CRR $\rightarrow$ SDE $\Rightarrow$ $\boxed{\dfrac{dS_t}{S_t} = r\,dt + \sigma\, dB_t}$

$dB_t \sim N(\underset{\mu}{0}, \underset{\sigma^2}{dt})$

$u = e^{\sigma\sqrt{dt}}$ $d = e^{-\sigma\sqrt{dt}}$

$$S_{t+dt} = S_t (u \text{ or } d)$$

$$S_{t+dt} = S_t\, e^{\sigma\sqrt{dt}\,\epsilon}$$

ϵ: 1 with p, -1 with $1-p$

$$p = \frac{e^{r\,dt} - e^{-\sigma\sqrt{dt}}}{e^{\sigma\sqrt{dt}} - e^{-\sigma\sqrt{dt}}}$$

return

$$\frac{dS_t}{S_t} = \frac{S_{t+dt} - S_t}{S_t}$$

$$= \frac{S_t e^{\sigma\sqrt{dt}\,\epsilon} - S_t}{S_t} = e^{\sigma\sqrt{dt}\,\epsilon} - 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$\frac{dS_t}{S_t} = \sigma\sqrt{dt}\,\epsilon + \frac{1}{2!}\sigma^2 (dt)\underset{1}{\epsilon^2} + \frac{1}{3!}\sigma^3(\sqrt{dt})^3\epsilon^3 + \cdots$$

$\rightarrow 0 \quad 0$

$(dt)^\alpha = 0 \quad \alpha > 1$

$$\boxed{\frac{dS_t}{S_t} = \frac{\sigma^2}{2}dt + \sigma\sqrt{dt}\,\epsilon}$$

ϵ: 1 with p, -1 with $1-p$

$(dt)^2 = 0$

$$p = \frac{e^{r\,dt} - e^{-\sigma\sqrt{dt}}}{e^{\sigma\sqrt{dt}} - e^{-\sigma\sqrt{dt}}}$$

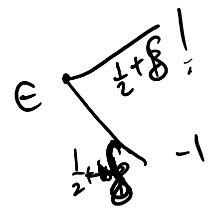
$$e^x = 1 + x + \frac{x^2}{2!} \cdots$$

$$p = \frac{r\,dt + \sigma\sqrt{dt} - \frac{\sigma^2}{2}dt}{} = \left(r - \frac{\sigma^2}{2}\right)dt + \sigma\sqrt{dt}$$

$$p = \frac{r\,dt + \sigma\sqrt{dt} - \frac{\sigma^2}{2}dt}{2\sigma\sqrt{dt}} = \frac{\left(r - \frac{\sigma^2}{2}\right)dt + \sigma\sqrt{dt}}{2\sigma\sqrt{dt}}$$

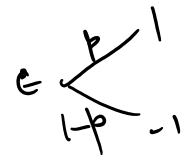
$$p = \left[\frac{\left(r - \frac{\sigma^2}{2}\right)dt}{2\sigma\sqrt{dt}}\right] + \left(\frac{1}{2}\right)$$

distorting

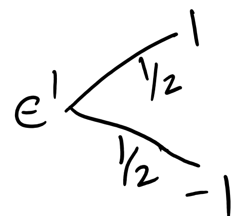


$$E(\epsilon) = 2p - 1 \quad \text{(check!)}$$

$$= \frac{\left(r - \frac{\sigma^2}{2}\right)\sqrt{dt}}{\sigma}$$



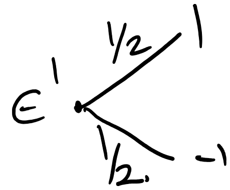
$$\epsilon = \frac{\left(r - \frac{\sigma^2}{2}\right)\sqrt{dt}}{\sigma} + \epsilon'$$



$$\frac{dS}{S} = \frac{\sigma^2}{2}dt + \sigma\sqrt{dt}\left[\frac{\left(r - \frac{\sigma^2}{2}\right)\sqrt{dt}}{\sigma} + \epsilon'\right]$$

$$\frac{dS}{S} = \frac{\sigma^2}{2}dt + \left(r - \frac{\sigma^2}{2}\right)dt + \sigma\sqrt{dt}\,\epsilon'$$

$$\frac{dS}{S} = r\,dt + \sigma\sqrt{dt}\,\epsilon'$$



$\Uparrow$??

Need $\frac{dS}{S} = r\,dt + \sigma\,dB_t$ $\quad dB_t \sim N(0, dt)$, $\mu = 0$, $= \sigma^2$

$$\|$$

$$\frac{dS}{S} = r\,dt + \sigma\sqrt{dt}\,r \qquad r \sim N(0,1) \qquad r = \frac{dB_t}{\sqrt{dt}}$$

Central Limit Thm:

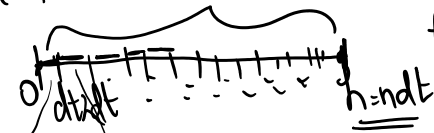
$X_1, \ldots, X_n$ i.i.d $\mu = 0;\ \sigma^2 = 1$

$X_1 + \cdots + X_n \sim N(0,1)$

$x_1, \ldots, x_n$ ...

$$\frac{x_1 + \cdots + x_n}{\sqrt{n}} \sim \underline{N(0,1)}$$

Consider a time interval h (slightly larger than dt)



$$(1 + r\,dt + \sigma\sqrt{dt}\,\epsilon_1)(1 + r\,dt + \sigma\sqrt{dt}\,\epsilon_2) \cdots (1 + r\,dt + \sigma\sqrt{dt}\,\epsilon_n) - 1$$

$$\cancel{S_0}(1+r_1)(1+r_2)(1+r_3)\cdots(1+r_n) = \cancel{S_0}(1+r)$$

$$(\ )(\ )(\ )(\ ) - 1 = r$$

$(dt)^\alpha = 0 \quad \alpha > 1$

$$1 - 1 + n\,r\,dt + \sigma\sqrt{dt}\,[\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n]$$

$$+ \sum_{i,j} \sigma^2\, dt\, \epsilon_i \epsilon_j + 0 + 0 + 0 \ldots$$

$$= r(h) + \sqrt{n}\,\sigma\sqrt{dt}\left[\frac{\epsilon_1 + \cdots + \epsilon_n}{\sqrt{n}}\right]$$

$$+ \sum \sigma^2\, dt\, \epsilon_i \epsilon_j$$

$n\,dt = h$

$\sqrt{n\,dt} = \sqrt{h}$

$$\frac{ds}{s} = r h + \sigma\sqrt{h}\, \underset{\substack{\sim \\ N(0,1)}}{B} + \sum \sigma^2 dt\, \epsilon_i \epsilon_j$$

on interval h

$\sum \sigma^2 dt\, \epsilon_i\epsilon_j = 0$

$\epsilon_i \epsilon_j = \epsilon$, $\epsilon_i = \pm 1$, $\epsilon_j = \pm 1$

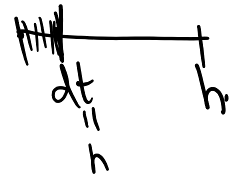
$dt.dB = 0 \leftarrow \sum \sigma^2 dt\, \epsilon = 0$

$E(\ \cdot\ ) = 0$

$V(\ \cdot\ ) = dt^2 = 0$

$$\frac{ds}{s} = r h + \sigma\sqrt{h}\, B$$

$$\frac{dS}{S} = rh + \sigma\sqrt{h}\,\underset{N(0,1)}{\underset{\wr}{B}}$$



Use Fractal Nature

$$\boxed{\begin{aligned}\frac{dS}{S} &= r\,dt + \sigma\sqrt{dt}\,\epsilon \\ &= r\,dt + \sigma\,dB_t\end{aligned}}$$

$\epsilon \sim \underline{\underline{N(0,1)}}$

$dB_t \sim \underline{\underline{N(0,dt)}}$

CRR

8:10

Rate

$$\frac{dS}{S} = r\,dt + \sigma\sqrt{dt}\,\epsilon \qquad \epsilon \sim N(0,1)$$

$$\text{Rate} \sim N(0,1)$$

$$S_T = S_0\, e^{(\text{rate})\,t}$$

$$\frac{S_T}{S_0} = e^{(\text{rate})\,t}$$

$$\underline{\underline{\ln(S_T)}} - \ln S_0 = \underset{\text{Normal}}{\underset{\wr}{(\text{rate})}}\,t$$

log of S_T is normal

S_T is log<u>normal</u>

---

1. Discretize the SDE suitably
2. Simulate paths
3. Apply Derivative logic on each path

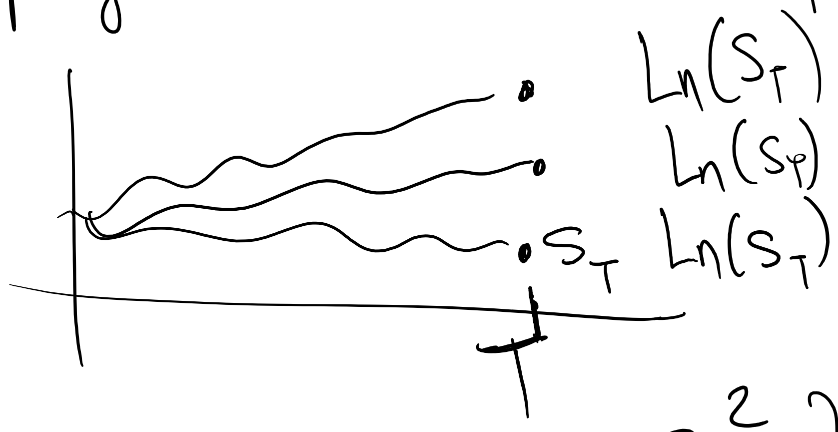
and calculate payoff

4. Average the payoffs
5. Discount $\equiv$ Price of the derivative. *

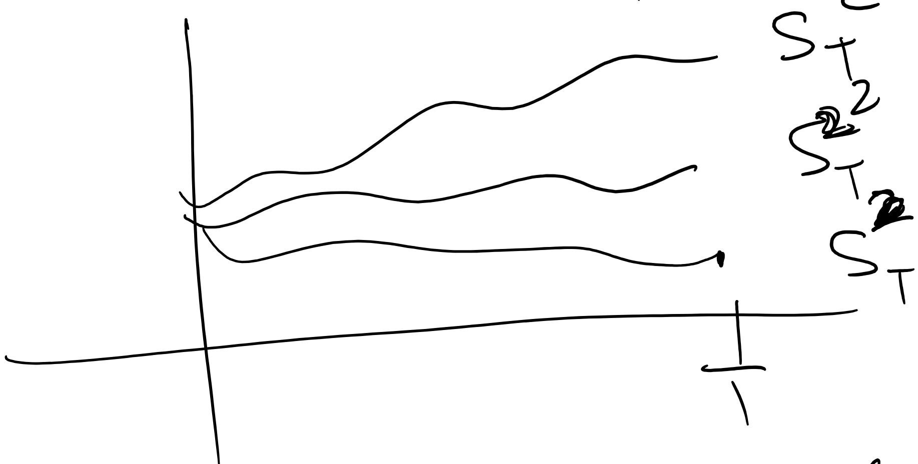
* Something needs to be done

Log Contract

pays $Ln(S_T)$ at expiration



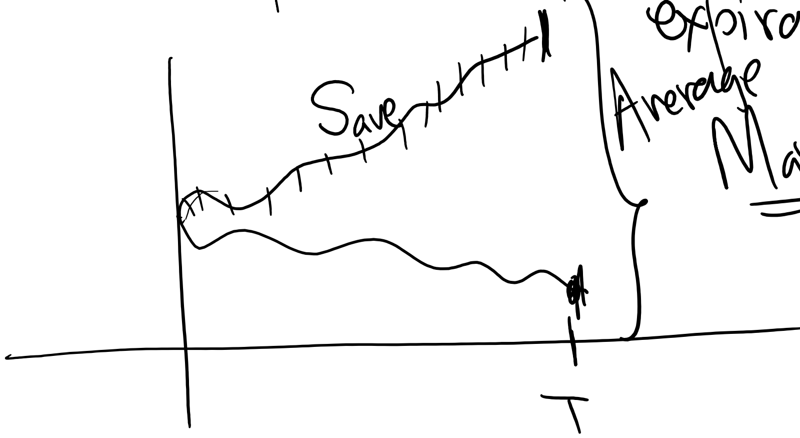
Average $\downarrow$ discount



Average $\downarrow$ discount

Asian Option pays $Max(S_{ave} - K, 0)$ at expiration ave =

expiration



Average

$$\text{Max}\left(\frac{S_1+S_2+\cdots+S_{100}}{100}-K,\ 0\right)$$

↓ Discount

Cash Barrier: Pays \$75 if stock crosses \$110 during 2-years

75 0 otherwise