CHAPTER 15

The Black–Scholes–Merton Model

In the early 1970s, Fischer Black, Myron Scholes, and Robert Merton achieved a major breakthrough in the pricing of European stock options.[1] This was the development of what has become known as the Black–Scholes–Merton (or Black–Scholes) model. The model has had a huge influence on the way that traders price and hedge derivatives. In 1997, the importance of the model was recognized when Robert Merton and Myron Scholes were awarded the Nobel prize for economics. Sadly, Fischer Black died in 1995; otherwise he too would undoubtedly have been one of the recipients of this prize.

How did Black, Scholes, and Merton make their breakthrough? Previous researchers had made similar assumptions and had correctly calculated the expected payoff from a European option. However, as explained in Section 13.2, it is difficult to know the correct discount rate to use for this payoff. Black and Scholes used the capital asset pricing model (see the appendix to Chapter 3) to determine a relationship between the market's required return on the option and the required return on the stock. This was not easy because the relationship depends on both the stock price and time. Merton's approach was different from that of Black and Scholes. It involved setting up a riskless portfolio consisting of the option and the underlying stock and arguing that the return on the portfolio over a short period of time must be the risk-free return. This is similar to what we did in Section 13.1—but more complicated because the portfolio changes continuously through time. Merton's approach was more general than that of Black and Scholes because it did not rely on the assumptions of the capital asset pricing model.

This chapter covers Merton's approach to deriving the Black–Scholes–Merton model. It explains how volatility can be either estimated from historical data or implied from option prices using the model. It shows how the risk-neutral valuation argument introduced in Chapter 13 can be used. It also shows how the Black–Scholes–Merton model can be extended to deal with European call and put options on dividend-paying stocks and presents some results on the pricing of American call options on dividend-paying stocks.

[1] See F. Black and M. Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81 (May/June 1973): 637–59; R.C. Merton, "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*, 4 (Spring 1973): 141–83.

15.1 LOGNORMAL PROPERTY OF STOCK PRICES

The model of stock price behavior used by Black, Scholes, and Merton is the model we developed in Chapter 14. It assumes that percentage changes in the stock price in a very short period of time are normally distributed. Define

μ: Expected return in a short period of time (annualized)
σ: Volatility of the stock price.

The mean and standard deviation of the return in time Δt are approximately $\mu\,\Delta t$ and $\sigma\sqrt{\Delta t}$, so that

$$\frac{\Delta S}{S} \sim \phi(\mu\,\Delta t,\ \sigma^2 \Delta t) \tag{15.1}$$

where ΔS is the change in the stock price S in time Δt, and $\phi(m, v)$ denotes a normal distribution with mean m and variance v. (This is equation (14.9).)

As shown in Section 14.7, the model implies that

$$\ln S_T - \ln S_0 \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right]$$

so that

$$\ln \frac{S_T}{S_0} \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right] \tag{15.2}$$

and

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right] \tag{15.3}$$

where S_T is the stock price at a future time T and S_0 is the stock price at time 0. There is no approximation here. The variable $\ln S_T$ is normally distributed, so that S_T has a lognormal distribution. The mean of $\ln S_T$ is $\ln S_0 + (\mu - \sigma^2/2)T$ and the standard deviation of $\ln S_T$ is $\sigma\sqrt{T}$.

Example 15.1

Consider a stock with an initial price of \$40, an expected return of 16% per annum, and a volatility of 20% per annum. From equation (15.3), the probability distribution of the stock price S_T in 6 months' time is given by

$$\ln S_T \sim \phi[\ln 40 + (0.16 - 0.2^2/2) \times 0.5,\ 0.2^2 \times 0.5]$$
$$\ln S_T \sim \phi(3.759,\ 0.02)$$

There is a 95% probability that a normally distributed variable has a value within 1.96 standard deviations of its mean. In this case, the standard deviation is $\sqrt{0.02} = 0.141$. Hence, with 95% confidence,

$$3.759 - 1.96 \times 0.141 < \ln S_T < 3.759 + 1.96 \times 0.141$$

This can be written

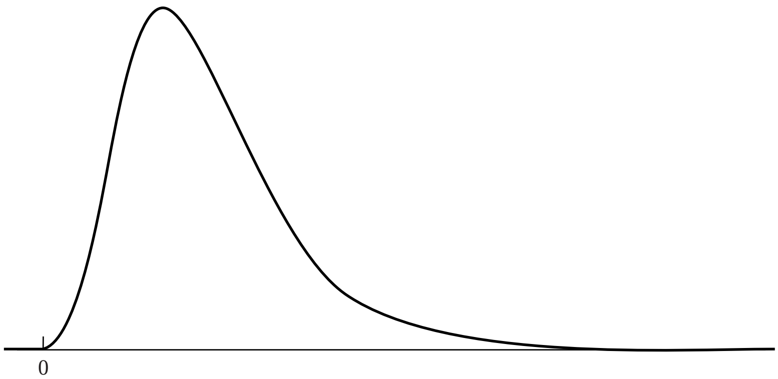
$$e^{3.759-1.96\times0.141} < S_T < e^{3.759+1.96\times0.141}$$

or

$$32.55 < S_T < 56.56$$

Thus, there is a 95% probability that the stock price in 6 months will lie between 32.55 and 56.56.

Figure 15.1 Lognormal distribution.



A variable that has a lognormal distribution can take any value between zero and infinity. Figure 15.1 illustrates the shape of a lognormal distribution. Unlike the normal distribution, it is skewed so that the mean, median, and mode are all different. From equation (15.3) and the properties of the lognormal distribution, it can be shown that the expected value $E(S_T)$ of S_T is given by

$$E(S_T) = S_0 e^{\mu T} \tag{15.4}$$

The variance $\text{var}(S_T)$ of S_T, can be shown to be given by[2]

$$\text{var}(S_T) = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1) \tag{15.5}$$

Example 15.2

Consider a stock where the current price is \$20, the expected return is 20% per annum, and the volatility is 40% per annum. The expected stock price, $E(S_T)$, and the variance of the stock price, $\text{var}(S_T)$, in 1 year are given by

$$E(S_T) = 20e^{0.2\times 1} = 24.43 \quad \text{and} \quad \text{var}(S_T) = 400e^{2\times 0.2\times 1}(e^{0.4^2\times 1} - 1) = 103.54$$

The standard deviation of the stock price in 1 year is $\sqrt{103.54}$, or 10.18.

15.2 THE DISTRIBUTION OF THE RATE OF RETURN

The lognormal property of stock prices can be used to provide information on the probability distribution of the continuously compounded rate of return earned on a stock between times 0 and T. If we define the continuously compounded rate of return per annum realized between times 0 and T as x, then

$$S_T = S_0 e^{xT}$$

[2] See Technical Note 2 at www-2.rotman.utoronto.ca/~hull/TechnicalNotes for a proof of the results in equations (15.4) and (15.5). For a more extensive discussion of the properties of the lognormal distribution, see J. Aitchison and J. A. C. Brown, *The Lognormal Distribution*. Cambridge University Press, 1966.

so that

$$x = \frac{1}{T} \ln \frac{S_T}{S_0} \tag{15.6}$$

From equation (15.2), it follows that

$$x \sim \phi\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T}\right) \tag{15.7}$$

Thus, the continuously compounded rate of return per annum is normally distributed with mean $\mu - \sigma^2/2$ and standard deviation $\sigma/\sqrt{T}$. As T increases, the standard deviation of x declines. To understand the reason for this, consider two cases: $T = 1$ and $T = 20$. We are more certain about the average return per year over 20 years than we are about the return in any one year.

Example 15.3

Consider a stock with an expected return of 17% per annum and a volatility of 20% per annum. The probability distribution for the average rate of return (continuously compounded) realized over 3 years is normal, with mean

$$0.17 - \frac{0.2^2}{2} = 0.15$$

or 15% per annum, and standard deviation

$$\sqrt{\frac{0.2^2}{3}} = 0.1155$$

or 11.55% per annum. Because there is a 95% chance that a normally distributed variable will lie within 1.96 standard deviations of its mean, we can be 95% confident that the average continuously compounded return realized over 3 years will be between $15 - 1.96 \times 11.55 = -7.6\%$ and $15 + 1.96 \times 11.55 = +37.6\%$ per annum.

15.3 THE EXPECTED RETURN

The expected return, μ, required by investors from a stock depends on the riskiness of the stock. The higher the risk, the higher the expected return. It also depends on the level of interest rates in the economy. The higher the level of interest rates, the higher the expected return required on any given stock. Fortunately, we do not have to concern ourselves with the determinants of μ in any detail. It turns out that the value of a stock option, when expressed in terms of the value of the underlying stock, does not depend on μ at all. Nevertheless, there is one aspect of the expected return from a stock that frequently causes confusion and needs to be explained.

Our model of stock price behavior implies that, in a very short period of time Δt, the mean return is $\mu \, \Delta t$. It is natural to assume from this that μ is the expected continuously compounded return on the stock. However, this is not the case. The continuously compounded return, x, actually realized over a period of time of length T

is given by equation (15.6) as

$$x = \frac{1}{T}\ln\frac{S_T}{S_0}$$

and, as indicated in equation (15.7), the expected value $E(x)$ of x is $\mu - \sigma^2/2$.

The reason why the expected continuously compounded return is different from μ is subtle, but important. Suppose we consider a very large number of very short periods of time of length Δt. Define S_i as the stock price at the end of the ith interval and ΔS_i as $S_{i+1} - S_i$. Under the assumptions we are making for stock price behavior, the arithmetic average of the returns on the stock in each interval is close to μ. In other words, $\mu\,\Delta t$ is close to the arithmetic mean of the $\Delta S_i/S_i$. However, the expected return over the whole period covered by the data, expressed with a compounding interval of Δt, is a geometric average and is close to $\mu - \sigma^2/2$, not μ.[3] Business Snapshot 15.1 provides a numerical example concerning the mutual fund industry to illustrate why this is so.

For another explanation of what is going on, we start with equation (15.4):

$$E(S_T) = S_0 e^{\mu T}$$

Taking logarithms, we get

$$\ln[E(S_T)] = \ln(S_0) + \mu T$$

It is now tempting to set $\ln[E(S_T)] = E[\ln(S_T)]$, so that $E[\ln(S_T)] - \ln(S_0) = \mu T$, or $E[\ln(S_T/S_0)] = \mu T$, which leads to $E(x) = \mu$. However, we cannot do this because ln is a nonlinear function. In fact, $\ln[E(S_T)] > E[\ln(S_T)]$, so that $E[\ln(S_T/S_0)] < \mu T$, which leads to $E(x) < \mu$. (As shown above, $E(x) = \mu - \sigma^2/2$.)

15.4 VOLATILITY

The volatility, σ, of a stock is a measure of our uncertainty about the returns provided by the stock. Stocks typically have a volatility between 15% and 60%.

From equation (15.7), the volatility of a stock price can be defined as the standard deviation of the return provided by the stock in 1 year when the return is expressed using continuous compounding.

When Δt is small, equation (15.1) shows that $\sigma^2\Delta t$ is approximately equal to the variance of the percentage change in the stock price in time Δt. This means that $\sigma\sqrt{\Delta t}$ is approximately equal to the standard deviation of the percentage change in the stock price in time Δt. Suppose that $\sigma = 0.3$, or 30%, per annum and the current stock price is \$50. The standard deviation of the percentage change in the stock price in 1 week is approximately

$$30 \times \sqrt{\frac{1}{52}} = 4.16\%$$

A 1-standard-deviation move in the stock price in 1 week is therefore $50 \times 0.0416 = 2.08$.

Uncertainty about a future stock price, as measured by its standard deviation, increases—at least approximately—with the square root of how far ahead we are looking. For example, the standard deviation of the stock price in 4 weeks is approximately twice the standard deviation in 1 week.

[3] The arguments in this section show that the term "expected return" is ambiguous. It can refer either to μ or to $\mu - \sigma^2/2$. Unless otherwise stated, it will be used to refer to μ throughout this book.

Business Snapshot 15.1 Mutual Fund Returns Can Be Misleading

The difference between μ and $\mu - \sigma^2/2$ is closely related to an issue in the reporting of mutual fund returns. Suppose that the following is a sequence of returns per annum reported by a mutual fund manager over the last five years (measured using annual compounding): 15%, 20%, 30%, −20%, 25%.

The arithmetic mean of the returns, calculated by taking the sum of the returns and dividing by 5, is 14%. However, an investor would actually earn less than 14% per annum by leaving the money invested in the fund for 5 years. The dollar value of \$100 at the end of the 5 years would be

$$100 \times 1.15 \times 1.20 \times 1.30 \times 0.80 \times 1.25 = \$179.40$$

By contrast, a 14% return with annual compounding would give

$$100 \times 1.14^5 = \$192.54$$

The return that gives \$179.40 at the end of five years is 12.4%. This is because

$$100 \times (1.124)^5 = 179.40$$

What average return should the fund manager report? It is tempting for the manager to make a statement such as: "The average of the returns per year that we have realized in the last 5 years is 14%." Although true, this is misleading. It is much less misleading to say: "The average return realized by someone who invested with us for the last 5 years is 12.4% per year." In some jurisdictions, regulations require fund managers to report returns the second way.

This phenomenon is an example of a result that is well known in mathematics. The geometric mean of a set of numbers is always less than the arithmetic mean. In our example, the return multipliers each year are 1.15, 1.20, 1.30, 0.80, and 1.25. The arithmetic mean of these numbers is 1.140, but the geometric mean is only 1.124 and it is the geometric mean that equals 1 plus the return realized over the 5 years.

Estimating Volatility from Historical Data

To estimate the volatility of a stock price empirically, the stock price is usually observed at fixed intervals of time (e.g., every day, week, or month). Define:

$n+1$: Number of observations
S_i: Stock price at end of ith interval, with $i = 0, 1, \dots, n$
τ: Length of time interval in years

and let

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right) \quad \text{for } i = 1, 2, \dots, n$$

The usual estimate, s, of the standard deviation of the u_i is given by

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (u_i - \bar{u})^2}$$

or

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n} u_i^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n} u_i\right)^2}$$

where $\bar{u}$ is the mean of the u_i.[4]

From equation (15.2), the standard deviation of the u_i is $\sigma\sqrt{\tau}$. The variable s is therefore an estimate of $\sigma\sqrt{\tau}$. It follows that σ itself can be estimated as $\hat{\sigma}$, where

$$\hat{\sigma} = \frac{s}{\sqrt{\tau}}$$

The standard error of this estimate can be shown to be approximately $\hat{\sigma}/\sqrt{2n}$.

Choosing an appropriate value for n is not easy. More data generally lead to more accuracy, but σ does change over time and data that are too old may not be relevant for predicting the future volatility. A compromise that seems to work reasonably well is to use closing prices from daily data over the most recent 90 to 180 days. Alternatively, as a rule of thumb, n can be set equal to the number of days to which the volatility is to be applied. Thus, if the volatility estimate is to be used to value a 2-year option, daily data for the last 2 years are used. More sophisticated approaches to estimating volatility involving GARCH models are discussed in Chapter 23.

Example 15.4

Table 15.1 shows a possible sequence of stock prices during 21 consecutive trading days. In this case, $n = 20$, so that

$$\sum_{i=1}^{n} u_i = 0.09531 \quad \text{and} \quad \sum_{i=1}^{n} u_i^2 = 0.00326$$

and the estimate of the standard deviation of the daily return is

$$\sqrt{\frac{0.00326}{19} - \frac{0.09531^2}{20 \times 19}} = 0.01216$$

or 1.216%. Assuming that there are 252 trading days per year, $\tau = 1/252$ and the data give an estimate for the volatility per annum of $0.01216\sqrt{252} = 0.193$, or 19.3%. The standard error of this estimate is

$$\frac{0.193}{\sqrt{2 \times 20}} = 0.031$$

or 3.1% per annum.

The foregoing analysis assumes that the stock pays no dividends, but it can be adapted to accommodate dividend-paying stocks. The return, u_i, during a time interval that includes an ex-dividend day is given by

$$u_i = \ln\frac{S_i + D}{S_{i-1}}$$

where D is the amount of the dividend. The return in other time intervals is still

$$u_i = \ln\frac{S_i}{S_{i-1}}$$

[4] The mean $\bar{u}$ is often assumed to be zero when estimates of historical volatilities are made.

Table 15.1 Computation of volatility.

Day i	*Closing stock price (dollars),* S_i	*Price relative* S_i/S_{i-1}	*Daily return* $u_i = \ln(S_i/S_{i-1})$
0	20.00		
1	20.10	1.00500	0.00499
2	19.90	0.99005	−0.01000
3	20.00	1.00503	0.00501
4	20.50	1.02500	0.02469
5	20.25	0.98780	−0.01227
6	20.90	1.03210	0.03159
7	20.90	1.00000	0.00000
8	20.90	1.00000	0.00000
9	20.75	0.99282	−0.00720
10	20.75	1.00000	0.00000
11	21.00	1.01205	0.01198
12	21.10	1.00476	0.00475
13	20.90	0.99052	−0.00952
14	20.90	1.00000	0.00000
15	21.25	1.01675	0.01661
16	21.40	1.00706	0.00703
17	21.40	1.00000	0.00000
18	21.25	0.99299	−0.00703
19	21.75	1.02353	0.02326
20	22.00	1.01149	0.01143

However, as tax factors play a part in determining returns around an ex-dividend date, it is probably best to discard altogether data for intervals that include an ex-dividend date.

Trading Days vs. Calendar Days

An important issue is whether time should be measured in calendar days or trading days when volatility parameters are being estimated and used. As shown in Business Snapshot 15.2, research shows that volatility is much higher when the exchange is open for trading than when it is closed. As a result, practitioners tend to ignore days when the exchange is closed when estimating volatility from historical data and when calculating the life of an option. The volatility per annum is calculated from the volatility per trading day using the formula

$$\begin{matrix}\text{Volatility}\\ \text{per annum}\end{matrix} = \begin{matrix}\text{Volatility}\\ \text{per trading day}\end{matrix} \times \sqrt{\begin{matrix}\text{Number of trading days}\\ \text{per annum}\end{matrix}}$$

This is what we did in Example 15.4 when calculating volatility from the data in Table 15.1. The number of trading days in a year is usually assumed to be 252 for stocks.

Business Snapshot 15.2 What Causes Volatility?

It is natural to assume that the volatility of a stock is caused by new information reaching the market. This new information causes people to revise their opinions about the value of the stock. The price of the stock changes and volatility results. This view of what causes volatility is not supported by research. With several years of daily stock price data, researchers can calculate:

1. The variance of stock price returns between the close of trading on one day and the close of trading on the next day when there are no intervening nontrading days
2. The variance of the stock price returns between the close of trading on Friday and the close of trading on Monday

The second of these is the variance of returns over a 3-day period. The first is a variance over a 1-day period. We might reasonably expect the second variance to be three times as great as the first variance. Fama (1965), French (1980), and French and Roll (1986) show that this is not the case. These three research studies estimate the second variance to be, respectively, 22%, 19%, and 10.7% higher than the first variance.

At this stage one might be tempted to argue that these results are explained by more news reaching the market when the market is open for trading. But research by Roll (1984) does not support this explanation. Roll looked at the prices of orange juice futures. By far the most important news for orange juice futures prices is news about the weather and this is equally likely to arrive at any time. When Roll did a similar analysis to that just described for stocks, he found that the second (Friday-to-Monday) variance for orange juice futures is only 1.54 times the first variance.

The only reasonable conclusion from all this is that volatility is to a large extent caused by trading itself. (Traders usually have no difficulty accepting this conclusion!)

The life of an option is also usually measured using trading days rather than calendar days. It is calculated as T years, where

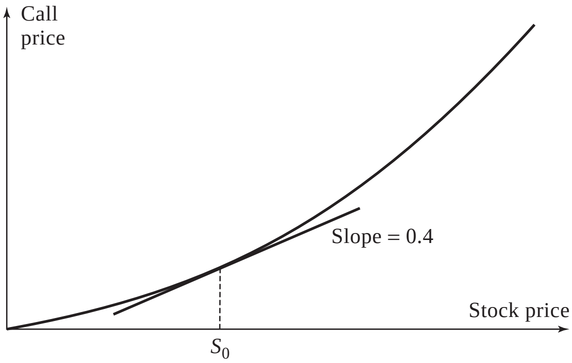
$$T = \frac{\text{Number of trading days until option maturity}}{252}$$

15.5 THE IDEA UNDERLYING THE BLACK–SCHOLES–MERTON DIFFERENTIAL EQUATION

The Black–Scholes–Merton differential equation is an equation that must be satisfied by the price of any derivative dependent on a non-dividend-paying stock. The equation is derived in the next section. Here we consider the nature of the arguments we will use.

These are similar to the no-arbitrage arguments we used to value stock options in Chapter 13 for the situation where stock price movements were assumed to be binomial. They involve setting up a riskless portfolio consisting of a position in the derivative and a position in the stock. In the absence of arbitrage opportunities, the return from the portfolio must be the risk-free interest rate, r. This leads to the Black-Scholes-Merton differential equation.

Figure 15.2 Relationship between call price and stock price. Current stock price is S_0.



The reason a riskless portfolio can be set up is that the stock price and the derivative price are both affected by the same underlying source of uncertainty: stock price movements. In any short period of time, the price of the derivative is perfectly correlated with the price of the underlying stock. When an appropriate portfolio of the stock and the derivative is established, the gain or loss from the stock position always offsets the gain or loss from the derivative position so that the overall value of the portfolio at the end of the short period of time is known with certainty.

Suppose, for example, that at a particular point in time the relationship between a small change ΔS in the stock price and the resultant small change Δc in the price of a European call option is given by

$$\Delta c = 0.4\,\Delta S$$

This means that the slope of the line representing the relationship between c and S is 0.4, as indicated in Figure 15.2. A riskless portfolio would consist of:

1. A long position in 40 shares
2. A short position in 100 call options.

Suppose, for example, that the stock price increases by 10 cents. The option price will increase by 4 cents and the $40 \times 0.1 = \$4$ gain on the shares is equal to the $100 \times 0.04 =$ \$4 loss on the short option position.

There is one important difference between the Black–Scholes–Merton analysis and our analysis using a binomial model in Chapter 13. In Black–Scholes–Merton, the position in the stock and the derivative is riskless for only a very short period of time. (Theoretically, it remains riskless only for an instantaneously short period of time.) To remain riskless, it must be adjusted, or *rebalanced*, frequently.[5] For example, the relationship between Δc and ΔS in our example might change from $\Delta c = 0.4\,\Delta S$ today to $\Delta c = 0.5\,\Delta S$ tomorrow. This would mean that, in order to maintain the riskless position, an extra 10 shares would have to be purchased for each 100 call options sold. It is nevertheless true that the return from the riskless portfolio in any very short period of time must be the risk-free interest rate. This is the key element in the Black–Scholes–Merton analysis and leads to their pricing formulas.

[5] We discuss the rebalancing of portfolios in more detail in Chapter 19.

Assumptions

The assumptions we use to derive the Black–Scholes–Merton differential equation are as follows:

1. The stock price follows the process developed in Chapter 14 with μ and σ constant.
2. The short selling of securities with full use of proceeds is permitted.
3. There are no transaction costs or taxes. All securities are perfectly divisible.
4. There are no dividends during the life of the derivative.
5. There are no riskless arbitrage opportunities.
6. Security trading is continuous.
7. The risk-free rate of interest, r, is constant and the same for all maturities.

As we discuss in later chapters, some of these assumptions can be relaxed. For example, σ and r can be known functions of t. We can even allow interest rates to be stochastic provided that the stock price distribution at maturity of the option is still lognormal.

15.6 DERIVATION OF THE BLACK–SCHOLES–MERTON DIFFERENTIAL EQUATION

In this section, the notation is different from elsewhere in the book. We consider a derivative's price at a general time t (not at time zero). If T is the maturity date, the time to maturity is $T - t$.

The stock price process we are assuming is the one we developed in Section 14.3:

$$dS = \mu S\,dt + \sigma S\,dz \tag{15.8}$$

Suppose that f is the price of a call option or other derivative contingent on S. The variable f must be some function of S and t. Hence, from equation (14.14),

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \tfrac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial f}{\partial S}\sigma S\,dz \tag{15.9}$$

The discrete versions of equations (15.8) and (15.9) are

$$\Delta S = \mu S\,\Delta t + \sigma S\,\Delta z \tag{15.10}$$

and

$$\Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \tfrac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right) \Delta t + \frac{\partial f}{\partial S}\sigma S\,\Delta z \tag{15.11}$$

where Δf and ΔS are the changes in f and S in a small time interval Δt. Recall from the discussion of Itô's lemma in Section 14.6 that the Wiener processes underlying f and S are the same. In other words, the Δz ($= \epsilon\sqrt{\Delta t}$) in equations (15.10) and (15.11) are the same. It follows that a portfolio of the stock and the derivative can be constructed so that the Wiener process is eliminated. The portfolio is

-1: derivative

$+\partial f/\partial S$: shares.

The holder of this portfolio is short one derivative and long an amount $\partial f/\partial S$ of shares. Define Π as the value of the portfolio. By definition

$$\Pi = -f + \frac{\partial f}{\partial S}S \tag{15.12}$$

The change $\Delta\Pi$ in the value of the portfolio in the time interval Δt is given by[6]

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S \tag{15.13}$$

Substituting equations (15.10) and (15.11) into equation (15.13) yields

$$\Delta\Pi = \left(-\frac{\partial f}{\partial t} - \tfrac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t \tag{15.14}$$

Because this equation does not involve Δz, the portfolio must be riskless during time Δt. The assumptions listed in the preceding section imply that the portfolio must instantaneously earn the same rate of return as other short-term risk-free securities. If it earned more than this return, arbitrageurs could make a riskless profit by borrowing money to buy the portfolio; if it earned less, they could make a riskless profit by shorting the portfolio and buying risk-free securities. It follows that

$$\Delta\Pi = r\Pi\,\Delta t \tag{15.15}$$

where r is the risk-free interest rate. Substituting from equations (15.12) and (15.14) into (15.15), we obtain

$$\left(\frac{\partial f}{\partial t} + \tfrac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t = r\left(f - \frac{\partial f}{\partial S}S\right)\Delta t$$

so that

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf \tag{15.16}$$

Equation (15.16) is the Black–Scholes–Merton differential equation. It has many solutions, corresponding to all the different derivatives that can be defined with S as the underlying variable. The particular derivative that is obtained when the equation is solved depends on the *boundary conditions* that are used. These specify the values of the derivative at the boundaries of possible values of S and t. In the case of a European call option, the key boundary condition is

$$f = \max(S - K,\ 0) \quad \text{when } t = T$$

In the case of a European put option, it is

$$f = \max(K - S,\ 0) \quad \text{when } t = T$$

Example 15.5

A forward contract on a non-dividend-paying stock is a derivative dependent on the stock. As such, it should satisfy equation (15.16). From equation (5.5), we

[6] This derivation of equation (15.16) is not completely rigorous. We need to justify ignoring changes in $\partial f/\partial S$ in time Δt in equation (15.13). A more rigorous derivation involves setting up a self-financing portfolio (i.e., a portfolio that requires no infusion or withdrawal of money).

know that the value of the forward contract, f, at a general time t is given in terms of the stock price S at this time by

$$f = S - Ke^{-r(T-t)}$$

where K is the delivery price. This means that

$$\frac{\partial f}{\partial t} = -rKe^{-r(T-t)}, \qquad \frac{\partial f}{\partial S} = 1, \qquad \frac{\partial^2 f}{\partial S^2} = 0$$

When these are substituted into the left-hand side of equation (15.16), we obtain

$$-rKe^{-r(T-t)} + rS$$

This equals rf, showing that equation (15.16) is indeed satisfied.

A Perpetual Derivative

Consider a perpetual derivative that pays off a fixed amount Q when the stock price equals H for the first time. In this case, the value of the derivative for a particular S has no dependence on t, so the $\partial f/\partial t$ term vanishes and the partial differential equation (15.16) becomes an ordinary differential equation.

Suppose first that $S < H$. The boundary conditions for the derivative are $f = 0$ when $S = 0$ and $f = Q$ when $S = H$. The simple solution $f = QS/H$ satisfies both the boundary conditions and the differential equation. It must therefore be the value of the derivative.

Suppose next that $S > H$. The boundary conditions are now $f = 0$ as S tends to infinity and $f = Q$ when $S = H$. The derivative price

$$f = Q\left(\frac{S}{H}\right)^{-\alpha}$$

where α is positive, satisfies the boundary conditions. It also satisfies the differential equation when

$$-r\alpha + \tfrac{1}{2}\sigma^2\alpha(\alpha+1) - r = 0$$

or $\alpha = 2r/\sigma^2$. The value of the derivative is therefore

$$f = Q\left(\frac{S}{H}\right)^{-2r/\sigma^2} \tag{15.17}$$

Problem 15.23 shows how equation (15.17) can be used to price a perpetual American put option. Section 26.2 extends the analysis to show how perpetual American call and put options can be priced when the underlying asset provides a yield at rate q.

The Prices of Tradeable Derivatives

Any function $f(S, t)$ that is a solution of the differential equation (15.16) is the theoretical price of a derivative that could be traded. If a derivative with that price existed, it would not create any arbitrage opportunities. Conversely, if a function $f(S, t)$ does not satisfy the differential equation (15.16), it cannot be the price of a derivative without creating arbitrage opportunities for traders.

To illustrate this point, consider first the function e^S. This does not satisfy the differential equation (15.16). It is therefore not a candidate for being the price of a derivative dependent on the stock price. If an instrument whose price was always e^S existed, there would be an arbitrage opportunity. As a second example, consider the function

$$\frac{e^{(\sigma^2-2r)(T-t)}}{S}$$

This does satisfy the differential equation, and so is, in theory, the price of a tradeable security. (It is the price of a derivative that pays off $1/S_T$ at time T.) For other examples of tradeable derivatives, see Problems 15.11, 15.12, 15.23, and 15.29.

15.7 RISK-NEUTRAL VALUATION

We introduced risk-neutral valuation in connection with the binomial model in Chapter 13. It is without doubt the single most important tool for the analysis of derivatives. It arises from one key property of the Black–Scholes–Merton differential equation (15.16). This property is that the equation does not involve any variables that are affected by the risk preferences of investors. The variables that do appear in the equation are the current stock price, time, stock price volatility, and the risk-free rate of interest. All are independent of risk preferences.

The Black–Scholes–Merton differential equation would not be independent of risk preferences if it involved the expected return, μ, on the stock. This is because the value of μ does depend on risk preferences. The higher the level of risk aversion by investors, the higher μ will be for any given stock. It is fortunate that μ happens to drop out in the derivation of the differential equation.

Because the Black–Scholes–Merton differential equation is independent of risk preferences, an ingenious argument can be used. If risk preferences do not enter the equation, they cannot affect its solution. Any set of risk preferences can, therefore, be used when evaluating f. In particular, the very simple assumption that all investors are risk neutral can be made.

In a world where investors are risk neutral, the expected return on all investment assets is the risk-free rate of interest, r. The reason is that risk-neutral investors do not require a premium to induce them to take risks. It is also true that the present value of any cash flow in a risk-neutral world can be obtained by discounting its expected value at the risk-free rate. The assumption that the world is risk neutral does, therefore, considerably simplify the analysis of derivatives.

Consider a derivative that provides a payoff at one particular time. It can be valued using risk-neutral valuation by using the following procedure:

1. Assume that the expected return from the underlying asset is the risk-free interest rate, r (i.e., assume $\mu = r$).
2. Calculate the expected payoff from the derivative.
3. Discount the expected payoff at the risk-free interest rate.

It is important to appreciate that risk-neutral valuation (or the assumption that all investors are risk neutral) is merely an artificial device for obtaining solutions to the

Black–Scholes–Merton differential equation. The solutions that are obtained are valid in all worlds, not just those where investors are risk neutral. When we move from a risk-neutral world to a risk-averse world, two things happen. The expected payoff from the derivative changes and the discount rate that must be used for this payoff changes. It happens that these two changes always offset each other exactly.

Application to Forward Contracts on a Stock

We valued forward contracts on a non-dividend-paying stock in Section 5.7. In Example 15.5, we verified that the pricing formula satisfies the Black–Scholes–Merton differential equation. In this section we derive the pricing formula from risk-neutral valuation. We make the assumption that interest rates are constant and equal to r. This is somewhat more restrictive than the assumption in Chapter 5.

Consider a long forward contract that matures at time T with delivery price, K. As indicated in Figure 1.2, the value of the contract at maturity is

$$S_T - K$$

where S_T is the stock price at time T. From the risk-neutral valuation argument, the value of the forward contract at time 0 is its expected value at time T in a risk-neutral world discounted at the risk-free rate of interest. Denoting the value of the forward contract at time zero by f, this means that

$$f = e^{-rT}\hat{E}(S_T - K)$$

where $\hat{E}$ denotes the expected value in a risk-neutral world. Since K is a constant, this equation becomes

$$f = e^{-rT}\hat{E}(S_T) - Ke^{-rT} \tag{15.18}$$

The expected return μ on the stock becomes r in a risk-neutral world. Hence, from equation (15.4), we have

$$\hat{E}(S_T) = S_0 e^{rT} \tag{15.19}$$

Substituting equation (15.19) into equation (15.18) gives

$$f = S_0 - Ke^{-rT}$$

This is in agreement with equation (5.5).

15.8 BLACK–SCHOLES–MERTON PRICING FORMULAS

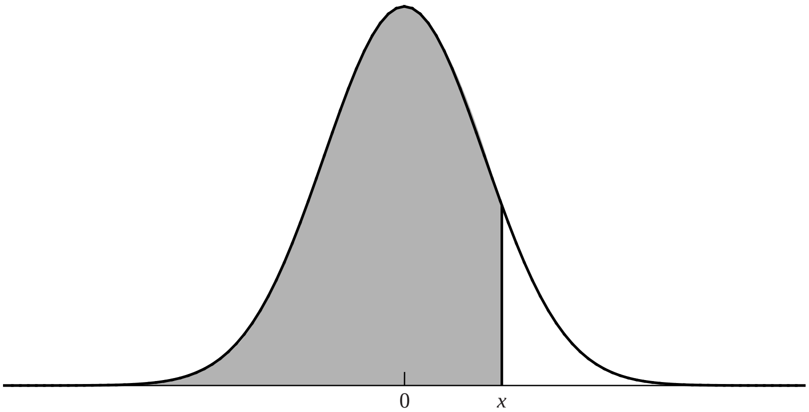
The most famous solutions to the differential equation (15.16) are the Black–Scholes–Merton formulas for the prices of European call and put options. These formulas are:

$$c = S_0 N(d_1) - Ke^{-rT}N(d_2) \tag{15.20}$$

and

$$p = Ke^{-rT}N(-d_2) - S_0 N(-d_1) \tag{15.21}$$

Figure 15.3 Shaded area represents $N(x)$.



where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

The function $N(x)$ is the cumulative probability distribution function for a variable with a standard normal distribution. In other words, it is the probability that a variable with a standard normal distribution will be less than x. It is illustrated in Figure 15.3. The remaining variables should be familiar. The variables c and p are the European call and European put price, S_0 is the stock price at time zero, K is the strike price, r is the continuously compounded risk-free rate, σ is the stock price volatility, and T is the time to maturity of the option.

One way of deriving the Black–Scholes–Merton formulas is by solving the differential equation (15.16) subject to the boundary condition mentioned in Section 15.6.[7] (See Problem 15.17 to prove that the call price in equation (15.20) satisfies the differential equation.) Another approach is to use risk-neutral valuation. Consider a European call option. The expected value of the option at maturity in a risk-neutral world is

$$\hat{E}[\max(S_T - K,\ 0)]$$

where, as before, $\hat{E}$ denotes the expected value in a risk-neutral world. From the risk-neutral valuation argument, the European call option price c is this expected value discounted at the risk-free rate of interest, that is,

$$c = e^{-rT}\hat{E}[\max(S_T - K,\ 0)] \tag{15.22}$$

[7] The differential equation gives the call and put prices at a general time t. For example, the call price that satisfies the differential equation is $c = SN(d_1) - Ke^{-r(T-t)}N(d_2)$, where

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

and $d_2 = d_1 - \sigma\sqrt{T - t}$.

The appendix at the end of this chapter shows that this equation leads to the result in equation (15.20).

Since it is never optimal to exercise early an American call option on a non-dividend-paying stock (see Section 11.5), equation (15.20) is the value of an American call option on a non-dividend-paying stock. Unfortunately, no exact analytic formula for the value of an American put option on a non-dividend-paying stock has been produced. Numerical procedures for calculating American put values are discussed in Chapter 21.

When the Black–Scholes–Merton formula is used in practice the interest rate r is set equal to the zero-coupon risk-free interest rate for a maturity T. As we show in later chapters, this is theoretically correct when r is a known function of time. It is also theoretically correct when the interest rate is stochastic provided that the stock price at time T is lognormal and the volatility parameter is chosen appropriately. As mentioned earlier, time is normally measured as the number of trading days left in the life of the option divided by the number of trading days in 1 year.

Understanding $N(d_1)$ and $N(d_2)$

The term $N(d_2)$ in equation (15.20) has a fairly simple interpretation. It is the probability that a call option will be exercised in a risk-neutral world. The $N(d_1)$ term is not quite so easy to interpret. The expression $S_0N(d_1)e^{rT}$ is the expected stock price at time T in a risk-neutral world when stock prices less than the strike price are counted as zero. The strike price is only paid if the stock price is greater than K and as just mentioned this has a probability of $N(d_2)$. The expected payoff in a risk-neutral world is therefore

$$S_0N(d_1)e^{rT} - KN(d_2)$$

Present-valuing this from time T to time zero gives the Black–Scholes–Merton equation for a European call option:

$$c = S_0N(d_1) - Ke^{-rT}N(d_2)$$

For another way of looking at the Black–Scholes–Merton equation for the value of a European call option, note that it can be written as

$$c = e^{-rT}N(d_2)[S_0e^{rT}N(d_1)/N(d_2) - K]$$

The terms here have the following interpretation:

e^{-rT}: Present value factor
$N(d_2)$: Probability of exercise
$S_0e^{rT}N(d_1)/N(d_2)$: Expected stock price in a risk-neutral world if option is exercised
K: Strike price paid if option is exercised.

Properties of the Black–Scholes–Merton Formulas

We now show that the Black–Scholes–Merton formulas have the right general properties by considering what happens when some of the parameters take extreme values.

When the stock price, S_0, becomes very large, a call option is almost certain to be exercised. It then becomes very similar to a forward contract with delivery price K.

From equation (5.5), we expect the call price to be

$$S_0 - Ke^{-rT}$$

This is, in fact, the call price given by equation (15.20) because, when S_0 becomes very large, both d_1 and d_2 become very large, and $N(d_1)$ and $N(d_2)$ become close to 1.0. When the stock price becomes very large, the price of a European put option, p, approaches zero. This is consistent with equation (15.21) because $N(-d_1)$ and $N(-d_2)$ are both close to zero in this case.

Consider next what happens when the volatility σ approaches zero. Because the stock is virtually riskless, its price will grow at rate r to S_0e^{rT} at time T and the payoff from a call option is

$$\max(S_0e^{rT} - K,\ 0)$$

Discounting at rate r, the value of the call today is

$$e^{-rT}\max(S_0e^{rT} - K,\ 0) = \max(S_0 - Ke^{-rT},\ 0)$$

To show that this is consistent with equation (15.20), consider first the case where $S_0 > Ke^{-rT}$. This implies that $\ln(S_0/K) + rT > 0$. As σ tends to zero, d_1 and d_2 tend to $+\infty$, so that $N(d_1)$ and $N(d_2)$ tend to 1.0 and equation (15.20) becomes

$$c = S_0 - Ke^{-rT}$$

When $S_0 < Ke^{-rT}$, it follows that $\ln(S_0/K) + rT < 0$. As σ tends to zero, d_1 and d_2 tend to $-\infty$, so that $N(d_1)$ and $N(d_2)$ tend to zero and equation (15.20) gives a call price of zero. The call price is therefore always $\max(S_0 - Ke^{-rT},\ 0)$ as σ tends to zero. Similarly, it can be shown that the put price is always $\max(Ke^{-rT} - S_0,\ 0)$ as σ tends to zero.

15.9 CUMULATIVE NORMAL DISTRIBUTION FUNCTION

When implementing equations (15.20) and (15.21), it is necessary to evaluate the cumulative normal distribution function $N(x)$. Tables for $N(x)$ are provided at the end of this book. The NORMSDIST function in Excel also provides a convenient way of calculating $N(x)$.

Example 15.6

The stock price 6 months from the expiration of an option is \$42, the exercise price of the option is \$40, the risk-free interest rate is 10% per annum, and the volatility is 20% per annum. This means that $S_0 = 42$, $K = 40$, $r = 0.1$, $\sigma = 0.2$, $T = 0.5$,

$$d_1 = \frac{\ln(42/40) + (0.1 + 0.2^2/2) \times 0.5}{0.2\sqrt{0.5}} = 0.7693$$

$$d_2 = \frac{\ln(42/40) + (0.1 - 0.2^2/2) \times 0.5}{0.2\sqrt{0.5}} = 0.6278$$

and

$$Ke^{-rT} = 40e^{-0.05} = 38.049$$

Hence, if the option is a European call, its value c is given by

$$c = 42N(0.7693) - 38.049N(0.6278)$$

If the option is a European put, its value p is given by

$$p = 38.049N(-0.6278) - 42N(-0.7693)$$

Using the NORMSDIST function in Excel gives

$$N(0.7693) = 0.7791, \qquad N(-0.7693) = 0.2209$$
$$N(0.6278) = 0.7349, \qquad N(-0.6278) = 0.2651$$

so that

$$c = 4.76, \qquad p = 0.81$$

Ignoring the time value of money, the stock price has to rise by \$2.76 for the purchaser of the call to break even. Similarly, the stock price has to fall by \$2.81 for the purchaser of the put to break even.

15.10 WARRANTS AND EMPLOYEE STOCK OPTIONS

The exercise of a regular call option on a company has no effect on the number of the company's shares outstanding. If the writer of the option does not own the company's shares, he or she must buy them in the market in the usual way and then sell them to the option holder for the strike price. As explained in Chapter 10, warrants and employee stock options are different from regular call options in that exercise leads to the company issuing more shares and then selling them to the option holder for the strike price. As the strike price is less than the market price, this dilutes the interest of the existing shareholders.

How should potential dilution affect the way we value outstanding warrants and employee stock options? The answer is that it should not! Assuming markets are efficient the stock price will reflect potential dilution from all outstanding warrants and employee stock options. This is explained in Business Snapshot 15.3.[8]

Consider next the situation a company is in when it is contemplating a new issue of warrants (or employee stock options). We suppose that the company is interested in calculating the cost of the issue assuming that there are no compensating benefits. We assume that the company has N shares worth S_0 each and the number of new options contemplated is M, with each option giving the holder the right to buy one share for K. The value of the company today is NS_0. This value does not change as a result of the warrant issue. Suppose that without the warrant issue the share price will be S_T at the warrant's maturity. This means that (with or without the warrant issue) the total value of the equity and the warrants at time T will NS_T. If the warrants are exercised, there is a cash inflow from the strike price increasing this to $NS_T + MK$. This value is distributed

---

[8] Analysts sometimes assume that the sum of the values of the warrants and the equity (rather than just the value of the equity) is lognormal. The result is a Black–Scholes type of equation for the value of the warrant in terms of the value of the warrant. See Technical Note 3 at www-2.rotman.utoronto.ca/~hull/TechnicalNotes for an explanation of this model.

Business Snapshot 15.3 Warrants, Employee Stock Options, and Dilution

Consider a company with 100,000 shares each worth \$50. It surprises the market with an announcement that it is granting 100,000 stock options to its employees with a strike price of \$50. If the market sees little benefit to the shareholders from the employee stock options in the form of reduced salaries and more highly motivated managers, the stock price will decline immediately after the announcement of the employee stock options. If the stock price declines to \$45, the dilution cost to the current shareholders is \$5 per share or \$500,000 in total.

Suppose that the company does well so that by the end of three years the share price is \$100. Suppose further that all the options are exercised at this point. The payoff to the employees is \$50 per option. It is tempting to argue that there will be further dilution in that 100,000 shares worth \$100 per share are now merged with 100,000 shares for which only \$50 is paid, so that (a) the share price reduces to \$75 and (b) the payoff to the option holders is only \$25 per option. However, this argument is flawed. The exercise of the options is anticipated by the market and already reflected in the share price. The payoff from each option exercised is \$50.

This example illustrates the general point that when markets are efficient the impact of dilution from executive stock options or warrants is reflected in the stock price as soon as they are announced and does not need to be taken into account again when the options are valued.

among $N + M$ shares, so that the share price immediately after exercise becomes

$$\frac{NS_T + MK}{N + M}$$

Therefore the payoff to an option holder if the option is exercised is

$$\frac{NS_T + MK}{N + M} - K$$

or

$$\frac{N}{N + M}(S_T - K)$$

This shows that the value of each option is the value of

$$\frac{N}{N + M}$$

regular call options on the company's stock. Therefore the total cost of the options is M times this. Since we are assuming that there are no benefits to the company from the warrant issue, the total value of the company's equity will decline by the total cost of the options as soon as the decision to issue the warrants becomes generally known. This means that the reduction in the stock price is

$$\frac{M}{N + M}$$

times the value of a regular call option with strike price K and maturity T.

Example 15.7

A company with 1 million shares worth \$40 each is considering issuing 200,000 warrants each giving the holder the right to buy one share with a strike price of \$60 in 5 years. It wants to know the cost of this. The interest rate is 3% per annum, and the volatility is 30% per annum. The company pays no dividends. From equation (15.20), the value of a 5-year European call option on the stock is \$7.04. In this case, $N = 1{,}000{,}000$ and $M = 200{,}000$, so that the value of each warrant is

$$\frac{1{,}000{,}000}{1{,}000{,}000 + 200{,}000} \times 7.04 = 5.87$$

or \$5.87. The total cost of the warrant issue is $200{,}000 \times 5.87 = \$1.17$ million. Assuming the market perceives no benefits from the warrant issue, we expect the stock price to decline by \$1.17 to \$38.83.

15.11 IMPLIED VOLATILITIES

The one parameter in the Black–Scholes–Merton pricing formulas that cannot be directly observed is the volatility of the stock price. In Section 15.4, we discussed how this can be estimated from a history of the stock price. In practice, traders usually work with what are known as *implied volatilities*. These are the volatilities implied by option prices observed in the market.[9]

To illustrate how implied volatilities are calculated, suppose that the market price of a European call option on a non-dividend-paying stock is 1.875 when $S_0 = 21$, $K = 20$, $r = 0.1$, and $T = 0.25$. The implied volatility is the value of σ that, when substituted into equation (15.20), gives $c = 1.875$. Unfortunately, it is not possible to invert equation (15.20) so that σ is expressed as a function of S_0, K, r, T, and c. However, an iterative search procedure can be used to find the implied σ. For example, we can start by trying $\sigma = 0.20$. This gives a value of c equal to 1.76, which is too low. Because c is an increasing function of σ, a higher value of σ is required. We can next try a value of 0.30 for σ. This gives a value of c equal to 2.10, which is too high and means that σ must lie between 0.20 and 0.30. Next, a value of 0.25 can be tried for σ. This also proves to be too high, showing that σ lies between 0.20 and 0.25. Proceeding in this way, we can halve the range for σ at each iteration and the correct value of σ can be calculated to any required accuracy.[10] In this example, the implied volatility is 0.235, or 23.5%, per annum. A similar procedure can be used in conjunction with binomial trees to find implied volatilities for American options.

Implied volatilities are used to monitor the market's opinion about the volatility of a particular stock. Whereas historical volatilities (see Section 15.4) are backward looking, implied volatilities are forward looking. Traders often quote the implied volatility of an option rather than its price. This is convenient because the implied volatility tends to be less variable than the option price. The implied volatilities of actively traded options on an asset are often used by traders to estimate appropriate implied volatilities for other options on the asset.

[9] Implied volatilities for European and American options can be calculated using DerivaGem.

[10] This method is presented for illustration. Other more powerful methods, such as the Newton–Raphson method, are often used in practice (see footnote 3 of Chapter 4).

The VIX Index

The CBOE publishes indices of implied volatility. The most popular index, the SPX VIX, is an index of the implied volatility of 30-day options on the S&P 500 calculated from a wide range of calls and puts. It is sometimes referred to as the "fear factor." An index value of 15 indicates that the implied volatility of 30-day options on the S&P 500 is estimated as 15%. Information on the way the index is calculated is in Section 26.15. Trading in futures on the VIX started in 2004 and trading in options on the VIX started in 2006. One contract is on 1,000 times the index.

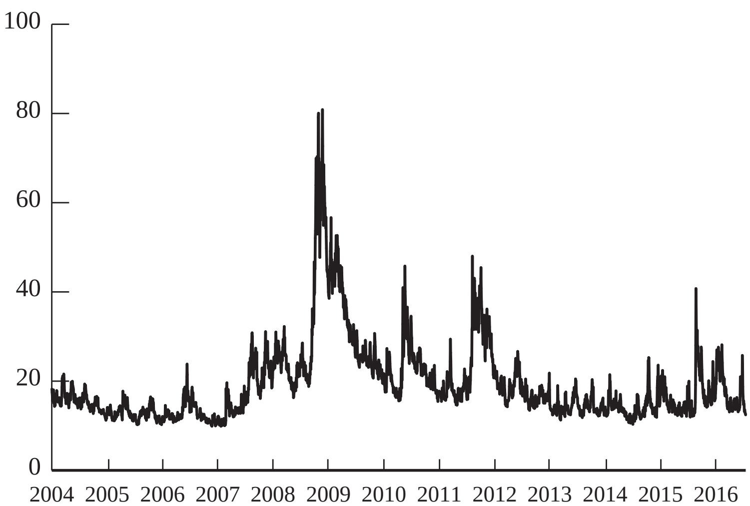
Example 15.8

> Suppose that a trader buys an April futures contract on the VIX when the futures price is 18.5 (corresponding to a 30-day S&P 500 volatility of 18.5%) and closes out the contract when the futures price is 19.3 (corresponding to an S&P 500 volatility of 19.3%). The trader makes a gain of $800.

A trade involving options on the S&P 500 is a bet on the future level of the S&P 500, which depends on the volatility of the S&P 500. By contrast, a futures or options contract on the VIX is a bet only on volatility. Figure 15.4 shows the VIX index between January 2004 and July 2016. Between 2004 and mid-2007 it tended to stay between 10 and 20. It reached 30 during the second half of 2007 and a record 80 in October and November 2008 after Lehman's bankruptcy. By early 2010, it had declined to a more normal levels, but it spiked again in May 2010 and the second half of 2011 because of stresses and uncertainties in financial markets.

VIX monitors the volatility of the S&P 500. The CBOE publishes a range of other volatility indices. These are on other stock indices, commodity indices, interest rates, currencies, and some individual stocks (for example, Amazon and Goldman Sachs). There is even a volatility index of the VIX index (VVIX).

Figure 15.4 The VIX index, January 2004 to July 2016.



15.12 DIVIDENDS

Up to now, we have assumed that the stock on which the option is written pays no dividends. In this section, we modify the Black–Scholes–Merton model to take account of dividends. We assume that the amount and timing of the dividends during the life of an option can be predicted with certainty. When options last for relatively short periods of time, this assumption is not too unreasonable. (For long-life options it is usual to assume that the dividend yield rather the dollar dividend payments are known. Options can then be valued as will be described in the Chapter 17.) The date on which the dividend is paid should be assumed to be the ex-dividend date. On this date the stock price declines by the amount of the dividend.[11]

European Options

European options can be analyzed by assuming that the stock price is the sum of two components: a riskless component that corresponds to the known dividends during the life of the option and a risky component. The riskless component, at any given time, is the present value of all the dividends during the life of the option discounted from the ex-dividend dates to the present at the risk-free rate. By the time the option matures, the dividends will have been paid and the riskless component will no longer exist. The Black–Scholes–Merton formula is therefore correct if S_0 is equal to the risky component of the stock price and σ is the volatility of the process followed by the risky component.[12]

Operationally, this means that the Black–Scholes–Merton formulas can be used provided that the stock price is reduced by the present value of all the dividends during the life of the option, the discounting being done from the ex-dividend dates at the risk-free rate. As already mentioned, a dividend is counted as being during the life of the option only if its ex-dividend date occurs during the life of the option.

Example 15.9

Consider a European call option on a stock when there are ex-dividend dates in two months and five months. The dividend on each ex-dividend date is expected to be \$0.50. The current share price is \$40, the exercise price is \$40, the stock price volatility is 30% per annum, the risk-free rate of interest is 9% per annum, and the time to maturity is six months. The present value of the dividends is

$$0.5e^{-0.09\times 2/12} + 0.5e^{-0.09\times 5/12} = 0.9742$$

The option price can therefore be calculated from the Black–Scholes–Merton

[11] For tax reasons the stock price may go down by somewhat less than the cash amount of the dividend. To take account of this phenomenon, we need to interpret the word 'dividend' in the context of option pricing as the reduction in the stock price on the ex-dividend date caused by the dividend. Thus, if a dividend of \$1 per share is anticipated and the share price normally goes down by 80% of the dividend on the ex-dividend date, the dividend should be assumed to be \$0.80 for the purpose of the analysis.

[12] This is not quite the same as the volatility of the whole stock price. (In theory, they cannot both follow geometric Brownian motion.) At time zero, the volatility of the risky component is approximately equal to the volatility of the whole stock price multiplied by $S_0/(S_0 - D)$, where D is the present value of the dividends.

formula, with $S_0 = 40 - 0.9742 = 39.0258$, $K = 40$, $r = 0.09$, $\sigma = 0.3$, and $T = 0.5$:

$$d_1 = \frac{\ln(39.0258/40) + (0.09 + 0.3^2/2) \times 0.5}{0.3\sqrt{0.5}} = 0.2020$$

$$d_2 = \frac{\ln(39.0258/40) + (0.09 - 0.3^2/2) \times 0.5}{0.3\sqrt{0.5}} = -0.0102$$

Using the NORMSDIST function in Excel gives

$$N(d_1) = 0.5800, \quad N(d_2) = 0.4959$$

and, from equation (15.20), the call price is

$$39.0258 \times 0.5800 - 40e^{-0.09 \times 0.5} \times 0.4959 = 3.67$$

or \$3.67.

Some researchers have criticized the approach just described for calculating the value of a European option on a dividend-paying stock. They argue that volatility should be applied to the stock price, not to the stock price less the present value of dividends. A number of different numerical procedures have been suggested for doing this.[13] When volatility is calculated from historical data, it might make sense to use one of these procedures. However, in practice the volatility used to price an option is nearly always implied from the prices of other options using procedures we will outline in Chapter 20. If an analyst uses the same model for both implying and applying volatilities, the resulting prices should be accurate and not highly model dependent. Another important point is that in practice, as will be explained in Chapter 18, practitioners usually value a European option in terms of the forward price of the underlying asset. This avoids the need to estimate explicitly the income that is expected from the asset. The volatility of the forward stock price is the same as the volatility of a variable equal to the stock price minus the present value of dividends.

The model we have proposed where the stock price is divided into two components is internally consistent and widely used in practice. We will use the same model when valuing American options in Chapter 21.

American Call Options

Consider next American call options. Chapter 11 showed that in the absence of dividends American options should never be exercised early. An extension to the argument shows that, when there are dividends, it can only be optimal to exercise at a time immediately before the stock goes ex-dividend. We assume that n ex-dividend dates are anticipated and that they are at times $t_1, t_2, \ldots, t_n$, with $t_1 < t_2 < \cdots < t_n$. The dividends corresponding to these times will be denoted by $D_1, D_2, \ldots, D_n$, respectively.

We start by considering the possibility of early exercise just prior to the final ex-dividend date (i.e., at time t_n). If the option is exercised at time t_n, the investor receives

$$S(t_n) - K$$

[13] See, for example, N. Areal and A. Rodrigues, "Fast Trees for Options with Discrete Dividends," *Journal of Derivatives*, 21, 1 (Fall 2013), 49–63.

where $S(t)$ denotes the stock price at time t. If the option is not exercised, the stock price drops to $S(t_n) - D_n$. As shown by equation (11.4), the value of the option is then greater than

$$S(t_n) - D_n - Ke^{-r(T-t_n)}$$

It follows that, if

$$S(t_n) - D_n - Ke^{-r(T-t_n)} \geqslant S(t_n) - K$$

that is,

$$D_n \leqslant K\left[1 - e^{-r(T-t_n)}\right] \qquad \textbf{(15.23)}$$

it cannot be optimal to exercise at time t_n. On the other hand, if

$$D_n > K\left[1 - e^{-r(T-t_n)}\right] \qquad \textbf{(15.24)}$$

for any reasonable assumption about the stochastic process followed by the stock price, it can be shown that it is always optimal to exercise at time t_n for a sufficiently high value of $S(t_n)$. The inequality in (15.24) will tend to be satisfied when the final ex-dividend date is fairly close to the maturity of the option (i.e., $T - t_n$ is small) and the dividend is large.

Consider next time t_{n-1}, the penultimate ex-dividend date. If the option is exercised immediately prior to time t_{n-1}, the investor receives $S(t_{n-1}) - K$. If the option is not exercised at time t_{n-1}, the stock price drops to $S(t_{n-1}) - D_{n-1}$ and the earliest subsequent time at which exercise could take place is t_n. Hence, from equation (11.4), a lower bound to the option price if it is not exercised at time t_{n-1} is

$$S(t_{n-1}) - D_{n-1} - Ke^{-r(t_n-t_{n-1})}$$

It follows that if

$$S(t_{n-1}) - D_{n-1} - Ke^{-r(t_n-t_{n-1})} \geqslant S(t_{n-1}) - K$$

or

$$D_{n-1} \leqslant K\left[1 - e^{-r(t_n-t_{n-1})}\right]$$

it is not optimal to exercise immediately prior to time t_{n-1}. Similarly, for any $i < n$, if

$$D_i \leqslant K\left[1 - e^{-r(t_{i+1}-t_i)}\right] \qquad \textbf{(15.25)}$$

it is not optimal to exercise immediately prior to time t_i.

The inequality in (15.25) is approximately equivalent to

$$D_i \leqslant Kr(t_{i+1} - t_i)$$

Assuming that K is fairly close to the current stock price, this inequality is satisfied when the dividend yield on the stock is less than the risk-free rate of interest. This is often the case.

We can conclude from this analysis that, in many circumstances, the most likely time for the early exercise of an American call is immediately before the final ex-dividend date, t_n. Furthermore, if inequality (15.25) holds for $i = 1, 2, \ldots, n-1$ and inequality (15.23) holds, we can be certain that early exercise is never optimal, and the American option can be treated as a European option.

Black's Approximation

Black suggests an approximate procedure for taking account of early exercise in call options.[14] This involves calculating, as described earlier in this section, the prices of European options that mature at times T and t_n, and then setting the American price equal to the greater of the two.[15] This is an approximation because it in effect assumes the option holder has to decide at time zero whether the option will be exercised at time T or t_n.

SUMMARY

We started this chapter by examining the properties of the process for stock prices introduced in Chapter 14. The process implies that the price of a stock at some future time, given its price today, is lognormal. It also implies that the continuously compounded return from the stock in a period of time is normally distributed. Our uncertainty about future stock prices increases as we look further ahead. The standard deviation of the logarithm of the stock price is proportional to the square root of how far ahead we are looking.

To estimate the volatility σ of a stock price empirically, the stock price is observed at fixed intervals of time (e.g., every day, every week, or every month). For each time period, the natural logarithm of the ratio of the stock price at the end of the time period to the stock price at the beginning of the time period is calculated. The volatility is estimated as the standard deviation of these numbers divided by the square root of the length of the time period in years. Usually, days when the exchanges are closed are ignored in measuring time for the purposes of volatility calculations.

The differential equation for the price of any derivative dependent on a stock can be obtained by creating a riskless portfolio of the derivative and the stock. Because the derivative's price and the stock price both depend on the same underlying source of uncertainty, this can always be done. The portfolio that is created remains riskless for only a very short period of time. However, the return on a riskless porfolio must always be the risk-free interest rate if there are to be no arbitrage opportunities.

The expected return on the stock does not enter into the Black–Scholes–Merton differential equation. This leads to an extremely useful result known as risk-neutral valuation. This result states that when valuing a derivative dependent on a stock price, we can assume that the world is risk neutral. This means that we can assume that the expected return from the stock is the risk-free interest rate, and then discount expected payoffs at the risk-free interest rate. The Black–Scholes–Merton equations for European call and put options can be derived by either solving their differential equation or by using risk-neutral valuation.

An implied volatility is the volatility that, when used in conjunction with the Black–Scholes–Merton option pricing formula, gives the market price of the option. Traders

[14] See F. Black, "Fact and Fantasy in the Use of Options," *Financial Analysts Journal*, 31 (July/August 1975): 36–41, 61–72.

[15] For an exact formula, suggested by Roll, Geske, and Whaley, for valuing American calls when there is only one ex-dividend date, see Technical Note 4 at www-2.rotman.utoronto.ca/~hull/TechnicalNotes. This involves the cumulative bivariate normal distribution function. A procedure for calculating this function is given in Technical Note 5 and a worksheet for calculating the cumulative bivariate normal distribution can be found on the author's website.

monitor implied volatilities. They often quote the implied volatility of an option rather than its price. They have developed procedures for using the volatilities implied by the prices of actively traded options to estimate volatilities for other options on the same asset.

The Black–Scholes–Merton results can be extended to cover European call and put options on dividend-paying stocks. The procedure is to use the Black–Scholes–Merton formula with the stock price reduced by the present value of the dividends anticipated during the life of the option, and the volatility equal to the volatility of the stock price net of the present value of these dividends.

In theory, it can be optimal to exercise American call options immediately before any ex-dividend date. In practice, it is often only necessary to consider the final ex-dividend date. Fischer Black has suggested an approximation. This involves setting the American call option price equal to the greater of two European call option prices. The first European call option expires at the same time as the American call option; the second expires immediately prior to the final ex-dividend date.

FURTHER READING

On the Distribution of Stock Price Changes

Blattberg, R., and N. Gonedes, "A Comparison of the Stable and Student Distributions as Statistical Models for Stock Prices," *Journal of Business*, 47 (April 1974): 244–80.

Fama, E. F., "The Behavior of Stock Market Prices," *Journal of Business*, 38 (January 1965): 34–105.

Kon, S. J., "Models of Stock Returns—A Comparison," *Journal of Finance*, 39 (March 1984): 147–65.

Richardson, M., and T. Smith, "A Test for Multivariate Normality in Stock Returns," *Journal of Business*, 66 (1993): 295–321.

On the Black–Scholes–Merton Analysis

Black, F. "Fact and Fantasy in the Use of Options and Corporate Liabilities," *Financial Analysts Journal*, 31 (July/August 1975): 36–41, 61–72.

Black, F. "How We Came Up with the Option Pricing Formula," *Journal of Portfolio Management*, 15, 2 (1989): 4–8.

Black, F., and M. Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81 (May/June 1973): 637–59.

Merton, R. C., "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*, 4 (Spring 1973): 141–83.

On Risk-Neutral Valuation

Cox, J. C., and S. A. Ross, "The Valuation of Options for Alternative Stochastic Processes," *Journal of Financial Economics*, 3 (1976): 145–66.

Smith, C. W., "Option Pricing: A Review," *Journal of Financial Economics*, 3 (1976): 3–54.

On the Causes of Volatility

Fama, E. F. "The Behavior of Stock Market Prices." *Journal of Business*, 38 (January 1965): 34–105.

French, K. R. "Stock Returns and the Weekend Effect." *Journal of Financial Economics*, 8 (March 1980): 55–69.

French, K. R., and R. Roll "Stock Return Variances: The Arrival of Information and the Reaction of Traders." *Journal of Financial Economics*, 17 (September 1986): 5–26.

Roll R. "Orange Juice and Weather," *American Economic Review*, 74, 5 (December 1984): 861–80.

Practice Questions (Answers in Solutions Manual)

15.1. What does the Black–Scholes–Merton stock option pricing model assume about the probability distribution of the stock price in one year? What does it assume about the probability distribution of the continuously compounded rate of return on the stock during the year?

15.2. The volatility of a stock price is 30% per annum. What is the standard deviation of the percentage price change in one trading day?

15.3. Explain the principle of risk-neutral valuation.

15.4. Calculate the price of a 3-month European put option on a non-dividend-paying stock with a strike price of \$50 when the current stock price is \$50, the risk-free interest rate is 10% per annum, and the volatility is 30% per annum.

15.5. What difference does it make to your calculations in Problem 15.4 if a dividend of \$1.50 is expected in 2 months?

15.6. What is *implied volatility*? How can it be calculated?

15.7. A stock price is currently \$40. Assume that the expected return from the stock is 15% and that its volatility is 25%. What is the probability distribution for the rate of return (with continuous compounding) earned over a 2-year period?

15.8. A stock price follows geometric Brownian motion with an expected return of 16% and a volatility of 35%. The current price is \$38.
(a) What is the probability that a European call option on the stock with an exercise price of \$40 and a maturity date in 6 months will be exercised?
(b) What is the probability that a European put option on the stock with the same exercise price and maturity will be exercised?

15.9. Using the notation in this chapter, prove that a 95% confidence interval for S_T is between $S_0e^{(\mu-\sigma^2/2)T-1.96\sigma\sqrt{T}}$ and $S_0e^{(\mu-\sigma^2/2)T+1.96\sigma\sqrt{T}}$.

15.10. A portfolio manager announces that the average of the returns realized in each year of the last 10 years is 20% per annum. In what respect is this statement misleading?

15.11. Assume that a non-dividend-paying stock has an expected return of μ and a volatility of σ. An innovative financial institution has just announced that it will trade a security that pays off a dollar amount equal to $\ln S_T$ at time T, where S_T denotes the value of the stock price at time T.
(a) Use risk-neutral valuation to calculate the price of the security at time t in terms of the stock price, S, at time t. The risk-free rate is r.
(b) Confirm that your price satisfies the differential equation (15.16).

15.12. Consider a derivative that pays off S_T^n at time T, where S_T is the stock price at that time. When the stock pays no dividends and its price follows geometric Brownian motion, it can be shown that its price at time t ($t \leqslant T$) has the form $h(t, T)S^n$, where S is the stock price at time t and h is a function only of t and T.
(a) By substituting into the Black–Scholes–Merton partial differential equation, derive an ordinary differential equation satisfied by $h(t, T)$.
(b) What is the boundary condition for the differential equation for $h(t, T)$?
(c) Show that $h(t, T) = e^{[0.5\sigma^2 n(n-1)+r(n-1)](T-t)}$, where r is the risk-free interest rate and σ is the stock price volatility.

15.13. What is the price of a European call option on a non-dividend-paying stock when the stock price is \$52, the strike price is \$50, the risk-free interest rate is 12% per annum, the volatility is 30% per annum, and the time to maturity is 3 months?

15.14. What is the price of a European put option on a non-dividend-paying stock when the stock price is \$69, the strike price is \$70, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is 6 months?

15.15. Consider an American call option on a stock. The stock price is \$70, the time to maturity is 8 months, the risk-free rate of interest is 10% per annum, the exercise price is \$65, and the volatility is 32%. A dividend of \$1 is expected after 3 months and again after 6 months. Show that it can never be optimal to exercise the option on either of the two dividend dates. Use DerivaGem to calculate the price of the option.

15.16. A call option on a non-dividend-paying stock has a market price of $\$2\frac{1}{2}$. The stock price is \$15, the exercise price is \$13, the time to maturity is 3 months, and the risk-free interest rate is 5% per annum. What is the implied volatility?

15.17. With the notation used in this chapter:

(a) What is $N'(x)$?

(b) Show that $SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$, where S is the stock price at time t and

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

(c) Calculate $\partial d_1/\partial S$ and $\partial d_2/\partial S$.

(d) Show that when $c = SN(d_1) - Ke^{-r(T-t)}N(d_2)$, it follows that

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}$$

where c is the price of a call option on a non-dividend-paying stock.

(e) Show that $\partial c/\partial S = N(d_1)$.

(f) Show that c satisfies the Black–Scholes–Merton differential equation.

(g) Show that c satisfies the boundary condition for a European call option, i.e., that $c = \max(S - K, 0)$ as $t \to T$.

15.18. Show that the Black–Scholes–Merton formulas for call and put options satisfy put–call parity.

15.19. A stock price is currently \$50 and the risk-free interest rate is 5%. Use the DerivaGem software to translate the following table of European call options on the stock into a table of implied volatilities, assuming no dividends. Are the option prices consistent with the assumptions underlying Black–Scholes–Merton?

	Maturity (months)		
Strike price (\$)	*3*	*6*	*12*
45	7.0	8.3	10.5
50	3.7	5.2	7.5
55	1.6	2.9	5.1

15.20. Explain carefully why Black's approach to evaluating an American call option on a dividend-paying stock may give an approximate answer even when only one dividend is anticipated. Does the answer given by Black's approach understate or overstate the true option value? Explain your answer.

15.21. Consider an American call option on a stock. The stock price is \$50, the time to maturity is 15 months, the risk-free rate of interest is 8% per annum, the exercise price is \$55, and the volatility is 25%. Dividends of \$1.50 are expected in 4 months and 10 months. Show that it can never be optimal to exercise the option on either of the two dividend dates. Calculate the price of the option.

15.22. Show that the probability that a European call option will be exercised in a risk-neutral world is, with the notation introduced in this chapter, $N(d_2)$. What is an expression for the value of a derivative that pays off \$100 if the price of a stock at time T is greater than K?

15.23. Use the result in equation (15.17) to determine the value of a perpetual American put option on a non-dividend-paying stock with strike price K if it is exercised when the stock price equals H where $H < K$. Assume that the current stock price S is greater than H. What is the value of H that maximizes the option value? Deduce the value of a perpetual American put with strike price K.

15.24. A company has an issue of executive stock options outstanding. Should dilution be taken into account when the options are valued? Explain your answer.

15.25. A company's stock price is \$50 and 10 million shares are outstanding. The company is considering giving its employees 3 million at-the-money 5-year call options. Option exercises will be handled by issuing more shares. The stock price volatility is 25%, the 5-year risk-free rate is 5%, and the company does not pay dividends. Estimate the cost to the company of the employee stock option issue.

Further Questions

15.26. If the volatility of a stock is 18% per annum, estimate the standard deviation of the percentage price change in (a) 1 day, (b) 1 week, and (c) 1 month.

15.27. A stock price is currently \$50. Assume that the expected return from the stock is 18% and its volatility is 30%. What is the probability distribution for the stock price in 2 years? Calculate the mean and standard deviation of the distribution. Determine the 95% confidence interval.

15.28. Suppose that observations on a stock price (in dollars) at the end of each of 15 consecutive weeks are as follows:

30.2, 32.0, 31.1, 30.1, 30.2, 30.3, 30.6, 33.0, 32.9, 33.0, 33.5, 33.5, 33.7, 33.5, 33.2

Estimate the stock price volatility. What is the standard error of your estimate?

15.29. A financial institution plans to offer a security that pays off a dollar amount equal to S_T^2 at time T, where S_T is the price at time T of a stock that pays no dividends.
(a) Use risk-neutral valuation to calculate the price of the security at time t in terms of the stock price S at time t and other variables. (*Hint*: The expected value of S_T^2 can be calculated from the mean and variance of S_T given in Section 15.1.)
(b) Confirm that your price satisfies the differential equation (15.16).

15.30. Consider an option on a non-dividend-paying stock when the stock price is \$30, the exercise price is \$29, the risk-free interest rate is 5%, the volatility is 25% per annum, and the time to maturity is 4 months.
(a) What is the price of the option if it is a European call?
(b) What is the price of the option if it is an American call?

(c) What is the price of the option if it is a European put?
(d) Verify that put–call parity holds.

15.31. Assume that the stock in Problem 15.30 is due to go ex-dividend in $1\frac{1}{2}$ months. The expected dividend is 50 cents.
(a) What is the price of the option if it is a European call?
(b) What is the price of the option if it is a European put?
(c) If the option is an American call, are there any circumstances under which it will be exercised early?

15.32. Consider an American call option when the stock price is \$18, the exercise price is \$20, the time to maturity is 6 months, the volatility is 30% per annum, and the risk-free interest rate is 10% per annum. Two equal dividends are expected during the life of the option with ex-dividend dates at the end of 2 months and 5 months. Assume the dividends are 40 cents. Use Black's approximation and the DerivaGem software to value the option. How high can the dividends be without the American option being worth more than the corresponding European option?

15.33. The appendix derives the key result

$$E[\max(V-K,\ 0)] = E(V)N(d_1) - KN(d_2)$$

Show that

$$E[\max(K-V,\ 0)] = KN(-d_1) - E(V)N(-d_2)$$

and use this to derive the Black–Scholes–Merton formula for the price of a European put option on a non-dividend-paying stock.

APPENDIX

PROOF OF THE BLACK–SCHOLES–MERTON FORMULA USING RISK-NEUTRAL VALUATION

We will prove the Black–Scholes result by first proving another key result that will also be useful in future chapters.

Key Result

If V is lognormally distributed and the standard deviation of $\ln V$ is w, then

$$E[\max(V-K,\ 0)] = E(V)N(d_1) - KN(d_2) \tag{15A.1}$$

where

$$d_1 = \frac{\ln[E(V)/K] + w^2/2}{w}$$

$$d_2 = \frac{\ln[E(V)/K] - w^2/2}{w}$$

and E denotes the expected value. (See Problem 15.33 for a similar result for puts.)

Proof of Key Result

Define $g(V)$ as the probability density function of V. It follows that

$$E[\max(V-K, 0)] = \int_K^{\infty} (V-K)g(V)\,dV \tag{15A.2}$$

The variable $\ln V$ is normally distributed with standard deviation w. From the properties of the lognormal distribution, the mean of $\ln V$ is m, where[16]

$$m = \ln[E(V)] - w^2/2 \tag{15A.3}$$

Define a new variable

$$Q = \frac{\ln V - m}{w} \tag{15A.4}$$

This variable is normally distributed with a mean of zero and a standard deviation of 1.0. Denote the density function for Q by $h(Q)$ so that

$$h(Q) = \frac{1}{\sqrt{2\pi}} e^{-Q^2/2}$$

Using equation (15A.4) to convert the expression on the right-hand side of equation (15A.2) from an integral over V to an integral over Q, we get

$$E[\max(V-K,\ 0)] = \int_{(\ln K-m)/w}^{\infty} (e^{Qw+m} - K)\,h(Q)\,dQ$$

or

$$E[\max(V-K,\ 0)] = \int_{(\ln K-m)/w}^{\infty} e^{Qw+m} h(Q)dQ - K\int_{(\ln K-m)/w}^{\infty} h(Q)dQ \tag{15A.5}$$

[16] For a proof of this, see Technical Note 2 at www-2.rotman.utoronto.ca/~hull/TechnicalNotes.

Now

$$e^{Qw+m}h(Q) = \frac{1}{\sqrt{2\pi}}e^{(-Q^2+2Qw+2m)/2} = \frac{1}{\sqrt{2\pi}}e^{[-(Q-w)^2+2m+w^2]/2}$$

$$= \frac{e^{m+w^2/2}}{\sqrt{2\pi}}e^{[-(Q-w)^2]/2} = e^{m+w^2/2}h(Q-w)$$

This means that equation (15A.5) becomes

$$E[\max(V-K,\ 0)] = e^{m+w^2/2}\int_{(\ln K-m)/w}^{\infty} h(Q-w)dQ - K\int_{(\ln K-m)/w}^{\infty} h(Q)dQ \qquad \textbf{(15A.6)}$$

If we define $N(x)$ as the probability that a variable with a mean of zero and a standard deviation of 1.0 is less than x, the first integral in equation (15A.6) is

$$1 - N[(\ln K - m)/w - w] = N[(-\ln K + m)/w + w]$$

Substituting for m from equation (15A.3) leads to

$$N\left(\frac{\ln[E(V)/K] + w^2/2}{w}\right) = N(d_1)$$

Similarly the second integral in equation (15A.6) is $N(d_2)$. Equation (15A.6), therefore, becomes

$$E[\max(V-K,\ 0)] = e^{m+w^2/2}N(d_1) - KN(d_2)$$

Substituting for m from equation (15A.3) gives the key result.

The Black–Scholes–Merton Result

We now consider a call option on a non-dividend-paying stock maturing at time T. The strike price is K, the risk-free rate is r, the current stock price is S_0, and the volatility is σ. As shown in equation (15.22), the call price c is given by

$$c = e^{-rT}\hat{E}[\max(S_T - K,\ 0)] \qquad \textbf{(15A.7)}$$

where S_T is the stock price at time T and $\hat{E}$ denotes the expectation in a risk-neutral world. Under the stochastic process assumed by Black–Scholes–Merton, S_T is log-normal. Also, from equations (15.3) and (15.4), $\hat{E}(S_T) = S_0e^{rT}$ and the standard deviation of $\ln S_T$ is $\sigma\sqrt{T}$.

From the key result just proved, equation (15A.7) implies

$$c = e^{-rT}[S_0e^{rT}N(d_1) - KN(d_2)] = S_0N(d_1) - Ke^{-rT}N(d_2)$$

where

$$d_1 = \frac{\ln[\hat{E}(S_T)/K] + \sigma^2T/2}{\sigma\sqrt{T}} = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln[\hat{E}(S_T)/K] - \sigma^2T/2}{\sigma\sqrt{T}} = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$$

This is the Black–Scholes–Merton result.

CHAPTER 17 Options on Stock Indices and Currencies

Options on stock indices and currencies were introduced in Chapter 10. This chapter discusses them in more detail. It explains how they work and reviews some of the ways they can be used. In the second half of the chapter, the valuation results in Chapter 15 are extended to cover European options on a stock paying a known dividend yield. It is then argued that both stock indices and currencies are analogous to stocks paying dividend yields. This enables the results for options on a stock paying a dividend yield to be applied to these types of options as well.

17.1 OPTIONS ON STOCK INDICES

Several exchanges trade options on stock indices. Some of the indices track the movement of the market as a whole. Others are based on the performance of a particular sector (e.g., computer technology, oil and gas, transportation, or telecoms). Among the index options traded on the Chicago Board Options Exchange (CBOE) are American and European options on the S&P 100 (OEX and XEO), European options on the S&P 500 (SPX), European options on the Dow Jones Industrial Average (DJX), and European options on the Nasdaq 100 (NDX). In Chapter 10, we explained that the CBOE trades LEAPS and flex options on individual stocks. It also offers these option products on indices.

One index option contract is usually on 100 times the index. (Note that the Dow Jones index used for index options is 0.01 times the usually quoted Dow Jones index.) Index options are settled in cash. This means that, on exercise of the option, the holder of a call option contract receives $(S - K) \times 100$ in cash and the writer of the option pays this amount in cash, where S is the value of the index at the close of trading on the day of the exercise and K is the strike price. Similarly, the holder of a put option contract receives $(K - S) \times 100$ in cash and the writer of the option pays this amount in cash.

Portfolio Insurance

Portfolio managers can use options on a well-diversified index to limit their downside risk. Suppose that the value of the index today is S_0. Consider a manager in charge of a well-diversified portfolio whose beta is 1.0. A beta of 1.0 implies that the returns from the portfolio mirror those from the index. Assuming the dividend yield from the portfolio is the same as the dividend yield from the index, the percentage changes in the value of the portfolio can be expected to be approximately the same as the percentage changes in the value of the index. Since each contract is on 100 times the index, it follows that the value of the portfolio is protected against the possibility of the index falling below K if, for each $100S_0$ dollars in the portfolio, the manager buys one put option contract with strike price K. Suppose that the manager's portfolio is worth \$500,000 and the value of the index is currently 1,000. The portfolio is worth 500 times the index. The manager can obtain insurance against the value of the portfolio dropping below \$450,000 in the next three months by buying five three-month put option contracts on the index with a strike price of 900.

To illustrate how the insurance works, consider the situation where the index drops to 880 in three months. The portfolio will be worth about \$440,000. The payoff from the options will be $5 \times (900 - 880) \times 100 = \$10{,}000$, bringing the total value of the portfolio up to the insured value of \$450,000.

When the Portfolio's Beta Is Not 1.0

If the portfolio's beta (β) is not 1.0, β put options must be purchased for each $100S_0$ dollars in the portfolio, where S_0 is the current value of the index. Suppose that the \$500,000 portfolio just considered has a beta of 2.0 instead of 1.0. We continue to assume that the index is 1,000. The number of put options required is

$$2.0 \times \frac{500{,}000}{1{,}000 \times 100} = 10$$

rather than 5 as before.

To calculate the appropriate strike price, the capital asset pricing model can be used (see the appendix to Chapter 3). Suppose that the risk free rate is 12%, the dividend yield on both the index and the portfolio is 4%, and protection is required against the value of the portfolio dropping below \$450,000 in the next three months. Under the

Table 17.1 Calculation of expected value of portfolio when the index is 1,040 in three months and $\beta = 2.0$.

Value of index in three months:	1,040
Return from change in index:	40/1,000, or 4% per three months
Dividends from index:	$0.25 \times 4 = 1\%$ per three months
Total return from index:	$4 + 1 = 5\%$ per three months
Risk-free interest rate:	$0.25 \times 12 = 3\%$ per three months
Excess return from index over risk-free interest rate:	$5 - 3 = 2\%$ per three months
Expected excess return from portfolio over risk-free interest rate:	$2 \times 2 = 4\%$ per three months
Expected return from portfolio:	$3 + 4 = 7\%$ per three months
Dividends from portfolio:	$0.25 \times 4 = 1\%$ per three months
Expected increase in value of portfolio:	$7 - 1 = 6\%$ per three months
Expected value of portfolio:	$\$500{,}000 \times 1.06 = \$530{,}000$

Table 17.2 Relationship between value of index and value of portfolio for $\beta = 2.0$.

Value of index in three months	*Value of portfolio in three months (\$)*
1,080	570,000
1,040	530,000
1,000	490,000
960	450,000
920	410,000
880	370,000

capital asset pricing model, the expected excess return of a portfolio over the risk-free rate is assumed to equal beta times the excess return of the index portfolio over the risk-free rate. The model enables the expected value of the portfolio to be calculated for different values of the index at the end of three months. Table 17.1 shows the calculations for the case where the index is 1,040. In this case, the expected value of the portfolio at the end of the three months is \$530,000. Similar calculations can be carried out for other values of the index at the end of the three months. The results are shown in Table 17.2. The strike price for the options that are purchased should be the index level corresponding to the protection level required on the portfolio. In this case, the protection level is \$450,000 and so the correct strike price for the 10 put option contracts that are purchased is 960.[1]

To illustrate how the insurance works, consider what happens if the value of the index falls to 880. As shown in Table 17.2, the value of the portfolio is then about \$370,000. The put options pay off $(960 - 880) \times 10 \times 100 = \$80{,}000$, and this is exactly what is necessary to move the total value of the portfolio manager's position up from \$370,000 to the required level of \$450,000.

The examples in this section show that there are two reasons why the cost of hedging increases as the beta of a portfolio increases. More put options are required and they have a higher strike price.

17.2 CURRENCY OPTIONS

Currency options are primarily traded in the over-the-counter market. The advantage of this market is that large trades are possible, with strike prices, expiration dates, and other features tailored to meet the needs of corporate treasurers. Although currency options do trade on NASDAQ OMX in the United States, the exchange-traded market for these options is much smaller than the over-the-counter market.

An example of a European call option is a contract that gives the holder the right to buy one million euros with U.S. dollars at an exchange rate of 1.1000 U.S. dollars per euro. If the actual exchange rate at the maturity of the option is 1.1500, the payoff is

[1] Approximately 1% of \$500,000, or \$5,000, will be earned in dividends over the next three months. If we want the insured level of \$450,000 to include dividends, we can choose a strike price corresponding to \$445,000 rather than \$450,000. This is 955.

$1{,}000{,}000 \times (1.1500 - 1.1000) = \$50{,}000$. Similarly, an example of a European put option is a contract that gives the holder the right to sell ten million Australian dollars for U.S. dollars at an exchange rate of 0.7000 U.S. dollars per Australian dollar. If the actual exchange rate at the maturity of the option is 0.6700, the payoff is $10{,}000{,}000 \times (0.7000 - 0.6700) = \$300{,}000$.

For a corporation wishing to hedge a foreign exchange exposure, foreign currency options are an alternative to forward contracts. A U.S. company due to receive sterling at a known time in the future can hedge its risk by buying put options on sterling that mature at that time. The hedging strategy guarantees that the exchange rate applicable to the sterling will not be less than the strike price, while allowing the company to benefit from any favorable exchange-rate movements. Similarly, a U.S. company due to pay sterling at a known time in the future can hedge by buying calls on sterling that mature at that time. This hedging strategy guarantees that the cost of the sterling will not be greater than a certain amount while allowing the company to benefit from favorable exchange-rate movements. Whereas a forward contract locks in the exchange rate for a future transaction, an option provides a type of insurance. This is not free. It costs nothing to enter into a forward transaction, but options require a premium to be paid up front.

Range Forwards

A *range forward contract* is a variation on a standard forward contract for hedging foreign exchange risk. Consider a U.S. company that knows it will receive one million pounds sterling in three months. Suppose that the three-month forward exchange rate is 1.3200 dollars per pound. The company could lock in this exchange rate for the dollars it receives by entering into a short forward contract to sell one million pounds sterling in three months. This would ensure that the amount received for the one million pounds is \$1,320,000.

An alternative is to buy a European put option with a strike price of K_1 and sell a European call option with a strike price K_2, where $K_1 < 1.3200 < K_2$. This is known as a short position in a range forward contract. The payoff is shown in Figure 17.1a. Both options are on one million pounds. If the exchange rate in three months proves to be less than K_1, the put option is exercised and as a result the company is able to sell the

Figure 17.1 Payoffs from (a) short and (b) long position in a range forward contract.

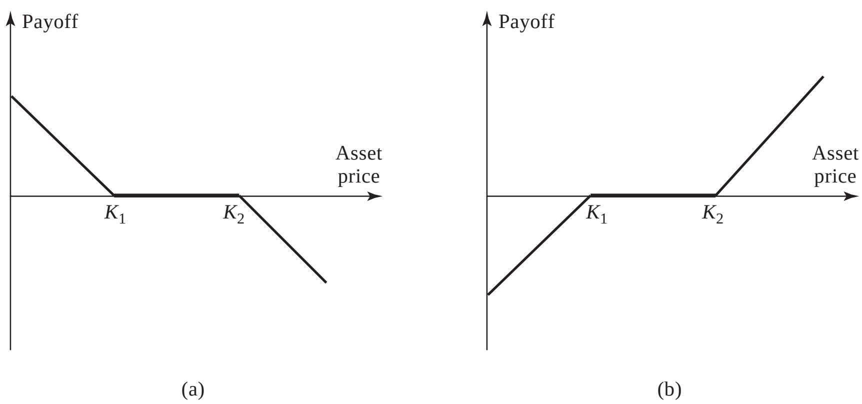
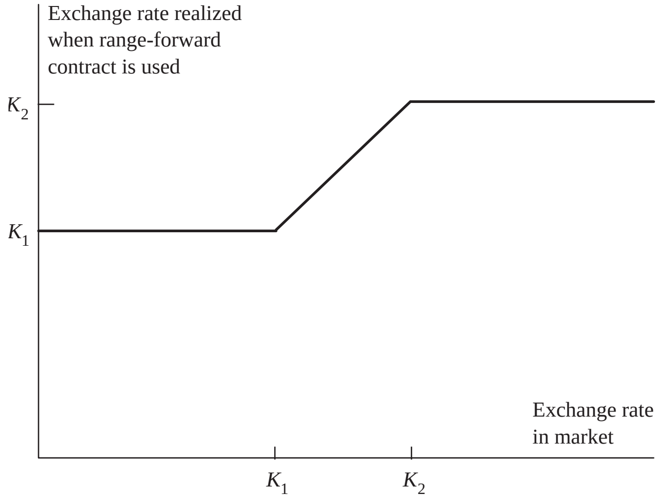


Figure 17.2 Exchange rate realized when a range forward contract is used to hedge either a future foreign currency inflow or a future foreign currency outflow.



one million pounds at an exchange rate of K_1. If the exchange rate is between K_1 and K_2, neither option is exercised and the company gets the current exchange rate for the one million pounds. If the exchange rate is greater than K_2, the call option is exercised against the company and the one million pounds is sold at an exchange rate of K_2. The exchange rate realized for the one million pounds is shown in Figure 17.2.

If the company knew it was due to pay rather than receive one million pounds in three months, it could sell a European put option with strike price K_1 and buy a European call option with strike price K_2. This is a long position in a range forward contract and the payoff is shown in Figure 17.1b. If the exchange rate in three months proves to be less than K_1, the put option is exercised against the company and as a result the company buys the one million pounds it needs at an exchange rate of K_1. If the exchange rate is between K_1 and K_2, neither option is exercised and the company buys the one million pounds at the current exchange rate. If the exchange rate is greater than K_2, the call option is exercised and the company is able to buy the one million pounds at an exchange rate of K_2. The exchange rate paid for the one million pounds is the same as that received for the one million pounds in the earlier example and is shown in Figure 17.2.

In practice, a range forward contract is set up so that the price of the put option equals the price of the call option. This means that it costs nothing to set up the range forward contract, just as it costs nothing to set up a regular forward contract. Suppose that the U.S. and British interest rates are both 2%, so that the spot exchange rate is 1.3200 (the same as the forward exchange rate). Suppose further that the exchange rate volatility is 14%. We can use DerivaGem to show that a European put with strike price 1.3000 to sell one pound has the same price as a European call option with a strike price of 1.3414 to buy one pound. (Both are worth 0.0273.) Setting $K_1 = 1.3000$ and $K_2 = 1.3414$ therefore leads to a contract with zero cost in our example.

As the strike prices of the call and put options in a range forward contract are moved together, the range forward contract becomes a regular forward contract. The (short) range forward contract in Figure 17.1a becomes a short forward contract and the (long) range forward contract in Figure 17.1b becomes a long forward contract.

17.3 OPTIONS ON STOCKS PAYING KNOWN DIVIDEND YIELDS

In this section we produce a simple rule that enables valuation results for European options on a non-dividend-paying stock to be extended so that they apply to European options on a stock paying a known dividend yield. Later we show how this enables us to value options on stock indices and currencies.

Suppose that the dividend yield per year (measured with continuous compounding) is q. Dividends cause stock prices to reduce on the ex-dividend date by the amount of the dividend payment. The payment of a dividend yield at rate q therefore causes the growth rate in the stock price to be less than it would otherwise be by an amount q. If, with a dividend yield of q, the stock price grows from S_0 today to S_T at time T, then in the absence of dividends it would grow from S_0 today to $S_T e^{qT}$ at time T. Alternatively, in the absence of dividends it would grow from $S_0 e^{-qT}$ today to S_T at time T.

This argument shows that we get the same probability distribution for the stock price at time T in each of the following two cases:

1. The stock starts at price S_0 and provides a dividend yield at rate q.
2. The stock starts at price $S_0 e^{-qT}$ and pays no dividends.

This leads to a simple rule. When valuing a European option lasting for time T on a stock paying a known dividend yield at rate q, we reduce the current stock price from S_0 to $S_0 e^{-qT}$ and then value the option as though the stock pays no dividends.[2]

Lower Bounds for Option Prices

As a first application of this rule, consider the problem of determining bounds for the price of a European option on a stock paying a dividend yield at rate q. Substituting $S_0 e^{-qT}$ for S_0 in equation (11.4), we see that a lower bound for the European call option price, c, is given by

$$c \geqslant \max\left(S_0 e^{-qT} - K e^{-rT},\ 0\right) \tag{17.1}$$

We can also prove this directly by considering the following two portfolios:

Portfolio A: one European call option plus an amount of cash equal to $K e^{-rT}$

Portfolio B: e^{-qT} shares with dividends being reinvested in additional shares.

To obtain a lower bound for a European put option, we can similarly replace S_0 by $S_0 e^{-qT}$ in equation (11.5) to get

$$p \geqslant \max\left(K e^{-rT} - S_0 e^{-qT},\ 0\right) \tag{17.2}$$

[2] This rule is analogous to the one developed in Section 15.12 for valuing a European option on a stock paying known cash dividends. (In that case we concluded that it is correct to reduce the stock price by the present value of the dividends; in this case we discount the stock price at the dividend yield rate.)

This result can also be proved directly by considering the following portfolios:

Portfolio C: one European put option plus e^{-qT} shares with dividends on the shares being reinvested in additional shares

Portfolio D: an amount of cash equal to Ke^{-rT}.

Put–Call Parity

Replacing S_0 by S_0e^{-qT} in equation (11.6) we obtain put–call parity for an option on a stock paying a dividend yield at rate q:

$$c + Ke^{-rT} = p + S_0e^{-qT} \tag{17.3}$$

This result can also be proved directly by considering the following two portfolios:

Portfolio A: one European call option plus an amount of cash equal to Ke^{-rT}

Portfolio C: one European put option plus e^{-qT} shares with dividends on the shares being reinvested in additional shares.

Both portfolios are both worth $\max(S_T,\ K)$ at time T. They must therefore be worth the same today, and the put–call parity result in equation (17.3) follows. For American options, the put–call parity relationship is (see Problem 17.12)

$$S_0e^{-qT} - K \leqslant C - P \leqslant S_0 - Ke^{-rT}$$

Pricing Formulas

By replacing S_0 by S_0e^{-qT} in the Black–Scholes–Merton formulas, equations (15.20) and (15.21), we obtain the price, c, of a European call and the price, p, of a European put on a stock paying a dividend yield at rate q as

$$c = S_0e^{-qT}N(d_1) - Ke^{-rT}N(d_2) \tag{17.4}$$

$$p = Ke^{-rT}N(-d_2) - S_0e^{-qT}N(-d_1) \tag{17.5}$$

Since

$$\ln\frac{S_0e^{-qT}}{K} = \ln\frac{S_0}{K} - qT$$

it follows that d_1 and d_2 are given by

$$d_1 = \frac{\ln(S_0/K) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - q - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

These results were first derived by Merton.[3] As discussed in Chapter 15, the word *dividend* should, for the purposes of option valuation, be defined as the reduction in the stock price on the ex-dividend date arising from any dividends declared. If the dividend yield rate is known but not constant during the life of the option, equations (17.4)

[3] See R.C. Merton, "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*, 4 (Spring 1973): 141–83.

and (17.5) are still true, with q equal to the average annualized dividend yield during the option's life.

Differential Equation and Risk-Neutral Valuation

To prove the results in equations (17.4) and (17.5) more formally, we can either solve the differential equation that the option price must satisfy or use risk-neutral valuation.

When we include a dividend yield of q in the analysis in Section 15.6, the differential equation (15.16) becomes[4]

$$\frac{\partial f}{\partial t} + (r-q)S\frac{\partial f}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{17.6}$$

Like equation (15.16), this does not involve any variable affected by risk preferences. Therefore the risk-neutral valuation procedure described in Section 15.7 can be used.

In a risk-neutral world, the total return from the stock must be r. The dividends provide a return of q. The expected growth rate in the stock price must therefore be $r-q$. It follows that the risk-neutral process for the stock price is

$$dS = (r-q)S\,dt + \sigma S\,dz \tag{17.7}$$

To value a derivative dependent on a stock that provides a dividend yield equal to q, we set the expected growth rate of the stock equal to $r-q$ and discount the expected payoff at rate r. When the expected growth rate in the stock price is $r-q$, the expected stock price at time T is $S_0e^{(r-q)T}$. A similar analysis to that in the appendix to Chapter 15 gives the expected payoff for a call option in a risk-neutral world as

$$e^{(r-q)T}S_0N(d_1) - KN(d_2)$$

where d_1 and d_2 are defined as above. Discounting at rate r for time T leads to equation (17.4).

17.4 VALUATION OF EUROPEAN STOCK INDEX OPTIONS

In valuing index futures in Chapter 5, we assumed that the index could be treated as an asset paying a known yield. In valuing index options, we make similar assumptions. This means that equations (17.1) and (17.2) provide a lower bound for European index options; equation (17.3) is the put–call parity result for European index options; equations (17.4) and (17.5) can be used to value European options on an index; and the binomial tree approach can be used for American options. In all cases, S_0 is equal to the current value of the index, σ is equal to the volatility of the index, and q is equal to the average annualized dividend yield on the index during the life of the option expressed with continuous compounding.

Example 17.1

Consider a European call option on an index that is two months from maturity. The current value of the index is 930, the exercise price is 900, the risk-free interest rate is

[4] See Technical Note 6 at www-2.rotman.utoronto.ca/~hull/TechnicalNotes for a proof of this.

8% per annum, and the volatility of the index is 20% per annum. Dividend yields of 0.2% and 0.3% (expressed with continuous compounding) are expected in the first month and the second month, respectively. In this case $S_0 = 930$, $K = 900$, $r = 0.08$, $\sigma = 0.2$, and $T = 2/12$. The total dividend yield during the option's life is $0.2\% + 0.3\% = 0.5\%$. This corresponds to 3% per annum. Hence, $q = 0.03$ and

$$d_1 = \frac{\ln(930/900) + (0.08 - 0.03 + 0.2^2/2) \times 2/12}{0.2\sqrt{2/12}} = 0.5444$$

$$d_2 = \frac{\ln(930/900) + (0.08 - 0.03 - 0.2^2/2) \times 2/12}{0.2\sqrt{2/12}} = 0.4628$$

$$N(d_1) = 0.7069, \qquad N(d_2) = 0.6782$$

so that the call price, c, is given by equation (17.4) as

$$c = 930 \times 0.7069e^{-0.03\times 2/12} - 900 \times 0.6782e^{-0.08\times 2/12} = 51.83$$

One contract, if on 100 times the index, would cost \$5,183.

The calculation of q should include only dividends for which the ex-dividend dates occur during the life of the option. In the United States ex-dividend dates tend to occur during the first week of February, May, August, and November. At any given time the correct value of q is therefore likely to depend on the life of the option. This is even more true for indices in other countries. In Japan, for example, all companies tend to use the same ex-dividend dates.

If the absolute amount of the dividend that will be paid on the stocks underlying the index (rather than the dividend yield) is assumed to be known, the basic Black–Scholes–Merton formulas can be used with the initial stock price being reduced by the present value of the dividends. This is the approach recommended in Chapter 15 for a stock paying known dividends. However, it may be difficult to implement for a broadly based stock index because it requires a knowledge of the dividends expected on every stock underlying the index.

It is sometimes argued that, in the long run, the return from investing a certain amount of money in a well-diversified stock portfolio is almost certain to beat the return from investing the same amount of money in a bond portfolio. If this were so, a long-dated put option allowing the stock portfolio to be sold for the value of the bond portfolio should not cost very much. In fact, as indicated by Business Snapshot 17.1, it is quite expensive.

Forward Prices

Define F_0 as the forward price of the index for a contract with maturity T. As shown by equation (5.3), $F_0 = S_0e^{(r-q)T}$. This means that the equations for the European call price c and the European put price p in equations (17.4) and (17.5) can be written

$$c = F_0e^{-rT}N(d_1) - Ke^{-rT}N(d_2) \tag{17.8}$$

$$p = Ke^{-rT}N(-d_2) - F_0e^{-rT}N(-d_1) \tag{17.9}$$

where

$$d_1 = \frac{\ln(F_0/K) + \sigma^2T/2}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = \frac{\ln(F_0/K) - \sigma^2T/2}{\sigma\sqrt{T}}$$

Business Snapshot 17.1 Can We Guarantee that Stocks Will Beat Bonds in the Long Run?

It is often said that if you are a long-term investor you should buy stocks rather than bonds. Consider a U.S. fund manager who is trying to persuade investors to buy, as a long-term investment, an equity fund that is expected to mirror the S&P 500. The manager might be tempted to offer purchasers of the fund a guarantee that their return will be at least as good as the return on risk-free bonds over the next 10 years. Historically stocks have outperformed bonds in the United States over almost any 10-year period. It appears that the fund manager would not be giving much away.

In fact, this type of guarantee is surprisingly expensive. Suppose that an equity index is 1,000 today, the dividend yield on the index is 1% per annum, the volatility of the index is 15% per annum, and the 10-year risk-free rate is 5% per annum. To outperform bonds, the stocks underlying the index must earn more than 5% per annum. The dividend yield will provide 1% per annum. The capital gains on the stocks must therefore provide 4% per annum. This means that we require the index level to be at least $1{,}000e^{0.04\times 10} = 1{,}492$ in 10 years.

A guarantee that the return on \$1,000 invested in the index will be greater than the return on \$1,000 invested in bonds over the next 10 years is therefore equivalent to the right to sell the index for 1,492 in 10 years. This is a European put option on the index and can be valued from equation (17.5) with $S_0 = 1{,}000$, $K = 1{,}492$, $r = 5\%$, $\sigma = 15\%$, $T = 10$, and $q = 1\%$. The value of the put option is 169.7. This shows that the guarantee contemplated by the fund manager is worth about 17% of the fund—hardly something that should be given away!

The put–call parity relationship in equation (17.3) can be written

$$c + Ke^{-rT} = p + F_0e^{-rT}$$

or

$$F_0 = K + (c - p)e^{rT} \tag{17.10}$$

If, as is not uncommon in the exchange-traded markets, pairs of puts and calls with the same strike price are traded actively for a particular maturity date, this equation can be used to estimate the forward price of the index for that maturity date. Once the forward prices of the index for a number of different maturity dates have been obtained, the term structure of forward prices can be estimated, and other options can be valued using equations (17.8) and (17.9). The advantage of this approach is that the dividend yield on the index does not have to be estimated explicitly.

Implied Dividend Yields

If estimates of the dividend yield are required (e.g., because an American option is being valued), European calls and puts with the same strike price and time to maturity can be used. From equation (17.3),

$$q = -\frac{1}{T}\ln\frac{c - p + Ke^{-rT}}{S_0}$$

For a particular strike price and time to maturity, the estimates of q calculated from this equation are liable to be unreliable. But when the results from many matched pairs of calls and puts are combined, a clearer picture of the term structure of dividend yields being assumed by the market emerges.

17.5 VALUATION OF EUROPEAN CURRENCY OPTIONS

To value currency options, we define S_0 as the spot exchange rate. To be precise, S_0 is the value of one unit of the foreign currency in U.S. dollars. As explained in Section 5.10, a foreign currency is analogous to a stock paying a known dividend yield. The owner of foreign currency receives a yield equal to the risk-free interest rate, r_f, in the foreign currency. Equations (17.1) and (17.2), with q replaced by r_f, provide bounds for the European call price, c, and the European put price, p:

$$c \geqslant \max\left(S_0 e^{-r_f T} - K e^{-rT},\ 0\right)$$
$$p \geqslant \max\left(K e^{-rT} - S_0 e^{-r_f T},\ 0\right)$$

Equation (17.3), with q replaced by r_f, provides the put–call parity result for European currency options:

$$c + Ke^{-rT} = p + S_0 e^{-r_f T}$$

Finally, equations (17.4) and (17.5) provide the pricing formulas for European currency options when q is replaced by r_f:

$$c = S_0 e^{-r_f T} N(d_1) - K e^{-rT} N(d_2) \qquad \textbf{(17.11)}$$
$$p = K e^{-rT} N(-d_2) - S_0 e^{-r_f T} N(-d_1) \qquad \textbf{(17.12)}$$

where

$$d_1 = \frac{\ln(S_0/K) + (r - r_f + \sigma^2/2)T}{\sigma\sqrt{T}}$$
$$d_2 = \frac{\ln(S_0/K) + (r - r_f - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Both the domestic interest rate, r, and the foreign interest rate, r_f, are the rates for a maturity T.

Example 17.2

Consider a 4-month European call option on the British pound. Suppose that the current exchange rate is 1.6000, the exercise price is 1.6000, the risk-free interest rate in the United States is 8% per annum, the risk-free interest rate in Britain is 11% per annum, and the option price is 4.3 cents. In this case, $S_0 = 1.6$, $K = 1.6$, $r = 0.08$, $r_f = 0.11$, $T = 0.3333$, and $c = 0.043$. The implied volatility can be calculated by trial and error. A volatility of 20% gives an option price of 0.0639; a volatility of 10% gives an option price of 0.0285; and so on. The implied volatility is 14.1%.

Put and call options on a currency are symmetrical in that a put option to sell one unit of currency A for currency B at strike price K is the same as a call option to buy K units of B with currency A at strike price $1/K$ (see Problem 17.8).

Using Forward Exchange Rates

Because banks and other financial institutions trade forward contracts on foreign exchange rates actively, forward exchange rates are often used for valuing options. From equation (5.9), the forward exchange rate, F_0, for a maturity T is given by

$$F_0 = S_0 e^{(r-r_f)T}$$

This relationship allows equations (17.11) and (17.12) to be simplified to

$$c = e^{-rT}[F_0 N(d_1) - KN(d_2)] \tag{17.13}$$

$$p = e^{-rT}[KN(-d_2) - F_0 N(-d_1)] \tag{17.14}$$

where

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(F_0/K) - \sigma^2 T/2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Equations (17.13) and (17.14) are the same as equations (17.8) and (17.9). As we shall see in Chapter 18, a European option on the spot price of any asset can be valued in terms of the price of a forward or futures contract on the asset using equations (17.13) and (17.14). The maturity of the forward or futures contract must be the same as the maturity of the European option.

17.6 AMERICAN OPTIONS

As described in Chapter 13, binomial trees can be used to value American options on indices and currencies. As in the case of American options on a non-dividend-paying stock, the parameter determining the size of up movements, u, is set equal to $e^{\sigma\sqrt{\Delta t}}$, where σ is the volatility and Δt is the length of time steps. The parameter determining the size of down movements, d, is set equal to $1/u$, or $e^{-\sigma\sqrt{\Delta t}}$. For a non-dividend-paying stock, the probability of an up movement is

$$p = \frac{a-d}{u-d}$$

where $a = e^{r\Delta t}$. For options on indices and currencies, the formula for p is the same, but a is defined differently. In the case of options on an index,

$$a = e^{(r-q)\Delta t} \tag{17.15}$$

where q is the dividend yield on the index. In the case of options on a currency,

$$a = e^{(r-r_f)\Delta t} \tag{17.16}$$

where r_f is the foreign risk-free rate. Example 13.1 in Section 13.11 shows how a two-step tree can be constructed to value an option on an index. Example 13.2 shows how a three-step tree can be constructed to value an option on a currency. Further examples of the use of binomial trees to value options on indices and currencies are given in Chapter 21.

In some circumstances, it is optimal to exercise American currency and index options prior to maturity. Thus, American currency and index options are worth more than their European counterparts. In general, call options on high-interest currencies and put options on low-interest currencies are the most likely to be exercised prior to maturity. The reason is that a high-interest currency is expected to depreciate and a low-interest currency is expected to appreciate. Similarly, call options on indices with high-dividend yields and put options on indices with low-dividend yields are most likely to be exercised early.

SUMMARY

The index options trading on exchanges are settled in cash. On exercise of an index call option contract, the holder typically receives 100 times the amount by which the index exceeds the strike price. Similarly, on exercise of an index put option contract, the holder receives 100 times the amount by which the strike price exceeds the index. Index options can be used for portfolio insurance. If the value of the portfolio mirrors the index, it is appropriate to buy one put option contract for each $100S_0$ dollars in the portfolio, where S_0 is the value of the index. If the portfolio does not mirror the index, β put option contracts should be purchased for each $100S_0$ dollars in the portfolio, where β is the beta of the portfolio from the capital asset pricing model. The strike price of the put options purchased should reflect the level of insurance required.

Most currency options are traded in the over-the-counter market. They can be used by corporate treasurers to hedge a foreign exchange exposure. For example, a U.S. corporate treasurer who knows that the company will be receiving sterling at a certain time in the future can hedge by buying put options that mature at that time. Similarly, a U.S. corporate treasurer who knows that the company will be paying sterling at a certain time in the future can hedge by buying call options that mature at that time. Currency options can also be used to create a range forward contract. This is a zero-cost contract that can be used to provide downside protection while giving up some of the upside for a company with a known foreign exchange exposure.

The Black–Scholes–Merton formula for valuing European options on a non-dividend-paying stock can be extended to cover European options on a stock paying a known dividend yield. The extension can be used to value European options on stock indices and currencies because:

1. A stock index is analogous to a stock paying a dividend yield. The dividend yield is the dividend yield on the stocks that make up the index.
2. A foreign currency is analogous to a stock paying a dividend yield. The foreign risk-free interest rate plays the role of the dividend yield.

Binomial trees can be used to value American options on stock indices and currencies.

FURTHER READING

Biger, N., and J. C. Hull. "The Valuation of Currency Options," *Financial Management*, 12 (Spring 1983): 24–28.

Bodie, Z. "On the Risk of Stocks in the Long Run," *Financial Analysts Journal*, 51, 3 (1995): 18–22.

Garman, M. B., and S. W. Kohlhagen. "Foreign Currency Option Values," *Journal of International Money and Finance*, 2 (December 1983): 231–37.

Giddy, I. H., and G. Dufey. "Uses and Abuses of Currency Options," *Journal of Applied Corporate Finance*, 8, 3 (1995): 49–57.

Grabbe, J. O. "The Pricing of Call and Put Options on Foreign Exchange," *Journal of International Money and Finance*, 2 (December 1983): 239–53.

Merton, R. C. "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*, 4 (Spring 1973): 141–83.

Practice Questions (Answers in Solutions Manual)

17.1. A portfolio is currently worth \$10 million and has a beta of 1.0. An index is currently standing at 800. Explain how a put option on the index with a strike price of 700 can be used to provide portfolio insurance.

17.2. "Once we know how to value options on a stock paying a dividend yield, we know how to value options on stock indices and currencies." Explain this statement.

17.3. A stock index is currently 300, the dividend yield on the index is 3% per annum, and the risk-free interest rate is 8% per annum. What is a lower bound for the price of a six-month European call option on the index when the strike price is 290?

17.4. A currency is currently worth \$0.80 and has a volatility of 12%. The domestic and foreign risk-free interest rates are 6% and 8%, respectively. Use a two-step binomial tree to value (a) a European four-month call option with a strike price of 0.79 and (b) an American four-month call option with the same strike price.

17.5. Explain how corporations can use range forward contracts to hedge their foreign exchange risk when they are due to receive a certain amount of a foreign currency in the future.

17.6. Calculate the value of a three-month at-the-money European call option on a stock index when the index is at 250, the risk-free interest rate is 10% per annum, the volatility of the index is 18% per annum, and the dividend yield on the index is 3% per annum.

17.7. Calculate the value of an eight-month European put option on a currency with a strike price of 0.50. The current exchange rate is 0.52, the volatility of the exchange rate is 12%, the domestic risk-free interest rate is 4% per annum, and the foreign risk-free interest rate is 8% per annum.

17.8. Show that the formula in equation (17.12) for a put option to sell one unit of currency A for currency B at strike price K gives the same value as equation (17.11) for a call option to buy K units of currency B for currency A at strike price $1/K$.

17.9. A foreign currency is currently worth \$1.50. The domestic and foreign risk-free interest rates are 5% and 9%, respectively. Calculate a lower bound for the value of a six-month call option on the currency with a strike price of \$1.40 if it is (a) European and (b) American.

17.10. Consider a stock index currently standing at 250. The dividend yield on the index is 4% per annum, and the risk-free rate is 6% per annum. A three-month European call option on the index with a strike price of 245 is currently worth \$10. What is the value of a three-month put option on the index with a strike price of 245?

17.11. An index currently stands at 696 and has a volatility of 30% per annum. The risk-free rate of interest is 7% per annum and the index provides a dividend yield of 4% per annum. Calculate the value of a three-month European put with an exercise price of 700.

17.12. Show that, if C is the price of an American call with exercise price K and maturity T on a stock paying a dividend yield of q, and P is the price of an American put on the same stock with the same strike price and exercise date, then

$$S_0e^{-qT} - K < C - P < S_0 - Ke^{-rT},$$

where S_0 is the stock price, r is the risk-free rate, and $r > 0$. (*Hint*: To obtain the first half of the inequality, consider possible values of:

Portfolio A: a European call option plus an amount K invested at the risk-free rate
Portfolio B: an American put option plus e^{-qT} of stock with dividends being reinvested in the stock.

To obtain the second half of the inequality, consider possible values of:

Portfolio C: an American call option plus an amount Ke^{-rT} invested at the risk-free rate
Portfolio D: a European put option plus one stock with dividends being reinvested in the stock.)

17.13. Show that a European call option on a currency has the same price as the corresponding European put option on the currency when the forward price equals the strike price.

17.14. Would you expect the volatility of a stock index to be greater or less than the volatility of a typical stock? Explain your answer.

17.15. Does the cost of portfolio insurance increase or decrease as the beta of a portfolio increases? Explain your answer.

17.16. Suppose that a portfolio is worth \$60 million and a stock index stands at 1,200. If the value of the portfolio mirrors the value of the index, what options should be purchased to provide protection against the value of the portfolio falling below \$54 million in one year's time?

17.17. Consider again the situation in Problem 17.16. Suppose that the portfolio has a beta of 2.0, the risk-free interest rate is 5% per annum, and the dividend yield on both the portfolio and the index is 3% per annum. What options should be purchased to provide protection against the value of the portfolio falling below \$54 million in one year's time?

17.18. An index currently stands at 1,500. European call and put options with a strike price of 1,400 and time to maturity of six months have market prices of 154.00 and 34.25, respectively. The six-month risk-free rate is 5%. What is the implied dividend yield?

17.19. A total return index tracks the return, including dividends, on a certain portfolio. Explain how you would value (a) forward contracts and (b) European options on the index.

17.20. What is the put–call parity relationship for European currency options?

17.21. Prove the results in equations (17.1), (17.2), and (17.3) using the portfolios indicated.

17.22. Can an option on the yen/euro exchange rate be created from two options, one on the dollar/euro exchange rate, and the other on the dollar/yen exchange rate? Explain your answer.

Further Questions

17.23. The Dow Jones Industrial Average on July 20, 2016, was 18,580 and the price of a September 185 (European) call option on the index was \$3.35. Use the DerivaGem software to calculate the implied volatility of this option. Assume the risk-free rate was 0.7% and the dividend yield was 2.75%. The option expires on September 16, 2016. Estimate the price of a September 185 put option. What is the volatility implied by the price you estimate for this option? (Note that options are on the Dow Jones index divided by 100.)

17.24. A stock index currently stands at 300 and has a volatility of 20%. The risk-free interest rate is 8% and the dividend yield on the index is 3%. Use a three-step binomial tree to value a six-month put option on the index with a strike price of 300 if it is (a) European and (b) American?

17.25. Suppose that the spot price of the Canadian dollar is U.S. \$0.95 and that the Canadian dollar/U.S. dollar exchange rate has a volatility of 8% per annum. The risk-free rates of interest in Canada and the United States are 4% and 5% per annum, respectively. Calculate the value of a European call option to buy one Canadian dollar for U.S. \$0.95 in nine months. Use put–call parity to calculate the price of a European put option to sell one Canadian dollar for U.S. \$0.95 in nine months. What is the price of a call option to buy U.S. \$0.95 with one Canadian dollar in nine months?

17.26. The spot price of an index is 1,000 and the risk-free rate is 4%. The prices of 3-month European call and put options when the strike price is 950 are 78 and 26. Estimate (a) the dividend yield and (b) the implied volatility.

17.27. Assume that the price of currency A expressed in terms of the price of currency B follows the process $dS = (r_B - r_A)S\,dt + \sigma S\,dz$, where r_A is the risk-free interest rate in currency A and r_B is the risk-free interest rate in currency B. What is the process followed by the price of currency B expressed in terms of currency A?

17.28. Suppose the USD/euro exchange rate is 1.3000. The exchange rate volatility is 15%. A U.S. company will receive 1 million euros in three months. The euro and USD risk-free rates are 5% and 4%, respectively. The company decides to use a range forward contract with the lower strike price equal to 1.2500.
(a) What should the higher strike price be to create a zero-cost contract?
(b) What position in calls and puts should the company take?
(c) Show that your answer to (a) does not depend on interest rates provided that the interest rate differential between the two currencies, $r - r_f$, remains the same.

17.29. In Business Snapshot 17.1, what is the cost of a guarantee that the return on the fund will not be negative over the next 10 years?

17.30. The one-year forward price of the Mexican peso is \$0.0750 per MXN. The U.S. risk-free rate is 1.25% and the Mexican risk-free rate is 4.5%. The exchange rate volatility is 13%. What are the values of one-year European and American put options with a strike price of \$0.0800.

CHAPTER 18 Futures Options and Black's Model

The options we have considered so far provide the holder with the right to buy or sell a certain asset by a certain date for a certain price. They are sometimes termed *options on spot* or *spot options* because, when the options are exercised, the sale or purchase of the asset at the agreed-on price takes place immediately. In this chapter we move on to consider *options on futures*, also known as *futures options*. In these contracts, exercise of the option gives the holder a position in a futures contract.

The Commodity Futures Trading Commission in the U.S. authorized the trading of options on futures on an experimental basis in 1982. Permanent trading was approved in 1987, and since then the popularity of the contract has grown very fast.

In this chapter we consider how futures options work and the differences between these options and spot options. We examine how futures options can be priced, explore the relative pricing of futures options and spot options, and discuss what are known as futures-style options.

In 1976, Fischer Black proposed a model, now known as Black's model, for valuing European options on futures.[1] As this chapter will show, the model has proved to be an important alternative to the Black–Scholes–Merton model for valuing a wide range of European spot options.

18.1 NATURE OF FUTURES OPTIONS

A futures option is the right, but not the obligation, to enter into a futures contract at a certain futures price by a certain date. Specifically, a futures call option is the right to enter into a long futures contract at a certain price; a futures put option is the right to enter into a short futures contract at a certain price. Futures options are generally American; that is, they can be exercised any time during the life of the contract.

If a futures call option is exercised, the holder acquires a long position in the underlying futures contract plus a cash amount equal to the most recent settlement futures price minus the strike price. If a futures put option is exercised, the holder acquires a short position in the underlying futures contract plus a cash amount equal to the strike price minus the most recent settlement futures price. As the following examples show, the effective payoff from a futures call option is $\max(F - K, 0)$ and

[1] See F. Black, "The Pricing of Commodity Contracts," *Journal of Financial Economics*, 3 (1976), 167–79.

the effective payoff from a futures put option is $\max(K - F, 0)$, where F is the futures price at the time of exercise and K is the strike price.

Example 18.1

Suppose it is August 15 and a trader has one September futures call option contract on copper with a strike price of 320 cents per pound. One futures contract is on 25,000 pounds of copper. Suppose that the futures price of copper for delivery in September is currently 331 cents, and at the close of trading on August 14 (the last settlement) it was 330 cents. If the option is exercised, the trader receives a cash amount of

$$25{,}000 \times (330 - 320)\,\text{cents} = \$2{,}500$$

plus a long position in a futures contract to buy 25,000 pounds of copper in September. If desired, the position in the futures contract can be closed out immediately. This would leave the trader with the \$2,500 cash payoff plus an amount

$$25{,}000 \times (331 - 330)\,\text{cents} = \$250$$

reflecting the change in the futures price since the last settlement. The total payoff from exercising the option on August 15 is \$2,750, which equals $25{,}000(F - K)$, where F is the futures price at the time of exercise and K is the strike price.

Example 18.2

A trader has one December futures put option on corn with a strike price of 600 cents per bushel. One futures contract is on 5,000 bushels of corn. Suppose that the current futures price of corn for delivery in December is 580, and the most recent settlement price is 579 cents. If the option is exercised, the trader receives a cash amount of

$$5{,}000 \times (600 - 579)\,\text{cents} = \$1{,}050$$

plus a short position in a futures contract to sell 5,000 bushels of corn in December. If desired, the position in the futures contract can be closed out. This would leave the trader with the \$1,050 cash payoff minus an amount

$$5{,}000 \times (580 - 579)\,\text{cents} = \$50$$

reflecting the change in the futures price since the last settlement. The net payoff from exercise is \$1,000, which equals $5{,}000(K - F)$, where F is the futures price at the time of exercise and K is the strike price.

Expiration Months

Futures options are referred to by the delivery month of the underlying futures contract —not by the expiration month of the option. As mentioned earlier, most futures options are American. The expiration date of a futures option contract is usually a short period of time before the last trading day of the underlying futures contract. (For example, the CME Group Treasury bond futures option expires on the latest Friday that precedes by at least two business days the end of the month before the futures delivery month.) An exception is the CME Group mid-curve Eurodollar contract where the futures contract expires much later than the options contract.

Popular contracts trading in the United States are those on corn, soybeans, cotton, sugar-world, crude oil, natural gas, gold, Treasury bonds, Treasury notes, five-year Treasury notes, 30-day federal funds, Eurodollars, one-year and two-year mid-curve Eurodollars, Euribor, Eurobunds, and the S&P 500.

Options on Interest Rate Futures

The most actively traded interest rate options offered by exchanges in the United States are those on Treasury bond futures, Treasury note futures, and Eurodollar futures.

A Treasury bond futures option, which is traded by the CME Group, is an option to enter a Treasury bond futures contract. As mentioned in Chapter 6, one Treasury bond futures contract is for the delivery of \$100,000 of Treasury bonds. The price of a Treasury bond futures option is quoted as a percentage of the face value of the underlying Treasury bonds to the nearest sixty-fourth of 1%.

An option on Eurodollar futures, which is traded by the CME Group, is an option to enter into a Eurodollar futures contract. As explained in Chapter 6, when the Eurodollar futures quote changes by 1 basis point, or 0.01%, there is a gain or loss on a Eurodollar futures contract of \$25. Similarly, in the pricing of options on Eurodollar futures, 1 basis point represents \$25.

Interest rate futures option contracts work in the same way as the other futures options contracts discussed in this chapter. For example, in addition to the cash payoff, the holder of a call option obtains a long position in the futures contract when the option is exercised and the option writer obtains a corresponding short position. The total payoff from the call, including the value of the futures position, is $\max(F - K, 0)$, where F is the futures price at the time of exercise and K is the strike price.

Interest rate futures prices increase when bond prices increase (i.e., when interest rates fall). They decrease when bond prices decrease (i.e., when interest rates rise). An investor who thinks that short-term interest rates will rise can speculate by buying put options on Eurodollar futures, whereas an investor who thinks the rates will fall can speculate by buying call options on Eurodollar futures. An investor who thinks that long-term interest rates will rise can speculate by buying put options on Treasury note futures or Treasury bond futures, whereas an investor who thinks the rates will fall can speculate by buying call options on these instruments.

Example 18.3

It is February and the futures price for the June Eurodollar contract is 96.82 (corresponding to a 3-month Eurodollar interest rate of 3.18% per annum). The price of a call option on the contract with a strike price of 97.00 is quoted as 0.1, or 10 basis points. This option could be attractive to an investor who feels that interest rates are likely to come down. Suppose that short-term interest rates do drop by about 100 basis points and the investor exercises the call when the Eurodollar futures price is 97.78 (corresponding to a 3-month Eurodollar interest rate of 3.22% per annum). The payoff is $25 \times (97.78 - 97.00) \times 100 = \$1{,}950$. The cost of the contract is $10 \times 25 = \$250$. The investor's profit is therefore \$1,700.

Example 18.4

It is August and the futures price for the December Treasury bond contract is 96-09 (or $96\frac{9}{32} = 96.28125$). The yield on long-term government bonds is about 6.4% per annum. An investor who feels that this yield will fall by December might choose to buy December calls with a strike price of 98. Assume that the price of these calls is 1-04 (or $1\frac{4}{64} = 1.0625\%$ of the principal). If long-term rates fall to 6% per annum and the Treasury bond futures price rises to 100-00, the investor will make a net profit per \$100 of bond futures of

$$100.00 - 98.00 - 1.0625 = 0.9375$$

Since one option contract is for the purchase or sale of instruments with a face value of \$100,000, the investor's profit is \$937.50 per option contract bought.

18.2 REASONS FOR THE POPULARITY OF FUTURES OPTIONS

It is natural to ask why people choose to trade options on futures rather than options on the underlying asset. The main reason appears to be that a futures contract is, in many circumstances, more liquid and easier to trade than the underlying asset. Furthermore, a futures price is known immediately from trading on the futures exchange, whereas the spot price of the underlying asset may not be so readily available.

Consider Treasury bonds. The market for Treasury bond futures is much more active than the market for any particular Treasury bond. Also, a Treasury bond futures price is known immediately from exchange trading. By contrast, the current market price of a bond can be obtained only by contacting one or more dealers. It is not surprising that investors would rather take delivery of a Treasury bond futures contract than Treasury bonds.

Futures on commodities are also often easier to trade than the commodities themselves. For example, it is much easier and more convenient to make or take delivery of a live-cattle futures contract than it is to make or take delivery of the cattle.

An important point about a futures option is that exercising it does not usually lead to delivery of the underlying asset, as in most circumstances the underlying futures contract is closed out prior to delivery. Futures options are therefore normally eventually settled in cash. This is appealing to many investors, particularly those with limited capital who may find it difficult to come up with the funds to buy the underlying asset when an option on spot is exercised. Another advantage sometimes cited for futures options is that futures and futures options are traded on the same exchange. This facilitates hedging, arbitrage, and speculation. It also tends to make the markets more efficient. A final point is that futures options entail lower transaction costs than spot options in many situations.

18.3 EUROPEAN SPOT AND FUTURES OPTIONS

The payoff from a European call option with strike price K on the spot price of an asset is

$$\max(S_T - K,\ 0)$$

where S_T is the spot price at the option's maturity. The payoff from a European call option with the same strike price on the futures price of the asset is

$$\max(F_T - K,\ 0)$$

where F_T is the futures price at the option's maturity. If the futures contract matures at the same time as the option, then $F_T = S_T$ and the two options are equivalent. Similarly, a European futures put option is worth the same as its spot put option counterpart when the futures contract matures at the same time as the option.

Most of the futures options that trade are American-style. However, as we shall see, it is useful to study European futures options because the results that are obtained can be used to value the corresponding European spot options.

18.4 PUT–CALL PARITY

In Chapter 11, we derived a put–call parity relationship for European stock options. We now consider a similar argument to derive a put–call parity relationship for European futures options. Consider European futures call and put options, both with strike price K and time to expiration T. We can form two portfolios:

Portfolio A: a European futures call option plus an amount of cash equal to Ke^{-rT}
Portfolio B: a European futures put option plus a long futures contract plus an amount of cash equal to F_0e^{-rT}, where F_0 is the futures price

In portfolio A, the cash can be invested at the risk-free rate, r, and grows to K at time T. Let F_T be the futures price at maturity of the option. If $F_T > K$, the call option in portfolio A is exercised and portfolio A is worth F_T. If $F_T \leqslant K$, the call is not exercised and portfolio A is worth K. The value of portfolio A at time T is therefore

$$\max(F_T,\ K)$$

In portfolio B, the cash can be invested at the risk-free rate to grow to F_0 at time T. The put option provides a payoff of $\max(K - F_T,\ 0)$. The futures contract provides a payoff of $F_T - F_0$.[2] The value of portfolio B at time T is therefore

$$F_0 + (F_T - F_0) + \max(K - F_T,\ 0) = \max(F_T,\ K)$$

Because the two portfolios have the same value at time T and European options cannot be exercised early, it follows that they are worth the same today. The value of portfolio A today is

$$c + Ke^{-rT}$$

where c is the price of the futures call option. The daily settlement process ensures that the futures contract in portfolio B is worth zero today. Portfolio B is therefore worth

$$p + F_0e^{-rT}$$

where p is the price of the futures put option. Hence

$$c + Ke^{-rT} = p + F_0e^{-rT} \qquad \textbf{(18.1)}$$

The difference between this put–call parity relationship and the one for a non-dividend-paying stock in equation (11.6) is that the stock price, S_0, is replaced by the discounted futures price, F_0e^{-rT}.

---

[2] This analysis assumes that a futures contract is like a forward contract and settled at the end of its life rather than on a day-to-day basis.

As shown in Section 18.3, when the underlying futures contract matures at the same time as the option, European futures and spot options are the same. Equation (18.1) therefore gives a relationship between the price of a call option on the spot price, the price of a put option on the spot price, and the futures price when both options mature at the same time as the futures contract.

Example 18.5

Suppose that the price of a European call option on a commodity for delivery in six months is \$0.56 per ounce when the strike price is \$8.50. Assume that the futures price for delivery in six months is currently \$8.00, and the risk-free interest rate for an investment that matures in six months is 10% per annum. From a rearrangement of equation (18.1), the price of a European put option on the spot price with the same maturity and exercise date as the call option is

$$0.56 + 8.50e^{-0.1\times 6/12} - 8.00e^{-0.1\times 6/12} = 1.04$$

For American futures options, the put–call relationship is (see Problem 18.19)

$$F_0e^{-rT} - K < C - P < F_0 - Ke^{-rT} \tag{18.2}$$

18.5 BOUNDS FOR FUTURES OPTIONS

The put–call parity relationship in equation (18.1) provides bounds for European call and put options. Because the price of a put, p, cannot be negative, it follows from equation (18.1) that

$$c + Ke^{-rT} \geqslant F_0e^{-rT}$$

so that

$$c \geqslant \max\big((F_0 - K)e^{-rT},\ 0\big) \tag{18.3}$$

Similarly, because the price of a call option cannot be negative, it follows from equation (18.1) that

$$Ke^{-rT} \leqslant F_0e^{-rT} + p$$

so that

$$p \geqslant \max\big((K - F_0)e^{-rT},\ 0\big) \tag{18.4}$$

These bounds are analogous to the ones derived for European stock options in Chapter 11. The prices of European call and put options are very close to their lower bounds when the options are deep in the money. To see why this is so, we return to the put–call parity relationship in equation (18.1). When a call option is deep in the money, a put option with the same strike price is deep out of the money. This means that p is very close to zero. The difference between c and its lower bound equals p, so that the price of the call option must be very close to its lower bound. A similar argument applies to put options.

Because American futures options can be exercised at any time, we must have

$$C \geqslant \max(F_0 - K,\ 0)$$

and

$$P \geqslant \max(K - F_0,\ 0)$$

Thus, assuming interest rates are positive, the lower bound for an American option price is always higher than the lower bound for the corresponding European option price. There is always some chance that an American futures option will be exercised early.

18.6 DRIFT OF A FUTURES PRICE IN A RISK-NEUTRAL WORLD

There is a general result that allows us to use the analysis in Section 17.3 for futures options. This result is that in a risk-neutral world a futures price behaves in the same way as a stock paying a dividend yield at the domestic risk-free interest rate r.

One clue that this might be so is given by noting that the put–call parity relationship for futures options prices is the same as that for options on a stock paying a dividend yield at rate q when the stock price is replaced by the futures price and $q = r$ (compare equations (18.1) and (17.3)).

To prove the result formally, we calculate the drift of a futures price in a risk-neutral world. We define F_t as the futures price at time t and suppose the settlement dates are at times 0, Δt, $2\Delta t, \ldots$ If we enter into a long futures contract at time 0, its value is zero. At time Δt, it provides a payoff of $F_{\Delta t} - F_0$. If r is the very-short-term (Δt-period) interest rate at time 0, risk-neutral valuation gives the value of the contract at time 0 as

$$e^{-r\Delta t}\hat{E}[F_{\Delta t} - F_0]$$

where $\hat{E}$ denotes expectations in a risk-neutral world. We must therefore have

$$e^{-r\Delta t}\hat{E}(F_{\Delta t} - F_0) = 0$$

showing that

$$\hat{E}(F_{\Delta t}) = F_0$$

Similarly, $\hat{E}(F_{2\Delta t}) = F_{\Delta t}$, $\hat{E}(F_{3\Delta t}) = F_{2\Delta t}$, and so on. Putting many results like this together, we see that

$$\hat{E}(F_T) = F_0$$

for any time T.

The drift of the futures price in a risk-neutral world is therefore zero. From equation (17.7), the futures price behaves like a stock providing a dividend yield q equal to r. This result is a very general one. It is true for all futures prices and does not depend on any assumptions about interest rates, volatilities, etc.[3]

The usual assumption made for the process followed by a futures price F in the risk-neutral world is

$$dF = \sigma F\,dz \tag{18.5}$$

where σ is a constant.

Differential Equation

For another way of seeing that a futures price behaves like a stock paying a dividend yield at rate q, we can derive the differential equation satisfied by a derivative dependent

[3] As we will discover in Chapter 28, a more precise statement of the result is: "A futures price has zero drift in the traditional risk-neutral world where the numeraire is the money market account." A zero-drift stochastic process is known as a martingale. A forward price is a martingale in a different risk-neutral world. This is one where the numeraire is a zero-coupon bond maturing at time T.

on a futures price in the same way as we derived the differential equation for a derivative dependent on a non-dividend-paying stock in Section 15.6. This is[4]

$$\frac{\partial f}{\partial t} + \tfrac{1}{2}\frac{\partial^2 f}{\partial F^2}\sigma^2 F^2 = rf \tag{18.6}$$

It has the same form as equation (17.6) with q set equal to r. This confirms that, for the purpose of valuing derivatives, a futures price can be treated in the same way as a stock providing a dividend yield at rate r.

18.7 BLACK'S MODEL FOR VALUING FUTURES OPTIONS

European futures options can be valued by extending the results we have produced. Fischer Black was the first to show this in a paper published in 1976.[5] Assuming that the futures price follows the (lognormal) process in equation (18.5), the European call price c and the European put price p for a futures option are given by equations (17.4) and (17.5) with S_0 replaced by F_0 and $q = r$:

$$c = e^{-rT}[F_0 N(d_1) - KN(d_2)] \tag{18.7}$$

$$p = e^{-rT}[KN(-d_2) - F_0 N(-d_1)] \tag{18.8}$$

where

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(F_0/K) - \sigma^2 T/2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

and σ is the volatility of the futures price. When the cost of carry and the convenience yield are functions only of time, it can be shown that the volatility of the futures price is the same as the volatility of the underlying asset.

Example 18.6

Consider a European put futures option on a commodity. The time to the option's maturity is 4 months, the current futures price is \$20, the exercise price is \$20, the risk-free interest rate is 9% per annum, and the volatility of the futures price is 25% per annum. In this case, $F_0 = 20$, $K = 20$, $r = 0.09$, $T = 4/12$, $\sigma = 0.25$, and $\ln(F_0/K) = 0$, so that

$$d_1 = \frac{\sigma\sqrt{T}}{2} = 0.07216$$

$$d_2 = -\frac{\sigma\sqrt{T}}{2} = -0.07216$$

$$N(-d_1) = 0.4712, \quad N(-d_2) = 0.5288$$

[4] See Technical Note 7 at www-2.rotman.utoronto.ca/~hull/TechnicalNotes for a proof of this.

[5] See F. Black, "The Pricing of Commodity Contracts," *Journal of Financial Economics*, 3 (March 1976): 167–79.

and the put price p is given by

$$p = e^{-0.09\times 4/12}(20 \times 0.5288 - 20 \times 0.4712) = 1.12$$

or \$1.12.

18.8 USING BLACK'S MODEL INSTEAD OF BLACK–SCHOLES–MERTON

The results in Section 18.3 show that European futures options and European spot options are equivalent when the option contract matures at the same time as the futures contract. Specifically, equations (18.7) and (18.8) provide the price of a European option on the spot price of a asset where F_0 is the futures price of the asset for a contract maturing at the same time as the option.

Example 18.7

Consider a six-month European call option on the spot price of gold, that is, an option to buy one ounce of gold in the spot market in six months. The strike price is \$1,200, the six-month futures price of gold is \$1,240, the risk-free rate of interest is 5% per annum, and the volatility of the futures price is 20%. The option is the same as a six-month European option on the six-month futures price. The value of the option is therefore given by equation (18.7) as

$$e^{-0.05\times 0.5}[1{,}240N(d_1) - 1{,}200N(d_2)]$$

where

$$d_1 = \frac{\ln(1{,}240/1{,}200) + 0.2^2 \times 0.5/2}{0.2 \times \sqrt{0.5}} = 0.3026$$

$$d_2 = \frac{\ln(1{,}240/1{,}200) - 0.2^2 \times 0.5/2}{0.2 \times \sqrt{0.5}} = 0.1611$$

It is \$88.37.

Traders like to use Black's model rather than Black–Scholes–Merton to value European spot options. It has a fairly general applicability. The underlying asset can be a consumption or investment asset and it can provide income to the holder. Often F_0 is set equal to the forward price rather than the futures price. When interest rates are assumed to be deterministic, forward and futures prices are equal and so this is valid. As we will see later in the book, when interest rates are stochastic it is valid to set F_0 equal to the forward price provided that r is the risk-free rate for maturity T.

The big advantage of Black's model is that it avoids the need to estimate the income (or convenience yield) on the underlying asset. The futures or forward price that is used in the model incorporate the market's estimate of this income.

Equations (17.13) and (17.14) show Black's model being used to value European options on the spot value of a currency. In this case, Black's model avoids the need to estimate the foreign risk-free rate explicity because all information needed about this rate is captured by F_0. Equations (17.8) and (17.9) show Black's model being used to value European options on the spot value of an index. In this case, dividends on the underlying portfolio of stocks do not have to be estimated explicitly because all information needed about dividends is captured by F_0.

When considering stock indices in Section 17.4, we explained that put–call parity is used to imply the forward prices for maturities for which there are actively traded options. Interpolation is then used to estimate forward prices for other maturities. The same approach can be used for a wide range of other underlying assets.

18.9 VALUATION OF FUTURES OPTIONS USING BINOMIAL TREES

This section examines, more formally than in Chapter 13, how binomial trees can be used to price futures options. A key difference between futures options and stock options is that there are no up-front costs when a futures contract is entered into.

Suppose that the current futures price is 30 and that it will move either up to 33 or down to 28 over the next month. We consider a one-month call option on the futures with a strike price of 29 and ignore daily settlement. The situation is as indicated in Figure 18.1. If the futures price proves to be 33, the payoff from the option is 4 and the value of the futures contract is 3. If the futures price proves to be 28, the payoff from the option is zero and the value of the futures contract is -2.[6]

To set up a riskless hedge, we consider a portfolio consisting of a short position in one option contract and a long position in Δ futures contracts. If the futures price moves up to 33, the value of the portfolio is $3\Delta - 4$; if it moves down to 28, the value of the portfolio is -2Δ. The portfolio is riskless when these are the same, that is, when

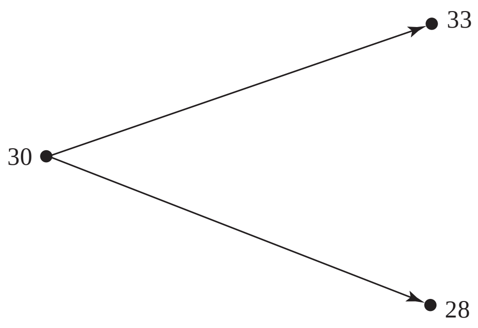
$$3\Delta - 4 = -2\Delta$$

or $\Delta = 0.8$.

For this value of Δ, we know the portfolio will be worth $3 \times 0.8 - 4 = -1.6$ in one month. Assume a risk-free interest rate of 6%. The value of the portfolio today must be

$$-1.6e^{-0.06\times 1/12} = -1.592$$

Figure 18.1 Futures price movements in numerical example.



[6] There is an approximation here in that the gain or loss on the futures contract is not realized at time T. It is realized day by day between time 0 and time T. However, as the length of the time step in a multistep binomial tree becomes shorter, the approximation becomes better.

The portfolio consists of one short option and Δ futures contracts. Because the value of the futures contract today is zero, the value of the option today must be 1.592.

A Generalization

We can generalize this analysis by considering a futures price that starts at F_0 and is anticipated to rise to F_0u or move down to F_0d over the time period T. We consider an option maturing at time T and suppose that its payoff is f_u if the futures price moves up and f_d if it moves down. The situation is summarized in Figure 18.2.

The riskless portfolio in this case consists of a short position in one option combined with a long position in Δ futures contracts, where

$$\Delta = \frac{f_u - f_d}{F_0u - F_0d}$$

The value of the portfolio at time T is then always

$$(F_0u - F_0)\Delta - f_u$$

Denoting the risk-free interest rate by r, we obtain the value of the portfolio today as

$$[(F_0u - F_0)\Delta - f_u]e^{-rT}$$

Another expression for the present value of the portfolio is $-f$, where f is the value of the option today. It follows that

$$-f = [(F_0u - F_0)\Delta - f_u]e^{-rT}$$

Substituting for Δ and simplifying reduces this equation to

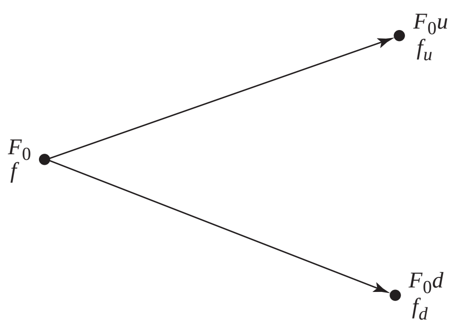
$$f = e^{-rT}[pf_u + (1-p)f_d] \tag{18.9}$$

where

$$p = \frac{1-d}{u-d} \tag{18.10}$$

This agrees with the result in Section 13.9. Equation (18.10) gives the risk-neutral probability of an up movement.

Figure 18.2 Futures price and option price in a general situation.



In the numerical example considered previously (see Figure 18.1), $u = 1.1$, $d = 0.9333$, $r = 0.06$, $T = 1/12$, $f_u = 4$, and $f_d = 0$. From equation (18.10),

$$p = \frac{1 - 0.9333}{1.1 - 0.9333} = 0.4$$

and, from equation (18.9),

$$f = e^{-0.06 \times 1/12}[0.4 \times 4 + 0.6 \times 0] = 1.592$$

This result agrees with the answer obtained for this example earlier.

Multistep Trees

Multistep binomial trees are used to value American-style futures options in much the same way that they are used to value options on stocks. This is explained in Section 13.9. The parameter u defining up movements in the futures price is $e^{\sigma\sqrt{\Delta t}}$, where σ is the volatility of the futures price and Δt is the length of one time step. The probability of an up movement in the future price is that in equation (18.10):

$$p = \frac{1 - d}{u - d}$$

Example 13.3 illustrates the use of multistep binomial trees for valuing a futures option. Example 21.3 in Chapter 21 provides a further illustration.

18.10 AMERICAN FUTURES OPTIONS vs. AMERICAN SPOT OPTIONS

Traded futures options are in practice usually American. Assuming that the risk-free rate of interest, r, is positive, there is always some chance that it will be optimal to exercise an American futures option early. American futures options are therefore worth more than their European counterparts.

It is not generally true that an American futures option is worth the same as the corresponding American spot option when the futures and options contracts have the same maturity.[7] Suppose, for example, that there is a normal market with futures prices consistently higher than spot prices prior to maturity. An American futures call option must be worth more than the corresponding American spot call option. The reason is that in some situations the futures option will be exercised early, in which case it will provide a greater profit to the holder. Similarly, an American futures put option must be worth less than the corresponding American spot put option. If there is an inverted market with futures prices consistently lower than spot prices, the reverse must be true. American futures call options are worth less than the corresponding American spot call option, whereas American futures put options are worth more than the corresponding American spot put option.

The differences just described between American futures options and American spot options hold true when the futures contract expires later than the options contract as well as when the two expire at the same time. In fact, the later the futures contract expires the greater the differences tend to be.

[7] The spot option "corresponding" to a futures option is defined here as one with the same strike price and the same expiration date.

18.11 FUTURES-STYLE OPTIONS

Some exchanges, particularly those in Europe, trade what are termed *futures-style options*. These are futures contracts on the payoff from an option. Normally a trader who buys (sells) an option, whether on the spot price of an asset or on the futures price of an asset, pays (receives) cash up front. By contrast, traders who buy or sell a futures-style option post margin in the same way that they do on a regular futures contract (see Chapter 2). The contract is settled daily as with any other futures contract and the final settlement price is the payoff from the option. Just as a futures contract is a bet on what the future price of an asset will be, a futures-style option is a bet on what the payoff from an option will be.[8] If interest rates are constant, the futures price in a futures-style option is the same as the forward price in a forward contract on the option payoff. Our analysis of forward contracts in Chapter 5 shows that this is the current option price compounded forward at the risk-free rate.

Black's model in equations (18.7) and (18.8) gives the current European option price. The futures price in a futures-style call option is therefore

$$F_0N(d_1) - KN(d_2)$$

and the futures price in a futures-style put option is

$$KN(-d_2) - F_0N(-d_1)$$

where d_1 and d_2 are as defined in equations (18.7) and (18.8). These formulas do not depend on the level of interest rates. They are correct for a futures-style option on a futures contract and a futures-style option on the spot value of an asset. In the first case, F_0 is the current futures price for the contract underlying the option; in the second case, it is the current futures price for a futures contract on the underlying asset maturing at the same time as the option.

The put–call parity relationship for a futures-style options is

$$p + F_0 = c + K$$

An American futures-style option can be exercised early, in which case there is an immediate final settlement at the option's intrinsic value. As it turns out, it is never optimal to exercise an American futures-style option early because the futures price of the option is always greater than the intrinsic value. This type of American futures-style option can therefore be treated as though it is European.

SUMMARY

Futures options require delivery of the underlying futures contract on exercise. When a call is exercised, the holder acquires a long futures position plus a cash amount equal to the excess of the futures price over the strike price. Similarly, when a put is exercised the holder acquires a short position plus a cash amount equal to the excess of the strike

[8] For a more detailed discussion of futures-style options, see D. Lieu, "Option Pricing with Futures-Style Margining," *Journal of Futures Markets*, 10, 4 (1990), 327–38. For pricing when interest rates are stochastic, see R.-R. Chen and L. Scott, "Pricing Interest Rate Futures Options with Futures-Style Margining." *Journal of Futures Markets*, 13, 1 (1993) 15–22.

price over the futures price. The futures contract that is delivered usually expires slightly later than the option.

A futures price behaves in the same way as a stock that provides a dividend yield equal to the risk-free rate, r. This means that the results produced in Chapter 17 for options on a stock paying a dividend yield apply to futures options if we replace the stock price by the futures price and set the dividend yield equal to the risk-free interest rate. Pricing formulas for European futures options were first produced by Fischer Black in 1976. They assume that the futures price is lognormally distributed at the option's expiration.

If the expiration dates for the option and futures contracts are the same, a European futures option is worth exactly the same as the corresponding European spot option. This result is often used to value European spot options. The result is not true for American options. If the futures market is normal, an American futures call is worth more than the corresponding American spot call option, while an American futures put is worth less than the corresponding American spot put option. If the futures market is inverted, the reverse is true.

FURTHER READING

Black, F. "The Pricing of Commodity Contracts," *Journal of Financial Economics*, 3 (1976): 167–79.

Practice Questions (Answers in Solutions Manual)

18.1. Explain the difference between a call option on yen and a call option on yen futures.

18.2. Why are options on bond futures more actively traded than options on bonds?

18.3. "A futures price is like a stock paying a dividend yield." What is the dividend yield?

18.4. A futures price is currently 50. At the end of six months it will be either 56 or 46. The risk-free interest rate is 6% per annum. What is the value of a six-month European call option on the futures with a strike price of 50?

18.5. How does the put–call parity formula for a futures option differ from put–call parity for an option on a non-dividend-paying stock?

18.6. Consider an American futures call option where the futures contract and the option contract expire at the same time. Under what circumstances is the futures option worth more than the corresponding American option on the underlying asset?

18.7. Calculate the value of a five-month European futures put option when the futures price is \$19, the strike price is \$20, the risk-free interest rate is 12% per annum, and the volatility of the futures price is 20% per annum.

18.8. Suppose you buy a put option contract on October gold futures with a strike price of \$1,400 per ounce. Each contract is for the delivery of 100 ounces. What happens if you exercise when the October futures price is \$1,380?

18.9. Suppose you sell a call option contract on April live cattle futures with a strike price of 130 cents per pound. Each contract is for the delivery of 40,000 pounds. What happens if the contract is exercised when the futures price is 135 cents?

18.10. Consider a two-month futures call option with a strike price of 40 when the risk-free interest rate is 10% per annum. The current futures price is 47. What is a lower bound for the value of the futures option if it is (a) European and (b) American?

18.11. Consider a four-month futures put option with a strike price of 50 when the risk-free interest rate is 10% per annum. The current futures price is 47. What is a lower bound for the value of the futures option if it is (a) European and (b) American?

18.12. A futures price is currently 60 and its volatility is 30%. The risk-free interest rate is 8% per annum. Use a two-step binomial tree to calculate the value of a six-month European call option on the futures with a strike price of 60. If the call were American, would it ever be worth exercising it early?

18.13. In Problem 18.12, what does the binomial tree give for the value of a six-month European put option on futures with a strike price of 60? If the put were American, would it ever be worth exercising it early? Verify that the call prices calculated in Problem 18.12 and the put prices calculated here satisfy put–call parity relationships.

18.14. A futures price is currently 25, its volatility is 30% per annum, and the risk-free interest rate is 10% per annum. What is the value of a nine-month European call on the futures with a strike price of 26?

18.15. A futures price is currently 70, its volatility is 20% per annum, and the risk-free interest rate is 6% per annum. What is the value of a five-month European put on the futures with a strike price of 65?

18.16. Suppose that a one-year futures price is currently 35. A one-year European call option and a one-year European put option on the futures with a strike price of 34 are both priced at 2 in the market. The risk-free interest rate is 10% per annum. Identify an arbitrage opportunity.

18.17. "The price of an at-the-money European futures call option always equals the price of a similar at-the-money European futures put option." Explain why this statement is true.

18.18. Suppose that a futures price is currently 30. The risk-free interest rate is 5% per annum. A three-month American futures call option with a strike price of 28 is worth 4. Calculate bounds for the price of a three-month American futures put option with a strike price of 28.

18.19. Show that, if C is the price of an American call option on a futures contract when the strike price is K and the maturity is T, and P is the price of an American put on the same futures contract with the same strike price and exercise date, then

$$F_0 e^{-rT} - K < C - P < F_0 - Ke^{-rT}$$

where F_0 is the futures price and r is the risk-free rate. Assume that $r > 0$ and that there is no difference between forward and futures contracts. (*Hint*: Use an analogous approach to that indicated for Problem 17.12.)

18.20. Calculate the price of a three-month European call option on the spot value of silver. The three-month futures price is \$12, the strike price is \$13, the risk-free rate is 4% and the volatility of the price of silver is 25%.

18.21. A corporation knows that in three months it will have \$5 million to invest for 90 days at LIBOR minus 50 basis points and wishes to ensure that the rate obtained will be at least 6.5%. What position in exchange-traded options should it take to hedge?

Further Questions

18.22. A futures price is currently 40. It is known that at the end of three months the price will be either 35 or 45. What is the value of a three-month European call option on the futures with a strike price of 42 if the risk-free interest rate is 7% per annum?

18.23. The futures price of an asset is currently 78 and the risk-free rate is 3%. A six-month put on the futures with a strike price of 80 is currently worth 6.5. What is the value of a six-month call on the futures with a strike price of 80 if both the put and call are European? What is the range of possible values of the six-month call with a strike price of 80 if both put and call are American?

18.24. Use a three-step tree to value an American futures put option when the futures price is 50, the life of the option is 9 months, the strike price is 50, the risk-free rate is 3%, and the volatility is 25%.

18.25. It is February 4. July call options on corn futures with strike prices of 260, 270, 280, 290, and 300 cost 26.75, 21.25, 17.25, 14.00, and 11.375, respectively. July put options with these strike prices cost 8.50, 13.50, 19.00, 25.625, and 32.625, respectively. The options mature on June 19, the current July corn futures price is 278.25, and the risk-free interest rate is 1.1%. Calculate implied volatilities for the options using DerivaGem. Comment on the results you get.

18.26. Calculate the implied volatility of soybean futures prices from the following information concerning a European put on soybean futures:

Current futures price	525
Exercise price	525
Risk-free rate	6% per annum
Time to maturity	5 months
Put price	20

18.27. Calculate the price of a six-month European put option on the spot value of the S&P 500. The six-month forward price of the index is 1,400, the strike price is 1,450, the risk-free rate is 5%, and the volatility of the index is 15%.

18.28. The strike price of a futures option is 550 cents, the risk-free interest rate is 3%, the volatility of the futures price is 20%, and the time to maturity of the option is 9 months. The futures price is 500 cents.

(a) What is the price of the option if it is a European call?
(b) What is the price of the option if it is a European put?
(c) Verify that put–call parity holds.
(d) What is the futures price for a futures-style option if it is a call?
(e) What is the futures price for a futures-style option if it is a put?

CHAPTER 19 The Greek Letters

A financial institution that sells an option to a client in the over-the-counter markets is faced with the problem of managing its risk. If the option happens to be the same as one that is traded actively on an exchange or in the OTC market, the financial institution can neutralize its exposure by buying the same option as it has sold. But when the option has been tailored to the needs of a client and does not correspond to the standardized products traded by exchanges, hedging the exposure is far more difficult.

In this chapter we discuss some of the alternative approaches to this problem. We cover what are commonly referred to as the "Greek letters", or simply the "Greeks". Each Greek letter measures a different dimension to the risk in an option position and the aim of a trader is to manage the Greeks so that all risks are acceptable. The analysis presented in this chapter is applicable to market makers in options on an exchange as well as to traders working in the over-the-counter market for financial institutions.

Toward the end of the chapter, we will consider the creation of options synthetically. This turns out to be very closely related to the hedging of options. Creating an option position synthetically is essentially the same task as hedging the opposite option position. For example, creating a long call option synthetically is the same as hedging a short position in the call option.

19.1 ILLUSTRATION

In the next few sections we use as an example the position of a financial institution that has sold for \$300,000 a European call option on 100,000 shares of a non-dividend-paying stock. We assume that the stock price is \$49, the strike price is \$50, the risk-free interest rate is 5% per annum, the stock price volatility is 20% per annum, the time to maturity is 20 weeks (0.3846 years), and the expected return from the stock is 13% per annum.[1] With our usual notation, this means that

$$S_0 = 49, \quad K = 50, \quad r = 0.05, \quad \sigma = 0.20, \quad T = 0.3846, \quad \mu = 0.13$$

The Black–Scholes–Merton price of the option is about \$240,000. (This is because the

[1] As shown in Chapters 13 and 15, the expected return is irrelevant to the pricing of an option. It is given here because it can have some bearing on the effectiveness of a hedging procedure.

value of an option to buy one share is \$2.40.) The financial institution has therefore sold a product for \$60,000 more than its theoretical value. But it is faced with the problem of hedging the risks.[2]

19.2 NAKED AND COVERED POSITIONS

One strategy open to the financial institution is to do nothing. This is sometimes referred to as a *naked position*. It is a strategy that works well if the stock price is below \$50 at the end of the 20 weeks. The option then costs the financial institution nothing and it makes a profit of \$300,000. A naked position works less well if the call is exercised because the financial institution then has to buy 100,000 shares at the market price prevailing in 20 weeks to cover the call. The cost to the financial institution is 100,000 times the amount by which the stock price exceeds the strike price. For example, if after 20 weeks the stock price is \$60, the option costs the financial institution \$1,000,000. This is considerably greater than the \$300,000 charged for the option.

As an alternative to a naked position, the financial institution can adopt a *covered position*. This involves buying 100,000 shares as soon as the option has been sold. If the option is exercised, this strategy works well, but in other circumstances it could lead to a significant loss. For example, if the stock price drops to \$40, the financial institution loses \$900,000 on its stock position. This is also considerably greater than the \$300,000 charged for the option.[3]

Neither a naked position nor a covered position provides a good hedge. If the assumptions underlying the Black–Scholes–Merton formula hold, the cost to the financial institution should always be \$240,000 on average for both approaches.[4] But on any one occasion the cost is liable to range from zero to over \$1,000,000. A good hedge would ensure that the cost is always close to \$240,000.

A Stop-Loss Strategy

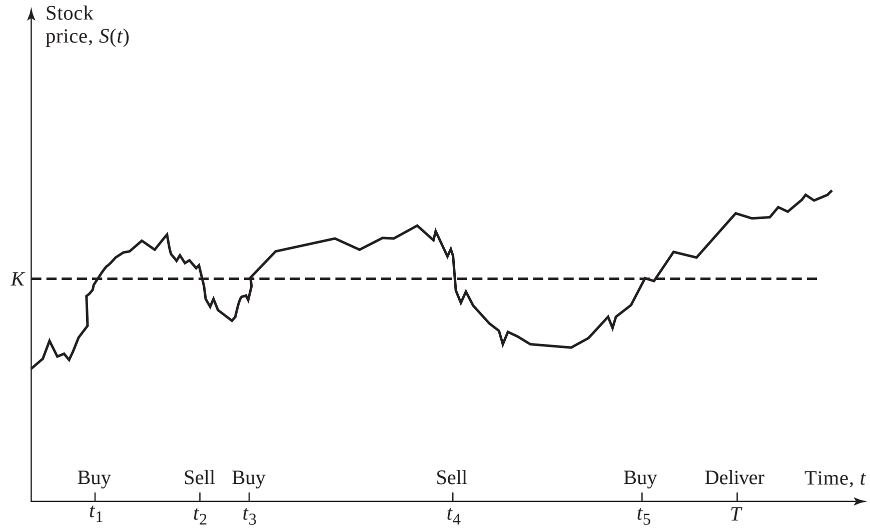
One interesting hedging procedure that is sometimes proposed involves a *stop-loss strategy*. To illustrate the basic idea, consider an institution that has written a call option with strike price K to buy one unit of a stock. The hedging procedure involves buying one unit of the stock as soon as its price rises above K and selling it as soon as its price falls below K. The objective is to hold a naked position whenever the stock price is less than K and a covered position whenever the stock price is greater than K. The procedure is designed to ensure that at time T the institution owns the stock if the option closes in the money and does not own it if the option closes out of the money. In the situation illustrated in Figure 19.1, it involves buying the stock at time t_1, selling it at time t_2, buying it at time t_3, selling it at time t_4, buying it at time t_5, and delivering it at time T.

[2] A call option on a non-dividend-paying stock is a convenient example with which to develop our ideas. The points that will be made apply to other types of options and to other derivatives.

[3] Put–call parity shows that the exposure from writing a covered call is the same as the exposure from writing a naked put.

[4] More precisely, the present value of the expected cost is \$240,000 for both approaches assuming that appropriate risk-adjusted discount rates are used.

Figure 19.1 A stop-loss strategy.



As usual, we denote the initial stock price by S_0. The cost of setting up the hedge initially is S_0 if $S_0 > K$ and zero otherwise. It seems as though the total cost, Q, of writing and hedging the option is the option's initial intrinsic value:

$$Q = \max(S_0 - K,\ 0) \tag{19.1}$$

This is because all purchases and sales subsequent to time 0 are made at price K. If this were in fact correct, the hedging procedure would work perfectly in the absence of transaction costs. Furthermore, the cost of hedging the option would always be less than its Black–Scholes–Merton price. Thus, a trader could earn riskless profits by writing options and hedging them.

There are two key reasons why equation (19.1) is incorrect. The first is that the cash flows to the hedger occur at different times and must be discounted. The second is that purchases and sales cannot be made at exactly the same price K. This second point is critical. If we assume a risk-neutral world with zero interest rates, we can justify ignoring the time value of money. But we cannot legitimately assume that both purchases and sales are made at the same price. If markets are efficient, the hedger cannot know whether, when the stock price equals K, it will continue above or below K.

As a practical matter, purchases must be made at a price $K + \epsilon$ and sales must be made at a price $K - \epsilon$, for some small positive number ϵ. Thus, every purchase and subsequent sale involves a cost (apart from transaction costs) of 2ϵ. A natural response on the part of the hedger is to monitor price movements more closely, so that ϵ is reduced. Assuming that stock prices change continuously, ϵ can be made arbitrarily small by monitoring the stock prices closely. But as ϵ is made smaller, trades tend to occur more frequently. Thus, the lower cost per trade is offset by the increased frequency of trading. As $\epsilon \to 0$, the expected number of trades tends to infinity.[5]

[5] As mentioned in Section 14.2, the expected number of times a Wiener process equals any particular value in a given time interval is infinite.

Table 19.1 Performance of stop-loss strategy. The performance measure is the ratio of the standard deviation of the cost of writing the option and hedging it to the theoretical price of the option.

Δt (weeks)	5	4	2	1	0.5	0.25
Hedge performance	0.98	0.93	0.83	0.79	0.77	0.76

A stop-loss strategy, although superficially attractive, does not work particularly well as a hedging procedure. Consider its use for an out-of-the-money option. If the stock price never reaches the strike price K, the hedging procedure costs nothing. If the path of the stock price crosses the strike price level many times, the procedure is quite expensive. Monte Carlo simulation can be used to assess the overall performance of stop-loss hedging. This involves randomly sampling paths for the stock price and observing the results of using the procedure. Table 19.1 shows the results for the option considered in Section 19.1. It assumes that the stock price is observed at the end of time intervals of length Δt.[6] The hedge performance measure in Table 19.1 is the ratio of the standard deviation of the cost of hedging the option to the Black–Scholes–Merton price. (The cost of hedging was calculated as the cumulative cost excluding the impact of interest payments and discounting.) Each result is based on one million sample paths for the stock price. An effective hedging scheme should have a hedge performance measure close to zero. In this case, it seems to stay above 0.7 regardless of how small Δt is. This emphasizes that the stop-loss strategy is not a good hedging procedure.

19.3 GREEK LETTER CALCULATION

Most traders use more sophisticated hedging procedures than those mentioned so far. These hedging procedures involve calculating measures such as delta, gamma, and vega. The measures are collectively referred to as *Greek letters*. They quantify different aspects of the risk in an option position. This chapter considers the properties of some of most important Greek letters.

In order to calculate a Greek letter, it is necessary to assume an option pricing model. Traders usually assume the Black–Scholes–Merton model (or its extensions in Chapters 17 and 18) for European options and the binomial tree model (introduced in Chapter 13) for American options. (As has been pointed out, the latter makes the same assumptions as Black–Scholes–Merton model.) When calculating Greek letters, traders normally set the volatility equal to the current implied volatility. This approach, which is sometimes referred to as using the ‘‘practitioner Black–Scholes model,’’ is appealing. When volatility is set equal to the implied volatility, the model gives the option price at a particular time as an exact function of the price of the underlying asset, the implied volatility, interest rates, and (possibly) dividends. The only way the option price can change in a short time period is if one of these variables changes. A trader naturally feels confident if the risks of changes in all these variables have been adequately hedged.

[6] The precise hedging rule used was as follows. If the stock price moves from below K to above K in a time interval of length Δt, it is bought at the end of the interval. If it moves from above K to below K in the time interval, it is sold at the end of the interval; otherwise, no action is taken.

In this chapter, we first consider the calculation of Greek letters for a European option on a non-dividend-paying stock. We then present results for other European options. Chapter 21 will show how Greek letters can be calculated for American-style options.

19.4 DELTA HEDGING

The *delta* (Δ) of an option was introduced in Chapter 13. It is defined as the rate of change of the option price with respect to the price of the underlying asset. It is the slope of the curve that relates the option price to the underlying asset price. Suppose that the delta of a call option on a stock is 0.6. This means that when the stock price changes by a small amount, the option price changes by about 60% of that amount. Figure 19.2 shows the relationship between a call price and the underlying stock price. When the stock price corresponds to point A, the option price corresponds to point B, and Δ is the slope of the line indicated. In general,

$$\Delta = \frac{\partial c}{\partial S}$$

where c is the price of the call option and S is the stock price.

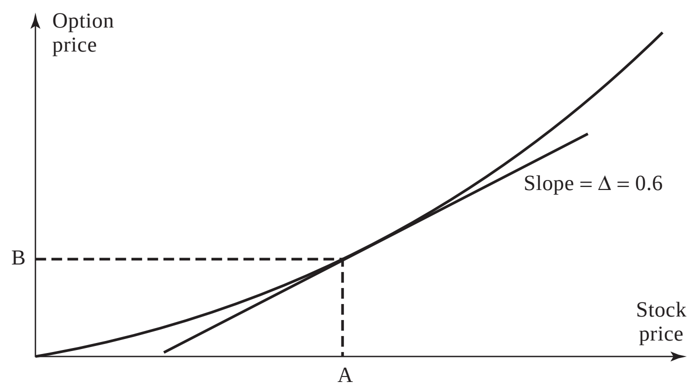
Suppose that, in Figure 19.2, the stock price is \$100 and the option price is \$10. Imagine an investor who has sold call options to buy 2,000 shares of a stock. The investor's position could be hedged by buying $0.6 \times 2{,}000 = 1{,}200$ shares. The gain (loss) on the stock position would then tend to offset the loss (gain) on the option position. For example, if the stock price goes up by \$1 (producing a gain of \$1,200 on the shares purchased), the option price will tend to go up by $0.6 \times \$1 = \$0.60$ (producing a loss of \$1,200 on the options written); if the stock price goes down by \$1 (producing a loss of \$1,200 on the shares purchased), the option price will tend to go down by \$0.60 (producing a gain of \$1,200 on the options written).

In this example, the delta of the trader's short position in 2,000 options is

$$0.6 \times (-2{,}000) = -1{,}200$$

This means that the trader loses $1{,}200\Delta S$ on the option position when the stock price

Figure 19.2 Calculation of delta.



increases by ΔS. The delta of one share of the stock is 1.0, so that the long position in 1,200 shares has a delta of +1,200. The delta of the trader's overall position in our example is, therefore, zero. The delta of the stock position offsets the delta of the option position. A position with a delta of zero is referred to as *delta neutral*.

It is important to realize that, since the delta of an option does not remain constant, the trader's position remains delta hedged (or delta neutral) for only a relatively short period of time. The hedge has to be adjusted periodically. This is known as *rebalancing*. In our example, by the end of 1 day the stock price might have increased to \$110. As indicated by Figure 19.2, an increase in the stock price leads to an increase in delta. Suppose that delta rises from 0.60 to 0.65. An extra $0.05 \times 2{,}000 = 100$ shares would then have to be purchased to maintain the hedge. A procedure such as this, where the hedge is adjusted on a regular basis, is referred to as *dynamic hedging*. It can be contrasted with *static hedging*, where a hedge is set up initially and never adjusted. Static hedging is sometimes also referred to as "hedge-and-forget."

Delta is closely related to the Black–Scholes–Merton analysis. As explained in Chapter 15, the Black–Scholes–Merton differential equation can be derived by setting up a riskless portfolio consisting of a position in an option on a stock and a position in the stock. Expressed in terms of Δ, the portfolio is

-1: option

$+\Delta$: shares of the stock.

Using our new terminology, we can say that options can be valued by setting up a delta-neutral position and arguing that the return on the position should (instantaneously) be the risk-free interest rate.

Delta of European Stock Options

For a European call option on a non-dividend-paying stock, it can be shown (see Problem 15.17) that the Black–Scholes–Merton model gives

$$\Delta(\text{call}) = N(d_1)$$

Figure 19.3 Variation of delta with stock price for (a) a call option and (b) a put option on a non-dividend-paying stock ($K = 50$, $r = 0$, $\sigma = 25\%$, $T = 2$).

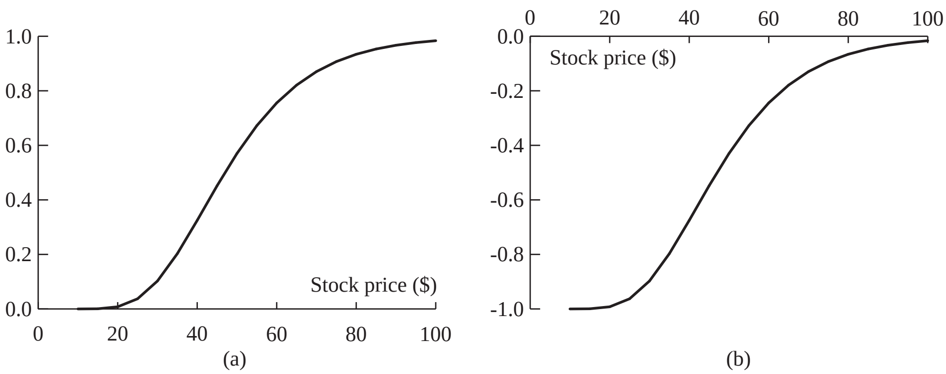
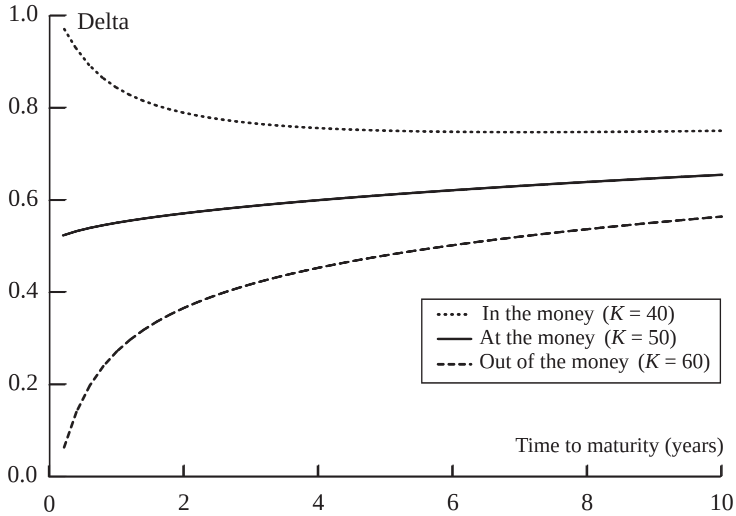


Figure 19.4 Typical patterns for variation of delta with time to maturity for a call option ($S_0 = 50$, $r = 0$, $\sigma = 25\%$).



where d_1 is defined as in equation (15.20) and $N(x)$ is the cumulative distribution function for a standard normal distribution. The formula gives the delta of a long position in one call option. The delta of a short position in one call option is $-N(d_1)$. Using delta hedging for a short position in a European call option involves maintaining a long position of $N(d_1)$ for each option sold. Similarly, using delta hedging for a long position in a European call option involves maintaining a short position of $N(d_1)$ shares for each option purchased.

For a European put option on a non-dividend-paying stock, delta is given by

$$\Delta(\text{put}) = N(d_1) - 1$$

Delta is negative, which means that a long position in a put option should be hedged with a long position in the underlying stock, and a short position in a put option should be hedged with a short position in the underlying stock. Figure 19.3 shows the variation of the delta of a call option and a put option with the stock price. Figure 19.4 shows the variation of delta with the time to maturity for in-the-money, at-the-money, and out-of-the-money call options.

Example 19.1

Consider again the call option on a non-dividend-paying stock in Section 19.1 where the stock price is \$49, the strike price is \$50, the risk-free rate is 5%, the time to maturity is 20 weeks (= 0.3846 years), and the volatility is 20%. In this case,

$$d_1 = \frac{\ln(49/50) + (0.05 + 0.2^2/2) \times 0.3846}{0.2 \times \sqrt{0.3846}} = 0.0542$$

Delta is $N(d_1)$, or 0.522. When the stock price changes by ΔS, the option price changes by $0.522\Delta S$.

Dynamic Aspects of Delta Hedging

Tables 19.2 and 19.3 provide two examples of the operation of delta hedging for the example in Section 19.1, where 100,000 call options are sold. The hedge is assumed to be adjusted or rebalanced weekly and the assumptions underlying the Black–Scholes–Merton model are assumed to hold with the volatility staying constant at 20%. The initial value of delta for a single option is calculated in Example 19.1 as 0.522. This means that the delta of the option position is initially $-100{,}000 \times 0.522$, or $-52{,}200$. As soon as the option is written, \$2,557,800 must be borrowed to buy 52,200 shares at a price of \$49 to create a delta-neutral position. The rate of interest is 5%. An interest cost of approximately \$2,500 is therefore incurred in the first week.

In Table 19.2, the stock price falls by the end of the first week to \$48.12. The delta of the option declines to 0.458, so that the new delta of the option position is $-45{,}800$. This means that 6,400 of the shares initially purchased are sold to maintain the delta-neutral hedge. The strategy realizes \$308,000 in cash, and the cumulative borrowings at the end of Week 1 are reduced to \$2,252,300. During the second week, the stock price reduces to \$47.37, delta declines again, and so on. Toward the end of the life of the option, it becomes apparent that the option will be exercised and the delta of the option approaches 1.0. By Week 20, therefore, the hedger has a fully covered position. The

Table 19.2 Simulation of delta hedging. Option closes in the money and cost of hedging is \$263,300.

Week	*Stock price*	*Delta*	*Shares purchased*	*Cost of shares purchased (\$000)*	*Cumulative cost including interest (\$000)*	*Interest cost (\$000)*
0	49.00	0.522	52,200	2,557.8	2,557.8	2.5
1	48.12	0.458	(6,400)	(308.0)	2,252.3	2.2
2	47.37	0.400	(5,800)	(274.7)	1,979.8	1.9
3	50.25	0.596	19,600	984.9	2,966.6	2.9
4	51.75	0.693	9,700	502.0	3,471.5	3.3
5	53.12	0.774	8,100	430.3	3,905.1	3.8
6	53.00	0.771	(300)	(15.9)	3,893.0	3.7
7	51.87	0.706	(6,500)	(337.2)	3,559.5	3.4
8	51.38	0.674	(3,200)	(164.4)	3,398.5	3.3
9	53.00	0.787	11,300	598.9	4,000.7	3.8
10	49.88	0.550	(23,700)	(1,182.2)	2,822.3	2.7
11	48.50	0.413	(13,700)	(664.4)	2,160.6	2.1
12	49.88	0.542	12,900	643.5	2,806.2	2.7
13	50.37	0.591	4,900	246.8	3,055.7	2.9
14	52.13	0.768	17,700	922.7	3,981.3	3.8
15	51.88	0.759	(900)	(46.7)	3,938.4	3.8
16	52.87	0.865	10,600	560.4	4,502.6	4.3
17	54.87	0.978	11,300	620.0	5,126.9	4.9
18	54.62	0.990	1,200	65.5	5,197.3	5.0
19	55.87	1.000	1,000	55.9	5,258.2	5.1
20	57.25	1.000	0	0.0	5,263.3	

Table 19.3 Simulation of delta hedging. Option closes out of the money and cost of hedging is $256,600.

Week	*Stock price*	*Delta*	*Shares purchased*	*Cost of shares purchased ($000)*	*Cumulative cost including interest ($000)*	*Interest cost ($000)*
0	49.00	0.522	52,200	2,557.8	2,557.8	2.5
1	49.75	0.568	4,600	228.9	2,789.2	2.7
2	52.00	0.705	13,700	712.4	3,504.3	3.4
3	50.00	0.579	(12,600)	(630.0)	2,877.7	2.8
4	48.38	0.459	(12,000)	(580.6)	2,299.9	2.2
5	48.25	0.443	(1,600)	(77.2)	2,224.9	2.1
6	48.75	0.475	3,200	156.0	2,383.0	2.3
7	49.63	0.540	6,500	322.6	2,707.9	2.6
8	48.25	0.420	(12,000)	(579.0)	2,131.5	2.1
9	48.25	0.410	(1,000)	(48.2)	2,085.4	2.0
10	51.12	0.658	24,800	1,267.8	3,355.2	3.2
11	51.50	0.692	3,400	175.1	3,533.5	3.4
12	49.88	0.542	(15,000)	(748.2)	2,788.7	2.7
13	49.88	0.538	(400)	(20.0)	2,771.4	2.7
14	48.75	0.400	(13,800)	(672.7)	2,101.4	2.0
15	47.50	0.236	(16,400)	(779.0)	1,324.4	1.3
16	48.00	0.261	2,500	120.0	1,445.7	1.4
17	46.25	0.062	(19,900)	(920.4)	526.7	0.5
18	48.13	0.183	12,100	582.4	1,109.6	1.1
19	46.63	0.007	(17,600)	(820.7)	290.0	0.3
20	48.12	0.000	(700)	(33.7)	256.6	

hedger receives $5 million for the stock held, so that the total cost of writing the option and hedging it is $263,300.

Table 19.3 illustrates an alternative sequence of events such that the option closes out of the money. As it becomes clear that the option will not be exercised, delta approaches zero. By Week 20 the hedger has a naked position and has incurred costs totaling $256,600.

In Tables 19.2 and 19.3, the costs of hedging the option, when discounted to the beginning of the period, are close to but not exactly the same as the Black–Scholes–Merton price of $240,000. If the hedging worked perfectly, the cost of hedging would, after discounting, be exactly equal to the Black–Scholes–Merton price for every simulated stock price path. The reason for the variation in the hedging cost is that the hedge is rebalanced only once a week. As rebalancing takes place more frequently, the variation in the hedging cost is reduced. Of course, the examples in Tables 19.2 and 19.3 are idealized in that they assume that the volatility is constant and there are no transaction costs.

Table 19.4 shows statistics on the performance of delta hedging obtained from one million random stock price paths in our example. The performance measure is calculated, similarly to Table 19.1, as the ratio of the standard deviation of the cost of hedging the option to the Black–Scholes–Merton price of the option. It is clear that delta hedging is a

Table 19.4 Performance of delta hedging. The performance measure is the ratio of the standard deviation of the cost of writing the option and hedging it to the theoretical price of the option.

Time between hedge rebalancing (weeks):	5	4	2	1	0.5	0.25
Performance measure:	0.42	0.38	0.28	0.21	0.16	0.13

great improvement over a stop-loss strategy. Unlike a stop-loss strategy, the performance of delta-hedging gets steadily better as the hedge is monitored more frequently.

Delta hedging aims to keep the value of the financial institution's position as close to unchanged as possible. Initially, the value of the written option is \$240,000. In the situation depicted in Table 19.2, the value of the option can be calculated as \$414,500 in Week 9. (This value is obtained from the Black–Scholes–Merton model by setting the stock price equal to \$53 and the time to maturity equal to 11 weeks.) Thus, the financial institution has lost \$174,500 on its short option position. Its cash position, as measured by the cumulative cost, is \$1,442,900 worse in Week 9 than in Week 0. The value of the shares held has increased from \$2,557,800 to \$4,171,100. The net effect of all this is that the value of the financial institution's position has changed by only \$4,100 between Week 0 and Week 9.

Where the Cost Comes From

The delta-hedging procedure in Tables 19.2 and 19.3 creates the equivalent of a long position in the option. This neutralizes the short position the financial institution created by writing the option. As the tables illustrate, delta hedging a short position generally involves selling stock just after the price has gone down and buying stock just after the price has gone up. It might be termed a buy-high, sell-low trading strategy! The average cost of \$240,000 comes from the present value of the difference between the price at which stock is purchased and the price at which it is sold.

Delta of a Portfolio

The delta of a portfolio of options or other derivatives dependent on a single asset whose price is S is

$$\frac{\partial \Pi}{\partial S}$$

where Π is the value of the portfolio.

The delta of the portfolio can be calculated from the deltas of the individual options in the portfolio. If a portfolio consists of a quantity w_i of option i $(1 \leqslant i \leqslant n)$, the delta of the portfolio is given by

$$\Delta = \sum_{i=1}^{n} w_i \, \Delta_i$$

where Δ_i is the delta of the ith option. The formula can be used to calculate the position in the underlying asset necessary to make the delta of the portfolio zero. When this position has been taken, the portfolio is delta neutral.

Suppose a financial institution has the following three positions in options on a stock:

1. A long position in 100,000 call options with strike price \$55 and an expiration date in 3 months. The delta of each option is 0.533.
2. A short position in 200,000 call options with strike price \$56 and an expiration date in 5 months. The delta of each option is 0.468.
3. A short position in 50,000 put options with strike price \$56 and an expiration date in 2 months. The delta of each option is −0.508.

The delta of the whole portfolio is

$$100{,}000 \times 0.533 - 200{,}000 \times 0.468 - 50{,}000 \times (-0.508) = -14{,}900$$

This means that the portfolio can be made delta neutral by buying 14,900 shares.

Transaction Costs

Derivatives dealers usually rebalance their positions once a day to maintain delta neutrality. When the dealer has a small number of options on a particular asset, this is liable to be prohibitively expensive because of the bid–offer spreads the dealer is subject to on trades. For a large portfolio of options, it is more feasible. Only one trade in the underlying asset is necessary to zero out delta for the whole portfolio. The bid–offer spread transaction costs are absorbed by the profits on many different trades.

19.5 THETA

The *theta* (Θ) of a portfolio of options is the rate of change of the value of the portfolio with respect to the passage of time with all else remaining the same. Theta is sometimes referred to as the *time decay* of the portfolio. For a European call option on a non-dividend-paying stock, it can be shown from the Black–Scholes–Merton formula (see Problem 15.17) that

$$\Theta(\text{call}) = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2)$$

where d_1 and d_2 are defined as in equation (15.20) and

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \qquad \textbf{(19.2)}$$

is the probability density function for a standard normal distribution.

For a European put option on the stock,

$$\Theta(\text{put}) = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2)$$

Because $N(-d_2) = 1 - N(d_2)$, the theta of a put exceeds the theta of the corresponding call by rKe^{-rT}.

In these formulas, time is measured in years. Usually, when theta is quoted, time is measured in days, so that theta is the change in the portfolio value when 1 day passes with all else remaining the same. We can measure theta either "per calendar day" or "per trading day." To obtain the theta per calendar day, the formula for theta must be divided by 365; to obtain theta per trading day, it must be divided by 252. (DerivaGem measures theta per calendar day.)

Example 19.2

As in Example 19.1, consider a call option on a non-dividend-paying stock where the stock price is \$49, the strike price is \$50, the risk-free rate is 5%, the time to maturity is 20 weeks (= 0.3846 years), and the volatility is 20%. In this case, $S_0 = 49$, $K = 50$, $r = 0.05$, $\sigma = 0.2$, and $T = 0.3846$.

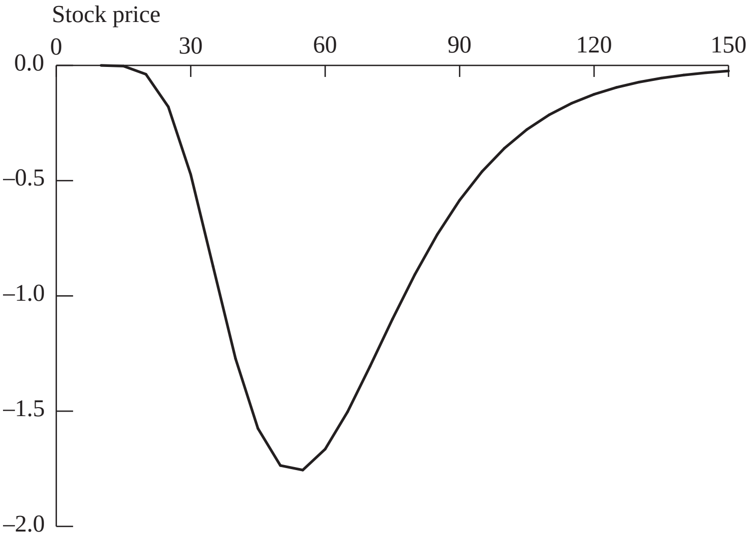
The option's theta is

$$-\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2) = -4.31$$

The theta is $-4.31/365 = -0.0118$ per calendar day, or $-4.31/252 = -0.0171$ per trading day.

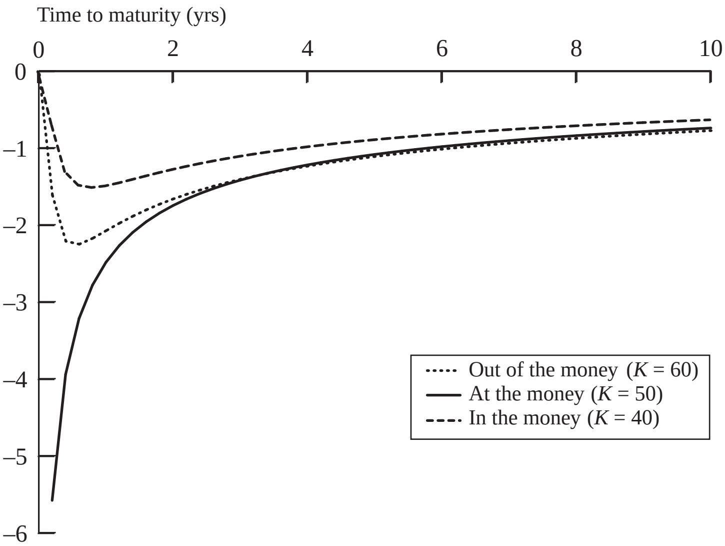
Theta is usually negative for an option.[7] This is because, as time passes with all else remaining the same, the option tends to become less valuable. The variation of Θ with stock price for a call option on a stock is shown in Figure 19.5. When the stock price is very low, theta is close to zero. For an at-the-money call option, theta is large and negative. As the stock price becomes larger, theta tends to $-rKe^{-rT}$. (In our example, $r = 0$.) Figure 19.6 shows typical patterns for the variation of Θ with the time to maturity for in-the-money, at-the-money, and out-of-the-money call options.

Figure 19.5 Variation of theta of a European call option with stock price ($K = 50$, $r = 0$, $\sigma = 0.25$, $T = 2$).



[7] An exception to this could be an in-the-money European put option on a non-dividend-paying stock or an in-the-money European call option on a currency with a very high interest rate.

Figure 19.6 Typical patterns for variation of theta of a European call option with time to maturity ($S_0 = 50,\ K = 50,\ r = 0,\ \sigma = 25\%$).



Theta is not the same type of hedge parameter as delta. There is uncertainty about the future stock price, but there is no uncertainty about the passage of time. It makes sense to hedge against changes in the price of the underlying asset, but it does not make any sense to hedge against the passage of time. In spite of this, many traders regard theta as a useful descriptive statistic for a portfolio. This is because, as we shall see later, in a delta-neutral portfolio theta is a proxy for gamma.

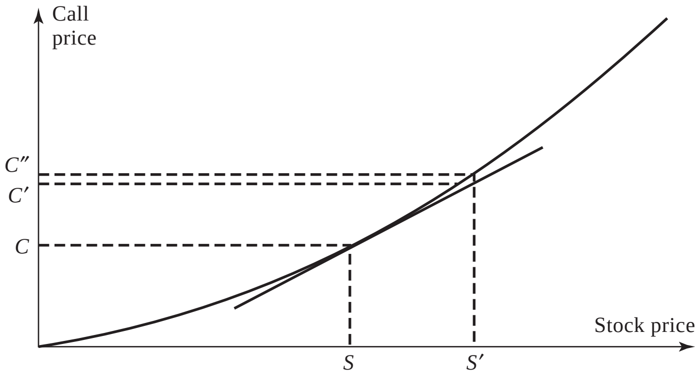
19.6 GAMMA

The *gamma* (Γ) of a portfolio of options on an underlying asset is the rate of change of the portfolio's delta with respect to the price of the underlying asset. It is the second partial derivative of the portfolio with respect to asset price:

$$\Gamma = \frac{\partial^2 \Pi}{\partial S^2}$$

If gamma is small, delta changes slowly, and adjustments to keep a portfolio delta neutral need to be made only relatively infrequently. However, if gamma is highly negative or highly positive, delta is very sensitive to the price of the underlying asset. It is then quite risky to leave a delta-neutral portfolio unchanged for any length of time. Figure 19.7 illustrates this point. When the stock price moves from S to S', delta hedging assumes that the option price moves from C to C', when in fact it moves from C to C''. The difference between C' and C'' leads to a hedging error. The size of the error depends on the curvature of the relationship between the option price and the stock price. Gamma measures this curvature.

Figure 19.7 Hedging error introduced by nonlinearity.



Suppose that ΔS is the price change of an underlying asset during a small interval of time, Δt, and $\Delta \Pi$ is the corresponding price change in the portfolio. The appendix at the end of this chapter shows that, if terms of order higher than Δt are ignored,

$$\Delta \Pi = \Theta\, \Delta t + \tfrac{1}{2}\Gamma\, \Delta S^2 \tag{19.3}$$

for a delta-neutral portfolio, where Θ is the theta of the portfolio. Figure 19.8 shows the nature of the relationship between $\Delta \Pi$ and ΔS. When gamma is positive, theta tends to be negative. The portfolio declines in value if there is no change in S, but increases in value if there is a large positive or negative change in S. When gamma is negative, theta tends to be positive and the reverse is true: the portfolio increases in value if there is no change in S but decreases in value if there is a large positive or negative change in S. As the absolute value of gamma increases, the sensitivity of the value of the portfolio to S increases.

Example 19.3

Suppose that the gamma of a delta-neutral portfolio of options on an asset is $-10{,}000$. Equation (19.3) shows that, if a change of $+2$ or -2 in the price of the asset occurs over a short period of time, there is an unexpected decrease in the value of the portfolio of approximately $0.5 \times 10{,}000 \times 2^2 = \$20{,}000$.

Making a Portfolio Gamma Neutral

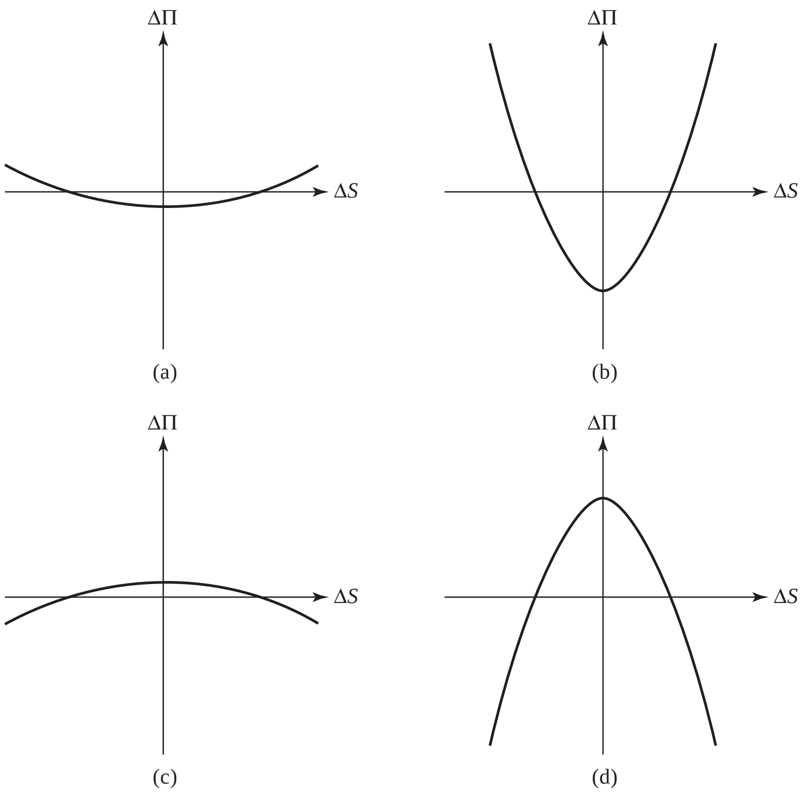
A position in the underlying asset has zero gamma and cannot be used to change the gamma of a portfolio. What is required is a position in an instrument such as an option that is not linearly dependent on the underlying asset.

Suppose that a delta-neutral portfolio has a gamma equal to Γ, and a traded option has a gamma equal to Γ_T. If the number of traded options added to the portfolio is w_T, the gamma of the portfolio is

$$w_T\, \Gamma_T + \Gamma$$

Hence, the position in the traded option necessary to make the portfolio gamma neutral is $-\Gamma/\Gamma_T$. Including the traded option is likely to change the delta of the portfolio, so

Figure 19.8 Relationship between $\Delta\Pi$ and ΔS in time Δt for a delta-neutral portfolio with (a) slightly positive gamma, (b) large positive gamma, (c) slightly negative gamma, and (d) large negative gamma.



the position in the underlying asset then has to be changed to maintain delta neutrality. Note that the portfolio is gamma neutral only for a short period of time. As time passes, gamma neutrality can be maintained only if the position in the traded option is adjusted so that it is always equal to $-\Gamma/\Gamma_T$.

Making a portfolio gamma neutral as well as delta-neutral can be regarded as a correction for the hedging error illustrated in Figure 19.7. Delta neutrality provides protection against relatively small stock price moves between rebalancing. Gamma neutrality provides protection against larger movements in this stock price between hedge rebalancing. Suppose that a portfolio is delta neutral and has a gamma of $-3{,}000$. The delta and gamma of a particular traded call option are 0.62 and 1.50, respectively. The portfolio can be made gamma neutral by including in the portfolio a long position of

$$\frac{3{,}000}{1.5} = 2{,}000$$

in the call option. However, the delta of the portfolio will then change from zero to $2{,}000 \times 0.62 = 1{,}240$. Therefore 1,240 units of the underlying asset must be sold from the portfolio to keep it delta neutral.

Calculation of Gamma

For a European call or put option on a non-dividend-paying stock, the gamma given by the Black–Scholes–Merton model is

$$\Gamma = \frac{N'(d_1)}{S_0\sigma\sqrt{T}}$$

where d_1 is defined as in equation (15.20) and $N'(x)$ is as given by equation (19.2). The gamma of a long position is always positive and varies with S_0 in the way indicated in Figure 19.9. The variation of gamma with time to maturity for out-of-the-money, at-the-money, and in-the-money options is shown in Figure 19.10. For an at-the-money option, gamma increases as the time to maturity decreases. Short-life at-the-money options have very high gammas, which means that the value of the option holder's position is highly sensitive to jumps in the stock price.

Example 19.4

As in Example 19.1, consider a call option on a non-dividend-paying stock where the stock price is \$49, the strike price is \$50, the risk-free rate is 5%, the time to maturity is 20 weeks (= 0.3846 years), and the volatility is 20%. In this case, $S_0 = 49$, $K = 50$, $r = 0.05$, $\sigma = 0.2$, and $T = 0.3846$.

The option's gamma is

$$\frac{N'(d_1)}{S_0\sigma\sqrt{T}} = 0.066$$

When the stock price changes by ΔS, the delta of the option changes by $0.066\Delta S$.

Figure 19.9 Variation of gamma with stock price for an option ($K = 50$, $r = 0$, $\sigma = 25\%$, $T = 2$).

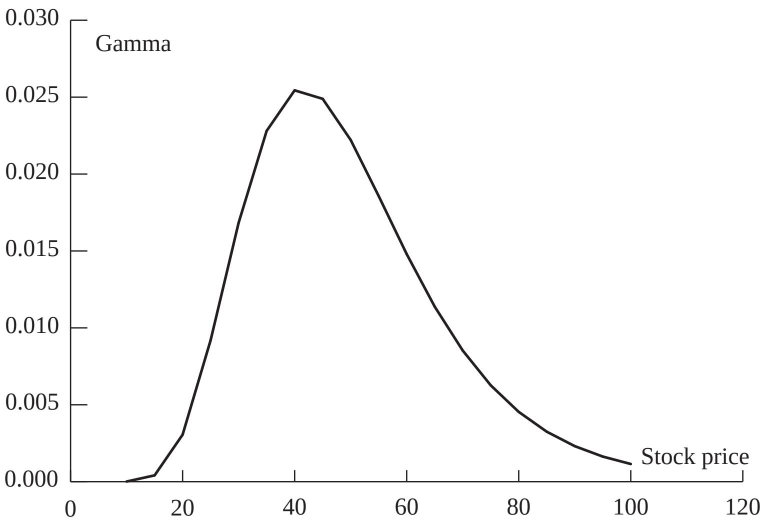
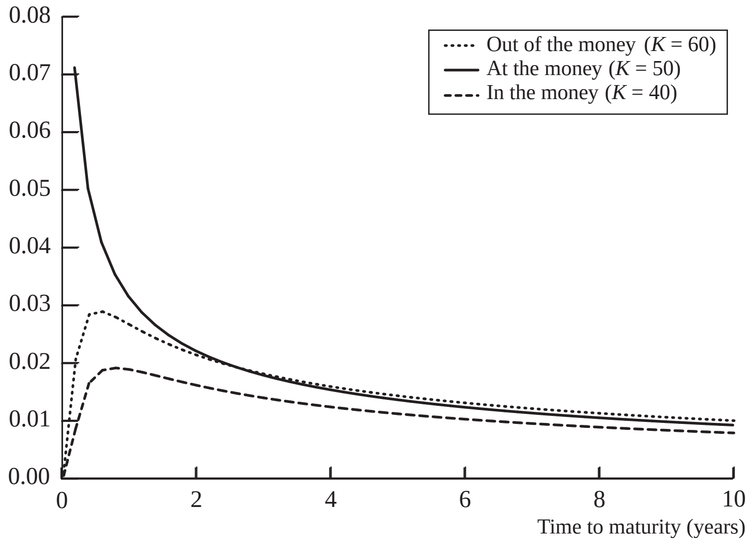


Figure 19.10 Variation of gamma with time to maturity for a stock option ($S_0 = 50$, $K = 50$, $r = 0$, $\sigma = 25\%$).



19.7 RELATIONSHIP BETWEEN DELTA, THETA, AND GAMMA

The price of a single derivative dependent on a non-dividend-paying stock must satisfy the differential equation (15.16). It follows that the value of Π of a portfolio of such derivatives also satisfies the differential equation

$$\frac{\partial \Pi}{\partial t} + rS\frac{\partial \Pi}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 \Pi}{\partial S^2} = r\Pi$$

Since

$$\Theta = \frac{\partial \Pi}{\partial t}, \qquad \Delta = \frac{\partial \Pi}{\partial S}, \qquad \Gamma = \frac{\partial^2 \Pi}{\partial S^2}$$

it follows that

$$\Theta + rS\Delta + \tfrac{1}{2}\sigma^2 S^2 \Gamma = r\Pi \tag{19.4}$$

Similar results can be produced for other underlying assets (see Problem 19.19).

For a delta-neutral portfolio, $\Delta = 0$ and

$$\Theta + \tfrac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

This shows that, when Θ is large and positive, gamma of a portfolio tends to be large and negative, and vice versa. This is consistent with the way in which Figure 19.8 has been drawn and explains why theta can to some extent be regarded as a proxy for gamma in a delta-neutral portfolio.

19.8 VEGA

As mentioned in Section 19.3, when Greek letters are calculated the volatility of the asset is in practice usually set equal to its implied volatility. The Black–Scholes–Merton model assumes that the volatility of the asset underlying an option is constant. This means that the implied volatilities of all options on the asset are constant and equal to this assumed volatility.

But in practice the volatility of an asset changes over time. As a result, the value of an option is liable to change because of movements in volatility as well as because of changes in the asset price and the passage of time. The vega of an option, $\mathcal{V}$, is the rate of change in its value with respect to the volatility of the underlying asset:[8]

$$\mathcal{V} = \frac{\partial f}{\partial \sigma}$$

where f is the option price and the volatility measure, σ, is usually the option's implied volatility. When vega is highly positive or highly negative, there is a high sensitivity to changes in volatility. If the vega of an option position is close to zero, volatility changes have very little effect on the value of the position.

A position in the underlying asset has zero vega. Vega cannot therefore be changed by taking a position in the underlying asset. In this respect, vega is like gamma. A complication is that different options in a portfolio are liable to have different implied volatilities. If all implied volatilities are assumed to change by the same amount during any short period of time, vega can be treated like gamma and the vega risk in a portfolio of options can be hedged by taking a position in a single option. If $\mathcal{V}$ is the vega of a portfolio and $\mathcal{V}_T$ is the vega of a traded option, a position of $-\mathcal{V}/\mathcal{V}_T$ in the traded option makes the portfolio instantaneously vega neutral. Unfortunately, a portfolio that is gamma neutral will not in general be vega neutral, and vice versa. If a hedger requires a portfolio to be both gamma and vega neutral, at least two traded options dependent on the underlying asset must be used.

Example 19.5

Consider a portfolio that is delta neutral, with a gamma of −5,000 and a vega (measuring sensitivity to implied volatility) of −8,000. The options shown in the following table can be traded. The portfolio can be made vega neutral by including a long position in 4,000 of Option 1. This would increase delta to 2,400 and require that 2,400 units of the asset be sold to maintain delta neutrality. The gamma of the portfolio would change from −5,000 to −3,000.

	Delta	*Gamma*	*Vega*
Portfolio	0	−5000	−8000
Option 1	0.6	0.5	2.0
Option 2	0.5	0.8	1.2

To make the portfolio gamma and vega neutral, both Option 1 and Option 2 can be used. If w_1 and w_2 are the quantities of Option 1 and Option 2 that are

[8] Vega is the name given to one of the "Greek letters" in option pricing, but it is not one of the letters in the Greek alphabet.

added to the portfolio, we require that

$$-5{,}000 + 0.5w_1 + 0.8w_2 = 0$$

and

$$-8{,}000 + 2.0w_1 + 1.2w_2 = 0$$

The solution to these equations is $w_1 = 400$, $w_2 = 6{,}000$. The portfolio can therefore be made gamma and vega neutral by including 400 of Option 1 and 6,000 of Option 2. The delta of the portfolio, after the addition of the positions in the two traded options, is $400 \times 0.6 + 6{,}000 \times 0.5 = 3{,}240$. Hence, 3,240 units of the asset would have to be sold to maintain delta neutrality.

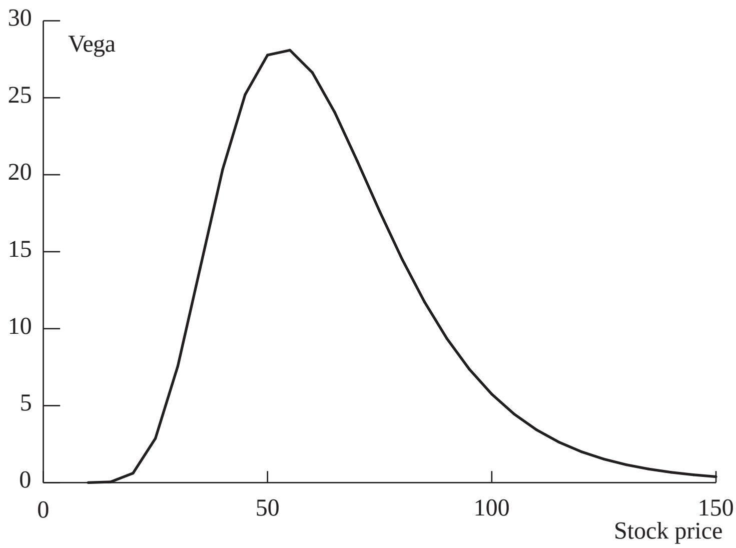
Hedging in the way indicated in Example 19.5 assumes that the implied volatilities of all options in a portfolio will change by the same amount during a short period of time. In practice, this is not necessarily true and a trader's hedging problem is more complex. As we will see in the next chapter, for any given underlying asset a trader monitors a "volatility surface" that describes the implied volatilities of options with different strike prices and times to maturity. The trader's total vega risk for a portfolio is related to the different ways in which the volatility surface can change.

For a European call or put option on a non-dividend-paying stock, vega given by the Black–Scholes–Merton model is

$$\mathcal{V} = S_0\sqrt{T}\,N'(d_1)$$

where d_1 is defined as in equation (15.20). The formula for $N'(x)$ is given in equation (19.2). The vega of a long position in a European or American option is always positive. The general way in which vega varies with S_0 is shown in Figure 19.11.

Figure 19.11 Variation of vega with stock price for an option ($K = 50$, $r = 0$, $\sigma = 25\%$, $T = 2$).



Example 19.6

As in Example 19.1, consider a call option on a non-dividend-paying stock where the stock price is \$49, the strike price is \$50, the risk-free rate is 5%, the time to maturity is 20 weeks (= 0.3846 years), and the implied volatility is 20%. In this case, $S_0 = 49$, $K = 50$, $r = 0.05$, $\sigma = 0.2$, and $T = 0.3846$.

The option's vega is

$$S_0\sqrt{T}N'(d_1) = 12.1$$

Thus a 1% (0.01) increase in the implied volatility from (20% to 21%) increases the value of the option by approximately $0.01 \times 12.1 = 0.121$.

Calculating vega from the Black–Scholes–Merton model and its extensions may seem strange because one of the assumptions underlying the model is that volatility is constant. It would be theoretically more correct to calculate vega from a model in which volatility is assumed to be stochastic.[9] However, traders prefer the simpler approach of measuring vega in terms of potential movements in the Black–Scholes–Merton implied volatility.

Gamma neutrality protects against large changes in the price of the underlying asset between hedge rebalancing. Vega neutrality protects against changes in volatility. As might be expected, whether it is best to use an available traded option for vega or gamma hedging depends on the time between hedge rebalancing and the volatility of the volatility.[10]

When volatilities change, the implied volatilities of short-dated options tend to change by more than the implied volatilities of long-dated options. The vega of a portfolio is therefore often calculated by changing the volatilities of long-dated options by less than that of short-dated options. One way of doing this is discussed in Section 23.6.

19.9 RHO

The *rho* of an option is the rate of change of its price f with respect to the interest rate r:

$$\frac{\partial f}{\partial r}$$

It measures the sensitivity of the value of a portfolio to a change in the interest rate when all else remains the same. In practice (at least for European options) r is usually set equal to the risk-free rate for a maturity equal to the option's maturity (see Section 28.6). This means that a trader has exposure to movements in the whole term structure when the options in the trader's portfolio have different maturities. For a European call option on a non-dividend-paying stock,

$$\text{rho (call)} = KTe^{-rT}N(d_2)$$

where d_2 is defined as in equation (15.20). For a European put option,

$$\text{rho (put)} = -KTe^{-rT}N(-d_2)$$

[9] See Chapter 27 for a discussion of stochastic volatility models.

[10] For a discussion of this issue, see J.C. Hull and A. White, "Hedging the Risks from Writing Foreign Currency Options," *Journal of International Money and Finance* 6 (June 1987): 131–52.

Example 19.7

As in Example 19.1, consider a call option on a non-dividend-paying stock where the stock price is \$49, the strike price is \$50, the risk-free rate is 5%, the time to maturity is 20 weeks (= 0.3846 years), and the volatility is 20%. In this case, $S_0 = 49$, $K = 50$, $r = 0.05$, $\sigma = 0.2$, and $T = 0.3846$.

The option's rho is

$$KTe^{-rT}N(d_2) = 8.91$$

This means that a 1% (0.01) increase in the risk-free rate (from 5% to 6%) increases the value of the option by approximately $0.01 \times 8.91 = 0.0891$.

19.10 THE REALITIES OF HEDGING

In an ideal world, traders working for financial institutions would be able to rebalance their portfolios very frequently in order to maintain all Greeks equal to zero. In practice, this is not possible. When managing a large portfolio dependent on a single underlying asset, traders usually make delta zero, or close to zero, at least once a day by trading the underlying asset. Unfortunately, a zero gamma and a zero vega are less easy to achieve because it is difficult to find options or other nonlinear derivatives that can be traded in the volume required at competitive prices. Business Snapshot 19.1 provides a discussion of how dynamic hedging is organized at financial institutions.

As already mentioned, there are big economies of scale in trading derivatives. Maintaining delta neutrality for a small number of options on an asset by trading daily is usually not economically feasible because the trading costs per option hedged are high.[11] But when a derivatives dealer maintains delta neutrality for a large portfolio of options on an asset, the trading costs per option hedged are more reasonable.

19.11 SCENARIO ANALYSIS

In addition to monitoring risks such as delta, gamma, and vega, option traders often also carry out a scenario analysis. The analysis involves calculating the gain or loss on their portfolio over a specified period under a variety of different scenarios. The time period chosen is likely to depend on the liquidity of the instruments. The scenarios can be either chosen by management or generated by a model.

Consider a bank with a portfolio of options dependent on the USD/EUR exchange rate. The two key variables on which the value of the portfolio depends are the exchange rate and the exchange-rate volatility. The bank could calculate a table such as Table 19.5 showing the profit or loss experienced during a 2-week period under different scenarios. This table considers seven different exchange rate movements and three different implied volatility movements. The table makes the simplifying assumption that the implied volatilities of all options in the portfolio change by the same amount. (Note: +2% would indicate a volatility change from 10% to 12%, not 10% to 10.2%.)

[11] The trading costs arise from the fact that each day the hedger buys some of the underlying asset at the offer price or sells some of the underlying asset at the bid price.

Business Snapshot 19.1 Dynamic Hedging in Practice

In a typical arrangement at a financial institution, the responsibility for a portfolio of derivatives dependent on a particular underlying asset is assigned to one trader or to a group of traders working together. For example, one trader at Goldman Sachs might be assigned responsibility for all derivatives dependent on the value of the Australian dollar. A computer system calculates the value of the portfolio and Greek letters for the portfolio. Limits are defined for each Greek letter and special permission is required if a trader wants to exceed a limit at the end of a trading day.

The delta limit is often expressed as the equivalent maximum position in the underlying asset. For example, the delta limit for a stock at a particular bank might be $1 million. If the stock price is $50, this means that the absolute value of delta as we have calculated it can be no more than 20,000. The vega limit is usually expressed as a maximum dollar exposure per 1% change in implied volatilities.

As a matter of course, options traders make themselves delta neutral—or close to delta neutral—at the end of each day. Gamma and vega are monitored, but are not usually managed on a daily basis. Financial institutions often find that their business with clients involves writing options and that as a result they accumulate negative gamma and vega. They are then always looking out for opportunities to manage their gamma and vega risks by buying options at competitive prices.

There is one aspect of an options portfolio that mitigates problems of managing gamma and vega somewhat. Options are often close to the money when they are first sold, so that they have relatively high gammas and vegas. But after some time has elapsed, the underlying asset price has often changed enough for them to become deep out of the money or deep in the money. Their gammas and vegas are then very small and of little consequence. A nightmare scenario for an options trader is where written options remain very close to the money as the maturity date is approached.

In Table 19.5, the greatest loss is in the lower right corner of the table. The loss corresponds to implied volatilities increasing by 2% and the exchange rate moving up by 0.06. Usually the greatest loss in a table such as Table 19.5 occurs at one of the corners, but this is not always so. Consider, for example, the situation where a bank's portfolio consists of a short position in a butterfly spread (see Section 12.3). The greatest loss will be experienced if the exchange rate stays where it is.

Table 19.5 Profit or loss realized in 2 weeks under different scenarios ($ million).

Implied volatility changes	*Exchange rate change*						
	−0.06	*−0.04*	*−0.02*	*0.00*	*+0.02*	*+0.04*	*+0.06*
−2%	+102	+55	+25	+6	−10	−34	−80
0%	+80	+40	+17	+2	−14	−38	−85
+2%	+60	+25	+9	−2	−18	−42	−90

19.12 EXTENSION OF FORMULAS

The formulas produced so far for delta, theta, gamma, vega, and rho have been for a European option on a non-dividend-paying stock. Table 19.6 shows how they change when the stock pays a continuous dividend yield at rate q. The expressions for d_1 and d_2 are as for equations (17.4) and (17.5). By setting q equal to the dividend yield on an index, we obtain the Greek letters for European options on indices. By setting q equal to the foreign risk-free rate, we obtain the Greek letters for European options on a currency. By setting $q = r$, we obtain delta, gamma, theta, and vega for European options on a futures contract. The rho for a call futures option is $-cT$ and the rho for a European put futures option is $-pT$.

In the case of currency options, there are two rhos corresponding to the two interest rates. The rho corresponding to the domestic interest rate is given by the formula in Table 19.6 (with d_2 as in equation (17.11)). The rho corresponding to the foreign interest rate for a European call on a currency is

$$\text{rho(call, foreign rate)} = -Te^{-r_f T} S_0 N(d_1)$$

For a European put, it is

$$\text{rho(put, foreign rate)} = Te^{-r_f T} S_0 N(-d_1)$$

with d_1 as in equation (17.11).

The calculation of Greek letters for American options is discussed in Chapter 21.

Delta of Forward Contracts

The concept of delta can be applied to financial instruments other than options. Consider a forward contract on a non-dividend-paying stock. Equation (5.5) shows that the value of a forward contract is $S_0 - Ke^{-rT}$, where K is the delivery price and T is the forward contract's time to maturity. When the price of the stock changes by ΔS, with all else remaining the same, the value of a forward contract on the stock also changes by ΔS. The

Table 19.6 Greek letters for European options on an asset providing a yield at rate q.

Greek letter	*Call option*	*Put option*
Delta	$e^{-qT}N(d_1)$	$e^{-qT}[N(d_1) - 1]$
Gamma	$\dfrac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$	$\dfrac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$
Theta	$-S_0N'(d_1)\sigma e^{-qT}\big/(2\sqrt{T})$ $+ qS_0N(d_1)e^{-qT} - rKe^{-rT}N(d_2)$	$-S_0N'(d_1)\sigma e^{-qT}\big/(2\sqrt{T})$ $- qS_0N(-d_1)e^{-qT} + rKe^{-rT}N(-d_2)$
Vega	$S_0\sqrt{T}N'(d_1)e^{-qT}$	$S_0\sqrt{T}N'(d_1)e^{-qT}$
Rho	$KTe^{-rT}N(d_2)$	$-KTe^{-rT}N(-d_2)$

delta of a long forward contract on one share of the stock is therefore always 1.0. This means that a long forward contract on one share can be hedged by shorting one share; a short forward contract on one share can be hedged by purchasing one share.[12]

For an asset providing a dividend yield at rate q, equation (5.7) shows that the forward contract's delta is e^{-qT}. For the delta of a forward contract on a stock index, q is set equal to the dividend yield on the index in this expression. For the delta of a forward foreign exchange contract, it is set equal to the foreign risk-free rate, r_f.

Delta of a Futures Contract

From equation (5.1), the futures price for a contract on a non-dividend-paying stock is S_0e^{rT}, where T is the time to maturity of the futures contract. This shows that when the price of the stock changes by ΔS, with all else remaining the same, the futures price changes by $\Delta S\, e^{rT}$. Since futures contracts are settled daily, the holder of a long futures position makes an almost immediate gain of this amount. The delta of a futures contract is therefore e^{rT}. For a futures position on an asset providing a dividend yield at rate q, equation (5.3) shows similarly that delta is $e^{(r-q)T}$.

It is interesting that daily settlement makes the deltas of futures and forward contracts slightly different. This is true even when interest rates are constant and the forward price equals the futures price. (A related point is made in Business Snapshot 5.2.)

Sometimes a futures contract is used to achieve a delta-neutral position. Define:

T: Maturity of futures contract

H_A: Required position in asset for delta hedging

H_F: Alternative required position in futures contracts for delta hedging.

If the underlying asset is a non-dividend-paying stock, the analysis we have just given shows that

$$H_F = e^{-rT} H_A \tag{19.5}$$

When the underlying asset pays a dividend yield q,

$$H_F = e^{-(r-q)T} H_A \tag{19.6}$$

For a stock index, we set q equal to the dividend yield on the index; for a currency, we set it equal to the foreign risk-free rate, r_f, so that

$$H_F = e^{-(r-r_f)T} H_A \tag{19.7}$$

Example 19.8

Suppose that a portfolio of currency options held by a U.S. bank can be made delta neutral with a short position of 458,000 pounds sterling. Risk-free rates are 4% in the United States and 7% in the United Kingdom. From equation (19.7), hedging using 9-month currency futures requires a short futures position

$$e^{-(0.04-0.07)\times 9/12} \times 458{,}000$$

or £468,442. Since each futures contract is for the purchase or sale of £62,500, seven contracts would be shorted. (Seven is the nearest whole number to 468,442/62,500.)

[12] These are hedge-and-forget schemes. Since delta is always 1.0, no changes need to be made to the position in the stock during the life of the contract.

19.13 PORTFOLIO INSURANCE

A portfolio manager is often interested in acquiring a put option on his or her portfolio. This provides protection against market declines while preserving the potential for a gain if the market does well. One approach (discussed in Section 17.1) is to buy put options on a market index such as the S&P 500. An alternative is to create the options synthetically.

Creating an option synthetically involves maintaining a position in the underlying asset (or futures on the underlying asset) so that the delta of the position is equal to the delta of the required option. The position necessary to create an option synthetically is the reverse of that necessary to hedge it. This is because the procedure for hedging an option involves the creation of an equal and opposite option synthetically.

There are two reasons why it may be more attractive for the portfolio manager to create the required put option synthetically than to buy it in the market. First, option markets do not always have the liquidity to absorb the trades required by managers of large funds. Second, fund managers often require strike prices and exercise dates that are different from those available in exchange-traded options markets.

The synthetic option can be created from trading the portfolio or from trading in index futures contracts. We first examine the creation of a put option by trading the portfolio. From Table 19.6, the delta of a European put on the portfolio is

$$\Delta = e^{-qT}[N(d_1) - 1] \tag{19.8}$$

where, with our usual notation,

$$d_1 = \frac{\ln(S_0/K) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}$$

The other variables are defined as usual: S_0 is the value of the portfolio, K is the strike price, r is the risk-free rate, q is the dividend yield on the portfolio, σ is the volatility of the portfolio, and T is the life of the option. The volatility of the portfolio can usually be assumed to be its beta times the volatility of a well-diversified market index.

To create the put option synthetically, the fund manager should ensure that at any given time a proportion

$$e^{-qT}[1 - N(d_1)]$$

of the stocks in the original portfolio has been sold and the proceeds invested in riskless assets. As the value of the original portfolio declines, the delta of the put given by equation (19.8) becomes more negative and the proportion of the original portfolio sold must be increased. As the value of the original portfolio increases, the delta of the put becomes less negative and the proportion of the original portfolio sold must be decreased (i.e., some of the original portfolio must be repurchased).

Using this strategy to create portfolio insurance means that at any given time funds are divided between the stock portfolio on which insurance is required and riskless assets. As the value of the stock portfolio increases, riskless assets are sold and the position in the stock portfolio is increased. As the value of the stock portfolio declines, the position in the stock portfolio is decreased and riskless assets are purchased. The cost of the insurance arises from the fact that the portfolio manager is always selling after a decline in the market and buying after a rise in the market.

Example 19.9

A portfolio is worth \$90 million. To protect against market downturns the managers of the portfolio require a 6-month European put option on the portfolio with a strike price of \$87 million. The risk-free rate is 9% per annum, the dividend yield is 3% per annum, and the volatility of the portfolio is estimated as 25% per annum. The S&P 500 index stands at 900. As the portfolio is considered to mimic the S&P 500 fairly closely, one alternative, discussed in Section 17.1, is to buy 1,000 put option contracts on the S&P 500 with a strike price of 870. Another alternative is to create the required option synthetically. In this case, $S_0 = 90$ million, $K = 87$ million, $r = 0.09$, $q = 0.03$, $\sigma = 0.25$, and $T = 0.5$, so that

$$d_1 = \frac{\ln(90/87) + (0.09 - 0.03 + 0.25^2/2)0.5}{0.25\sqrt{0.5}} = 0.4499$$

and the delta of the required option is

$$e^{-qT}[N(d_1) - 1] = -0.3215$$

This shows that 32.15% of the portfolio should be sold initially and invested in risk-free assets to match the delta of the required option. The amount of the portfolio sold must be monitored frequently. For example, if the value of the original portfolio reduces to \$88 million after 1 day, the delta of the required option changes to 0.3679 and a further 4.64% of the original portfolio should be sold and invested in risk-free assets. If the value of the portfolio increases to \$92 million, the delta of the required option changes to -0.2787 and 4.28% of the original portfolio should be repurchased.

Use of Index Futures

Using index futures to create options synthetically can be preferable to using the underlying stocks because the transaction costs associated with trades in index futures are generally lower than those associated with the corresponding trades in the underlying stocks. The dollar amount of the futures contracts shorted as a proportion of the value of the portfolio should from equations (19.6) and (19.8) be

$$e^{-qT}e^{-(r-q)T^*}[1 - N(d_1)] = e^{q(T^*-T)}e^{-rT^*}[1 - N(d_1)]$$

where T^* is the maturity of the futures contract. If the portfolio is worth A_1 times the index and each index futures contract is on A_2 times the index, the number of futures contracts shorted at any given time should be

$$e^{q(T^*-T)}e^{-rT^*}[1 - N(d_1)]A_1/A_2$$

Example 19.10

Suppose that in the previous example futures contracts on the S&P 500 maturing in 9 months are used to create the option synthetically. In this case initially $T = 0.5$, $T^* = 0.75$, $A_1 = 100{,}000$, and $d_1 = 0.4499$. Each index futures contract is on 250 times the index, so that $A_2 = 250$. The number of futures contracts shorted should be

$$e^{q(T^*-T)}e^{-rT^*}[1 - N(d_1)]A_1/A_2 = 122.96$$

or 123, rounding to the nearest whole number. As time passes and the index changes, the position in futures contracts must be adjusted.

This analysis assumes that the portfolio mirrors the index. When this is not the case, it is necessary to (a) calculate the portfolio's beta, (b) find the position in options on the index that gives the required protection, and (c) choose a position in index futures to create the options synthetically. As discussed in Section 17.1, the strike price for the options should be the expected level of the market index when the portfolio reaches its insured value. The number of options required is beta times the number that would be required if the portfolio had a beta of 1.0.

19.14 STOCK MARKET VOLATILITY

We discussed in Chapter 15 the issue of whether volatility is caused solely by the arrival of new information or whether trading itself generates volatility. Portfolio insurance strategies such as those just described have the potential to increase volatility. When the market declines, they cause portfolio managers either to sell stock or to sell index futures contracts. Either action may accentuate the decline (see Business Snapshot 19.2). The sale of stock is liable to drive down the market index further in a direct way. The sale of index futures contracts is liable to drive down futures prices. This creates selling pressure on stocks via the mechanism of index arbitrage (see Chapter 5), so that the market index is liable to be driven down in this case as well. Similarly, when the market rises, the portfolio insurance strategies cause portfolio managers either to buy stock or to buy futures contracts. This may accentuate the rise.

In addition to formal portfolio trading strategies, we can speculate that many investors consciously or subconsciously follow portfolio insurance rules of their own. For example, an investor may choose to sell when the market is falling to limit the downside risk.

Whether portfolio insurance trading strategies (formal or informal) affect volatility depends on how easily the market can absorb the trades that are generated by portfolio insurance. If portfolio insurance trades are a very small fraction of all trades, there is likely to be no effect. But if portfolio insurance becomes very popular, it is liable to have a destabilizing effect on the market, as it did in 1987.

SUMMARY

Financial institutions offer a variety of option products to their clients. Often the options do not correspond to the standardized products traded by exchanges. The financial institutions are then faced with the problem of hedging their exposure. Naked and covered positions leave them subject to an unacceptable level of risk. One course of action that is sometimes proposed is a stop-loss strategy. This involves holding a naked position when an option is out of the money and converting it to a covered position as soon as the option moves into the money. Although superficially attractive, the strategy does not provide a good hedge.

The delta (Δ) of an option is the rate of change of its price with respect to the price of the underlying asset. Delta hedging involves creating a position with zero delta (sometimes referred to as a delta-neutral position). Because the delta of the underlying asset

Business Snapshot 19.2 Was Portfolio Insurance to Blame for the Crash of 1987?

On Monday, October 19, 1987, the Dow Jones Industrial Average dropped by more than 20%. Many people feel that portfolio insurance played a major role in this crash. In October 1987 between $60 billion and $90 billion of equity assets were subject to portfolio insurance trading rules where put options were created synthetically in the way discussed in Section 19.13. During the period Wednesday, October 14, 1987, to Friday, October 16, 1987, the market declined by about 10%, with much of this decline taking place on Friday afternoon. The portfolio trading rules should have generated at least $12 billion of equity or index futures sales as a result of this decline. In fact, portfolio insurers had time to sell only $4 billion and they approached the following week with huge amounts of selling already dictated by their models. It is estimated that on Monday, October 19, sell programs by three portfolio insurers accounted for almost 10% of the sales on the New York Stock Exchange, and that portfolio insurance sales amounted to 21.3% of all sales in index futures markets. It is likely that the decline in equity prices was exacerbated by investors other than portfolio insurers selling heavily because they anticipated the actions of portfolio insurers.

Because the market declined so fast and the stock exchange systems were overloaded, many portfolio insurers were unable to execute the trades generated by their models and failed to obtain the protection they required. Needless to say, the popularity of portfolio insurance schemes has declined significantly since 1987. One of the morals of this story is that it is dangerous to follow a particular trading strategy—even a hedging strategy—when many other market participants are doing the same thing.

is 1.0, one way of hedging is to take a position of $-\Delta$ in the underlying asset for each long option being hedged. The delta of an option changes over time. This means that the position in the underlying asset has to be frequently adjusted.

Once an option position has been made delta neutral, the next stage is often to look at its gamma (Γ). The gamma of an option is the rate of change of its delta with respect to the price of the underlying asset. It is a measure of the curvature of the relationship between the option price and the asset price. The impact of this curvature on the performance of delta hedging can be reduced by making an option position gamma neutral. If Γ is the gamma of the position being hedged, this reduction is usually achieved by taking a position in a traded option that has a gamma of $-\Gamma$.

Delta and gamma hedging are both based on the assumption that the volatility of the underlying asset is constant. In practice, volatilities do change over time. The vega of an option or an option portfolio measures the rate of change of its value with respect to volatility, often implied volatility. Sometimes the same change is assumed to apply to all volatilities. A trader who wishes to hedge an option position against volatility changes can make the position vega neutral. As with the procedure for creating gamma neutrality, this usually involves taking an offsetting position in a traded option. If the trader wishes to achieve both gamma and vega neutrality, at least two traded options are usually required.

Two other measures of the risk of an option position are theta and rho. Theta measures the rate of change of the value of the position with respect to the passage of time, with all else remaining constant. Rho measures the rate of change of the value of the position with respect to the interest rate, with all else remaining constant.

In practice, option traders usually rebalance their portfolios at least once a day to maintain delta neutrality. It is usually not feasible to maintain gamma and vega neutrality on a regular basis. Typically a trader monitors these measures. If they get too large, either corrective action is taken or trading is curtailed.

Portfolio managers are sometimes interested in creating put options synthetically for the purposes of insuring an equity portfolio. They can do so either by trading the portfolio or by trading index futures on the portfolio. Trading the portfolio involves splitting the portfolio between equities and risk-free securities. As the market declines, more is invested in risk-free securities. As the market increases, more is invested in equities. Trading index futures involves keeping the equity portfolio intact and selling index futures. As the market declines, more index futures are sold; as it rises, fewer are sold. This type of portfolio insurance works well in normal market conditions. On Monday, October 19, 1987, when the Dow Jones Industrial Average dropped very sharply, it worked badly. Portfolio insurers were unable to sell either stocks or index futures fast enough to protect their positions.

FURTHER READING

Passarelli, D. *Trading Option Greeks: How Time, Volatility, and Other Factors Drive Profits*, 2nd edn. Hoboken, NJ: Wiley, 2012.

Taleb, N. N., *Dynamic Hedging: Managing Vanilla and Exotic Options*. New York: Wiley, 1996.

Practice Questions (Answers in Solutions Manual)

19.1. Explain how a stop-loss trading rule can be implemented for the writer of an out-of-the-money call option. Why does it provide a relatively poor hedge?

19.2. What does it mean to assert that the delta of a call option is 0.7? How can a short position in 1,000 options be made delta neutral when the delta of each option is 0.7?

19.3. Calculate the delta of an at-the-money six-month European call option on a non-dividend-paying stock when the risk-free interest rate is 10% per annum and the stock price volatility is 25% per annum.

19.4. What does it mean to assert that the theta of an option position is -0.1 when time is measured in years? If a trader feels that neither a stock price nor its implied volatility will change, what type of option position is appropriate?

19.5. What is meant by the gamma of an option position? What are the risks in the situation where the gamma of a position is highly negative and the delta is zero?

19.6. "The procedure for creating an option position synthetically is the reverse of the procedure for hedging the option position." Explain this statement.

19.7. Why did portfolio insurance not work well on October 19, 1987?

19.8. The Black–Scholes–Merton price of an out-of-the-money call option with an exercise price of \$40 is \$4. A trader who has written the option plans to use a stop-loss strategy. The trader's plan is to buy at \$40.10 and to sell at \$39.90. Estimate the expected number of times the stock will be bought or sold.

19.9. Suppose that a stock price is currently \$20 and that a call option with an exercise price of \$25 is created synthetically using a continually changing position in the stock. Consider the following two scenarios: (a) Stock price increases steadily from \$20 to \$35 during the life of the option; (b) Stock price oscillates wildly, ending up at \$35. Which scenario would make the synthetically created option more expensive? Explain your answer.

19.10. What is the delta of a short position in 1,000 European call options on silver futures? The options mature in 8 months, and the futures contract underlying the option matures in 9 months. The current 9-month futures price is \$8 per ounce, the exercise price of the options is \$8, the risk-free interest rate is 12% per annum, and the volatility of silver futures prices is 18% per annum.

19.11. In Problem 19.10, what initial position in 9-month silver futures is necessary for delta hedging? If silver itself is used, what is the initial position? If 1-year silver futures are used, what is the initial position? Assume no storage costs for silver.

19.12. A company uses delta hedging to hedge a portfolio of long positions in put and call options on a currency. Which of the following would give the most favorable result?
(a) A virtually constant spot rate
(b) Wild movements in the spot rate
Explain your answer.

19.13. Repeat Problem 19.12 for a financial institution with a portfolio of short positions in put and call options on a currency.

19.14. A financial institution has just sold 1,000 7-month European call options on the Japanese yen. Suppose that the spot exchange rate is 0.80 cent per yen, the exercise price is 0.81 cent per yen, the risk-free interest rate in the United States is 8% per annum, the risk-free interest rate in Japan is 5% per annum, and the volatility of the yen is 15% per annum. Calculate the delta, gamma, vega, theta, and rho of the financial institution's position. Interpret each number.

19.15. Under what circumstances is it possible to make a European option on a stock index both gamma neutral and vega neutral by adding a position in one other European option?

19.16. A fund manager has a well-diversified portfolio that mirrors the performance of the S&P 500 and is worth \$360 million. The value of the S&P 500 is 1,200, and the portfolio manager would like to buy insurance against a reduction of more than 5% in the value of the portfolio over the next 6 months. The risk-free interest rate is 6% per annum. The dividend yield on both the portfolio and the S&P 500 is 3%, and the volatility of the index is 30% per annum.
(a) If the fund manager buys traded European put options, how much would the insurance cost?
(b) Explain carefully alternative strategies open to the fund manager involving traded European call options, and show that they lead to the same result.
(c) If the fund manager decides to provide insurance by keeping part of the portfolio in risk-free securities, what should the initial position be?
(d) If the fund manager decides to provide insurance by using 9-month index futures, what should the initial position be?

19.17. Repeat Problem 19.16 on the assumption that the portfolio has a beta of 1.5. Assume that the dividend yield on the portfolio is 4% per annum.

19.18. Show by substituting for the various terms in equation (19.4) that the equation is true for:
(a) A single European call option on a non-dividend-paying stock
(b) A single European put option on a non-dividend-paying stock
(c) Any portfolio of European put and call options on a non-dividend-paying stock.

19.19. What is the equation corresponding to equation (19.4) for (a) a portfolio of derivatives on a currency and (b) a portfolio of derivatives on a futures price?

19.20. Suppose that \$70 billion of equity assets are the subject of portfolio insurance schemes. Assume that the schemes are designed to provide insurance against the value of the assets declining by more than 5% within 1 year. Making whatever estimates you find necessary, use the DerivaGem software to calculate the value of the stock or futures contracts that the administrators of the portfolio insurance schemes will attempt to sell if the market falls by 23% in a single day.

19.21. Does a forward contract on a stock index have the same delta as the corresponding futures contract? Explain your answer.

19.22. A bank's position in options on the dollar/euro exchange rate has a delta of 30,000 and a gamma of −80,000. Explain how these numbers can be interpreted. The exchange rate (dollars per euro) is 0.90. What position would you take to make the position delta neutral? After a short period of time, the exchange rate moves to 0.93. Estimate the new delta. What additional trade is necessary to keep the position delta neutral? Assuming the bank did set up a delta-neutral position originally, has it gained or lost money from the exchange-rate movement?

19.23. Use the put–call parity relationship to derive, for a non-dividend-paying stock, the relationship between:
(a) The delta of a European call and the delta of a European put
(b) The gamma of a European call and the gamma of a European put
(c) The vega of a European call and the vega of a European put
(d) The theta of a European call and the theta of a European put.

Further Questions

19.24. A financial institution has the following portfolio of over-the-counter options on sterling:

Type	*Position*	*Delta of option*	*Gamma of option*	*Vega of option*
Call	−1,000	0.50	2.2	1.8
Call	−500	0.80	0.6	0.2
Put	−2,000	−0.40	1.3	0.7
Call	−500	0.70	1.8	1.4

A traded option is available with a delta of 0.6, a gamma of 1.5, and a vega of 0.8.
(a) What position in the traded option and in sterling would make the portfolio both gamma neutral and delta neutral?
(b) What position in the traded option and in sterling would make the portfolio both vega neutral and delta neutral? Assume that all implied volatilities change by the same amount so that vegas can be aggregated.

19.25. Consider again the situation in Problem 19.24. Suppose that a second traded option with a delta of 0.1, a gamma of 0.5, and a vega of 0.6 is available. How could the portfolio be made delta, gamma, and vega neutral?

19.26. Consider a 1-year European call option on a stock when the stock price is \$30, the strike price is \$30, the risk-free rate is 5%, and the volatility is 25% per annum. Use the DerivaGem software to calculate the price, delta, gamma, vega, theta, and rho of the option. Verify that delta is correct by changing the stock price to \$30.1 and recomputing the option price. Verify that gamma is correct by recomputing the delta for the situation where the stock price is \$30.1. Carry out similar calculations to verify that vega, theta, and rho are correct. Use the DerivaGem Applications Builder functions to plot the option price, delta, gamma, vega, theta, and rho against the stock price for the stock option.

19.27. A deposit instrument offered by a bank guarantees that investors will receive a return during a 6-month period that is the greater of (a) zero and (b) 40% of the return provided by a market index. An investor is planning to put \$100,000 in the instrument. Describe the payoff as an option on the index. Assuming that the risk-free rate of interest is 8% per annum, the dividend yield on the index is 3% per annum, and the volatility of the index is 25% per annum, is the product a good deal for the investor?

19.28. The formula for the price c of a European call futures option in terms of the futures price F_0 is given in Chapter 18 as

$$c = e^{-rT}[F_0 N(d_1) - KN(d_2)]$$

where

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

and K, r, T, and σ are the strike price, interest rate, time to maturity, and volatility, respectively.

(a) Prove that $F_0 N'(d_1) = KN'(d_2)$.

(b) Prove that the delta of the call price with respect to the futures price is $e^{-rT}N(d_1)$.

(c) Prove that the vega of the call price is $F_0\sqrt{T}N'(d_1)e^{-rT}$.

(d) Prove the formula for the rho of a call futures option given in Section 19.12.

The delta, gamma, theta, and vega of a call futures option are the same as those for a call option on a stock paying dividends at rate q, with q replaced by r and S_0 replaced by F_0. Explain why the same is not true of the rho of a call futures option.

19.29. Use DerivaGem to check that equation (19.4) is satisfied for the option considered in Section 19.1. (*Note*: DerivaGem produces a value of theta "per calendar day." The theta in equation (19.4) is "per year.")

19.30. Use the DerivaGem Application Builder functions to reproduce Table 19.2. (In Table 19.2 the stock position is rounded to the nearest 100 shares.) Calculate the gamma and theta of the position each week. Calculate the change in the value of the portfolio each week and check whether equation (19.3) is approximately satisfied. (*Note*: DerivaGem produces a value of theta "per calendar day." The theta in equation (19.3) is "per year.")

APPENDIX

TAYLOR SERIES EXPANSIONS AND GREEK LETTERS

A Taylor series expansion of the change in the portfolio value in a short period of time shows the role played by different Greek letters. If the volatility of the underlying asset is assumed to be constant, the value Π of the portfolio is a function of the asset price S, and time t. The Taylor series expansion gives

$$\Delta\Pi = \frac{\partial\Pi}{\partial S}\Delta S + \frac{\partial\Pi}{\partial t}\Delta t + \tfrac{1}{2}\frac{\partial^2\Pi}{\partial S^2}\Delta S^2 + \tfrac{1}{2}\frac{\partial^2\Pi}{\partial t^2}\Delta t^2 + \frac{\partial^2\Pi}{\partial S\,\partial t}\Delta S\,\Delta t + \cdots \qquad \textbf{(19A.1)}$$

where $\Delta\Pi$ and ΔS are the change in Π and S in a small time interval Δt. Delta hedging eliminates the first term on the right-hand side. The second term is nonstochastic. The third term (which is of order Δt) can be made zero by ensuring that the portfolio is gamma neutral as well as delta neutral. Other terms are of order higher than Δt.

For a delta-neutral portfolio, the first term on the right-hand side of equation (19A.1) is zero, so that

$$\Delta\Pi = \Theta\,\Delta t + \tfrac{1}{2}\Gamma\,\Delta S^2$$

when terms of order higher than Δt are ignored. This is equation (19.3).

The Practitioner Black–Scholes Model

In practice, volatility is not constant. As explained in this chapter, practitioners usually set volatility equal to implied volatility when calculating Greek letters. From the definition of implied volatility, the option price is an exact function of the asset price, implied volatility, time, interest rates, and dividends. As an approximation, we can ignore changes in interest rates and dividends and assume that an option price, f, is at any given time a function of only two variables: the asset price, S, and the implied volatility, σ_{imp}. The change in the option price over a short period of time is then given by

$$\Delta f = \frac{\partial f}{\partial S}\Delta S + \frac{\partial f}{\partial \sigma_{\text{imp}}}\Delta\sigma_{\text{imp}} + \tfrac{1}{2}\frac{\partial^2 f}{\partial S^2}(\Delta S)^2 + \tfrac{1}{2}\frac{\partial^2 f}{\partial \sigma_{\text{imp}}^2}(\Delta\sigma_{\text{imp}})^2 + \frac{\partial^2 f}{\partial S\,\partial\sigma_{\text{imp}}}\Delta S\,\Delta\sigma_{\text{imp}} + \cdots$$

Delta, vega, and gamma hedging deal with the first three terms in this expansion (which are the most important ones). Traders sometimes define other Greek letters such as $\partial^2 f/\partial\sigma_{\text{imp}}^2$ and $\partial^2 f/\partial S\,\partial\sigma_{\text{imp}}$ to explore their exposure to later terms in the Taylor series.

When portfolios of options are considered, the trader's problem is more complicated because the implied volatility of an option on a particular asset depends on the option's strike price and time to maturity. The trader must consider the portfolio's exposure to the different ways the volatility surface can change over a short period of time. Volatility surfaces are discussed in the next chapter.