CHAPTER 14

Wiener Processes and Itô's Lemma

Any variable whose value changes over time in an uncertain way is said to follow a *stochastic process*. Stochastic processes can be classified as *discrete time* or *continuous time*. A discrete-time stochastic process is one where the value of the variable can change only at certain fixed points in time, whereas a continuous-time stochastic process is one where changes can take place at any time. Stochastic processes can also be classified as *continuous variable* or *discrete variable*. In a continuous-variable process, the underlying variable can take any value within a certain range, whereas in a discrete-variable process, only certain discrete values are possible.

This chapter develops a continuous-variable, continuous-time stochastic process for stock prices. A similar process is often assumed for the prices of other assets. Learning about this process is the first step to understanding the pricing of options and other more complicated derivatives. It should be noted that, in practice, we do not observe stock prices following continuous-variable, continuous-time processes. Stock prices are restricted to discrete values (e.g., multiples of a cent) and changes can be observed only when the exchange is open for trading. Nevertheless, the continuous-variable, continuous-time process proves to be a useful model for many purposes.

Many people feel that continuous-time stochastic processes are so complicated that they should be left entirely to "rocket scientists." This is not so. The biggest hurdle to understanding these processes is the notation. Here we present a step-by-step approach aimed at getting the reader over this hurdle. We also explain an important result known as *Itô's lemma* that is central to the pricing of derivatives.

14.1 THE MARKOV PROPERTY

A *Markov process* is a particular type of stochastic process where only the current value of a variable is relevant for predicting the future. The past history of the variable and the way that the present has emerged from the past are irrelevant.

Stock prices are usually assumed to follow a Markov process. Suppose that the price of a stock is \$100 now. If the stock price follows a Markov process, our predictions for the future should be unaffected by the price one week ago, one month ago, or one year ago. The only relevant piece of information is that the price is now \$100.[1] Predictions for the future are uncertain and must be expressed in terms of probability distributions. The Markov property implies that the probability distribution of the price at any particular future time is not dependent on the particular path followed by the price in the past.

The Markov property of stock prices is consistent with the weak form of market efficiency. This states that the present price of a stock impounds all the information contained in a record of past prices. If the weak form of market efficiency were not true, technical analysts could make above-average returns by interpreting charts of the past history of stock prices. There is very little evidence that they are in fact able to do this.

It is competition in the marketplace that tends to ensure that weak-form market efficiency and the Markov property hold. There are many investors watching the stock market closely. This leads to a situation where a stock price, at any given time, reflects the information in past prices. Suppose that it was discovered that a particular pattern in a stock price always gave a 65% chance of subsequent steep price rises. Investors would attempt to buy a stock as soon as the pattern was observed, and demand for the stock would immediately rise. This would lead to an immediate rise in its price and the observed effect would be eliminated, as would any profitable trading opportunities.

14.2 CONTINUOUS-TIME STOCHASTIC PROCESSES

Consider a variable that follows a Markov stochastic process. Suppose that its current value is 10 and that the change in its value during a year is $\phi(0, 1)$, where $\phi(m, v)$ denotes a probability distribution that is normally distributed with mean m and variance v.[2] What is the probability distribution of the change in the value of the variable during 2 years?

The change in 2 years is the sum of two normal distributions, each of which has a mean of zero and variance of 1.0. Because the variable is Markov, the two probability distributions are independent. When we add two independent normal distributions, the result is a normal distribution where the mean is the sum of the means and the variance is the sum of the variances. The mean of the change during 2 years in the variable we are considering is, therefore, zero and the variance of this change is 2.0. Hence, the change in the variable over 2 years has the distribution $\phi(0, 2)$. The standard deviation of the change is $\sqrt{2}$.

Consider next the change in the variable during 6 months. The variance of the change in the value of the variable during 1 year equals the variance of the change during the first 6 months plus the variance of the change during the second 6 months. We assume these are the same. It follows that the variance of the change during a 6-month period must be 0.5. Equivalently, the standard deviation of the change is $\sqrt{0.5}$. The probability distribution for the change in the value of the variable during 6 months is $\phi(0, 0.5)$.

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[1] Statistical properties of the stock price history may be useful in determining the characteristics of the stochastic process followed by the stock price (e.g., its volatility). The point being made here is that the particular path followed by the stock in the past is irrelevant.

[2] Variance is the square of standard deviation. The standard deviation of a 1-year change in the value of the variable we are considering is therefore 1.0.

A similar argument shows that the probability distribution for the change in the value of the variable during 3 months is $\phi(0, 0.25)$. More generally, the change during any time period of length T is $\phi(0, T)$. In particular, the change during a very short time period of length Δt is $\phi(0, \Delta t)$.

Note that, when Markov processes are considered, the variances of the changes in successive time periods are additive. The standard deviations of the changes in successive time periods are not additive. The variance of the change in the variable in our example is 1.0 per year, so that the variance of the change in 2 years is 2.0 and the variance of the change in 3 years is 3.0. The standard deviations of the changes in 2 and 3 years are $\sqrt{2}$ and $\sqrt{3}$, respectively. Uncertainty is often measured by standard deviation. These results therefore explain why uncertainty is sometimes referred to as being proportional to the square root of time.

Wiener Process

The process followed by the variable we have been considering is known as a *Wiener process*. It is a particular type of Markov stochastic process with a mean change of zero and a variance rate of 1.0 per year. It has been used in physics to describe the motion of a particle that is subject to a large number of small molecular shocks and is sometimes referred to as *Brownian motion*.

Expressed formally, a variable z follows a Wiener process if it has the following two properties:

Property 1. *The change Δz during a small period of time Δt is*

$$\Delta z = \epsilon \sqrt{\Delta t} \tag{14.1}$$

where ϵ has a standard normal distribution $\phi(0, 1)$.

Property 2. *The values of Δz for any two different short intervals of time, Δt, are independent.*

It follows from the first property that Δz itself has a normal distribution with

$$\text{mean of } \Delta z = 0$$
$$\text{standard deviation of } \Delta z = \sqrt{\Delta t}$$
$$\text{variance of } \Delta z = \Delta t$$

The second property implies that z follows a Markov process.

Consider the change in the value of z during a relatively long period of time, T. This can be denoted by $z(T) - z(0)$. It can be regarded as the sum of the changes in z in N small time intervals of length Δt, where

$$N = \frac{T}{\Delta t}$$

Thus,

$$z(T) - z(0) = \sum_{i=1}^{N} \epsilon_i \sqrt{\Delta t} \tag{14.2}$$

where the ϵ_i $(i = 1, 2, \dots, N)$ are distributed $\phi(0, 1)$. We know from the second property of Wiener processes that the ϵ_i are independent of each other. It follows from equation (14.2) that $z(T) - z(0)$ is normally distributed, with

$$\begin{aligned}\text{mean of } [z(T) - z(0)] &= 0\\ \text{variance of}[z(T) - z(0)] &= N\,\Delta t = T\\ \text{standard deviation of } [z(T) - z(0)] &= \sqrt{T}\end{aligned}$$

This is consistent with the discussion earlier in this section.

Example 14.1

Suppose that the value, z, of a variable that follows a Wiener process is initially 25 and that time is measured in years. At the end of 1 year, the value of the variable is normally distributed with a mean of 25 and a standard deviation of 1.0. At the end of 5 years, it is normally distributed with a mean of 25 and a standard deviation of $\sqrt{5}$, or 2.236. Our uncertainty about the value of the variable at a certain time in the future, as measured by its standard deviation, increases as the square root of how far we are looking ahead.

In ordinary calculus, it is usual to proceed from small changes to the limit as the small changes become closer to zero. Thus, $dx = a\,dt$ is the notation used to indicate that $\Delta x = a\,\Delta t$ in the limit as $\Delta t \to 0$. We use similar notational conventions in stochastic calculus. So, when we refer to dz as a Wiener process, we mean that it has the properties for Δz given above in the limit as $\Delta t \to 0$.

Figure 14.1 illustrates what happens to the path followed by z as the limit $\Delta t \to 0$ is approached. Note that the path is quite "jagged." This is because the standard deviation of the movement in z in time Δt equals $\sqrt{\Delta t}$ and, when Δt is small, $\sqrt{\Delta t}$ is much bigger than Δt. Two intriguing properties of Wiener processes, related to this $\sqrt{\Delta t}$ property, are as follows:

1. The expected length of the path followed by z in any time interval is infinite.
2. The expected number of times z equals any particular value in any time interval is infinite.[3]

Generalized Wiener Process

The mean change per unit time for a stochastic process is known as the *drift rate* and the variance per unit time is known as the *variance rate*. The basic Wiener process, dz, that has been developed so far has a drift rate of zero and a variance rate of 1.0. The drift rate of zero means that the expected value of z at any future time is equal to its current value. The variance rate of 1.0 means that the variance of the change in z in a time interval of length T equals T. A *generalized Wiener process* for a variable x can be defined in terms of dz as

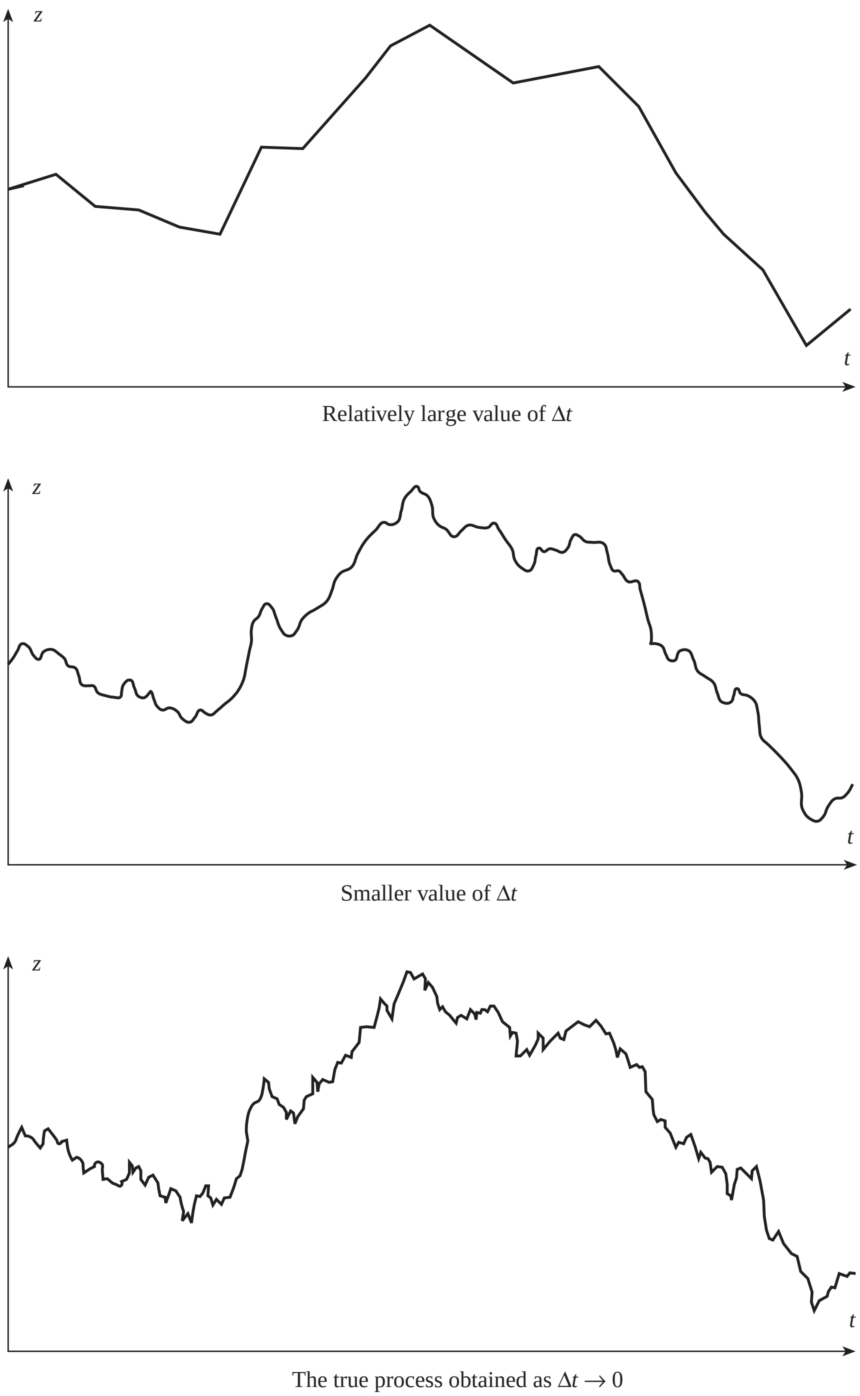
$$dx = a\,dt + b\,dz \tag{14.3}$$

where a and b are constants.

To understand equation (14.3), it is useful to consider the two components on the right-hand side separately. The $a\,dt$ term implies that x has an expected drift rate of a per unit of time. Without the $b\,dz$ term, the equation is $dx = a\,dt$, which implies that

[3] This is because z has some nonzero probability of equaling any value v in the time interval. If it equals v in time t, the expected number of times it equals v in the immediate vicinity of t is infinite.

Figure 14.1 How a Wiener process is obtained when $\Delta t \to 0$ in equation (14.1).



$dx/dt = a$. Integrating with respect to time, we get

$$x = x_0 + at$$

where x_0 is the value of x at time 0. In a period of time of length T, the variable x increases by an amount aT. The $b\,dz$ term on the right-hand side of equation (14.3) can be regarded as adding noise or variability to the path followed by x. The amount of this noise or variability is b times a Wiener process. A Wiener process has a variance rate per unit time of 1.0. It follows that b times a Wiener process has a variance rate per unit time of b^2. In a small time interval Δt, the change Δx in the value of x is given by equations (14.1) and (14.3) as

$$\Delta x = a\,\Delta t + b\epsilon\sqrt{\Delta t}$$

where, as before, ϵ has a standard normal distribution $\phi(0, 1)$. Thus Δx has a normal distribution with

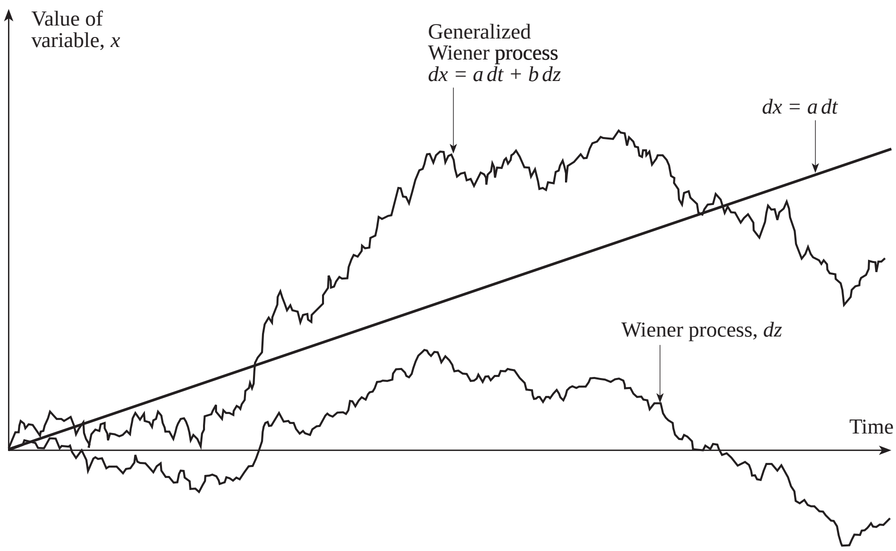
$$\text{mean of } \Delta x = a\,\Delta t$$
$$\text{standard deviation of } \Delta x = b\sqrt{\Delta t}$$
$$\text{variance of } \Delta x = b^2\Delta t$$

Similar arguments to those given for a Wiener process show that the change in the value of x in any time interval T is normally distributed with

$$\text{mean of change in } x = aT$$
$$\text{standard deviation of change in } x = b\sqrt{T}$$
$$\text{variance of change in } x = b^2 T$$

To summarize, the generalized Wiener process given in equation (14.3) has an expected drift rate (i.e., average drift per unit of time) of a and a variance rate (i.e., variance per unit of time) of b^2. It is illustrated in Figure 14.2.

Figure 14.2 Generalized Wiener process with $a = 0.3$ and $b = 1.5$.



Example 14.2

Consider the situation where the cash position of a company, measured in thousands of dollars, follows a generalized Wiener process with a drift of 20 per year and a variance rate of 900 per year. Initially, the cash position is 50. At the end of 1 year the cash position will have a normal distribution with a mean of 70 and a standard deviation of $\sqrt{900}$, or 30. At the end of 6 months it will have a normal distribution with a mean of 60 and a standard deviation of $30\sqrt{0.5} = 21.21$. Our uncertainty about the cash position at some time in the future, as measured by its standard deviation, increases as the square root of how far ahead we are looking. (Note that the cash position can become negative. We can interpret this as a situation where the company is borrowing funds.)

Itô Process

A further type of stochastic process, known as an *Itô process*, can be defined. This is a generalized Wiener process in which the parameters a and b are functions of the value of the underlying variable x and time t. An Itô process can therefore be written as

$$dx = a(x, t)\,dt + b(x, t)\,dz \tag{14.4}$$

Both the expected drift rate and variance rate of an Itô process are liable to change over time. They are functions of the current value of x and the current time, t. In a small time interval between t and $t + \Delta t$, the variable changes from x to $x + \Delta x$, where

$$\Delta x = a(x, t)\Delta t + b(x, t)\epsilon\sqrt{\Delta t}$$

This equation involves a small approximation. It assumes that the drift and variance rate of x remain constant, equal to their values at time t, in the time interval between t and $t + \Delta t$.

Note that the process in equation (14.4) is Markov because the change in x at time t depends only on the value of x at time t, not on its history. A non-Markov process could be defined by letting a and b in equation (14.4) depend on values of x prior to time t.

14.3 THE PROCESS FOR A STOCK PRICE

In this section we discuss the stochastic process usually assumed for the price of a non-dividend-paying stock.

It is tempting to suggest that a stock price follows a generalized Wiener process; that is, that it has a constant expected drift rate and a constant variance rate. However, this model fails to capture a key aspect of stock prices. This is that the expected percentage return required by investors from a stock is independent of the stock's price. If investors require a 14% per annum expected return when the stock price is \$10, then, *ceteris paribus*, they will also require a 14% per annum expected return when it is \$50.

Clearly, the assumption of constant expected drift rate is inappropriate and needs to be replaced by the assumption that the expected return (i.e., expected drift divided by the stock price) is constant. If S is the stock price at time t, then the expected drift rate in S should be assumed to be μS for some constant parameter μ. This means that in a short interval of time, Δt, the expected increase in S is $\mu S\,\Delta t$. The parameter μ is the expected rate of return on the stock.

If the coefficient of dz is zero, so that there is no uncertainty, then this model implies that

$$\Delta S = \mu S\,\Delta t$$

in the limit, as $\Delta t \to 0$, so that:

$$dS = \mu S\,dt$$

or

$$\frac{dS}{S} = \mu\,dt$$

Integrating between time 0 and time T, we get

$$S_T = S_0 e^{\mu T} \qquad \textbf{(14.5)}$$

where S_0 and S_T are the stock price at time 0 and time T. Equation (14.5) shows that, when there is no uncertainty, the stock price grows at a continuously compounded rate of μ per unit of time.

In practice, of course, there is uncertainty. A reasonable assumption is that the variability of the return in a short period of time, Δt, is the same regardless of the stock price. In other words, an investor is just as uncertain about the return when the stock price is \$50 as when it is \$10. This suggests that the standard deviation of the change in a short period of time Δt should be proportional to the stock price and leads to the model

$$dS = \mu S\,dt + \sigma S\,dz$$

or

$$\frac{dS}{S} = \mu\,dt + \sigma\,dz \qquad \textbf{(14.6)}$$

Equation (14.6) is the most widely used model of stock price behavior. The variable μ is the stock's expected rate of return. The variable σ is the volatility of the stock price. The variable σ^2 is referred to as its variance rate. The model in equation (14.6) represents the stock price process in the real world. In a risk-neutral world, μ equals the risk-free rate r.

Discrete-Time Model

The model of stock price behavior we have developed is known as *geometric Brownian motion*. The discrete-time version of the model is

$$\frac{\Delta S}{S} = \mu\,\Delta t + \sigma\epsilon\sqrt{\Delta t} \qquad \textbf{(14.7)}$$

or

$$\Delta S = \mu S\,\Delta t + \sigma S\epsilon\sqrt{\Delta t} \qquad \textbf{(14.8)}$$

The variable ΔS is the change in the stock price S in a small time interval Δt, and as before ϵ has a standard normal distribution (i.e., a normal distribution with a mean of zero and standard deviation of 1.0). The parameter μ is the expected rate of return per unit of time from the stock. The parameter σ is the stock price volatility. In this chapter we will assume these parameters are constant.

The left-hand side of equation (14.7) is the discrete approximation to the return provided by the stock in a short period of time, Δt. The term $\mu\,\Delta t$ is the expected value of this return, and the term $\sigma\epsilon\sqrt{\Delta t}$ is the stochastic component of the return. The variance of the stochastic component (and, therefore, of the whole return) is $\sigma^2 \Delta t$. This is consistent with the definition of the volatility σ given in Section 13.7; that is, σ is such that $\sigma\sqrt{\Delta t}$ is the standard deviation of the return in a short time period Δt.

Equation (14.7) shows that $\Delta S/S$ is approximately normally distributed with mean $\mu\,\Delta t$ and standard deviation $\sigma\sqrt{\Delta t}$. In other words,

$$\frac{\Delta S}{S} \sim \phi(\mu\,\Delta t,\ \sigma^2 \Delta t) \tag{14.9}$$

Example 14.3

Consider a stock that pays no dividends, has a volatility of 30% per annum, and provides an expected return of 15% per annum with continuous compounding. In this case, $\mu = 0.15$ and $\sigma = 0.30$. The process for the stock price is

$$\frac{dS}{S} = 0.15\,dt + 0.30\,dz$$

If S is the stock price at a particular time and ΔS is the increase in the stock price in the next small interval of time, the discrete approximation to the process is

$$\frac{\Delta S}{S} = 0.15\Delta t + 0.30\epsilon\sqrt{\Delta t}$$

where ϵ has a standard normal distribution. Consider a time interval of 1 week, or 0.0192 year, so that $\Delta t = 0.0192$. Then the approximation gives

$$\frac{\Delta S}{S} = 0.15 \times 0.0192 + 0.30 \times \sqrt{0.0192}\,\epsilon$$

or

$$\Delta S = 0.00288S + 0.0416S\epsilon$$

Monte Carlo Simulation

A Monte Carlo simulation of a stochastic process is a procedure for sampling random outcomes for the process. We will use it as a way of developing some understanding of the nature of the stock price process in equation (14.6).

Consider the situation in Example 14.3 where the expected return from a stock is 15% per annum and the volatility is 30% per annum. The stock price change over 1 week was shown to be approximately

$$\Delta S = 0.00288S + 0.0416S\epsilon \tag{14.10}$$

A path for the stock price over 10 weeks can be simulated by sampling repeatedly for ϵ from $\phi(0, 1)$ and substituting into equation (14.10). The expression =RAND( ) in Excel produces a random sample between 0 and 1. The inverse cumulative normal distribution is NORMSINV. The instruction to produce a random sample from a standard normal distribution in Excel is therefore =NORMSINV(RAND( )). Table 14.1 shows one path for a stock price that was sampled in this way. The initial stock price is assumed to be \$100. For the first period, ϵ is sampled as 0.52. From equation (14.10), the change during the first time period is

$$\Delta S = 0.00288 \times 100 + 0.0416 \times 100 \times 0.52 = 2.45$$

Therefore, at the beginning of the second time period, the stock price is \$102.45. The

Table 14.1 Simulation of stock price when $\mu = 0.15$ and $\sigma = 0.30$ during 1-week periods.

Stock price at start of period	*Random sample for ϵ*	*Change in stock price during period*
100.00	0.52	2.45
102.45	1.44	6.43
108.88	−0.86	−3.58
105.30	1.46	6.70
112.00	−0.69	−2.89
109.11	−0.74	−3.04
106.06	0.21	1.23
107.30	−1.10	−4.60
102.69	0.73	3.41
106.11	1.16	5.43
111.54	2.56	12.20

value of ϵ sampled for the next period is 1.44. From equation (14.10), the change during the second time period is

$$\Delta S = 0.00288 \times 102.45 + 0.0416 \times 102.45 \times 1.44 = 6.43$$

So, at the beginning of the next period, the stock price is \$108.88, and so on.[4] Note that, because the process we are simulating is Markov, the samples for ϵ should be independent of each other.

Table 14.1 assumes that stock prices are measured to the nearest cent. It is important to realize that the table shows only one possible pattern of stock price movements. Different random samples would lead to different price movements. Any small time interval Δt can be used in the simulation. In the limit as $\Delta t \to 0$, a perfect description of the stochastic process is obtained. The final stock price of 111.54 in Table 14.1 can be regarded as a random sample from the distribution of stock prices at the end of 10 weeks. By repeatedly simulating movements in the stock price, a complete probability distribution of the stock price at the end of this time is obtained. Monte Carlo simulation is discussed in more detail in Chapter 21.

14.4 THE PARAMETERS

The process for a stock price developed in this chapter involves two parameters, μ and σ. The parameter μ is the expected return (annualized) earned by an investor in a short period of time. Most investors require higher expected returns to induce them to take higher risks. It follows that the value of μ should depend on the risk of the return from the stock.[5] It should also depend on the level of interest rates in the economy. The higher the level of interest rates, the higher the expected return required on any given stock.

[4] In practice, it is more efficient to sample $\ln S$ rather than S, as will be discussed in Section 21.6.

[5] More precisely, μ depends on that part of the risk that cannot be diversified away by the investor.

Fortunately, we do not have to concern ourselves with the determinants of μ in any detail because the value of a derivative dependent on a stock is, in general, independent of μ. The parameter σ, the stock price volatility, is, by contrast, critically important to the determination of the value of many derivatives. We will discuss procedures for estimating σ in Chapter 15. Typical values of σ for a stock are in the range 0.15 to 0.60 (i.e., 15% to 60%).

The standard deviation of the proportional change in the stock price in a small interval of time Δt is $\sigma\sqrt{\Delta t}$. As a rough approximation, the standard deviation of the proportional change in the stock price over a relatively long period of time T is $\sigma\sqrt{T}$. This means that, as an approximation, volatility can be interpreted as the standard deviation of the change in the stock price in 1 year. In Chapter 15, we will show that the volatility of a stock price is exactly equal to the standard deviation of the continuously compounded return provided by the stock in 1 year.

14.5 CORRELATED PROCESSES

So far we have considered how the stochastic process for a single variable can be represented. We now extend the analysis to the situation where there are two or more variables following correlated stochastic processes. Suppose that the processes followed by two variables x_1 and x_2 are

$$dx_1 = a_1\,dt + b_1\,dz_1 \quad \text{and} \quad dx_2 = a_2\,dt + b_2\,dz_2$$

where dz_1 and dz_2 are Wiener processes.

As has been explained, the discrete-time approximations for these processes are

$$\Delta x_1 = a_1\,\Delta t + b_1\,\epsilon_1\sqrt{\Delta t} \quad \text{and} \quad \Delta x_2 = a_2\,\Delta t + b_2\,\epsilon_2\sqrt{\Delta t}$$

where ϵ_1 and ϵ_2 are samples from a standard normal distribution $\phi(0, 1)$.

The variables x_1 and x_2 can be simulated in the way described in Section 14.3. If they are uncorrelated with each other, the random samples ϵ_1 and ϵ_2 that are used to obtain movements in a particular period of time Δt should be independent of each other.

If x_1 and x_2 have a nonzero correlation ρ, then the ϵ_1 and ϵ_2 that are used to obtain movements in a particular period of time should be sampled from a bivariate normal distribution. Each variable in the bivariate normal distribution has a standard normal distribution and the correlation between the variables is ρ. In this situation, we would refer to the Wiener processes dz_1 and dz_2 as having a correlation ρ.

Obtaining samples for uncorrelated standard normal variables in cells in Excel involves putting the instruction "=NORMSINV(RAND())" in each of the cells. To sample standard normal variables ϵ_1 and ϵ_2 with correlation ρ, we can set

$$\epsilon_1 = u \quad \text{and} \quad \epsilon_2 = \rho u + \sqrt{1-\rho^2}\,v$$

where u and v are sampled as uncorrelated variables with standard normal distributions.

Note that, in the processes we have assumed for x_1 and x_2, the parameters a_1, a_2, b_1, and b_2 can be functions of x_1, x_2, and t. In particular, a_1 and b_1 can be functions of x_2 as well as x_1 and t; and a_2 and b_2 can be functions of x_1 as well as x_2 and t.

The results here can be generalized. When there are three different variables following correlated stochastic processes, we have to sample three different ϵ's. These have a trivariate normal distribution. When there are n correlated variables, we have n different ϵ's and these must be sampled from an appropriate multivariate normal distribution. The way this is done is explained in Chapter 21.

14.6 ITÔ'S LEMMA

The price of a stock option is a function of the underlying stock's price and time. More generally, we can say that the price of any derivative is a function of the stochastic variables underlying the derivative and time. A serious student of derivatives must, therefore, acquire some understanding of the behavior of functions of stochastic variables. An important result in this area was discovered by the mathematician K. Itô in 1951,[6] and is known as *Itô's lemma*.

Suppose that the value of a variable x follows the Itô process

$$dx = a(x,t)\,dt + b(x,t)\,dz \tag{14.11}$$

where dz is a Wiener process and a and b are functions of x and t. The variable x has a drift rate of a and a variance rate of b^2. Itô's lemma shows that a function G of x and t follows the process

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \tfrac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}b\,dz \tag{14.12}$$

where the dz is the same Wiener process as in equation (14.11). Thus, G also follows an Itô process, with a drift rate of

$$\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \tfrac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2$$

and a variance rate of

$$\left(\frac{\partial G}{\partial x}\right)^2 b^2$$

A completely rigorous proof of Itô's lemma is beyond the scope of this book. In the appendix to this chapter, we show that the lemma can be viewed as an extension of well-known results in differential calculus.

Earlier, we argued that

$$dS = \mu S\,dt + \sigma S\,dz \tag{14.13}$$

with μ and σ constant, is a reasonable model of stock price movements. From Itô's lemma, it follows that the process followed by a function G of S and t is

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \tfrac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial G}{\partial S}\sigma S\,dz \tag{14.14}$$

Note that both S and G are affected by the same underlying source of uncertainty, dz. This proves to be very important in the derivation of the Black–Scholes–Merton results.

[6] See K. Itô, "On Stochastic Differential Equations," *Memoirs of the American Mathematical Society*, 4 (1951): 1–51.

Application to Forward Contracts

To illustrate Itô's lemma, consider a forward contract on a non-dividend-paying stock. Assume that the risk-free rate of interest is constant and equal to r for all maturities. From equation (5.1),

$$F_0 = S_0 e^{rT}$$

where F_0 is the forward price at time zero, S_0 is the spot price at time zero, and T is the time to maturity of the forward contract.

We are interested in what happens to the forward price as time passes. We define F as the forward price at a general time t, and S as the stock price at time t, with $t < T$. The relationship between F and S is given by

$$F = Se^{r(T-t)} \tag{14.15}$$

Assuming that the process for S is given by equation (14.13), we can use Itô's lemma to determine the process for F. From equation (14.15),

$$\frac{\partial F}{\partial S} = e^{r(T-t)}, \qquad \frac{\partial^2 F}{\partial S^2} = 0, \qquad \frac{\partial F}{\partial t} = -rSe^{r(T-t)}$$

From equation (14.14), the process for F is given by

$$dF = \left[e^{r(T-t)}\mu S - rSe^{r(T-t)}\right] dt + e^{r(T-t)}\sigma S\, dz$$

Substituting F for $Se^{r(T-t)}$ gives

$$dF = (\mu - r)F\, dt + \sigma F\, dz \tag{14.16}$$

Like S, the forward price F follows geometric Brownian motion. It has the same volatility as S and an expected growth rate of $\mu - r$ rather than μ.

14.7 THE LOGNORMAL PROPERTY

We now use Itô's lemma to derive the process followed by $\ln S$ when S follows the process in equation (14.13). We define

$$G = \ln S$$

Since

$$\frac{\partial G}{\partial S} = \frac{1}{S}, \qquad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}, \qquad \frac{\partial G}{\partial t} = 0$$

it follows from equation (14.14) that the process followed by G is

$$dG = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dz \tag{14.17}$$

Since μ and σ are constant, this equation indicates that $G = \ln S$ follows a generalized Wiener process. It has constant drift rate $\mu - \sigma^2/2$ and constant variance rate σ^2. The change in $\ln S$ between time 0 and some future time T is therefore normally distributed, with mean $(\mu - \sigma^2/2)T$ and variance $\sigma^2 T$. This means that

$$\ln S_T - \ln S_0 \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right] \tag{14.18}$$

or

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right] \tag{14.19}$$

where S_T is the stock price at time T, S_0 is the stock price at time 0, and as before $\phi(m, v)$ denotes a normal distribution with mean m and variance v.

Equation (14.19) shows that $\ln S_T$ is normally distributed. A variable has a lognormal distribution if the natural logarithm of the variable is normally distributed. The model of stock price behavior we have developed in this chapter therefore implies that a stock's price at time T, given its price today, is lognormally distributed. The standard deviation of the logarithm of the stock price is $\sigma\sqrt{T}$. It is proportional to the square root of how far ahead we are looking.

SUMMARY

Stochastic processes describe the probabilistic evolution of the value of a variable through time. A Markov process is one where only the present value of the variable is relevant for predicting the future. The past history of the variable and the way in which the present has emerged from the past is irrelevant.

A Wiener process dz is a Markov process describing the evolution of a normally distributed variable. The drift of the process is zero and the variance rate is 1.0 per unit time. This means that, if the value of the variable is x_0 at time 0, then at time T it is normally distributed with mean x_0 and standard deviation $\sqrt{T}$.

A generalized Wiener process describes the evolution of a normally distributed variable with a drift of a per unit time and a variance rate of b^2 per unit time, where a and b are constants. This means that if, as before, the value of the variable is x_0 at time 0, it is normally distributed with a mean of $x_0 + aT$ and a standard deviation of $b\sqrt{T}$ at time T.

An Itô process is a process where the drift and variance rate of x can be a function of both x itself and time. The change in x in a very short period of time is, to a good approximation, normally distributed, but its change over longer periods of time is liable to be nonnormal.

One way of gaining an intuitive understanding of a stochastic process for a variable is to use Monte Carlo simulation. This involves dividing a time interval into many small time steps and randomly sampling possible paths for the variable. The future probability distribution for the variable can then be calculated. Monte Carlo simulation is discussed further in Chapter 21.

Itô's lemma is a way of calculating the stochastic process followed by a function of a variable from the stochastic process followed by the variable itself. As we shall see in Chapter 15, Itô's lemma plays a very important part in the pricing of derivatives. A key point is that the Wiener process dz underlying the stochastic process for the variable is exactly the same as the Wiener process underlying the stochastic process for the function of the variable. Both are subject to the same underlying source of uncertainty.

The stochastic process usually assumed for a stock price is geometric Brownian motion. Under this process the return to the holder of the stock in a small period of time is normally distributed and the returns in two nonoverlapping periods are independent. The value of the stock price at a future time has a lognormal distribution. The Black–Scholes–Merton model, which we cover in the next chapter, is based on the geometric Brownian motion assumption.

FURTHER READING

On Efficient Markets and the Markov Property of Stock Prices

Brealey, R. A. *An Introduction to Risk and Return from Common Stock*, 2nd edn. Cambridge, MA: MIT Press, 1986.

Cootner, P. H. (ed.) *The Random Character of Stock Market Prices*. Cambridge, MA: MIT Press, 1964.

On Stochastic Processes

Cox, D. R., and H. D. Miller. *The Theory of Stochastic Processes*. London: Chapman & Hall, 1977.

Feller, W. *Introduction to Probability Theory and Its Applications*. New York: Wiley, 1968.

Karlin, S., and H. M. Taylor. *A First Course in Stochastic Processes*, 2nd edn. New York: Academic Press, 1975.

Shreve, S. E. *Stochastic Calculus for Finance II: Continuous-Time Models*. New York: Springer, 2008.

Practice Questions (Answers in Solutions Manual)

14.1. What would it mean to assert that the temperature at a certain place follows a Markov process? Do you think that temperatures do, in fact, follow a Markov process?

14.2. Can a trading rule based on the past history of a stock's price ever produce returns that are consistently above average? Discuss.

14.3. A company's cash position, measured in millions of dollars, follows a generalized Wiener process with a drift rate of 0.5 per quarter and a variance rate of 4.0 per quarter. How high does the company's initial cash position have to be for the company to have a less than 5% chance of a negative cash position by the end of 1 year?

14.4. Variables X_1 and X_2 follow generalized Wiener processes, with drift rates μ_1 and μ_2 and variances σ_1^2 and σ_2^2. What process does $X_1 + X_2$ follow if:
(a) The changes in X_1 and X_2 in any short interval of time are uncorrelated?
(b) There is a correlation ρ between the changes in X_1 and X_2 in any short time interval?

14.5. Consider a variable S that follows the process

$$dS = \mu\, dt + \sigma\, dz$$

For the first three years, $\mu = 2$ and $\sigma = 3$; for the next three years, $\mu = 3$ and $\sigma = 4$. If the initial value of the variable is 5, what is the probability distribution of the value of the variable at the end of year 6?

14.6. Suppose that G is a function of a stock price S and time. Suppose that σ_S and σ_G are the volatilities of S and G. Show that, when the expected return of S increases by $\lambda\sigma_S$, the growth rate of G increases by $\lambda\sigma_G$, where λ is a constant.

14.7. Stock A and stock B both follow geometric Brownian motion. Changes in any short interval of time are uncorrelated with each other. Does the value of a portfolio consisting of one of stock A and one of stock B follow geometric Brownian motion? Explain your answer.

14.8. The process for the stock price in equation (14.8) is

$$\Delta S = \mu S\,\Delta t + \sigma S\epsilon\sqrt{\Delta t}$$

where μ and σ are constant. Explain carefully the difference between this model and each of the following:

$$\Delta S = \mu\,\Delta t + \sigma\epsilon\sqrt{\Delta t}$$
$$\Delta S = \mu S\,\Delta t + \sigma\epsilon\sqrt{\Delta t}$$
$$\Delta S = \mu\,\Delta t + \sigma S\epsilon\sqrt{\Delta t}$$

Why is the model in equation (14.8) a more appropriate model of stock price behavior than any of these three alternatives?

14.9. It has been suggested that the short-term interest rate r follows the stochastic process

$$dr = a(b - r)\,dt + rc\,dz$$

where a, b, c are positive constants and dz is a Wiener process. Describe the nature of this process.

14.10. Suppose that a stock price S follows geometric Brownian motion with expected return μ and volatility σ:

$$dS = \mu S\,dt + \sigma S\,dz$$

What is the process followed by the variable S^n? Show that S^n also follows geometric Brownian motion.

14.11. Suppose that x is the yield to maturity with continuous compounding on a zero-coupon bond that pays off \$1 at time T. Assume that x follows the process

$$dx = a(x_0 - x)\,dt + sx\,dz$$

where a, x_0, and s are positive constants and dz is a Wiener process. What is the process followed by the bond price?

Further Questions

14.12. A stock whose price is \$30 has an expected return of 9% and a volatility of 20%. In Excel, simulate the stock price path over 5 years using monthly time steps and random samples from a normal distribution. Chart the simulated stock price path. By hitting F9, observe how the path changes as the random samples change.

14.13. Suppose that a stock price has an expected return of 16% per annum and a volatility of 30% per annum. When the stock price at the end of a certain day is \$50, calculate the following:
(a) The expected stock price at the end of the next day
(b) The standard deviation of the stock price at the end of the next day
(c) The 95% confidence limits for the stock price at the end of the next day.

14.14. A company's cash position, measured in millions of dollars, follows a generalized Wiener process with a drift rate of 0.1 per month and a variance rate of 0.16 per month. The initial cash position is 2.0.
(a) What are the probability distributions of the cash position after 1 month, 6 months, and 1 year?
(b) What are the probabilities of a negative cash position at the end of 6 months and 1 year?
(c) At what time in the future is the probability of a negative cash position greatest?

14.15. Suppose that x is the yield on a perpetual government bond that pays interest at the rate of \$1 per annum. Assume that x is expressed with continuous compounding, that interest is paid continuously on the bond, and that x follows the process

$$dx = a(x_0 - x)\,dt + sx\,dz$$

where a, x_0, and s are positive constants, and dz is a Wiener process. What is the process followed by the bond price? What is the expected instantaneous return (including interest and capital gains) to the holder of the bond?

14.16. If S follows the geometric Brownian motion process in equation (14.6), what is the process followed by
(a) $y = 2S$
(b) $y = S^2$
(c) $y = e^S$
(d) $y = e^{r(T-t)}/S$.
In each case express the coefficients of dt and dz in terms of y rather than S.

14.17. A stock price is currently 50. Its expected return and volatility are 12% and 30%, respectively. What is the probability that the stock price will be greater than 80 in 2 years? (*Hint*: $S_T > 80$ when $\ln S_T > \ln 80$.)

14.18. Stock A, whose price is \$30, has an expected return of 11% and a volatility of 25%. Stock B, whose price is \$40, has an expected return of 15% and a volatility of 30%. The processes driving the returns are correlated with correlation parameter ρ. In Excel, simulate the two stock price paths over 3 months using daily time steps and random samples from normal distributions. Chart the results and by hitting F9 observe how the paths change as the random samples change. Consider values for ρ equal to 0.25, 0.75, and 0.95.

APPENDIX
A NONRIGOROUS DERIVATION OF ITÔ'S LEMMA

In this appendix, we show how Itô's lemma can be regarded as a natural extension of other, simpler results. Consider a continuous and differentiable function G of a variable x. If Δx is a small change in x and ΔG is the resulting small change in G, a well-known result from ordinary calculus is

$$\Delta G \approx \frac{dG}{dx} \Delta x \tag{14A.1}$$

In other words, ΔG is approximately equal to the rate of change of G with respect to x multiplied by Δx. The error involves terms of order Δx^2. If more precision is required, a Taylor series expansion of ΔG can be used:

$$\Delta G = \frac{dG}{dx}\Delta x + \tfrac{1}{2}\frac{d^2G}{dx^2}\Delta x^2 + \tfrac{1}{6}\frac{d^3G}{dx^3}\Delta x^3 + \cdots$$

For a continuous and differentiable function G of two variables x and y, the result analogous to equation (14A.1) is

$$\Delta G \approx \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial y}\Delta y \tag{14A.2}$$

and the Taylor series expansion of ΔG is

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial y}\Delta y + \tfrac{1}{2}\frac{\partial^2 G}{\partial x^2}\Delta x^2 + \frac{\partial^2 G}{\partial x\,\partial y}\Delta x\,\Delta y + \tfrac{1}{2}\frac{\partial^2 G}{\partial y^2}\Delta y^2 + \cdots \tag{14A.3}$$

In the limit, as Δx and Δy tend to zero, equation (14A.3) becomes

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy \tag{14A.4}$$

We now extend equation (14A.4) to cover functions of variables following Itô processes. Suppose that a variable x follows the Itô process

$$dx = a(x, t)\,dt + b(x, t)\,dz \tag{14A.5}$$

and that G is some function of x and of time t. By analogy with equation (14A.3), we can write

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \tfrac{1}{2}\frac{\partial^2 G}{\partial x^2}\Delta x^2 + \frac{\partial^2 G}{\partial x\,\partial t}\Delta x\,\Delta t + \tfrac{1}{2}\frac{\partial^2 G}{\partial t^2}\Delta t^2 + \cdots \tag{14A.6}$$

Equation (14A.5) can be discretized to

$$\Delta x = a(x, t)\,\Delta t + b(x, t)\epsilon\sqrt{\Delta t}$$

or, if arguments are dropped,

$$\Delta x = a\,\Delta t + b\epsilon\sqrt{\Delta t} \tag{14A.7}$$

This equation reveals an important difference between the situation in equation (14A.6) and the situation in equation (14A.3). When limiting arguments were used to move from equation (14A.3) to equation (14A.4), terms in Δx^2 were ignored because they were second-order terms. From equation (14A.7), we have

$$\Delta x^2 = b^2 \epsilon^2 \Delta t + \text{terms of higher order in } \Delta t \tag{14A.8}$$

This shows that the term involving Δx^2 in equation (14A.6) has a component that is of order Δt and cannot be ignored.

The variance of a standard normal distribution is 1.0. This means that

$$E(\epsilon^2) - [E(\epsilon)]^2 = 1$$

where E denotes expected value. Since $E(\epsilon) = 0$, it follows that $E(\epsilon^2) = 1$. The expected value of $\epsilon^2 \Delta t$, therefore, is Δt. The variance of $\epsilon^2 \Delta t$ is, from the properties of the standard normal distribution, $2\Delta t^2$. We know that the variance of the change in a stochastic variable in time Δt is proportional to Δt, not Δt^2. The variance of $\epsilon^2 \Delta t$ is therefore too small for it to have a stochastic component. As a result, we can treat $\epsilon^2 \Delta t$ as nonstochastic and equal to its expected value, Δt, as Δt tends to zero. It follows from equation (14A.8) that Δx^2 becomes nonstochastic and equal to $b^2 dt$ as Δt tends to zero. Taking limits as Δx and Δt tend to zero in equation (14A.6), and using this last result, we obtain

$$dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + \tfrac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 dt \tag{14A.9}$$

This is Itô's lemma. If we substitute for dx from equation (14A.5), equation (14A.9) becomes

$$dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \tfrac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b\, dz.$$

Technical Note 29 at www-2.rotman.utoronto.ca/~hull/TechnicalNotes provides proofs of extensions to Itô's lemma. When G is a function of variables $x_1, x_2, \ldots, x_n$ and

$$dx_i = a_i\, dt + b_i\, dz_i$$

we have

$$dG = \left(\sum_{i=1}^{n} \frac{\partial G}{\partial x_i} a_i + \frac{\partial G}{\partial t} + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 G}{\partial x_i\, \partial x_j} b_i b_j \rho_{ij} \right) dt + \sum_{i=1}^{n} \frac{\partial G}{\partial x_i} b_i\, dz_i \tag{14A.10}$$

Also, when G is a function of a variable x with several sources of uncertainty so that

$$dx = a\, dt + \sum_{i=1}^{m} b_i\, dz_i$$

we have

$$dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \tfrac{1}{2} \frac{\partial^2 G}{\partial x^2} \sum_{i=1}^{m} \sum_{j=1}^{m} b_i b_j \rho_{ij} \right) dt + \frac{\partial G}{\partial x} \sum_{i=1}^{m} b_i\, dz_i \tag{14A.11}$$

In these equations, ρ_{ij} is the correlation between dz_i and dz_j (see Section 14.5).

CHAPTER 15

The Black–Scholes–Merton Model

In the early 1970s, Fischer Black, Myron Scholes, and Robert Merton achieved a major breakthrough in the pricing of European stock options.[1] This was the development of what has become known as the Black–Scholes–Merton (or Black–Scholes) model. The model has had a huge influence on the way that traders price and hedge derivatives. In 1997, the importance of the model was recognized when Robert Merton and Myron Scholes were awarded the Nobel prize for economics. Sadly, Fischer Black died in 1995; otherwise he too would undoubtedly have been one of the recipients of this prize.

How did Black, Scholes, and Merton make their breakthrough? Previous researchers had made similar assumptions and had correctly calculated the expected payoff from a European option. However, as explained in Section 13.2, it is difficult to know the correct discount rate to use for this payoff. Black and Scholes used the capital asset pricing model (see the appendix to Chapter 3) to determine a relationship between the market's required return on the option and the required return on the stock. This was not easy because the relationship depends on both the stock price and time. Merton's approach was different from that of Black and Scholes. It involved setting up a riskless portfolio consisting of the option and the underlying stock and arguing that the return on the portfolio over a short period of time must be the risk-free return. This is similar to what we did in Section 13.1—but more complicated because the portfolio changes continuously through time. Merton's approach was more general than that of Black and Scholes because it did not rely on the assumptions of the capital asset pricing model.

This chapter covers Merton's approach to deriving the Black–Scholes–Merton model. It explains how volatility can be either estimated from historical data or implied from option prices using the model. It shows how the risk-neutral valuation argument introduced in Chapter 13 can be used. It also shows how the Black–Scholes–Merton model can be extended to deal with European call and put options on dividend-paying stocks and presents some results on the pricing of American call options on dividend-paying stocks.

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[1] See F. Black and M. Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81 (May/June 1973): 637–59; R.C. Merton, "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*, 4 (Spring 1973): 141–83.

15.1 LOGNORMAL PROPERTY OF STOCK PRICES

The model of stock price behavior used by Black, Scholes, and Merton is the model we developed in Chapter 14. It assumes that percentage changes in the stock price in a very short period of time are normally distributed. Define

μ: Expected return in a short period of time (annualized)
σ: Volatility of the stock price.

The mean and standard deviation of the return in time Δt are approximately $\mu\,\Delta t$ and $\sigma\sqrt{\Delta t}$, so that

$$\frac{\Delta S}{S} \sim \phi(\mu\,\Delta t,\ \sigma^2\Delta t) \tag{15.1}$$

where ΔS is the change in the stock price S in time Δt, and $\phi(m, v)$ denotes a normal distribution with mean m and variance v. (This is equation (14.9).)

As shown in Section 14.7, the model implies that

$$\ln S_T - \ln S_0 \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right]$$

so that

$$\ln\frac{S_T}{S_0} \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right] \tag{15.2}$$

and

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right] \tag{15.3}$$

where S_T is the stock price at a future time T and S_0 is the stock price at time 0. There is no approximation here. The variable $\ln S_T$ is normally distributed, so that S_T has a lognormal distribution. The mean of $\ln S_T$ is $\ln S_0 + (\mu - \sigma^2/2)T$ and the standard deviation of $\ln S_T$ is $\sigma\sqrt{T}$.

Example 15.1

Consider a stock with an initial price of \$40, an expected return of 16% per annum, and a volatility of 20% per annum. From equation (15.3), the probability distribution of the stock price S_T in 6 months' time is given by

$$\ln S_T \sim \phi[\ln 40 + (0.16 - 0.2^2/2) \times 0.5,\ 0.2^2 \times 0.5]$$
$$\ln S_T \sim \phi(3.759,\ 0.02)$$

There is a 95% probability that a normally distributed variable has a value within 1.96 standard deviations of its mean. In this case, the standard deviation is $\sqrt{0.02} = 0.141$. Hence, with 95% confidence,

$$3.759 - 1.96 \times 0.141 < \ln S_T < 3.759 + 1.96 \times 0.141$$

This can be written

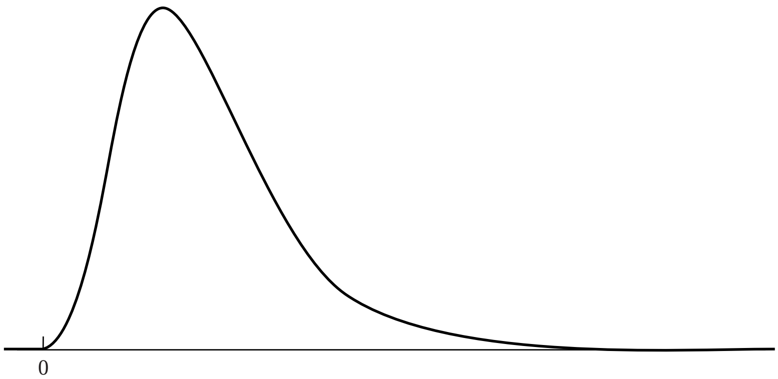
$$e^{3.759-1.96\times0.141} < S_T < e^{3.759+1.96\times0.141}$$

or

$$32.55 < S_T < 56.56$$

Thus, there is a 95% probability that the stock price in 6 months will lie between 32.55 and 56.56.

Figure 15.1 Lognormal distribution.



A variable that has a lognormal distribution can take any value between zero and infinity. Figure 15.1 illustrates the shape of a lognormal distribution. Unlike the normal distribution, it is skewed so that the mean, median, and mode are all different. From equation (15.3) and the properties of the lognormal distribution, it can be shown that the expected value $E(S_T)$ of S_T is given by

$$E(S_T) = S_0 e^{\mu T} \tag{15.4}$$

The variance $\mathrm{var}(S_T)$ of S_T, can be shown to be given by[2]

$$\mathrm{var}(S_T) = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1) \tag{15.5}$$

Example 15.2

Consider a stock where the current price is \$20, the expected return is 20% per annum, and the volatility is 40% per annum. The expected stock price, $E(S_T)$, and the variance of the stock price, $\mathrm{var}(S_T)$, in 1 year are given by

$$E(S_T) = 20e^{0.2\times 1} = 24.43 \quad \text{and} \quad \mathrm{var}(S_T) = 400e^{2\times 0.2\times 1}(e^{0.4^2\times 1} - 1) = 103.54$$

The standard deviation of the stock price in 1 year is $\sqrt{103.54}$, or 10.18.

15.2 THE DISTRIBUTION OF THE RATE OF RETURN

The lognormal property of stock prices can be used to provide information on the probability distribution of the continuously compounded rate of return earned on a stock between times 0 and T. If we define the continuously compounded rate of return per annum realized between times 0 and T as x, then

$$S_T = S_0 e^{xT}$$

[2] See Technical Note 2 at www-2.rotman.utoronto.ca/~hull/TechnicalNotes for a proof of the results in equations (15.4) and (15.5). For a more extensive discussion of the properties of the lognormal distribution, see J. Aitchison and J. A. C. Brown, *The Lognormal Distribution*. Cambridge University Press, 1966.

so that

$$x = \frac{1}{T}\ln\frac{S_T}{S_0} \tag{15.6}$$

From equation (15.2), it follows that

$$x \sim \phi\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T}\right) \tag{15.7}$$

Thus, the continuously compounded rate of return per annum is normally distributed with mean $\mu - \sigma^2/2$ and standard deviation $\sigma/\sqrt{T}$. As T increases, the standard deviation of x declines. To understand the reason for this, consider two cases: $T = 1$ and $T = 20$. We are more certain about the average return per year over 20 years than we are about the return in any one year.

Example 15.3

Consider a stock with an expected return of 17% per annum and a volatility of 20% per annum. The probability distribution for the average rate of return (continuously compounded) realized over 3 years is normal, with mean

$$0.17 - \frac{0.2^2}{2} = 0.15$$

or 15% per annum, and standard deviation

$$\sqrt{\frac{0.2^2}{3}} = 0.1155$$

or 11.55% per annum. Because there is a 95% chance that a normally distributed variable will lie within 1.96 standard deviations of its mean, we can be 95% confident that the average continuously compounded return realized over 3 years will be between $15 - 1.96 \times 11.55 = -7.6\%$ and $15 + 1.96 \times 11.55 = +37.6\%$ per annum.

15.3 THE EXPECTED RETURN

The expected return, μ, required by investors from a stock depends on the riskiness of the stock. The higher the risk, the higher the expected return. It also depends on the level of interest rates in the economy. The higher the level of interest rates, the higher the expected return required on any given stock. Fortunately, we do not have to concern ourselves with the determinants of μ in any detail. It turns out that the value of a stock option, when expressed in terms of the value of the underlying stock, does not depend on μ at all. Nevertheless, there is one aspect of the expected return from a stock that frequently causes confusion and needs to be explained.

Our model of stock price behavior implies that, in a very short period of time Δt, the mean return is $\mu\,\Delta t$. It is natural to assume from this that μ is the expected continuously compounded return on the stock. However, this is not the case. The continuously compounded return, x, actually realized over a period of time of length T

is given by equation (15.6) as

$$x = \frac{1}{T}\ln\frac{S_T}{S_0}$$

and, as indicated in equation (15.7), the expected value $E(x)$ of x is $\mu - \sigma^2/2$.

The reason why the expected continuously compounded return is different from μ is subtle, but important. Suppose we consider a very large number of very short periods of time of length Δt. Define S_i as the stock price at the end of the ith interval and ΔS_i as $S_{i+1} - S_i$. Under the assumptions we are making for stock price behavior, the arithmetic average of the returns on the stock in each interval is close to μ. In other words, $\mu\,\Delta t$ is close to the arithmetic mean of the $\Delta S_i/S_i$. However, the expected return over the whole period covered by the data, expressed with a compounding interval of Δt, is a geometric average and is close to $\mu - \sigma^2/2$, not μ.[3] Business Snapshot 15.1 provides a numerical example concerning the mutual fund industry to illustrate why this is so.

For another explanation of what is going on, we start with equation (15.4):

$$E(S_T) = S_0 e^{\mu T}$$

Taking logarithms, we get

$$\ln[E(S_T)] = \ln(S_0) + \mu T$$

It is now tempting to set $\ln[E(S_T)] = E[\ln(S_T)]$, so that $E[\ln(S_T)] - \ln(S_0) = \mu T$, or $E[\ln(S_T/S_0)] = \mu T$, which leads to $E(x) = \mu$. However, we cannot do this because ln is a nonlinear function. In fact, $\ln[E(S_T)] > E[\ln(S_T)]$, so that $E[\ln(S_T/S_0)] < \mu T$, which leads to $E(x) < \mu$. (As shown above, $E(x) = \mu - \sigma^2/2$.)

15.4 VOLATILITY

The volatility, σ, of a stock is a measure of our uncertainty about the returns provided by the stock. Stocks typically have a volatility between 15% and 60%.

From equation (15.7), the volatility of a stock price can be defined as the standard deviation of the return provided by the stock in 1 year when the return is expressed using continuous compounding.

When Δt is small, equation (15.1) shows that $\sigma^2\Delta t$ is approximately equal to the variance of the percentage change in the stock price in time Δt. This means that $\sigma\sqrt{\Delta t}$ is approximately equal to the standard deviation of the percentage change in the stock price in time Δt. Suppose that $\sigma = 0.3$, or 30%, per annum and the current stock price is \$50. The standard deviation of the percentage change in the stock price in 1 week is approximately

$$30 \times \sqrt{\frac{1}{52}} = 4.16\%$$

A 1-standard-deviation move in the stock price in 1 week is therefore $50 \times 0.0416 = 2.08$.

Uncertainty about a future stock price, as measured by its standard deviation, increases—at least approximately—with the square root of how far ahead we are looking. For example, the standard deviation of the stock price in 4 weeks is approximately twice the standard deviation in 1 week.

[3] The arguments in this section show that the term "expected return" is ambiguous. It can refer either to μ or to $\mu - \sigma^2/2$. Unless otherwise stated, it will be used to refer to μ throughout this book.

Business Snapshot 15.1 Mutual Fund Returns Can Be Misleading

The difference between μ and $\mu - \sigma^2/2$ is closely related to an issue in the reporting of mutual fund returns. Suppose that the following is a sequence of returns per annum reported by a mutual fund manager over the last five years (measured using annual compounding): 15%, 20%, 30%, −20%, 25%.

The arithmetic mean of the returns, calculated by taking the sum of the returns and dividing by 5, is 14%. However, an investor would actually earn less than 14% per annum by leaving the money invested in the fund for 5 years. The dollar value of \$100 at the end of the 5 years would be

$$100 \times 1.15 \times 1.20 \times 1.30 \times 0.80 \times 1.25 = \$179.40$$

By contrast, a 14% return with annual compounding would give

$$100 \times 1.14^5 = \$192.54$$

The return that gives \$179.40 at the end of five years is 12.4%. This is because

$$100 \times (1.124)^5 = 179.40$$

What average return should the fund manager report? It is tempting for the manager to make a statement such as: "The average of the returns per year that we have realized in the last 5 years is 14%." Although true, this is misleading. It is much less misleading to say: "The average return realized by someone who invested with us for the last 5 years is 12.4% per year." In some jurisdictions, regulations require fund managers to report returns the second way.

This phenomenon is an example of a result that is well known in mathematics. The geometric mean of a set of numbers is always less than the arithmetic mean. In our example, the return multipliers each year are 1.15, 1.20, 1.30, 0.80, and 1.25. The arithmetic mean of these numbers is 1.140, but the geometric mean is only 1.124 and it is the geometric mean that equals 1 plus the return realized over the 5 years.

Estimating Volatility from Historical Data

To estimate the volatility of a stock price empirically, the stock price is usually observed at fixed intervals of time (e.g., every day, week, or month). Define:

$n+1$: Number of observations
S_i: Stock price at end of ith interval, with $i = 0, 1, \dots, n$
τ: Length of time interval in years

and let

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right) \quad \text{for } i = 1, 2, \dots, n$$

The usual estimate, s, of the standard deviation of the u_i is given by

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (u_i - \bar{u})^2}$$

or

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n} u_i^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n} u_i\right)^2}$$

where $\bar{u}$ is the mean of the u_i.[4]

From equation (15.2), the standard deviation of the u_i is $\sigma\sqrt{\tau}$. The variable s is therefore an estimate of $\sigma\sqrt{\tau}$. It follows that σ itself can be estimated as $\hat{\sigma}$, where

$$\hat{\sigma} = \frac{s}{\sqrt{\tau}}$$

The standard error of this estimate can be shown to be approximately $\hat{\sigma}/\sqrt{2n}$.

Choosing an appropriate value for n is not easy. More data generally lead to more accuracy, but σ does change over time and data that are too old may not be relevant for predicting the future volatility. A compromise that seems to work reasonably well is to use closing prices from daily data over the most recent 90 to 180 days. Alternatively, as a rule of thumb, n can be set equal to the number of days to which the volatility is to be applied. Thus, if the volatility estimate is to be used to value a 2-year option, daily data for the last 2 years are used. More sophisticated approaches to estimating volatility involving GARCH models are discussed in Chapter 23.

Example 15.4

Table 15.1 shows a possible sequence of stock prices during 21 consecutive trading days. In this case, $n = 20$, so that

$$\sum_{i=1}^{n} u_i = 0.09531 \quad \text{and} \quad \sum_{i=1}^{n} u_i^2 = 0.00326$$

and the estimate of the standard deviation of the daily return is

$$\sqrt{\frac{0.00326}{19} - \frac{0.09531^2}{20 \times 19}} = 0.01216$$

or 1.216%. Assuming that there are 252 trading days per year, $\tau = 1/252$ and the data give an estimate for the volatility per annum of $0.01216\sqrt{252} = 0.193$, or 19.3%. The standard error of this estimate is

$$\frac{0.193}{\sqrt{2 \times 20}} = 0.031$$

or 3.1% per annum.

The foregoing analysis assumes that the stock pays no dividends, but it can be adapted to accommodate dividend-paying stocks. The return, u_i, during a time interval that includes an ex-dividend day is given by

$$u_i = \ln\frac{S_i + D}{S_{i-1}}$$

where D is the amount of the dividend. The return in other time intervals is still

$$u_i = \ln\frac{S_i}{S_{i-1}}$$

[4] The mean $\bar{u}$ is often assumed to be zero when estimates of historical volatilities are made.

Table 15.1 Computation of volatility.

Day i	*Closing stock price (dollars),* S_i	*Price relative* S_i/S_{i-1}	*Daily return* $u_i = \ln(S_i/S_{i-1})$
0	20.00		
1	20.10	1.00500	0.00499
2	19.90	0.99005	−0.01000
3	20.00	1.00503	0.00501
4	20.50	1.02500	0.02469
5	20.25	0.98780	−0.01227
6	20.90	1.03210	0.03159
7	20.90	1.00000	0.00000
8	20.90	1.00000	0.00000
9	20.75	0.99282	−0.00720
10	20.75	1.00000	0.00000
11	21.00	1.01205	0.01198
12	21.10	1.00476	0.00475
13	20.90	0.99052	−0.00952
14	20.90	1.00000	0.00000
15	21.25	1.01675	0.01661
16	21.40	1.00706	0.00703
17	21.40	1.00000	0.00000
18	21.25	0.99299	−0.00703
19	21.75	1.02353	0.02326
20	22.00	1.01149	0.01143

However, as tax factors play a part in determining returns around an ex-dividend date, it is probably best to discard altogether data for intervals that include an ex-dividend date.

Trading Days vs. Calendar Days

An important issue is whether time should be measured in calendar days or trading days when volatility parameters are being estimated and used. As shown in Business Snapshot 15.2, research shows that volatility is much higher when the exchange is open for trading than when it is closed. As a result, practitioners tend to ignore days when the exchange is closed when estimating volatility from historical data and when calculating the life of an option. The volatility per annum is calculated from the volatility per trading day using the formula

$$\begin{matrix}\text{Volatility}\\ \text{per annum}\end{matrix} = \begin{matrix}\text{Volatility}\\ \text{per trading day}\end{matrix} \times \sqrt{\begin{matrix}\text{Number of trading days}\\ \text{per annum}\end{matrix}}$$

This is what we did in Example 15.4 when calculating volatility from the data in Table 15.1. The number of trading days in a year is usually assumed to be 252 for stocks.

Business Snapshot 15.2 What Causes Volatility?

It is natural to assume that the volatility of a stock is caused by new information reaching the market. This new information causes people to revise their opinions about the value of the stock. The price of the stock changes and volatility results. This view of what causes volatility is not supported by research. With several years of daily stock price data, researchers can calculate:

1. The variance of stock price returns between the close of trading on one day and the close of trading on the next day when there are no intervening nontrading days
2. The variance of the stock price returns between the close of trading on Friday and the close of trading on Monday

The second of these is the variance of returns over a 3-day period. The first is a variance over a 1-day period. We might reasonably expect the second variance to be three times as great as the first variance. Fama (1965), French (1980), and French and Roll (1986) show that this is not the case. These three research studies estimate the second variance to be, respectively, 22%, 19%, and 10.7% higher than the first variance.

At this stage one might be tempted to argue that these results are explained by more news reaching the market when the market is open for trading. But research by Roll (1984) does not support this explanation. Roll looked at the prices of orange juice futures. By far the most important news for orange juice futures prices is news about the weather and this is equally likely to arrive at any time. When Roll did a similar analysis to that just described for stocks, he found that the second (Friday-to-Monday) variance for orange juice futures is only 1.54 times the first variance.

The only reasonable conclusion from all this is that volatility is to a large extent caused by trading itself. (Traders usually have no difficulty accepting this conclusion!)

The life of an option is also usually measured using trading days rather than calendar days. It is calculated as T years, where

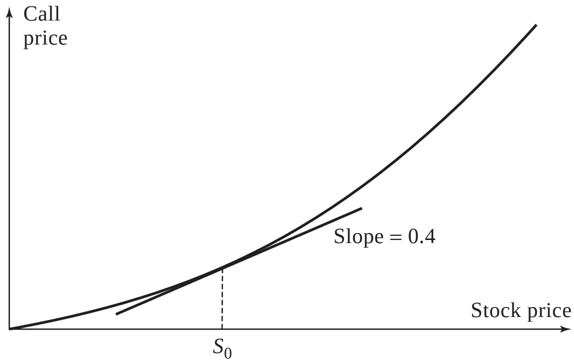
$$T = \frac{\text{Number of trading days until option maturity}}{252}$$

15.5 THE IDEA UNDERLYING THE BLACK–SCHOLES–MERTON DIFFERENTIAL EQUATION

The Black–Scholes–Merton differential equation is an equation that must be satisfied by the price of any derivative dependent on a non-dividend-paying stock. The equation is derived in the next section. Here we consider the nature of the arguments we will use.

These are similar to the no-arbitrage arguments we used to value stock options in Chapter 13 for the situation where stock price movements were assumed to be binomial. They involve setting up a riskless portfolio consisting of a position in the derivative and a position in the stock. In the absence of arbitrage opportunities, the return from the portfolio must be the risk-free interest rate, r. This leads to the Black-Scholes-Merton differential equation.

Figure 15.2 Relationship between call price and stock price. Current stock price is S_0.



The reason a riskless portfolio can be set up is that the stock price and the derivative price are both affected by the same underlying source of uncertainty: stock price movements. In any short period of time, the price of the derivative is perfectly correlated with the price of the underlying stock. When an appropriate portfolio of the stock and the derivative is established, the gain or loss from the stock position always offsets the gain or loss from the derivative position so that the overall value of the portfolio at the end of the short period of time is known with certainty.

Suppose, for example, that at a particular point in time the relationship between a small change ΔS in the stock price and the resultant small change Δc in the price of a European call option is given by

$$\Delta c = 0.4\,\Delta S$$

This means that the slope of the line representing the relationship between c and S is 0.4, as indicated in Figure 15.2. A riskless portfolio would consist of:

1. A long position in 40 shares
2. A short position in 100 call options.

Suppose, for example, that the stock price increases by 10 cents. The option price will increase by 4 cents and the $40 \times 0.1 = \$4$ gain on the shares is equal to the $100 \times 0.04 =$ \$4 loss on the short option position.

There is one important difference between the Black–Scholes–Merton analysis and our analysis using a binomial model in Chapter 13. In Black–Scholes–Merton, the position in the stock and the derivative is riskless for only a very short period of time. (Theoretically, it remains riskless only for an instantaneously short period of time.) To remain riskless, it must be adjusted, or *rebalanced*, frequently.[5] For example, the relationship between Δc and ΔS in our example might change from $\Delta c = 0.4\,\Delta S$ today to $\Delta c = 0.5\,\Delta S$ tomorrow. This would mean that, in order to maintain the riskless position, an extra 10 shares would have to be purchased for each 100 call options sold. It is nevertheless true that the return from the riskless portfolio in any very short period of time must be the risk-free interest rate. This is the key element in the Black–Scholes–Merton analysis and leads to their pricing formulas.

[5] We discuss the rebalancing of portfolios in more detail in Chapter 19.

Assumptions

The assumptions we use to derive the Black–Scholes–Merton differential equation are as follows:

1. The stock price follows the process developed in Chapter 14 with μ and σ constant.
2. The short selling of securities with full use of proceeds is permitted.
3. There are no transaction costs or taxes. All securities are perfectly divisible.
4. There are no dividends during the life of the derivative.
5. There are no riskless arbitrage opportunities.
6. Security trading is continuous.
7. The risk-free rate of interest, r, is constant and the same for all maturities.

As we discuss in later chapters, some of these assumptions can be relaxed. For example, σ and r can be known functions of t. We can even allow interest rates to be stochastic provided that the stock price distribution at maturity of the option is still lognormal.

15.6 DERIVATION OF THE BLACK–SCHOLES–MERTON DIFFERENTIAL EQUATION

In this section, the notation is different from elsewhere in the book. We consider a derivative's price at a general time t (not at time zero). If T is the maturity date, the time to maturity is $T - t$.

The stock price process we are assuming is the one we developed in Section 14.3:

$$dS = \mu S\,dt + \sigma S\,dz \tag{15.8}$$

Suppose that f is the price of a call option or other derivative contingent on S. The variable f must be some function of S and t. Hence, from equation (14.14),

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \tfrac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial f}{\partial S}\sigma S\,dz \tag{15.9}$$

The discrete versions of equations (15.8) and (15.9) are

$$\Delta S = \mu S\,\Delta t + \sigma S\,\Delta z \tag{15.10}$$

and

$$\Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \tfrac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right) \Delta t + \frac{\partial f}{\partial S}\sigma S\,\Delta z \tag{15.11}$$

where Δf and ΔS are the changes in f and S in a small time interval Δt. Recall from the discussion of Itô's lemma in Section 14.6 that the Wiener processes underlying f and S are the same. In other words, the Δz $(= \epsilon\sqrt{\Delta t})$ in equations (15.10) and (15.11) are the same. It follows that a portfolio of the stock and the derivative can be constructed so that the Wiener process is eliminated. The portfolio is

-1: derivative

$+\partial f/\partial S$: shares.

The holder of this portfolio is short one derivative and long an amount $\partial f/\partial S$ of shares. Define Π as the value of the portfolio. By definition

$$\Pi = -f + \frac{\partial f}{\partial S} S \tag{15.12}$$

The change $\Delta\Pi$ in the value of the portfolio in the time interval Δt is given by[6]

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S \tag{15.13}$$

Substituting equations (15.10) and (15.11) into equation (15.13) yields

$$\Delta\Pi = \left(-\frac{\partial f}{\partial t} - \tfrac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t \tag{15.14}$$

Because this equation does not involve Δz, the portfolio must be riskless during time Δt. The assumptions listed in the preceding section imply that the portfolio must instantaneously earn the same rate of return as other short-term risk-free securities. If it earned more than this return, arbitrageurs could make a riskless profit by borrowing money to buy the portfolio; if it earned less, they could make a riskless profit by shorting the portfolio and buying risk-free securities. It follows that

$$\Delta\Pi = r\Pi\,\Delta t \tag{15.15}$$

where r is the risk-free interest rate. Substituting from equations (15.12) and (15.14) into (15.15), we obtain

$$\left(\frac{\partial f}{\partial t} + \tfrac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t = r\left(f - \frac{\partial f}{\partial S} S\right)\Delta t$$

so that

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{15.16}$$

Equation (15.16) is the Black–Scholes–Merton differential equation. It has many solutions, corresponding to all the different derivatives that can be defined with S as the underlying variable. The particular derivative that is obtained when the equation is solved depends on the *boundary conditions* that are used. These specify the values of the derivative at the boundaries of possible values of S and t. In the case of a European call option, the key boundary condition is

$$f = \max(S - K,\, 0) \quad \text{when } t = T$$

In the case of a European put option, it is

$$f = \max(K - S,\, 0) \quad \text{when } t = T$$

Example 15.5

A forward contract on a non-dividend-paying stock is a derivative dependent on the stock. As such, it should satisfy equation (15.16). From equation (5.5), we

[6] This derivation of equation (15.16) is not completely rigorous. We need to justify ignoring changes in $\partial f/\partial S$ in time Δt in equation (15.13). A more rigorous derivation involves setting up a self-financing portfolio (i.e., a portfolio that requires no infusion or withdrawal of money).

know that the value of the forward contract, f, at a general time t is given in terms of the stock price S at this time by

$$f = S - Ke^{-r(T-t)}$$

where K is the delivery price. This means that

$$\frac{\partial f}{\partial t} = -rKe^{-r(T-t)}, \qquad \frac{\partial f}{\partial S} = 1, \qquad \frac{\partial^2 f}{\partial S^2} = 0$$

When these are substituted into the left-hand side of equation (15.16), we obtain

$$-rKe^{-r(T-t)} + rS$$

This equals rf, showing that equation (15.16) is indeed satisfied.

A Perpetual Derivative

Consider a perpetual derivative that pays off a fixed amount Q when the stock price equals H for the first time. In this case, the value of the derivative for a particular S has no dependence on t, so the $\partial f/\partial t$ term vanishes and the partial differential equation (15.16) becomes an ordinary differential equation.

Suppose first that $S < H$. The boundary conditions for the derivative are $f = 0$ when $S = 0$ and $f = Q$ when $S = H$. The simple solution $f = QS/H$ satisfies both the boundary conditions and the differential equation. It must therefore be the value of the derivative.

Suppose next that $S > H$. The boundary conditions are now $f = 0$ as S tends to infinity and $f = Q$ when $S = H$. The derivative price

$$f = Q\left(\frac{S}{H}\right)^{-\alpha}$$

where α is positive, satisfies the boundary conditions. It also satisfies the differential equation when

$$-r\alpha + \tfrac{1}{2}\sigma^2\alpha(\alpha + 1) - r = 0$$

or $\alpha = 2r/\sigma^2$. The value of the derivative is therefore

$$f = Q\left(\frac{S}{H}\right)^{-2r/\sigma^2} \tag{15.17}$$

Problem 15.23 shows how equation (15.17) can be used to price a perpetual American put option. Section 26.2 extends the analysis to show how perpetual American call and put options can be priced when the underlying asset provides a yield at rate q.

The Prices of Tradeable Derivatives

Any function $f(S, t)$ that is a solution of the differential equation (15.16) is the theoretical price of a derivative that could be traded. If a derivative with that price existed, it would not create any arbitrage opportunities. Conversely, if a function $f(S, t)$ does not satisfy the differential equation (15.16), it cannot be the price of a derivative without creating arbitrage opportunities for traders.

To illustrate this point, consider first the function e^S. This does not satisfy the differential equation (15.16). It is therefore not a candidate for being the price of a derivative dependent on the stock price. If an instrument whose price was always e^S existed, there would be an arbitrage opportunity. As a second example, consider the function

$$\frac{e^{(\sigma^2-2r)(T-t)}}{S}$$

This does satisfy the differential equation, and so is, in theory, the price of a tradeable security. (It is the price of a derivative that pays off $1/S_T$ at time T.) For other examples of tradeable derivatives, see Problems 15.11, 15.12, 15.23, and 15.29.

15.7 RISK-NEUTRAL VALUATION

We introduced risk-neutral valuation in connection with the binomial model in Chapter 13. It is without doubt the single most important tool for the analysis of derivatives. It arises from one key property of the Black–Scholes–Merton differential equation (15.16). This property is that the equation does not involve any variables that are affected by the risk preferences of investors. The variables that do appear in the equation are the current stock price, time, stock price volatility, and the risk-free rate of interest. All are independent of risk preferences.

The Black–Scholes–Merton differential equation would not be independent of risk preferences if it involved the expected return, μ, on the stock. This is because the value of μ does depend on risk preferences. The higher the level of risk aversion by investors, the higher μ will be for any given stock. It is fortunate that μ happens to drop out in the derivation of the differential equation.

Because the Black–Scholes–Merton differential equation is independent of risk preferences, an ingenious argument can be used. If risk preferences do not enter the equation, they cannot affect its solution. Any set of risk preferences can, therefore, be used when evaluating f. In particular, the very simple assumption that all investors are risk neutral can be made.

In a world where investors are risk neutral, the expected return on all investment assets is the risk-free rate of interest, r. The reason is that risk-neutral investors do not require a premium to induce them to take risks. It is also true that the present value of any cash flow in a risk-neutral world can be obtained by discounting its expected value at the risk-free rate. The assumption that the world is risk neutral does, therefore, considerably simplify the analysis of derivatives.

Consider a derivative that provides a payoff at one particular time. It can be valued using risk-neutral valuation by using the following procedure:

1. Assume that the expected return from the underlying asset is the risk-free interest rate, r (i.e., assume $\mu = r$).
2. Calculate the expected payoff from the derivative.
3. Discount the expected payoff at the risk-free interest rate.

It is important to appreciate that risk-neutral valuation (or the assumption that all investors are risk neutral) is merely an artificial device for obtaining solutions to the

Black–Scholes–Merton differential equation. The solutions that are obtained are valid in all worlds, not just those where investors are risk neutral. When we move from a risk-neutral world to a risk-averse world, two things happen. The expected payoff from the derivative changes and the discount rate that must be used for this payoff changes. It happens that these two changes always offset each other exactly.

Application to Forward Contracts on a Stock

We valued forward contracts on a non-dividend-paying stock in Section 5.7. In Example 15.5, we verified that the pricing formula satisfies the Black–Scholes–Merton differential equation. In this section we derive the pricing formula from risk-neutral valuation. We make the assumption that interest rates are constant and equal to r. This is somewhat more restrictive than the assumption in Chapter 5.

Consider a long forward contract that matures at time T with delivery price, K. As indicated in Figure 1.2, the value of the contract at maturity is

$$S_T - K$$

where S_T is the stock price at time T. From the risk-neutral valuation argument, the value of the forward contract at time 0 is its expected value at time T in a risk-neutral world discounted at the risk-free rate of interest. Denoting the value of the forward contract at time zero by f, this means that

$$f = e^{-rT}\hat{E}(S_T - K)$$

where $\hat{E}$ denotes the expected value in a risk-neutral world. Since K is a constant, this equation becomes

$$f = e^{-rT}\hat{E}(S_T) - Ke^{-rT} \tag{15.18}$$

The expected return μ on the stock becomes r in a risk-neutral world. Hence, from equation (15.4), we have

$$\hat{E}(S_T) = S_0 e^{rT} \tag{15.19}$$

Substituting equation (15.19) into equation (15.18) gives

$$f = S_0 - Ke^{-rT}$$

This is in agreement with equation (5.5).

15.8 BLACK–SCHOLES–MERTON PRICING FORMULAS

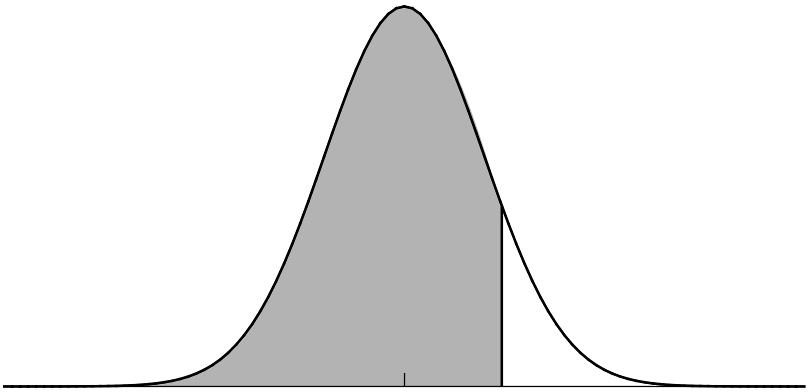
The most famous solutions to the differential equation (15.16) are the Black–Scholes–Merton formulas for the prices of European call and put options. These formulas are:

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2) \tag{15.20}$$

and

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1) \tag{15.21}$$

Figure 15.3 Shaded area represents $N(x)$.



where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

The function $N(x)$ is the cumulative probability distribution function for a variable with a standard normal distribution. In other words, it is the probability that a variable with a standard normal distribution will be less than x. It is illustrated in Figure 15.3. The remaining variables should be familiar. The variables c and p are the European call and European put price, S_0 is the stock price at time zero, K is the strike price, r is the continuously compounded risk-free rate, σ is the stock price volatility, and T is the time to maturity of the option.

One way of deriving the Black–Scholes–Merton formulas is by solving the differential equation (15.16) subject to the boundary condition mentioned in Section 15.6.[7] (See Problem 15.17 to prove that the call price in equation (15.20) satisfies the differential equation.) Another approach is to use risk-neutral valuation. Consider a European call option. The expected value of the option at maturity in a risk-neutral world is

$$\hat{E}[\max(S_T - K, 0)]$$

where, as before, $\hat{E}$ denotes the expected value in a risk-neutral world. From the risk-neutral valuation argument, the European call option price c is this expected value discounted at the risk-free rate of interest, that is,

$$c = e^{-rT}\hat{E}[\max(S_T - K, 0)] \tag{15.22}$$

[7] The differential equation gives the call and put prices at a general time t. For example, the call price that satisfies the differential equation is $c = SN(d_1) - Ke^{-r(T-t)}N(d_2)$, where

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

and $d_2 = d_1 - \sigma\sqrt{T-t}$.

The appendix at the end of this chapter shows that this equation leads to the result in equation (15.20).

Since it is never optimal to exercise early an American call option on a non-dividend-paying stock (see Section 11.5), equation (15.20) is the value of an American call option on a non-dividend-paying stock. Unfortunately, no exact analytic formula for the value of an American put option on a non-dividend-paying stock has been produced. Numerical procedures for calculating American put values are discussed in Chapter 21.

When the Black–Scholes–Merton formula is used in practice the interest rate r is set equal to the zero-coupon risk-free interest rate for a maturity T. As we show in later chapters, this is theoretically correct when r is a known function of time. It is also theoretically correct when the interest rate is stochastic provided that the stock price at time T is lognormal and the volatility parameter is chosen appropriately. As mentioned earlier, time is normally measured as the number of trading days left in the life of the option divided by the number of trading days in 1 year.

Understanding $N(d_1)$ and $N(d_2)$

The term $N(d_2)$ in equation (15.20) has a fairly simple interpretation. It is the probability that a call option will be exercised in a risk-neutral world. The $N(d_1)$ term is not quite so easy to interpret. The expression $S_0N(d_1)e^{rT}$ is the expected stock price at time T in a risk-neutral world when stock prices less than the strike price are counted as zero. The strike price is only paid if the stock price is greater than K and as just mentioned this has a probability of $N(d_2)$. The expected payoff in a risk-neutral world is therefore

$$S_0N(d_1)e^{rT} - KN(d_2)$$

Present-valuing this from time T to time zero gives the Black–Scholes–Merton equation for a European call option:

$$c = S_0N(d_1) - Ke^{-rT}N(d_2)$$

For another way of looking at the Black–Scholes–Merton equation for the value of a European call option, note that it can be written as

$$c = e^{-rT}N(d_2)[S_0e^{rT}N(d_1)/N(d_2) - K]$$

The terms here have the following interpretation:

$$\begin{aligned} e^{-rT}&: \text{ Present value factor} \\ N(d_2)&: \text{ Probability of exercise} \\ S_0e^{rT}N(d_1)/N(d_2)&: \text{ Expected stock price in a risk-neutral world if option is exercised} \\ K&: \text{ Strike price paid if option is exercised.} \end{aligned}$$

Properties of the Black–Scholes–Merton Formulas

We now show that the Black–Scholes–Merton formulas have the right general properties by considering what happens when some of the parameters take extreme values.

When the stock price, S_0, becomes very large, a call option is almost certain to be exercised. It then becomes very similar to a forward contract with delivery price K.

From equation (5.5), we expect the call price to be

$$S_0 - Ke^{-rT}$$

This is, in fact, the call price given by equation (15.20) because, when S_0 becomes very large, both d_1 and d_2 become very large, and $N(d_1)$ and $N(d_2)$ become close to 1.0. When the stock price becomes very large, the price of a European put option, p, approaches zero. This is consistent with equation (15.21) because $N(-d_1)$ and $N(-d_2)$ are both close to zero in this case.

Consider next what happens when the volatility σ approaches zero. Because the stock is virtually riskless, its price will grow at rate r to S_0e^{rT} at time T and the payoff from a call option is

$$\max(S_0e^{rT} - K,\ 0)$$

Discounting at rate r, the value of the call today is

$$e^{-rT}\max(S_0e^{rT} - K,\ 0) = \max(S_0 - Ke^{-rT},\ 0)$$

To show that this is consistent with equation (15.20), consider first the case where $S_0 > Ke^{-rT}$. This implies that $\ln(S_0/K) + rT > 0$. As σ tends to zero, d_1 and d_2 tend to $+\infty$, so that $N(d_1)$ and $N(d_2)$ tend to 1.0 and equation (15.20) becomes

$$c = S_0 - Ke^{-rT}$$

When $S_0 < Ke^{-rT}$, it follows that $\ln(S_0/K) + rT < 0$. As σ tends to zero, d_1 and d_2 tend to $-\infty$, so that $N(d_1)$ and $N(d_2)$ tend to zero and equation (15.20) gives a call price of zero. The call price is therefore always $\max(S_0 - Ke^{-rT},\ 0)$ as σ tends to zero. Similarly, it can be shown that the put price is always $\max(Ke^{-rT} - S_0,\ 0)$ as σ tends to zero.

15.9 CUMULATIVE NORMAL DISTRIBUTION FUNCTION

When implementing equations (15.20) and (15.21), it is necessary to evaluate the cumulative normal distribution function $N(x)$. Tables for $N(x)$ are provided at the end of this book. The NORMSDIST function in Excel also provides a convenient way of calculating $N(x)$.

Example 15.6

The stock price 6 months from the expiration of an option is \$42, the exercise price of the option is \$40, the risk-free interest rate is 10% per annum, and the volatility is 20% per annum. This means that $S_0 = 42$, $K = 40$, $r = 0.1$, $\sigma = 0.2$, $T = 0.5$,

$$d_1 = \frac{\ln(42/40) + (0.1 + 0.2^2/2) \times 0.5}{0.2\sqrt{0.5}} = 0.7693$$

$$d_2 = \frac{\ln(42/40) + (0.1 - 0.2^2/2) \times 0.5}{0.2\sqrt{0.5}} = 0.6278$$

and

$$Ke^{-rT} = 40e^{-0.05} = 38.049$$

Hence, if the option is a European call, its value c is given by

$$c = 42N(0.7693) - 38.049N(0.6278)$$

If the option is a European put, its value p is given by

$$p = 38.049N(-0.6278) - 42N(-0.7693)$$

Using the NORMSDIST function in Excel gives

$$N(0.7693) = 0.7791, \qquad N(-0.7693) = 0.2209$$
$$N(0.6278) = 0.7349, \qquad N(-0.6278) = 0.2651$$

so that

$$c = 4.76, \qquad p = 0.81$$

Ignoring the time value of money, the stock price has to rise by \$2.76 for the purchaser of the call to break even. Similarly, the stock price has to fall by \$2.81 for the purchaser of the put to break even.

15.10 WARRANTS AND EMPLOYEE STOCK OPTIONS

The exercise of a regular call option on a company has no effect on the number of the company's shares outstanding. If the writer of the option does not own the company's shares, he or she must buy them in the market in the usual way and then sell them to the option holder for the strike price. As explained in Chapter 10, warrants and employee stock options are different from regular call options in that exercise leads to the company issuing more shares and then selling them to the option holder for the strike price. As the strike price is less than the market price, this dilutes the interest of the existing shareholders.

How should potential dilution affect the way we value outstanding warrants and employee stock options? The answer is that it should not! Assuming markets are efficient the stock price will reflect potential dilution from all outstanding warrants and employee stock options. This is explained in Business Snapshot 15.3.[8]

Consider next the situation a company is in when it is contemplating a new issue of warrants (or employee stock options). We suppose that the company is interested in calculating the cost of the issue assuming that there are no compensating benefits. We assume that the company has N shares worth S_0 each and the number of new options contemplated is M, with each option giving the holder the right to buy one share for K. The value of the company today is NS_0. This value does not change as a result of the warrant issue. Suppose that without the warrant issue the share price will be S_T at the warrant's maturity. This means that (with or without the warrant issue) the total value of the equity and the warrants at time T will NS_T. If the warrants are exercised, there is a cash inflow from the strike price increasing this to $NS_T + MK$. This value is distributed

[8] Analysts sometimes assume that the sum of the values of the warrants and the equity (rather than just the value of the equity) is lognormal. The result is a Black–Scholes type of equation for the value of the warrant in terms of the value of the warrant. See Technical Note 3 at www-2.rotman.utoronto.ca/~hull/TechnicalNotes for an explanation of this model.

Business Snapshot 15.3 Warrants, Employee Stock Options, and Dilution

Consider a company with 100,000 shares each worth \$50. It surprises the market with an announcement that it is granting 100,000 stock options to its employees with a strike price of \$50. If the market sees little benefit to the shareholders from the employee stock options in the form of reduced salaries and more highly motivated managers, the stock price will decline immediately after the announcement of the employee stock options. If the stock price declines to \$45, the dilution cost to the current shareholders is \$5 per share or \$500,000 in total.

Suppose that the company does well so that by the end of three years the share price is \$100. Suppose further that all the options are exercised at this point. The payoff to the employees is \$50 per option. It is tempting to argue that there will be further dilution in that 100,000 shares worth \$100 per share are now merged with 100,000 shares for which only \$50 is paid, so that (a) the share price reduces to \$75 and (b) the payoff to the option holders is only \$25 per option. However, this argument is flawed. The exercise of the options is anticipated by the market and already reflected in the share price. The payoff from each option exercised is \$50.

This example illustrates the general point that when markets are efficient the impact of dilution from executive stock options or warrants is reflected in the stock price as soon as they are announced and does not need to be taken into account again when the options are valued.

among $N+M$ shares, so that the share price immediately after exercise becomes

$$\frac{NS_T + MK}{N+M}$$

Therefore the payoff to an option holder if the option is exercised is

$$\frac{NS_T + MK}{N+M} - K$$

or

$$\frac{N}{N+M}(S_T - K)$$

This shows that the value of each option is the value of

$$\frac{N}{N+M}$$

regular call options on the company's stock. Therefore the total cost of the options is M times this. Since we are assuming that there are no benefits to the company from the warrant issue, the total value of the company's equity will decline by the total cost of the options as soon as the decision to issue the warrants becomes generally known. This means that the reduction in the stock price is

$$\frac{M}{N+M}$$

times the value of a regular call option with strike price K and maturity T.

Example 15.7

A company with 1 million shares worth \$40 each is considering issuing 200,000 warrants each giving the holder the right to buy one share with a strike price of \$60 in 5 years. It wants to know the cost of this. The interest rate is 3% per annum, and the volatility is 30% per annum. The company pays no dividends. From equation (15.20), the value of a 5-year European call option on the stock is \$7.04. In this case, $N = 1{,}000{,}000$ and $M = 200{,}000$, so that the value of each warrant is

$$\frac{1{,}000{,}000}{1{,}000{,}000 + 200{,}000} \times 7.04 = 5.87$$

or \$5.87. The total cost of the warrant issue is $200{,}000 \times 5.87 = \$1.17$ million. Assuming the market perceives no benefits from the warrant issue, we expect the stock price to decline by \$1.17 to \$38.83.

15.11 IMPLIED VOLATILITIES

The one parameter in the Black–Scholes–Merton pricing formulas that cannot be directly observed is the volatility of the stock price. In Section 15.4, we discussed how this can be estimated from a history of the stock price. In practice, traders usually work with what are known as *implied volatilities*. These are the volatilities implied by option prices observed in the market.[9]

To illustrate how implied volatilities are calculated, suppose that the market price of a European call option on a non-dividend-paying stock is 1.875 when $S_0 = 21$, $K = 20$, $r = 0.1$, and $T = 0.25$. The implied volatility is the value of σ that, when substituted into equation (15.20), gives $c = 1.875$. Unfortunately, it is not possible to invert equation (15.20) so that σ is expressed as a function of S_0, K, r, T, and c. However, an iterative search procedure can be used to find the implied σ. For example, we can start by trying $\sigma = 0.20$. This gives a value of c equal to 1.76, which is too low. Because c is an increasing function of σ, a higher value of σ is required. We can next try a value of 0.30 for σ. This gives a value of c equal to 2.10, which is too high and means that σ must lie between 0.20 and 0.30. Next, a value of 0.25 can be tried for σ. This also proves to be too high, showing that σ lies between 0.20 and 0.25. Proceeding in this way, we can halve the range for σ at each iteration and the correct value of σ can be calculated to any required accuracy.[10] In this example, the implied volatility is 0.235, or 23.5%, per annum. A similar procedure can be used in conjunction with binomial trees to find implied volatilities for American options.

Implied volatilities are used to monitor the market's opinion about the volatility of a particular stock. Whereas historical volatilities (see Section 15.4) are backward looking, implied volatilities are forward looking. Traders often quote the implied volatility of an option rather than its price. This is convenient because the implied volatility tends to be less variable than the option price. The implied volatilities of actively traded options on an asset are often used by traders to estimate appropriate implied volatilities for other options on the asset.

[9] Implied volatilities for European and American options can be calculated using DerivaGem.

[10] This method is presented for illustration. Other more powerful methods, such as the Newton–Raphson method, are often used in practice (see footnote 3 of Chapter 4).

The VIX Index

The CBOE publishes indices of implied volatility. The most popular index, the SPX VIX, is an index of the implied volatility of 30-day options on the S&P 500 calculated from a wide range of calls and puts. It is sometimes referred to as the ‘‘fear factor.’’ An index value of 15 indicates that the implied volatility of 30-day options on the S&P 500 is estimated as 15%. Information on the way the index is calculated is in Section 26.15. Trading in futures on the VIX started in 2004 and trading in options on the VIX started in 2006. One contract is on 1,000 times the index.

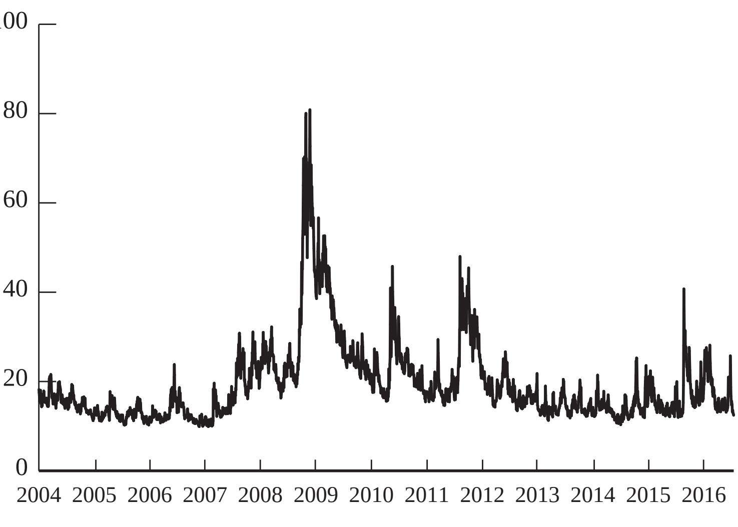
Example 15.8

> Suppose that a trader buys an April futures contract on the VIX when the futures price is 18.5 (corresponding to a 30-day S&P 500 volatility of 18.5%) and closes out the contract when the futures price is 19.3 (corresponding to an S&P 500 volatility of 19.3%). The trader makes a gain of $800.

A trade involving options on the S&P 500 is a bet on the future level of the S&P 500, which depends on the volatility of the S&P 500. By contrast, a futures or options contract on the VIX is a bet only on volatility. Figure 15.4 shows the VIX index between January 2004 and July 2016. Between 2004 and mid-2007 it tended to stay between 10 and 20. It reached 30 during the second half of 2007 and a record 80 in October and November 2008 after Lehman’s bankruptcy. By early 2010, it had declined to a more normal levels, but it spiked again in May 2010 and the second half of 2011 because of stresses and uncertainties in financial markets.

VIX monitors the volatility of the S&P 500. The CBOE publishes a range of other volatility indices. These are on other stock indices, commodity indices, interest rates, currencies, and some individual stocks (for example, Amazon and Goldman Sachs). There is even a volatility index of the VIX index (VVIX).

Figure 15.4 The VIX index, January 2004 to July 2016.



15.12 DIVIDENDS

Up to now, we have assumed that the stock on which the option is written pays no dividends. In this section, we modify the Black–Scholes–Merton model to take account of dividends. We assume that the amount and timing of the dividends during the life of an option can be predicted with certainty. When options last for relatively short periods of time, this assumption is not too unreasonable. (For long-life options it is usual to assume that the dividend yield rather the dollar dividend payments are known. Options can then be valued as will be described in the Chapter 17.) The date on which the dividend is paid should be assumed to be the ex-dividend date. On this date the stock price declines by the amount of the dividend.[11]

European Options

European options can be analyzed by assuming that the stock price is the sum of two components: a riskless component that corresponds to the known dividends during the life of the option and a risky component. The riskless component, at any given time, is the present value of all the dividends during the life of the option discounted from the ex-dividend dates to the present at the risk-free rate. By the time the option matures, the dividends will have been paid and the riskless component will no longer exist. The Black–Scholes–Merton formula is therefore correct if S_0 is equal to the risky component of the stock price and σ is the volatility of the process followed by the risky component.[12]

Operationally, this means that the Black–Scholes–Merton formulas can be used provided that the stock price is reduced by the present value of all the dividends during the life of the option, the discounting being done from the ex-dividend dates at the risk-free rate. As already mentioned, a dividend is counted as being during the life of the option only if its ex-dividend date occurs during the life of the option.

Example 15.9

Consider a European call option on a stock when there are ex-dividend dates in two months and five months. The dividend on each ex-dividend date is expected to be \$0.50. The current share price is \$40, the exercise price is \$40, the stock price volatility is 30% per annum, the risk-free rate of interest is 9% per annum, and the time to maturity is six months. The present value of the dividends is

$$0.5e^{-0.09\times 2/12} + 0.5e^{-0.09\times 5/12} = 0.9742$$

The option price can therefore be calculated from the Black–Scholes–Merton

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[11] For tax reasons the stock price may go down by somewhat less than the cash amount of the dividend. To take account of this phenomenon, we need to interpret the word 'dividend' in the context of option pricing as the reduction in the stock price on the ex-dividend date caused by the dividend. Thus, if a dividend of \$1 per share is anticipated and the share price normally goes down by 80% of the dividend on the ex-dividend date, the dividend should be assumed to be \$0.80 for the purpose of the analysis.

[12] This is not quite the same as the volatility of the whole stock price. (In theory, they cannot both follow geometric Brownian motion.) At time zero, the volatility of the risky component is approximately equal to the volatility of the whole stock price multiplied by $S_0/(S_0 - D)$, where D is the present value of the dividends.

formula, with $S_0 = 40 - 0.9742 = 39.0258$, $K = 40$, $r = 0.09$, $\sigma = 0.3$, and $T = 0.5$:

$$d_1 = \frac{\ln(39.0258/40) + (0.09 + 0.3^2/2) \times 0.5}{0.3\sqrt{0.5}} = 0.2020$$

$$d_2 = \frac{\ln(39.0258/40) + (0.09 - 0.3^2/2) \times 0.5}{0.3\sqrt{0.5}} = -0.0102$$

Using the NORMSDIST function in Excel gives

$$N(d_1) = 0.5800, \quad N(d_2) = 0.4959$$

and, from equation (15.20), the call price is

$$39.0258 \times 0.5800 - 40e^{-0.09\times 0.5} \times 0.4959 = 3.67$$

or \$3.67.

Some researchers have criticized the approach just described for calculating the value of a European option on a dividend-paying stock. They argue that volatility should be applied to the stock price, not to the stock price less the present value of dividends. A number of different numerical procedures have been suggested for doing this.[13] When volatility is calculated from historical data, it might make sense to use one of these procedures. However, in practice the volatility used to price an option is nearly always implied from the prices of other options using procedures we will outline in Chapter 20. If an analyst uses the same model for both implying and applying volatilities, the resulting prices should be accurate and not highly model dependent. Another important point is that in practice, as will be explained in Chapter 18, practitioners usually value a European option in terms of the forward price of the underlying asset. This avoids the need to estimate explicitly the income that is expected from the asset. The volatility of the forward stock price is the same as the volatility of a variable equal to the stock price minus the present value of dividends.

The model we have proposed where the stock price is divided into two components is internally consistent and widely used in practice. We will use the same model when valuing American options in Chapter 21.

American Call Options

Consider next American call options. Chapter 11 showed that in the absence of dividends American options should never be exercised early. An extension to the argument shows that, when there are dividends, it can only be optimal to exercise at a time immediately before the stock goes ex-dividend. We assume that n ex-dividend dates are anticipated and that they are at times $t_1, t_2, \dots, t_n$, with $t_1 < t_2 < \cdots < t_n$. The dividends corresponding to these times will be denoted by $D_1, D_2, \dots, D_n$, respectively.

We start by considering the possibility of early exercise just prior to the final ex-dividend date (i.e., at time t_n). If the option is exercised at time t_n, the investor receives

$$S(t_n) - K$$

[13] See, for example, N. Areal and A. Rodrigues, "Fast Trees for Options with Discrete Dividends," *Journal of Derivatives*, 21, 1 (Fall 2013), 49–63.

where $S(t)$ denotes the stock price at time t. If the option is not exercised, the stock price drops to $S(t_n) - D_n$. As shown by equation (11.4), the value of the option is then greater than

$$S(t_n) - D_n - Ke^{-r(T-t_n)}$$

It follows that, if

$$S(t_n) - D_n - Ke^{-r(T-t_n)} \geqslant S(t_n) - K$$

that is,

$$D_n \leqslant K\left[1 - e^{-r(T-t_n)}\right] \tag{15.23}$$

it cannot be optimal to exercise at time t_n. On the other hand, if

$$D_n > K\left[1 - e^{-r(T-t_n)}\right] \tag{15.24}$$

for any reasonable assumption about the stochastic process followed by the stock price, it can be shown that it is always optimal to exercise at time t_n for a sufficiently high value of $S(t_n)$. The inequality in (15.24) will tend to be satisfied when the final ex-dividend date is fairly close to the maturity of the option (i.e., $T - t_n$ is small) and the dividend is large.

Consider next time t_{n-1}, the penultimate ex-dividend date. If the option is exercised immediately prior to time t_{n-1}, the investor receives $S(t_{n-1}) - K$. If the option is not exercised at time t_{n-1}, the stock price drops to $S(t_{n-1}) - D_{n-1}$ and the earliest subsequent time at which exercise could take place is t_n. Hence, from equation (11.4), a lower bound to the option price if it is not exercised at time t_{n-1} is

$$S(t_{n-1}) - D_{n-1} - Ke^{-r(t_n - t_{n-1})}$$

It follows that if

$$S(t_{n-1}) - D_{n-1} - Ke^{-r(t_n - t_{n-1})} \geqslant S(t_{n-1}) - K$$

or

$$D_{n-1} \leqslant K\left[1 - e^{-r(t_n - t_{n-1})}\right]$$

it is not optimal to exercise immediately prior to time t_{n-1}. Similarly, for any $i < n$, if

$$D_i \leqslant K\left[1 - e^{-r(t_{i+1} - t_i)}\right] \tag{15.25}$$

it is not optimal to exercise immediately prior to time t_i.

The inequality in (15.25) is approximately equivalent to

$$D_i \leqslant Kr(t_{i+1} - t_i)$$

Assuming that K is fairly close to the current stock price, this inequality is satisfied when the dividend yield on the stock is less than the risk-free rate of interest. This is often the case.

We can conclude from this analysis that, in many circumstances, the most likely time for the early exercise of an American call is immediately before the final ex-dividend date, t_n. Furthermore, if inequality (15.25) holds for $i = 1, 2, \ldots, n-1$ and inequality (15.23) holds, we can be certain that early exercise is never optimal, and the American option can be treated as a European option.

Black's Approximation

Black suggests an approximate procedure for taking account of early exercise in call options.[14] This involves calculating, as described earlier in this section, the prices of European options that mature at times T and t_n, and then setting the American price equal to the greater of the two.[15] This is an approximation because it in effect assumes the option holder has to decide at time zero whether the option will be exercised at time T or t_n.

SUMMARY

We started this chapter by examining the properties of the process for stock prices introduced in Chapter 14. The process implies that the price of a stock at some future time, given its price today, is lognormal. It also implies that the continuously compounded return from the stock in a period of time is normally distributed. Our uncertainty about future stock prices increases as we look further ahead. The standard deviation of the logarithm of the stock price is proportional to the square root of how far ahead we are looking.

To estimate the volatility σ of a stock price empirically, the stock price is observed at fixed intervals of time (e.g., every day, every week, or every month). For each time period, the natural logarithm of the ratio of the stock price at the end of the time period to the stock price at the beginning of the time period is calculated. The volatility is estimated as the standard deviation of these numbers divided by the square root of the length of the time period in years. Usually, days when the exchanges are closed are ignored in measuring time for the purposes of volatility calculations.

The differential equation for the price of any derivative dependent on a stock can be obtained by creating a riskless portfolio of the derivative and the stock. Because the derivative's price and the stock price both depend on the same underlying source of uncertainty, this can always be done. The portfolio that is created remains riskless for only a very short period of time. However, the return on a riskless porfolio must always be the risk-free interest rate if there are to be no arbitrage opportunities.

The expected return on the stock does not enter into the Black–Scholes–Merton differential equation. This leads to an extremely useful result known as risk-neutral valuation. This result states that when valuing a derivative dependent on a stock price, we can assume that the world is risk neutral. This means that we can assume that the expected return from the stock is the risk-free interest rate, and then discount expected payoffs at the risk-free interest rate. The Black–Scholes–Merton equations for European call and put options can be derived by either solving their differential equation or by using risk-neutral valuation.

An implied volatility is the volatility that, when used in conjunction with the Black–Scholes–Merton option pricing formula, gives the market price of the option. Traders

[14] See F. Black, "Fact and Fantasy in the Use of Options," *Financial Analysts Journal*, 31 (July/August 1975): 36–41, 61–72.

[15] For an exact formula, suggested by Roll, Geske, and Whaley, for valuing American calls when there is only one ex-dividend date, see Technical Note 4 at www-2.rotman.utoronto.ca/~hull/TechnicalNotes. This involves the cumulative bivariate normal distribution function. A procedure for calculating this function is given in Technical Note 5 and a worksheet for calculating the cumulative bivariate normal distribution can be found on the author's website.

monitor implied volatilities. They often quote the implied volatility of an option rather than its price. They have developed procedures for using the volatilities implied by the prices of actively traded options to estimate volatilities for other options on the same asset.

The Black–Scholes–Merton results can be extended to cover European call and put options on dividend-paying stocks. The procedure is to use the Black–Scholes–Merton formula with the stock price reduced by the present value of the dividends anticipated during the life of the option, and the volatility equal to the volatility of the stock price net of the present value of these dividends.

In theory, it can be optimal to exercise American call options immediately before any ex-dividend date. In practice, it is often only necessary to consider the final ex-dividend date. Fischer Black has suggested an approximation. This involves setting the American call option price equal to the greater of two European call option prices. The first European call option expires at the same time as the American call option; the second expires immediately prior to the final ex-dividend date.

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Practice Questions (Answers in Solutions Manual)

15.1. What does the Black–Scholes–Merton stock option pricing model assume about the probability distribution of the stock price in one year? What does it assume about the probability distribution of the continuously compounded rate of return on the stock during the year?

15.2. The volatility of a stock price is 30% per annum. What is the standard deviation of the percentage price change in one trading day?

15.3. Explain the principle of risk-neutral valuation.

15.4. Calculate the price of a 3-month European put option on a non-dividend-paying stock with a strike price of $50 when the current stock price is $50, the risk-free interest rate is 10% per annum, and the volatility is 30% per annum.

15.5. What difference does it make to your calculations in Problem 15.4 if a dividend of $1.50 is expected in 2 months?

15.6. What is *implied volatility*? How can it be calculated?

15.7. A stock price is currently $40. Assume that the expected return from the stock is 15% and that its volatility is 25%. What is the probability distribution for the rate of return (with continuous compounding) earned over a 2-year period?

15.8. A stock price follows geometric Brownian motion with an expected return of 16% and a volatility of 35%. The current price is $38.
(a) What is the probability that a European call option on the stock with an exercise price of $40 and a maturity date in 6 months will be exercised?
(b) What is the probability that a European put option on the stock with the same exercise price and maturity will be exercised?

15.9. Using the notation in this chapter, prove that a 95% confidence interval for S_T is between $S_0e^{(\mu-\sigma^2/2)T-1.96\sigma\sqrt{T}}$ and $S_0e^{(\mu-\sigma^2/2)T+1.96\sigma\sqrt{T}}$.

15.10. A portfolio manager announces that the average of the returns realized in each year of the last 10 years is 20% per annum. In what respect is this statement misleading?

15.11. Assume that a non-dividend-paying stock has an expected return of μ and a volatility of σ. An innovative financial institution has just announced that it will trade a security that pays off a dollar amount equal to $\ln S_T$ at time T, where S_T denotes the value of the stock price at time T.
(a) Use risk-neutral valuation to calculate the price of the security at time t in terms of the stock price, S, at time t. The risk-free rate is r.
(b) Confirm that your price satisfies the differential equation (15.16).

15.12. Consider a derivative that pays off S_T^n at time T, where S_T is the stock price at that time. When the stock pays no dividends and its price follows geometric Brownian motion, it can be shown that its price at time t ($t \leqslant T$) has the form $h(t, T)S^n$, where S is the stock price at time t and h is a function only of t and T.
(a) By substituting into the Black–Scholes–Merton partial differential equation, derive an ordinary differential equation satisfied by $h(t, T)$.
(b) What is the boundary condition for the differential equation for $h(t, T)$?
(c) Show that $h(t, T) = e^{[0.5\sigma^2n(n-1)+r(n-1)](T-t)}$, where r is the risk-free interest rate and σ is the stock price volatility.

15.13. What is the price of a European call option on a non-dividend-paying stock when the stock price is \$52, the strike price is \$50, the risk-free interest rate is 12% per annum, the volatility is 30% per annum, and the time to maturity is 3 months?

15.14. What is the price of a European put option on a non-dividend-paying stock when the stock price is \$69, the strike price is \$70, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is 6 months?

15.15. Consider an American call option on a stock. The stock price is \$70, the time to maturity is 8 months, the risk-free rate of interest is 10% per annum, the exercise price is \$65, and the volatility is 32%. A dividend of \$1 is expected after 3 months and again after 6 months. Show that it can never be optimal to exercise the option on either of the two dividend dates. Use DerivaGem to calculate the price of the option.

15.16. A call option on a non-dividend-paying stock has a market price of $\$2\frac{1}{2}$. The stock price is \$15, the exercise price is \$13, the time to maturity is 3 months, and the risk-free interest rate is 5% per annum. What is the implied volatility?

15.17. With the notation used in this chapter:

(a) What is $N'(x)$?

(b) Show that $SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$, where S is the stock price at time t and

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

(c) Calculate $\partial d_1/\partial S$ and $\partial d_2/\partial S$.

(d) Show that when $c = SN(d_1) - Ke^{-r(T-t)}N(d_2)$, it follows that

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}$$

where c is the price of a call option on a non-dividend-paying stock.

(e) Show that $\partial c/\partial S = N(d_1)$.

(f) Show that c satisfies the Black–Scholes–Merton differential equation.

(g) Show that c satisfies the boundary condition for a European call option, i.e., that $c = \max(S - K, 0)$ as $t \to T$.

15.18. Show that the Black–Scholes–Merton formulas for call and put options satisfy put–call parity.

15.19. A stock price is currently \$50 and the risk-free interest rate is 5%. Use the DerivaGem software to translate the following table of European call options on the stock into a table of implied volatilities, assuming no dividends. Are the option prices consistent with the assumptions underlying Black–Scholes–Merton?

	Maturity (months)		
Strike price (\$)	*3*	*6*	*12*
45	7.0	8.3	10.5
50	3.7	5.2	7.5
55	1.6	2.9	5.1

15.20. Explain carefully why Black's approach to evaluating an American call option on a dividend-paying stock may give an approximate answer even when only one dividend is anticipated. Does the answer given by Black's approach understate or overstate the true option value? Explain your answer.

15.21. Consider an American call option on a stock. The stock price is \$50, the time to maturity is 15 months, the risk-free rate of interest is 8% per annum, the exercise price is \$55, and the volatility is 25%. Dividends of \$1.50 are expected in 4 months and 10 months. Show that it can never be optimal to exercise the option on either of the two dividend dates. Calculate the price of the option.

15.22. Show that the probability that a European call option will be exercised in a risk-neutral world is, with the notation introduced in this chapter, $N(d_2)$. What is an expression for the value of a derivative that pays off \$100 if the price of a stock at time T is greater than K?

15.23. Use the result in equation (15.17) to determine the value of a perpetual American put option on a non-dividend-paying stock with strike price K if it is exercised when the stock price equals H where $H < K$. Assume that the current stock price S is greater than H. What is the value of H that maximizes the option value? Deduce the value of a perpetual American put with strike price K.

15.24. A company has an issue of executive stock options outstanding. Should dilution be taken into account when the options are valued? Explain your answer.

15.25. A company's stock price is \$50 and 10 million shares are outstanding. The company is considering giving its employees 3 million at-the-money 5-year call options. Option exercises will be handled by issuing more shares. The stock price volatility is 25%, the 5-year risk-free rate is 5%, and the company does not pay dividends. Estimate the cost to the company of the employee stock option issue.

Further Questions

15.26. If the volatility of a stock is 18% per annum, estimate the standard deviation of the percentage price change in (a) 1 day, (b) 1 week, and (c) 1 month.

15.27. A stock price is currently \$50. Assume that the expected return from the stock is 18% and its volatility is 30%. What is the probability distribution for the stock price in 2 years? Calculate the mean and standard deviation of the distribution. Determine the 95% confidence interval.

15.28. Suppose that observations on a stock price (in dollars) at the end of each of 15 consecutive weeks are as follows:

30.2, 32.0, 31.1, 30.1, 30.2, 30.3, 30.6, 33.0, 32.9, 33.0, 33.5, 33.5, 33.7, 33.5, 33.2

Estimate the stock price volatility. What is the standard error of your estimate?

15.29. A financial institution plans to offer a security that pays off a dollar amount equal to S_T^2 at time T, where S_T is the price at time T of a stock that pays no dividends.
(a) Use risk-neutral valuation to calculate the price of the security at time t in terms of the stock price S at time t and other variables. (*Hint*: The expected value of S_T^2 can be calculated from the mean and variance of S_T given in Section 15.1.)
(b) Confirm that your price satisfies the differential equation (15.16).

15.30. Consider an option on a non-dividend-paying stock when the stock price is \$30, the exercise price is \$29, the risk-free interest rate is 5%, the volatility is 25% per annum, and the time to maturity is 4 months.
(a) What is the price of the option if it is a European call?
(b) What is the price of the option if it is an American call?

(c) What is the price of the option if it is a European put?
(d) Verify that put–call parity holds.

15.31. Assume that the stock in Problem 15.30 is due to go ex-dividend in $1\frac{1}{2}$ months. The expected dividend is 50 cents.
(a) What is the price of the option if it is a European call?
(b) What is the price of the option if it is a European put?
(c) If the option is an American call, are there any circumstances under which it will be exercised early?

15.32. Consider an American call option when the stock price is \$18, the exercise price is \$20, the time to maturity is 6 months, the volatility is 30% per annum, and the risk-free interest rate is 10% per annum. Two equal dividends are expected during the life of the option with ex-dividend dates at the end of 2 months and 5 months. Assume the dividends are 40 cents. Use Black's approximation and the DerivaGem software to value the option. How high can the dividends be without the American option being worth more than the corresponding European option?

15.33. The appendix derives the key result

$$E[\max(V - K,\ 0)] = E(V)N(d_1) - KN(d_2)$$

Show that

$$E[\max(K - V,\ 0)] = KN(-d_1) - E(V)N(-d_2)$$

and use this to derive the Black–Scholes–Merton formula for the price of a European put option on a non-dividend-paying stock.

APPENDIX

PROOF OF THE BLACK–SCHOLES–MERTON FORMULA USING RISK-NEUTRAL VALUATION

We will prove the Black–Scholes result by first proving another key result that will also be useful in future chapters.

Key Result

If V is lognormally distributed and the standard deviation of $\ln V$ is w, then

$$E[\max(V-K,\ 0)] = E(V)N(d_1) - KN(d_2) \tag{15A.1}$$

where

$$d_1 = \frac{\ln[E(V)/K] + w^2/2}{w}$$
$$d_2 = \frac{\ln[E(V)/K] - w^2/2}{w}$$

and E denotes the expected value. (See Problem 15.33 for a similar result for puts.)

Proof of Key Result

Define $g(V)$ as the probability density function of V. It follows that

$$E[\max(V-K, 0)] = \int_K^{\infty} (V-K)g(V)\,dV \tag{15A.2}$$

The variable $\ln V$ is normally distributed with standard deviation w. From the properties of the lognormal distribution, the mean of $\ln V$ is m, where[16]

$$m = \ln[E(V)] - w^2/2 \tag{15A.3}$$

Define a new variable

$$Q = \frac{\ln V - m}{w} \tag{15A.4}$$

This variable is normally distributed with a mean of zero and a standard deviation of 1.0. Denote the density function for Q by $h(Q)$ so that

$$h(Q) = \frac{1}{\sqrt{2\pi}} e^{-Q^2/2}$$

Using equation (15A.4) to convert the expression on the right-hand side of equation (15A.2) from an integral over V to an integral over Q, we get

$$E[\max(V-K,\ 0)] = \int_{(\ln K - m)/w}^{\infty} (e^{Qw+m} - K)\,h(Q)\,dQ$$

or

$$E[\max(V-K,\ 0)] = \int_{(\ln K - m)/w}^{\infty} e^{Qw+m} h(Q)dQ - K\int_{(\ln K - m)/w}^{\infty} h(Q)dQ \tag{15A.5}$$

[16] For a proof of this, see Technical Note 2 at www-2.rotman.utoronto.ca/~hull/TechnicalNotes.

Now

$$e^{Qw+m}h(Q) = \frac{1}{\sqrt{2\pi}}e^{(-Q^2+2Qw+2m)/2} = \frac{1}{\sqrt{2\pi}}e^{[-(Q-w)^2+2m+w^2]/2}$$

$$= \frac{e^{m+w^2/2}}{\sqrt{2\pi}}e^{[-(Q-w)^2]/2} = e^{m+w^2/2}h(Q-w)$$

This means that equation (15A.5) becomes

$$E[\max(V-K,\ 0)] = e^{m+w^2/2}\int_{(\ln K-m)/w}^{\infty} h(Q-w)dQ - K\int_{(\ln K-m)/w}^{\infty} h(Q)dQ \quad \textbf{(15A.6)}$$

If we define $N(x)$ as the probability that a variable with a mean of zero and a standard deviation of 1.0 is less than x, the first integral in equation (15A.6) is

$$1 - N[(\ln K - m)/w - w] = N[(-\ln K + m)/w + w]$$

Substituting for m from equation (15A.3) leads to

$$N\left(\frac{\ln[E(V)/K] + w^2/2}{w}\right) = N(d_1)$$

Similarly the second integral in equation (15A.6) is $N(d_2)$. Equation (15A.6), therefore, becomes

$$E[\max(V-K,\ 0)] = e^{m+w^2/2}N(d_1) - KN(d_2)$$

Substituting for m from equation (15A.3) gives the key result.

The Black–Scholes–Merton Result

We now consider a call option on a non-dividend-paying stock maturing at time T. The strike price is K, the risk-free rate is r, the current stock price is S_0, and the volatility is σ. As shown in equation (15.22), the call price c is given by

$$c = e^{-rT}\hat{E}[\max(S_T - K,\ 0)] \quad \textbf{(15A.7)}$$

where S_T is the stock price at time T and $\hat{E}$ denotes the expectation in a risk-neutral world. Under the stochastic process assumed by Black–Scholes–Merton, S_T is lognormal. Also, from equations (15.3) and (15.4), $\hat{E}(S_T) = S_0e^{rT}$ and the standard deviation of $\ln S_T$ is $\sigma\sqrt{T}$.

From the key result just proved, equation (15A.7) implies

$$c = e^{-rT}[S_0e^{rT}N(d_1) - KN(d_2)] = S_0N(d_1) - Ke^{-rT}N(d_2)$$

where

$$d_1 = \frac{\ln[\hat{E}(S_T)/K] + \sigma^2T/2}{\sigma\sqrt{T}} = \frac{\ln(S_0/K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln[\hat{E}(S_T)/K] - \sigma^2T/2}{\sigma\sqrt{T}} = \frac{\ln(S_0/K) + (r-\sigma^2/2)T}{\sigma\sqrt{T}}$$

This is the Black–Scholes–Merton result.