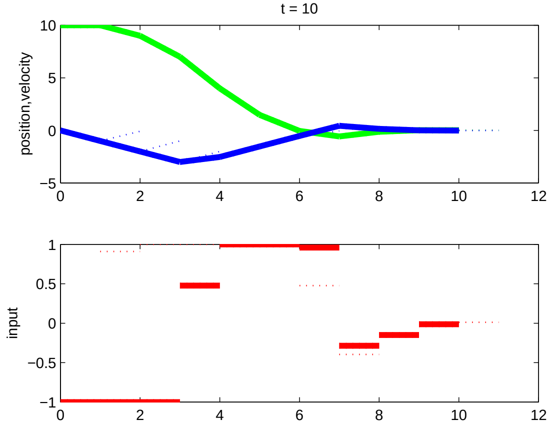

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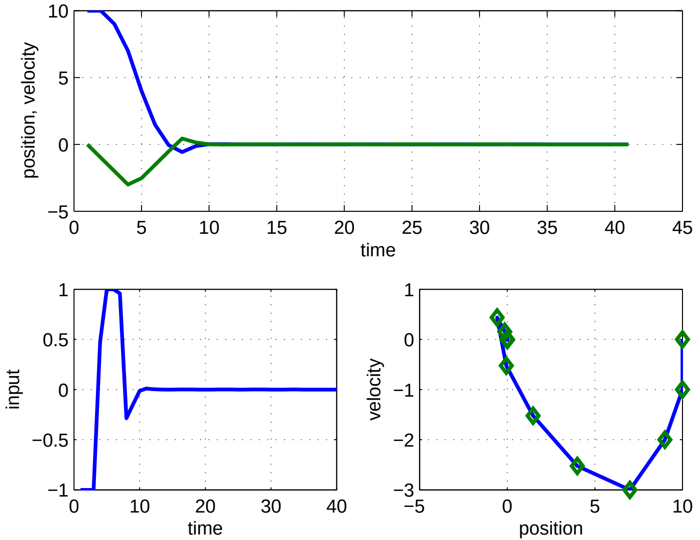
Example - Double Integrator System



Example - Double Integrator System

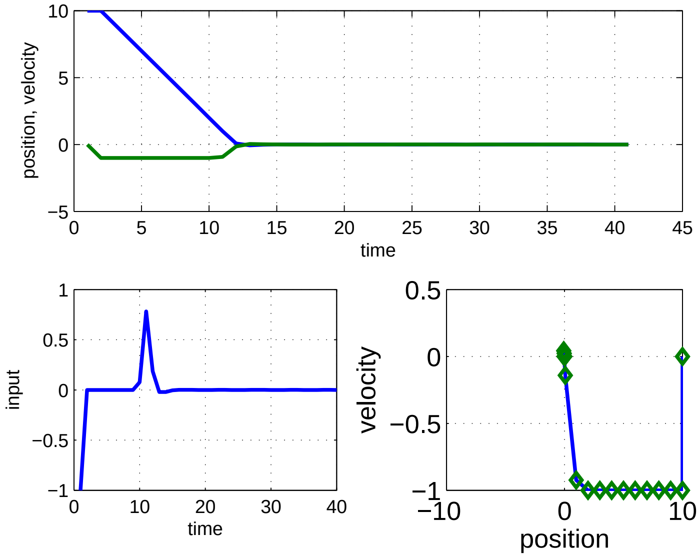
- add a state constraint $x_2(t+k) \geq -1, k = 1$
- $V(x_0) = \frac{1}{2}x_0' Y x_0 + \min\limits_{U} \frac{1}{2} U' H U + x_0' F U$, where

$$H = \begin{pmatrix} 4.2 & 2 \\ 2 & 2.2 \end{pmatrix}, F = \begin{pmatrix} 2 & 0 \\ 6 & 2 \end{pmatrix}, Y = \begin{pmatrix} 6 & 6 \\ 6 & 12 \end{pmatrix}$$

- subject to $GU \leq W + Sx_0$, where

$$G = \begin{pmatrix} -1 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}, W = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Example - Double Integrator System



MPC tracking

- Objective: make the output $y(t)$ track a reference signal $r(t)$
- Idea: parameterize the problem using input increments

$$\Delta u(t) = u(t) - u(t-1)$$

- Linear model:

$$\begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

- Prediction:

$$y(k) = CA^k x(0) + \sum_{i=0}^{k-1} CA^i Bu(k-1-i), \ k = 1, \dots, N$$
$$u(k) = u(-1) + \sum_{i=0}^{k-1} \Delta u(i), \ k = 0, \dots, N-1$$

Optimal Control Problem

$$\begin{aligned}
\min_{\Delta U} \quad & \sum_{k=0}^{N-1} ||W_y \left(y(k+1) - r(t)\right)||^2 + ||W_{\Delta u}\Delta u(k)||^2 \\
\text{s.t.} \quad & u_{min} \leq u(k) \leq u_{max}, \quad k = 0, \dots, N-1 \\
& \Delta u_{min} \leq \Delta u(k) \leq \Delta u_{max}, \quad k = 0, \dots, N-1 \\
& y_{min} \leq y(k) \leq y_{max}, \quad k = 1, \dots, N
\end{aligned}$$

where $\Delta u(k) = u(k) - u(k-1)$

Optimization Problem

$$\min_{\Delta U} \quad J(\Delta U, x(t)) = \frac{1}{2}\Delta U' H \Delta U + \begin{pmatrix} x(t)' & r(t)' & u(t-1)' \end{pmatrix} F \Delta U$$

$$\text{s.t.} \quad G\Delta U \leq W + K \begin{pmatrix} x(t) \\ r(t) \\ u(t-1) \end{pmatrix}$$

Convex Quadratic Program (QP)

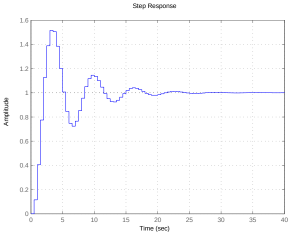
Tracking - example

- Plant:

$$G(s) = \frac{1}{s^2 + 0.4s + 1}$$

- For $T_s = 0.5sec.$

$$\begin{cases} x(t+1) & = & \begin{pmatrix} 1.597 & -0.4094 \\ 2 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0.5 \\ 0 \end{pmatrix} u(t) \\ y(t) & = & (0.2294 \;\; 0.1072)\, x(t) \end{cases}$$

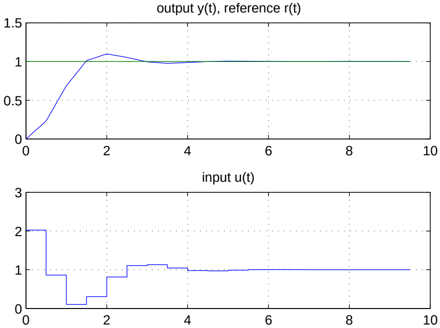


Tracking - example

- Performance index:

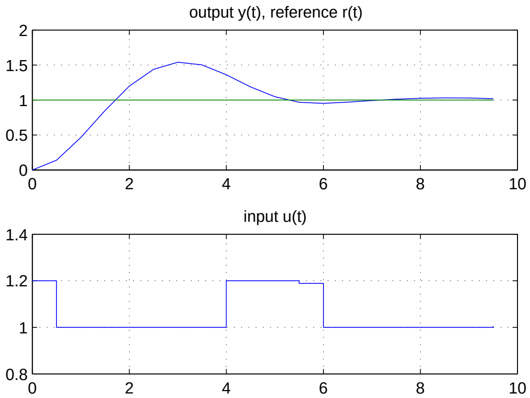
$$J(U,t) = \sum_{k=0}^{9} (y(t+k+1) - r(t))^2 + 0.04\Delta u^2(t)$$

- Closed Loop MPC:



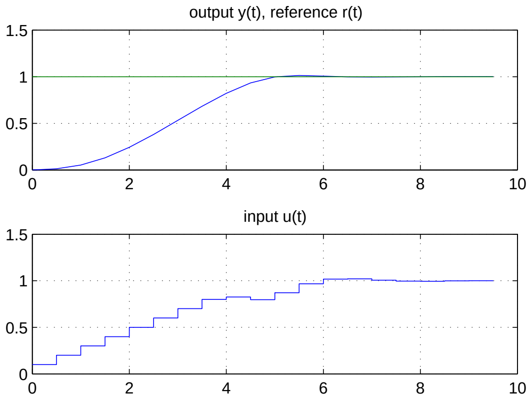
Tracking - example

- Constraint: $1 \leq u(t) \leq 1.2$



Tracking - example

- Constraint: $-0.1 \leq \Delta u(t) \leq 0.1$



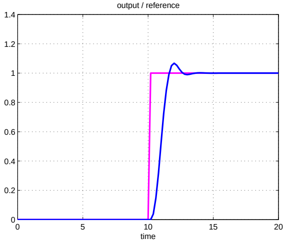
Anticipative Reference

Optimal Control Problem

$$\min_{\Delta U} \sum_{k=0}^{N-1} \left\| W_y \left(y(k+1) - r(t+k+1) \right) \right\|^2 + \left\| W_{\Delta u} \Delta u(k) \right\|^2$$

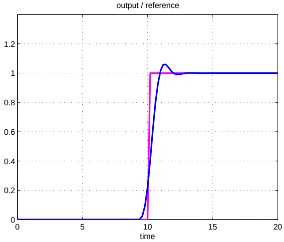
Reference not known in advance

$$r(t+k) = r(t), \forall k = 0, \ldots, N-1$$



Ref. (partially) known in advance

$$r(t+k) = \begin{cases} r(t+k) & \text{if } k = 0, \ldots, N_r \\ r(t+N_r) & \text{if } k > N_r \end{cases}$$



Soft Constrains

Optimal Control Problem

$$\min_{\Delta U} \quad \sum_{k=0}^{N-1} ||W_y \left(y(k+1) - r(t) \right)||^2 + ||W_{\Delta u} \Delta u(k)||^2 + \rho_\epsilon \epsilon^2$$

$$\text{s.t.} \quad u_{min} \leq u(k) \leq u_{max}, \quad k = 0, \dots, N-1$$

$$\Delta u_{min} \leq \Delta u(k) \leq \Delta u_{max}, \quad k = 0, \dots, N-1$$

$$y_{min} - \epsilon V_{min} \leq y(k) \leq y_{max} + \epsilon V_{max}, \quad k = 1, \dots, N$$

where

- ϵ - panic variable
- $\Delta U = (\Delta u'(0) \; \Delta u'(1) \; \dots \; \Delta u'(N-1) \; \epsilon)'$
- $\rho_\epsilon \gg W_y$ and $\rho_\epsilon \gg W_{\Delta u}$
- V_{min}, V_{max} positive vectors

Outline

1. Model Predictive Control
1.1 Linear MPC
1.2 Properties of MPC
1.3 MPC and LQR
1.4 Linear MPC based on LP
1.5 Explicit MPC

MPC feature

- multivariable constrained systems
- optimal delay compensation
- anticipating action for future reference changing
- "integral action", i.e., no offset for step-like input

Price to pay

- substantial online computation
- for simple/small systems other techniques are used, e.g. PID + anti-windup
- A possibility: explicit piecewise linear form

MPC theory

Areas

- Linear MPC: linear model
- Nonlinear MPC: nonlinear model
- Robust MPC: uncertain (linear) model
- Hybrid MPC: model integrating logic, dynamics and constrains

Issues

- Feasibility
- Stability (Convergence)
- Computations

Feasibility

- Feasibility: guarantees that the QP problem remains feasible at all sampling times t
- Input constraints only: no feasibility issues
- Hard output constraints:
  - when $N < \infty$ there is no guarantee that the QP problem will remain feasible at all future times steps t
  - $N = \infty \Longrightarrow$ infinite number of constraints
  - Maximum output admissible set theory: $N < \infty$ is enough

Convergence

MPC (*)

$$
\begin{aligned}
\min_U J(U, x(t)) \quad &= \quad \sum_{k=0}^{N-1} \left(x'(t+k)Qx(t+k) + u'(t+k)Ru(t+k)\right) \\
\text{s.t.} \quad & \quad y_{min} \leq Cx(t+k) \leq y_{max} \\
& \quad u_{min} \leq u(t+k) \leq u_{max}
\end{aligned}
$$

$Q = Q' \geq 0, R = R' > 0$

- stability is a complex function of the MPC parameters $N, Q, R, u_{min}, u_{max}, y_{min}, y_{max}$
- stability constraints and weights on terminal state can be imposed over the prediction horizon to ensure stability properties of MPC

Convergence - Example

- System:

$$\begin{aligned} x(k+1) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x(k) + \begin{pmatrix} -.1 & -.1 \\ .1 & 0 \end{pmatrix} u(k) \\ y(k) &= x(k) \end{aligned}$$

- Constraints:

$$\begin{aligned} &0 \leq u_1(k) \leq \tfrac{1}{2} x_1(x) \\ &0 \leq u_2(k) \leq 5 x_1(k) \\ &x_1(k) \geq -1 \\ &x_2(k) \geq -1 \end{aligned}$$

- Control objective:

$$\sum_{k=0}^{1} (y'(k)y(k)) + u'(t)u(t)$$

Convergence - Example

Put the problem in standard form:

- Let

$$y_3(k) = \frac{1}{2}x_1(k) - u_1(k)$$

hence

$$u_1(k) \leq \frac{1}{2}x_1(k) \Longleftrightarrow y_3(k) \geq 0$$

- Let

$$y_4(k) = 5x_1(k) - u_2(k)$$

hence

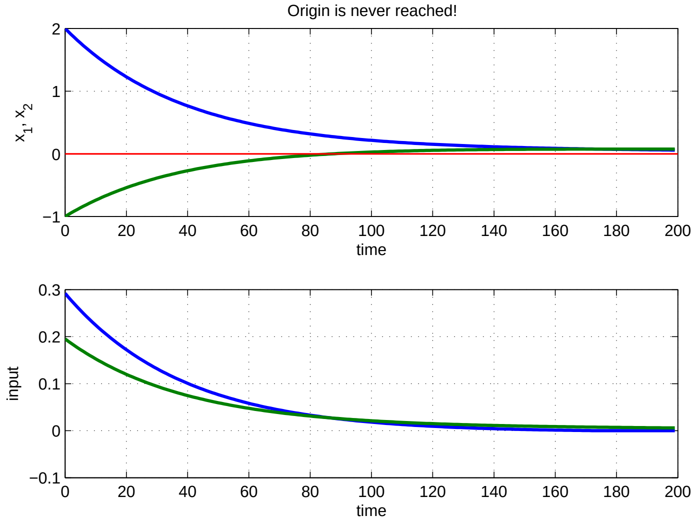
$$u_2(k) \leq 5x_2(k) \Longleftrightarrow y_4(k) \geq 0$$

Convergence - Example

Our System

$$
\begin{aligned}
x(k+1) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x(k) + \begin{pmatrix} -.1 & -.1 \\ .1 & 0 \end{pmatrix} u(k) \\
y(k) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & 0 \\ 5 & 0 \end{pmatrix} x(k) + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} u(k) \\
\begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \end{pmatrix} &\leq y(t+k) \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &\leq u(t+k)
\end{aligned}
$$

Convergence - Example



Lyapunov Functions for DT systems

- Consider a discrete time system

$$x(t+1) = f(x(t))$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$ continuous and $f(0) = 0$

Definition

A continuous function $V : S \to \mathbb{R}$ defined on a region $S \subset \mathbb{R}^n$ containing the origin in its interior is called a Lyapunov function if:

1. $V(0) = 0$
2. $V(x) > 0$ for all $x \in S$ with $x \neq 0$
3. $V(f(x)) - V(x) \leq 0$ for all $x \in S$

Lyapunov Functions for DT systems

Theorem

If there exists a Lyapunov function such that

$$V(f(x)) - V(x) < 0 \text{ for all } x \in S \text{ with } x \neq 0$$

and

$$V(x) \to \infty \text{ as } ||x|| \to \infty$$

then the origin is an asymptotically stable equilibrium point for

$$x(t+1) = f(x(t))$$

with region of attraction S.

Lyapunov Stability for Linear DT Systems

- Consider the linear DT system

$$x(t+1) = Ax(t)$$

and the candidate Lyapunov function

$$V(x) = x'Px$$

  - $V(0) = 0$
  - If $P > 0$, $V(x) > 0$ and $V(x) \to \infty$ as $||x|| \to \infty$
  - $V(Ax) - V(x) = (Ax)'P(Ax) - x'Px = x'(A'PA - P)x$

Lyapunov Stability for Linear DT Systems

Problem

To ensure asymptotic stability of the system

$$x(t+1) = Ax(t)$$

chose $P > 0$ such that $A'PA - P < 0$

Choosing P to satisfy the above is possible if, for some $Z > 0$, the discrete Lyapunov equation

$$A'PA - P = -Z$$

has a positive definite solution $P > 0$.

Conclusion

The system is stable if and only if $P > 0$ can be found for any $Z > 0$.

Application to MPC

Implication

Consider the MPC control scheme:

$$\min_U \sum_{k=0}^{N-1} (x'(t+k)Qx(t+k) + u'(t+k)Ru(t+k)) + \\ +x'(t+N)Px(t+N)$$

$$\text{s.t.} \quad \begin{cases} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \\ y_{min} &\leq y(t+k) \leq y_{max}, k = 0, \dots, N \\ u_{min} &\leq u(t+k) \leq u_{max}, k = 1, \dots, N-1 \\ & u(t+N) = 0 \end{cases}$$

where P is the solution of the Lyapunov equation $A'PA - P = -Q$. The resulted MPC control scheme is asymptotically stable.

Control Lyapunov Functions

- Consider a discrete time system

$$x(t+1) = f(x(t), u(t))$$

with $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ continuous and $f(0,0) = 0$

Definition

A continuous function $V : S \to \mathbb{R}$ defined on a region $S \subset \mathbb{R}^n$ containing the origin in its interior is called a Control Lyapunov function if:

1. $V(0) = 0$
2. $V(x) > 0$ for all $x \in S$ with $x \neq 0$
3. There exists a continuous control law $u = k(k)$ such that

$$V(f(x, k(x))) - V(x) \leq 0 \text{ for all } x \in S$$

Control Lyapunov Functions

Theorem

If there exists a control Lyapunov function and a continuous control law $u = k(x)$ such that

$$V(f(x, k(x))) - V(x) < 0 \text{ for all } x \in S \text{ with } x \neq 0$$

and

$$V(x) \to \infty \text{ as } ||x|| \to \infty$$

then the origin is an asymptotically stable equilibrium point for

$$x(t+1) = f(x(t), k(x))$$

with region of attraction S.

Lyapunov Stability for Linear DT Systems

- Consider the linear DT system

$$x(t+1) = Ax(t) + Bu(t)$$

and the candidate control Lyapunov function

$$V(x) = x'Px$$

and control law $u = Kx$

  - $V(0) = 0$
  - If $P > 0$, $V(x) > 0$ and $V(x) \to \infty$ as $||x|| \to \infty$
  - $$\begin{aligned} V((A+BK)x) - V(x) &= ((A+BK)x)'P((A+BK)x) - x'Px \\ &= x'((A+BK)'P(A+BK) - P)x \end{aligned}$$

Lyapunov Stability for Linear DT Systems

Problem

To ensure asymptotic stability of the system

$$x(t+1) = (A+BK)x(t)$$

chose $P > 0$ such that $(A+BK)'P(A+BK) - P < 0$

Choosing P to satisfy the above is possible if, for some $Z > 0$, the discrete Lyapunov equation

$$(A+BK)'P(A+BK) - P = -Z$$

has a positive definite solution $P > 0$.

Conclusion

$(A+BK)$ is stable if and only if $P > 0$ can be found for any $Z > 0$.

Application to MPC

Implication

Consider the MPC control scheme:

$$\min_{U} \sum_{k=0}^{N-1} \left(x'(t+k)Qx(t+k) + u'(t+k)Ru(t+k)\right) + \\ +x'(t+N)Px(t+N)$$

$$\text{s.t.} \quad \begin{cases} x(t+1) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) \\ y_{min} & \leq & y(t+k) \leq y_{max}, k = 0, \ldots, N \\ u_{min} & \leq & u(t+k) \leq u_{max}, k = 1, \ldots, N-1 \\ & & u(t+N) = Kx(N) \end{cases}$$

where K and P are, the solutions of the unconstrained infinite horizon LQR problem with weights Q, R

$$K = -(R + B'PB)^{-1}B'PA$$
$$P = (A + BK)'P(A + BK) + K'RK + Q$$

The resulted MPC control scheme is asymptotically stable.

This choice of K implies that, after N time steps, the control is switched to the unconstrained LQR.

Convergence

Theorem

Consider the linear system:

$$\begin{cases} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{cases}$$

and the MPC control law $(*)$.

Assume that the optimization problem is feasible at $t = 0$. Then, for either $N \to \infty$ or with an extra constraint $x(t+N) = 0$, for all $R > 0, Q \geq 0$:

$$\lim_{t\to\infty} u(t) = 0$$

while fulfilling the constraints. Moreover, if $(Q^{1/2}C, A)$ is detectable, then

$$\lim_{t\to\infty} x(t) = 0$$

Convergence Proof

- Assume we set terminal constraint:

$$x(t+N) = 0$$

- Let U_t^* denote the optimal control sequence at time t:

$$U_t^* = \{u_t^*(0), \dots, u_t^*(N-1)\}$$

- Let

$$V(t) = J(U_t^*, x(t))$$

we show that it is a Lyapunov function

Convergence Proof

- By construction $U_1 = \{u_t^*(1), \ldots, u_t^*(N-1), 0\}$ is feasible at $t+1$.
- Therefore,

$$\begin{aligned} V(t+1) = J(U_{t+1}^*, \quad & x(t+1)) \leq J(U_1, x(t+1)) = \\ & = V(t) - x'(t)Qx(t) - u'(t)Ru(t) \end{aligned}$$

- $V(t)$ is decreasing and lower-bounded by 0, therefore:

$$\lim_{t\to\infty} V(t) = 0 \Longrightarrow V(t+1) - V(t) \to 0,$$

which implies

$$\begin{aligned} x'(t)Qx(t) &\to 0 \\ u'(t)Ru(t) &\to 0 \end{aligned}$$

- Since $R > 0$, $\lim_{t\to\infty} u(t) = 0$

Convergence Proof

- If $Q > 0$, then also $\lim_{t\to\infty} x(t) = 0$
- If $Q \geq 0$,
- For all $k = 0, \ldots, N-1$, we have

$$\begin{aligned} \lim_{t\to\infty} \quad & x'(t+k)Qx(t+k) = \\ & = \lim_{t\to\infty} ||Q^{1/2}A^k x(t) + \sum_{j=0}^{k-1} A^j Bu(t+k-1-j)||^2 = 0 \end{aligned}$$

- As $u(t) \to 0$, also $Q^{1/2}A^k x(t) \to 0$
- If $(Q^{1/2}C, A)$ is detectable, through a canonical decomposition can be shown that, as $u(t) \to 0$, the modes go to zero spontaneously (*Bemporad et. al., 1994*)

Convergence Proof

- Similar argument for $N = \infty$
  - Let U_t^* denote the optimal control sequence at time t:

$$U_t^* = \{u_t^*(0), \ldots, u_t^*(N-1)\}$$

  - Let

$$V(t) = J(U_t^*, x(t))$$

    we shaw that it is a Lyapunov function
  - Because constraints were checked up to $t + k = \infty$, $U_1^* = \{u_t^*(1), u_t^*(2), \ldots\}$ is feasible at $t+1$ by construction
  - Hence,

$$\begin{aligned} V(t+1) = J(U_{t+1}^*, \quad & x(t+1)) \leq J(U_1, x(t+1)) = \\ & = V(t) - y'(t)Qy(t) - u'(t)Ru(t) \end{aligned}$$

  - Repeat same arguments as before

Stability Constraints

Ensuring convergence

1. No constraints, infinite horizon

$$N = \infty$$

2. End-point constraints

$$x(t+N) = 0$$

3. Relaxed terminal constraints

$$x(t+N) \in \Omega$$

4. Contraction constraints

$$||x(t+1)|| \leq \alpha ||x(t)||, \alpha < 1$$

All proofs use the value function $V(t) = \min_U J(U,t)$ as a Lyapunov function

Outline

1. Model Predictive Control
1.1 Linear MPC
1.2 Properties of MPC
1.3 MPC and LQR
1.4 Linear MPC based on LP
1.5 Explicit MPC

MPC control law

$$\begin{aligned}\min_U J(U,t) &= x'(t+N)Px(t+N)+ \\ &+ \sum_{k=0}^{N-1} \left(x'(t+k)Qx(t+k) + u'(t+k)Ru(t+k)\right)\end{aligned}$$

$R = R' > 0$, $Q = Q' \geq 0$ and P satisfies the Riccati equation

$$P = A'PA - A'PB(B'PB+R)^{-1}B'PA + Q$$

Conclusion

(Unconstrained) MPC $\equiv$ LQR

MPC control law

$$
\begin{aligned}
\min_U J(U,t) \quad &= \quad x'(t+N)Px(t+N)+ \\
&+ \quad \sum_{k=0}^{N-1} (x'(t+k)Qx(t+k) + u'(t+k)Ru(t+k)) \\
\text{s.t.} \quad & \quad y_{min} \leq y(t+k) \leq y_{max}, k = 1, \ldots N \\
& \quad u_{min} \leq u(t+k) \leq u_{max}, k = 0, \ldots, N-1 \\
& \quad u(t+N) = Kx(t+N)
\end{aligned}
$$

$R = R' > 0, Q = Q' \geq 0$ and P, K satisfy the Riccati equation

$$
\begin{aligned}
K &= -(R + B'PB)^{-1}B'PA \\
P &= (A+BK)'P(A+BK) + K'RK + Q
\end{aligned}
$$

- In a polyhedral region around the origin the MPC control law is equivalent to the LQR controller with weights Q and R.

Conclusion

$$\text{MPC} \equiv \text{constrained LQR}$$

Example - Double Integrator System

- System:

$$
\begin{aligned}
x(t+1) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) \\
y(t) &= (1\ 0)x(t)
\end{aligned}
$$

- Constraints:

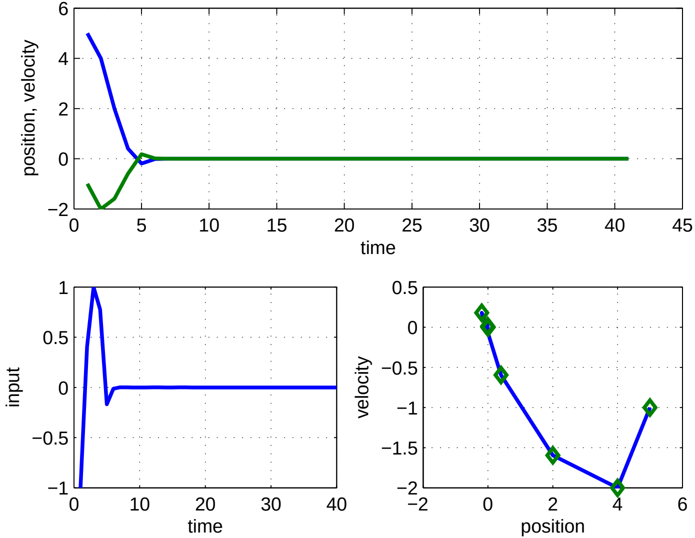
$$|u(t)| \leq 1$$

- Control objective:

$$\sum_{k=0}^{\infty} y^2(k) + \frac{1}{100}u^2(k), u(k) = K_{LQR}x(k), \forall k \geq 2$$

$$\Longrightarrow \left(\sum_{k=0}^{1} y^2(k) + \frac{1}{100}u^2(k) \right) + x'(2) \begin{pmatrix} 2.1429 & 1.2246 \\ 1.2246 & 1.3996 \end{pmatrix} x(2)$$

Example - Double Integrator System



Outline

1. Model Predictive Control
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Problem Definition

Discrete-time system

Linear Model: $\begin{cases} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{cases}$

Constraints: $\begin{cases} u_{min} \leq u(t) \leq u_{max} \\ y_{min} \leq Cx(t) \leq y_{max} \end{cases}$

Constraint Optimal Control:

$$\begin{aligned} \min_{u_0,\ldots,u_{N-1}} \quad & \left(\sum_{k=0}^{N-1} (||Qx(k)||_\infty + ||Ru(k)||_\infty) + ||Px(N)||_\infty \right) \\ \text{s.t.} \quad & x(k+1) = Ax(k) + Bu(k), k = 1, \ldots, N-1 \\ & u_{min} \leq u(k) \leq u_{max}, k = 0, \ldots, N-1 \\ & y_{min} \leq Cx(k) \leq y_{max}, k = 1, \ldots N \end{aligned}$$

Problem Definition

$$\begin{array}{l} \min |x| \\ x \in \mathbb{R} \end{array} \iff \begin{array}{ll} \min & \epsilon \\ \text{s.t.} & \epsilon \geq x \\ & \epsilon \geq -x \end{array}$$

- Introduce slack variables:

$$\begin{array}{l} \epsilon_k^x \geq ||Qx(k)||_\infty \\ \epsilon_k^u \geq ||Ru(k)||_\infty \\ \epsilon_N^x \geq ||Px(N)||_\infty \end{array} \Longrightarrow \begin{array}{l} \epsilon_k^x \geq Q(i,:)x(k), \forall i \\ \epsilon_k^x \geq -Q(i,:)x(k), \forall i \\ \epsilon_k^u \geq R(i,:)u(k), \forall i \\ \epsilon_k^u \geq -R(i,:)u(k), \forall i \\ \epsilon_N^x \geq P(i,:)x(N), \forall i \\ \epsilon_N^x \geq -P(i,:)x(N), \forall i \end{array}$$

Linear Program

- Substituting $x(t+k) = A^k x(t) + \sum_{j=0}^{k-1} A^j B u(t+k-1-j)$

Optimization Problem

$$\begin{aligned} V(x(t)) &= \min_z (1 \; \dots \; 1 \; 0 \; \dots \; 0) z \\ &\quad \text{s.t. } \; Gz \leq W + Sx(t) \end{aligned}$$

where

- $z = (\epsilon_0^u \; \dots \; \epsilon_{N-1}^u \; \epsilon_1^x \; \dots \; \epsilon_N^x \; u'(t) \; \dots \; u'(t+N-1))$
- G, W, S are obtained from weights matrices Q, R, P and model matrices A, B, C

Outline

1. Model Predictive Control

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1.2 Properties of MPC

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Basic Idea

Optimization Problem

$$\begin{array}{ll} \min\limits_{U} & \frac{1}{2}U'HU + x(t)'F'U + \frac{1}{2}x'(t)Yx(t) \\ \text{s.t.} & GU \leq W + Sx(t) \end{array}$$

- Online optimization: given $x(t)$ solve the problem at each step t (the control law $u = u(x)$ is implicitly defined by the QP solver)
  $\Longrightarrow$ Quadratic Program (QP)

- Offline optimization: solve the QP for all $x(t)$ to find the control law $u = u(x)$ explicitly
  $\Longrightarrow$ multi-parametric Quadratic Program (mp-QP)

Multiparametric Quadratic Programming

Optimization Problem

$$\begin{array}{ll} \min\limits_{U} & \frac{1}{2}U'HU + x(t)'F'U + \cancel{\frac{1}{2}x'(t)Yx(t)} \\ \text{s.t.} & GU \leq W + Sx(t) \end{array}$$

- Objective: solve the QP for all x
- Assumption: $\begin{cases} \begin{pmatrix} H & F \\ F' & Y \end{pmatrix} \geq 0 \\ H \geq 0 \end{cases}$

KKT conditions for optimality

$$\begin{aligned} HU + F'x(t) + G'\lambda &= 0 \\ \lambda'\left(GU - W - Sx(t)\right) &= 0 \\ \lambda &\geq 0 \\ GU - W - Sx(t) &\leq 0 \end{aligned}$$

Solution

- take a point $x_0 \in X$
- solve QP to find $U^*(x_0), \lambda^*(x_0)$
- identify the active constraints at $U^*(x_0)$
- form the matrices $\bar{G}, \bar{W}, \bar{S}$ by collecting the active constraints: $\bar{G}U^*(x_0) - \bar{W} - \bar{S}x_0$
- Write the KKT conditions for optimality:

$$
\begin{array}{llll}
(1) & HU + Fx + G'\lambda = 0 & (2) & \bar{G}U - \bar{W} - \bar{S}x = 0 \\
(3) & \lambda'(GU - W - Sx) = 0 & (4) & \hat{G}U \leq \hat{W} + \hat{S}x \\
(5) & \bar{\lambda} \geq 0, \hat{\lambda} = 0, \lambda = (\bar{\lambda}\ \hat{\lambda}) & &
\end{array}
$$

Solution

- Write the KKT conditions for optimality:

$$
\begin{array}{llll}
(1) & HU + Fx + G'\lambda = 0 & (2) & \bar{G}U - \bar{W} - \bar{S}x = 0 \\
(3) & \lambda'(GU - W - Sx) = 0 & (4) & \hat{G}U \leq \hat{W} + \hat{S}x \\
(5) & \bar{\lambda} \geq 0, \hat{\lambda} = 0, \lambda = (\bar{\lambda}\ \hat{\lambda}) & &
\end{array}
$$

- From (1): $U = -H^{-1}(Fx + \bar{G}'\bar{\lambda})$
- From (1) + (2):
  $\bar{\lambda}(x) = -(\bar{G}H^{-1}\bar{G}')^{-1}(\bar{W} + (\bar{S} + \bar{G}H^{-1}F)x)$
  $U(x) = H^{-1}\left(\bar{G}'(\bar{G}H^{-1}\bar{G}')^{-1}(\bar{W} + (\bar{S} + \bar{G}H^{-1}F)x) - Fx\right)$

What we obtained?

In some neighborhood of x_0, λ and U are explicit affine functions of x!!!

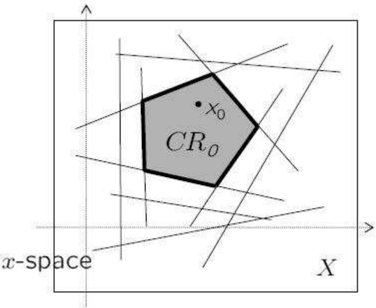
Determining a Critical Region

- Imposing primal and dual feasibility: $\begin{cases} \hat{G}U(x) \leq \hat{W} + \hat{S}x \\ \bar{\lambda} \geq 0 \end{cases}$

  $\Longrightarrow$ linear inequalities in x
- Remove redundant constraints (this requires solving LP's)

$\Longrightarrow$ critical region CR_0

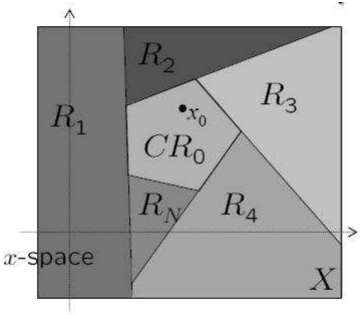
$CR_0 = \{x \in X | \mathcal{A}x \leq \mathcal{B}\}$



What is CR_0?

CR_0 is the set of all and only parameters x for which $\bar{G}, \bar{W}, \bar{S}$ is the optimal combination of active constraints at the optimizer.

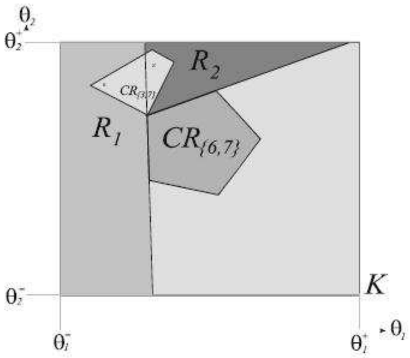
Multiparametric QP



$$
\begin{aligned}
CR_0 &= \{x \in X | \mathcal{A}x \leq \mathcal{B}\} \\
R_i &= \{x \in X | \quad \mathcal{A}^i x > \mathcal{B}^i, \\
&\qquad\qquad\quad \mathcal{A}^j x \leq \mathcal{B}^j, \forall j \neq i\}
\end{aligned}
$$

CR_0 is characterizing a set of active constraints, R_i is not

Multiparametric QP



1. Use the above splitting only as a search procedure, don't split the *CR*
2. Remove duplicates of *CR* already found

mp-QP properties

Theorem

The linear MPC controller is a continuous piecewise affine function of the state.

$$u(x) = \begin{cases} F_1 x + g_1 & \text{if} \quad H_1 x \leq K_1 \\ \vdots & \qquad \vdots \\ F_M x + g_M & \text{if} \quad H_M x \leq K_M \end{cases}$$

eMPC - Double Integrator System

- System:

$$x(t+1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$
$$y(t) = (1\ 0)x(t)$$

- Constraints:

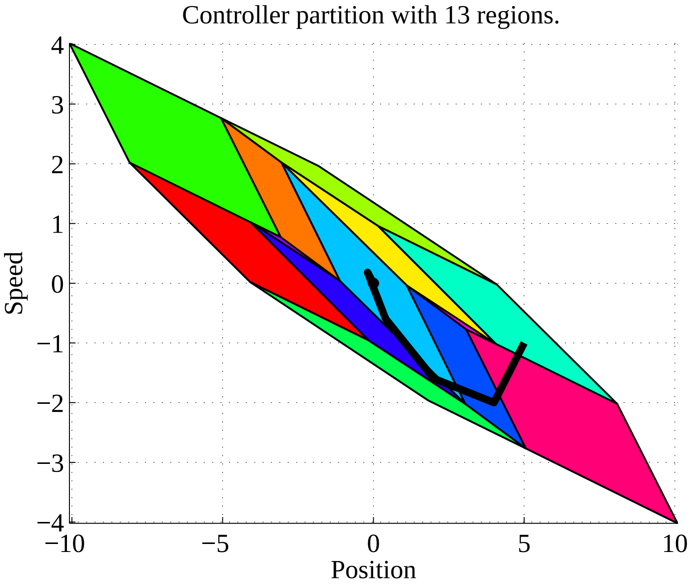
$$|u(t)| \leq 1$$

- Control objective:

$$\sum_{k=0}^{\infty} y^2(k) + \frac{1}{100}u^2(k), u(k) = K_{LQR}x(k), \forall k \geq 2$$

$$\Longrightarrow \left(\sum_{k=0}^{1} y^2(k) + \frac{1}{100}u^2(k)\right) + x'(2)\begin{pmatrix} 2.1429 & 1.2246 \\ 1.2246 & 1.3996 \end{pmatrix} x(2)$$

eMPC - Double Integrator System



eMPC based on LP

Optimization Problem

$$\begin{aligned} V(x(t)) &= \min_z (1 \ \ldots \ 1 \ 0 \ \ldots \ 0)z \\ &\text{s.t. } \ Gz \leq W + Sx(t) \end{aligned}$$

$$z = (\epsilon_0^u \ \ldots \ \epsilon_{N-1}^u \ \epsilon_1^x \ \ldots \ \epsilon_N^x \ u'(t) \ \ldots \ u'(t+N-1))$$

- Online optimization: given $x(t)$ solve the problem at each step t (the control law $u = u(x)$ is implicitly defined by the LP solver)
  $\Longrightarrow$ Linear Program (LP)
- Offline optimization: solve the QP for all $x(t)$ to find the control law $u = u(x)$ explicitly
  $\Longrightarrow$ multi-parametric Linear Program (mp-LP)

eMPC based on LP

Primal Problem

$$\begin{aligned} \min_{z} \quad & f'z \\ \text{s.t.} \quad & Gz \leq W + Sx \end{aligned}$$

Dual Problem

$$\begin{aligned} \max_{\lambda} \quad & (W + Sx)'\lambda \\ \text{s.t.} \quad & G'\lambda = f \\ & \lambda \leq 0 \end{aligned}$$

Optimality conditions:

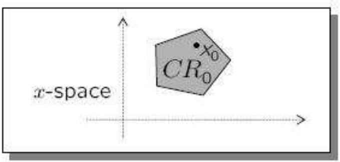
- Primal feasibility: $Gz \leq W + Sx$
- Dual feasibility: $G'\lambda = f, \lambda \leq 0$
- Complementary slackness: $\lambda_j(G_j z - W_j - S_j x) = 0, \forall j$

For a given parameter x_0:

- solve LP to find z_0^*, λ_0^* (suppose no degeneracy)
- identify active constraints
- form submatrices $\bar{G}, \bar{W}, \bar{S}$ of active constraints

Multiparametric LP

- Primal feasibility condition: $\bar{G}z \leq \bar{W} + \bar{S}x$
  - $z^*(x) = (\bar{G}^{-1}\bar{S})x + (\bar{G}^{-1}\bar{W})$ (optimizer)
  - $\hat{G}(\bar{G}^{-1}\bar{S})x + \hat{G}(\bar{G}^{-1}\bar{W}) \leq \hat{W} + \hat{S}x$ (critical region)



- Dual feasibility condition: $\bar{\lambda}^*(x) = (\bar{G}')^{-1}f, \hat{\lambda}(x) = 0$
- Primal cost = Dual cost:

$$V^*(x) = f'z^*(x) = f'\bar{G}^{-1}(\bar{W} + \bar{S}x)$$

eMPC - Double Integrator System

- System:

$$\begin{aligned} x(t+1) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) \\ y(t) &= (1\ 0)\ x(t) \end{aligned}$$

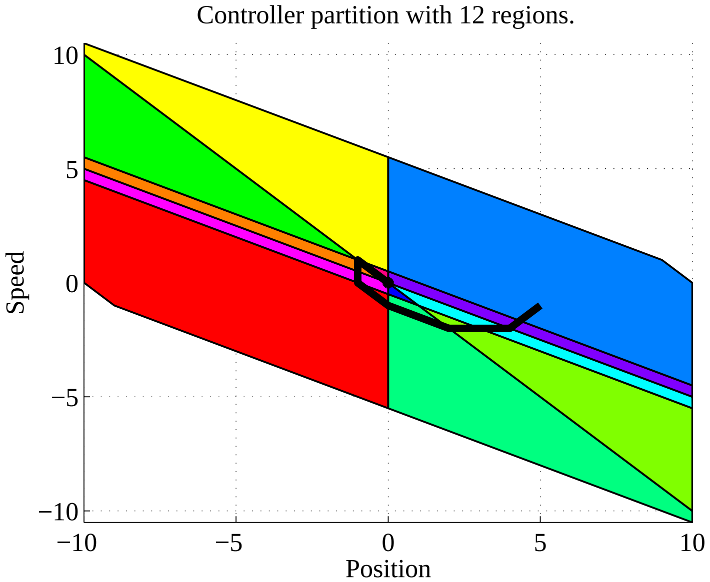
- Constraints:

$$|u(t)| \leq 1$$

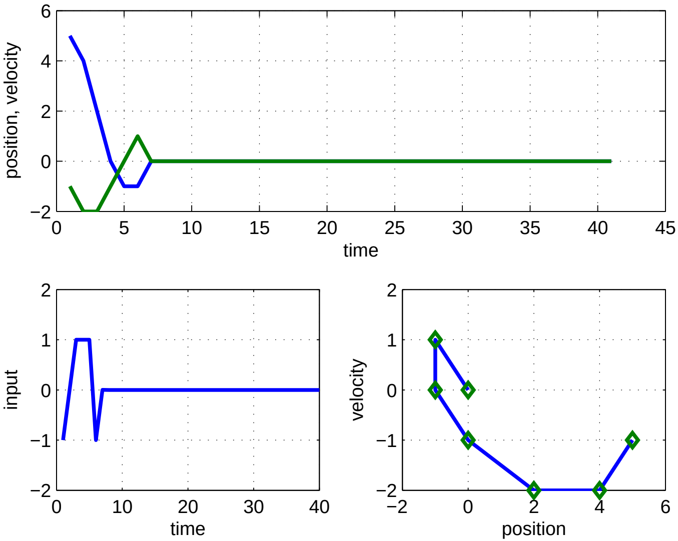
- Control objective:

$$\sum_{k=0}^{1} \left(||y(t+k)||_\infty + \frac{1}{100} ||u(t+k)||_\infty \right) + ||y(t+2)||_\infty$$

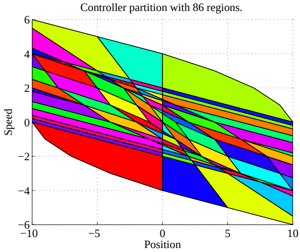
eMPC - Double Integrator System



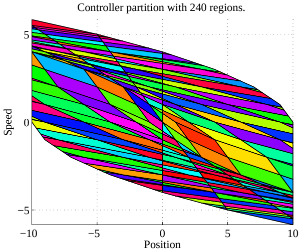
eMPC - Double Integrator System



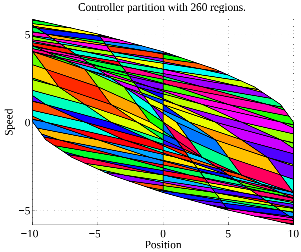
eMPC - Complexity (mp-LP)



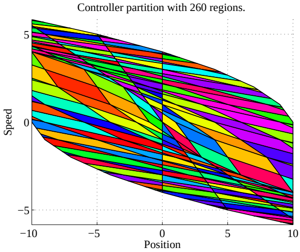
$N = 5$



$N = 10$



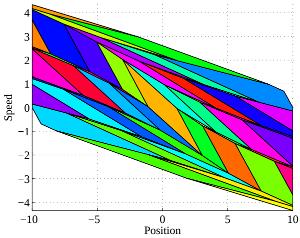
$N = 20$



$N = 50$

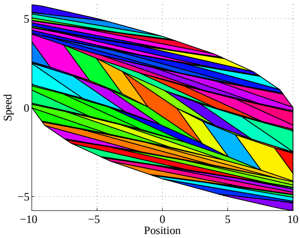
eMPC - Complexity (mp-QP)

Controller partition with 59 regions.



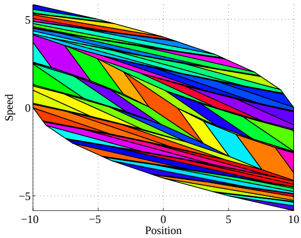
$N = 5$

Controller partition with 115 regions.



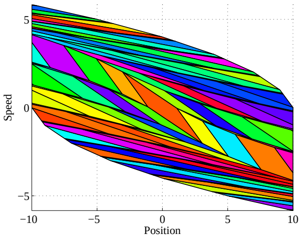
$N = 10$

Controller partition with 123 regions.



$N = 20$

Controller partition with 123 regions.



$N = 50$

Optimization and Optimal Control

Lecture 5: Optimal Control: Linear MPC

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