CHAPTER 26 Exotic Options

Derivatives such as European and American call and put options are what are termed *plain vanilla products*. They have standard well-defined properties and trade actively. Their prices or implied volatilities are quoted by exchanges or by interdealer brokers on a regular basis. One of the exciting aspects of the over-the-counter derivatives market is the number of nonstandard products that have been created by financial engineers. These products are termed *exotic options*, or simply *exotics*. Although they usually constitute a relatively small part of its portfolio, exotics are important to a derivatives dealer because they are generally much more profitable than plain vanilla products.

Exotic products are developed for a number of reasons. Sometimes they meet a genuine hedging need in the market; sometimes there are tax, accounting, legal, or regulatory reasons why corporate treasurers, fund managers, and financial institutions find exotic products attractive; sometimes the products are designed to reflect a view on potential future movements in particular market variables; occasionally an exotic product is designed by a derivatives dealer to appear more attractive than it is to an unwary corporate treasurer or fund manager.

In this chapter, we describe some of the more commonly occurring exotic options and discuss their valuation. We assume that the underlying asset provides a yield at rate q. As discussed in Chapters 17 and 18, for an option on a stock index q should be set equal to the dividend yield on the index, for an option on a currency it should be set equal to the foreign risk-free rate, and for an option on a futures contract it should be set equal to the domestic risk-free rate. Many of the options discussed in this chapter can be valued using the DerivaGem software.

26.1 PACKAGES

A *package* is a portfolio consisting of standard European calls, standard European puts, forward contracts, cash, and the underlying asset itself. We discussed a number of different types of packages in Chapter 12: bull spreads, bear spreads, butterfly spreads, calendar spreads, straddles, strangles, and so on.

Often a package is structured by traders so that it has zero cost initially. An example is a *range forward contract*.[1] This was discussed in Section 17.2. It consists of a long call and a short put or a short call and a long put. The call strike price is greater than the put strike price and the strike prices are chosen so that the value of the call equals the value of the put.

It is worth noting that any derivative can be converted into a zero-cost product by deferring payment until maturity. Consider a European call option. If c is the cost of the option when payment is made at time zero, then $A = ce^{rT}$ is the cost when payment is made at time T, the maturity of the option. The payoff is then $\max(S_T - K,\ 0) - A$ or $\max(S_T - K - A,\ -A)$. When the strike price, K, equals the forward price, other names for a deferred payment option are break forward, Boston option, forward with optional exit, and cancelable forward.

26.2 PERPETUAL AMERICAN CALL AND PUT OPTIONS

The differential equation that must be satisfied by the price of a derivative when there is a dividend at rate q is equation (17.6):

$$\frac{\partial f}{\partial t} + (r - q)S\frac{\partial f}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

Consider a derivative that pays off a fixed amount Q when $S = H$ for the first time. If $S < H$, the boundary conditions for the differential equation are that $f = Q$ when $S = H$ and $f = 0$ when $S = 0$. The solution $f = Q(S/H)^\alpha$ satisfies the boundary conditions when $\alpha > 0$. Furthermore, it satisfies the differential equation when

$$(r - q)\alpha + \tfrac{1}{2}\alpha(\alpha - 1)\sigma^2 = r$$

The positive solution to this equation is $\alpha = \alpha_1$, where

$$\alpha_1 = \frac{-w + \sqrt{w^2 + 2\sigma^2 r}}{\sigma^2}$$

and $w = r - q - \sigma^2/2$. It follows that the value of the derivative must be $Q(S/H)^{\alpha_1}$ because this satisfies the boundary conditions and the differential equation.

Consider next a perpetual American call option with strike price K. If the option is exercised when $S = H$, the payoff is $H - K$ and from the result just proved the value of the option is $(H - K)(S/H)^{\alpha_1}$. The holder of the call option can choose the asset price, H, at which the option is exercised. The optimal H is the one that maximizes the value we have just calculated. Using standard calculus methods, it is $H = H_1$, where

$$H_1 = K\frac{\alpha_1}{\alpha_1 - 1}$$

[1] Other names used for a range forward contract are zero-cost collar, flexible forward, cylinder option, option fence, min–max, and forward band.

The price of a perpetual call if $S < H_1$ is therefore

$$(H_1 - K)\left(\frac{S_0}{H_1}\right)^{\alpha_1} = \frac{K}{\alpha_1 - 1}\left(\frac{\alpha_1 - 1}{\alpha_1}\frac{S}{K}\right)^{\alpha_1}$$

If $S > H_1$, the call should be exercised immediately and is worth $S - K$.

To value an American put, we consider a derivative that pays off Q when $S = H$ in the situation where $S > H$ (so that the barrier H is reached from above). In this case, the boundary conditions for the differential equation are that $f = Q$ when $S = H$ and $f = 0$ as S tends to infinity. In this case, the solution $f = Q(S/H)^{-\alpha}$ satisfies the boundary conditions when $\alpha > 0$. As above, we can show that it also satisfies the differential equation when $\alpha = \alpha_2$, where

$$\alpha_2 = \frac{w + \sqrt{w^2 + 2\sigma^2 r}}{\sigma^2}$$

If the holder of the American put chooses to exercise when $S = H$, the value of the put is $(K - H)(S/H)^{-\alpha_2}$. The holder of the put will choose the exercise level $H = H_2$ to maximize this. This is

$$H_2 = K\frac{\alpha_2}{\alpha_2 + 1}$$

The price of a perpetual put if $S > H_2$ is therefore

$$(K - H_2)\left(\frac{S_0}{H_2}\right)^{-\alpha_2} = \frac{K}{\alpha_2 + 1}\left(\frac{\alpha_2 + 1}{\alpha_2}\frac{S}{K}\right)^{-\alpha_2}$$

If $S < H_2$, the put should be exercised immediately and is worth $K - S$.

Section 15.6 and Problem 15.23 give particular cases of the results here for $q = 0$.

26.3 NONSTANDARD AMERICAN OPTIONS

In a standard American option, exercise can take place at any time during the life of the option and the exercise price is always the same. The American options that are traded in the over-the-counter market sometimes have nonstandard features. For example:

1. Early exercise may be restricted to certain dates. The instrument is then known as a *Bermudan option*. (Bermuda is between Europe and America!)
2. Early exercise may be allowed during only part of the life of the option. For example, there may be an initial "lock out" period with no early exercise.
3. The strike price may change during the life of the option.

The warrants issued by corporations on their own stock often have some or all of these features. For example, in a 7-year warrant, exercise might be possible on particular dates during years 3 to 7, with the strike price being \$30 during years 3 and 4, \$32 during the next 2 years, and \$33 during the final year.

Nonstandard American options can usually be valued using a binomial tree. At each node, the test (if any) for early exercise is adjusted to reflect the terms of the option.

26.4 GAP OPTIONS

A gap call option is a European call options that pays off $S_T - K_1$ when $S_T > K_2$. The difference between a gap call option and a regular call option with a strike price of K_2 is that the payoff when $S_T > K_2$ is increased by $K_2 - K_1$. (This increase is positive or negative depending on whether $K_2 > K_1$ or $K_1 > K_2$.)

A gap call option can be valued by a small modification to the Black–Scholes–Merton formula. With our usual notation, the value is

$$S_0 e^{-qT} N(d_1) - K_1 e^{-rT} N(d_2) \tag{26.1}$$

where

$$d_1 = \frac{\ln(S_0/K_2) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

The price in this formula is greater than the price given by the Black–Scholes–Merton formula for a regular call option with strike price K_2 by

$$(K_2 - K_1)e^{-rT} N(d_2)$$

To understand this difference, note that the probability that the option will be exercised is $N(d_2)$ and, when it is exercised, the payoff to the holder of the gap option is greater than that to the holder of the regular option by $K_2 - K_1$.

For a gap put option, the payoff is $K_1 - S_T$ when $S_T < K_2$. The value of the option is

$$K_1 e^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1) \tag{26.2}$$

where d_1 and d_2 are defined as for equation (26.1).

Example 26.1

An asset is currently worth \$500,000. Over the next year, its price is expected to have a volatility of 20%. The risk-free rate is 5%, and no income is expected. Suppose that an insurance company agrees to buy the asset for \$400,000 if its value has fallen below \$400,000 at the end of one year. The payout will be $400{,}000 - S_T$ whenever the value of the asset is less than \$400,000. The insurance company has provided a regular put option where the policyholder has the right to sell the asset to the insurance company for \$400,000 in one year. This can be valued using equation (15.21), with $S_0 = 500{,}000$, $K = 400{,}000$, $r = 0.05$, $\sigma = 0.2$, $T = 1$. The value is \$3,436.

Suppose next that the cost of transferring the asset is \$50,000 and this cost is borne by the policyholder. The option is then exercised only if the value of the asset is less than \$350,000. In this case, the cost to the insurance company is $K_1 - S_T$ when $S_T < K_2$, where $K_2 = 350{,}000$, $K_1 = 400{,}000$, and S_T is the price of the asset in one year. This is a gap put option. The value is given by equation (26.2), with $S_0 = 500{,}000$, $K_1 = 400{,}000$, $K_2 = 350{,}000$, $r = 0.05$, $q = 0$, $\sigma = 0.2$, $T = 1$. It is \$1,896. Recognizing the costs to the policyholder of making a claim reduces the cost of the policy to the insurance company by about 45% in this case.

26.5 FORWARD START OPTIONS

Forward start options are options that will start at some time in the future. Sometimes employee stock options, which were discussed in Chapter 16, can be viewed as forward start options. This is because the company commits (implicitly or explicitly) to granting at-the-money options to employees in the future.

Consider a forward start at-the-money European call option that will start at time T_1 and mature at time T_2. Suppose that the asset price is S_0 at time zero and S_1 at time T_1. To value the option, we note from the European option pricing formulas in Chapters 15 and 17 that the value of an at-the-money call option on an asset is proportional to the asset price. The value of the forward start option at time T_1 is therefore cS_1/S_0, where c is the value at time zero of an at-the-money option that lasts for $T_2 - T_1$. Using risk-neutral valuation, the value of the forward start option at time zero is

$$e^{-rT_1}\hat{E}\left[c\frac{S_1}{S_0}\right]$$

where $\hat{E}$ denotes the expected value in a risk-neutral world. Since c and S_0 are known and $\hat{E}[S_1] = S_0 e^{(r-q)T_1}$, the value of the forward start option is ce^{-qT_1}. For a non-dividend-paying stock, $q = 0$ and the value of the forward start option is exactly the same as the value of a regular at-the-money option with the same life as the forward start option.

26.6 CLIQUET OPTIONS

A cliquet option (which is also called a ratchet or strike reset option) is a series of call or put options with rules for determining the strike price. Suppose that the reset dates are at times $t_1, t_2, \dots, t_{n-1}$, with t_n being the end of the cliquet's life. A simple structure would be as follows. The first option has a strike price K equal to the initial asset price and lasts between times 0 and t_1; the second option provides a payoff at time t_2 with a strike price equal to the value of the asset at time t_1; the third option provides a payoff at time t_3 with a strike price equal to the value of the asset at time t_2; and so on. This is a regular option plus $n - 1$ forward start options. The latter can be valued as described in Section 26.5.

Some cliquet options are much more complicated than the one described here. For example, sometimes there are upper and lower limits on the total payoff over the whole period; sometimes cliquets terminate at the end of a period if the asset price is in a certain range. When analytic results are not available, Monte Carlo simulation is often the best approach for valuation.

26.7 COMPOUND OPTIONS

Compound options are options on options. There are four main types of compound options: a call on a call, a put on a call, a call on a put, and a put on a put. Compound options have two strike prices and two exercise dates. Consider, for example, a call on a call. On the first exercise date, T_1, the holder of the compound option is entitled to pay the first strike price, K_1, and receive a call option. The call option gives the holder the right to buy the underlying asset for the second strike price, K_2, on the second exercise date, T_2. The compound option will be exercised on the first exercise date only if the value of the option on that date is greater than the first strike price.

When the usual geometric Brownian motion assumption is made, European-style compound options can be valued analytically in terms of integrals of the bivariate normal distribution.[2] With our usual notation, the value at time zero of a European call option on a call option is

$$S_0e^{-qT_2}M(a_1, b_1; \sqrt{T_1/T_2}) - K_2e^{-rT_2}M(a_2, b_2; \sqrt{T_1/T_2}) - e^{-rT_1}K_1N(a_2)$$

where

$$a_1 = \frac{\ln(S_0/S^*) + (r - q + \sigma^2/2)T_1}{\sigma\sqrt{T_1}}, \quad a_2 = a_1 - \sigma\sqrt{T_1}$$

$$b_1 = \frac{\ln(S_0/K_2) + (r - q + \sigma^2/2)T_2}{\sigma\sqrt{T_2}}, \quad b_2 = b_1 - \sigma\sqrt{T_2}$$

The function $M(a, b : \rho)$ is the cumulative bivariate normal distribution function that the first variable will be less than a and the second will be less than b when the coefficient of correlation between the two is ρ.[3] The variable S^* is the asset price at time T_1 for which the option price at time T_1 equals K_1. If the actual asset price is above S^* at time T_1, the first option will be exercised; if it is not above S^*, the option expires worthless.

With similar notation, the value of a European put on a call is

$$K_2e^{-rT_2}M(-a_2, b_2; -\sqrt{T_1/T_2}) - S_0e^{-qT_2}M(-a_1, b_1; -\sqrt{T_1/T_2}) + e^{-rT_1}K_1N(-a_2)$$

The value of a European call on a put is

$$K_2e^{-rT_2}M(-a_2, -b_2; \sqrt{T_1/T_2}) - S_0e^{-qT_2}M(-a_1, -b_1; \sqrt{T_1/T_2}) - e^{-rT_1}K_1N(-a_2)$$

The value of a European put on a put is

$$S_0e^{-qT_2}M(a_1, -b_1; -\sqrt{T_1/T_2}) - K_2e^{-rT_2}M(a_2, -b_2; -\sqrt{T_1/T_2}) + e^{-rT_1}K_1N(a_2)$$

26.8 CHOOSER OPTIONS

A *chooser* option (sometimes referred to as an *as you like it* option) has the feature that, after a specified period of time, the holder can choose whether the option is a call or a put. Suppose that the time when the choice is made is T_1. The value of the chooser option at this time is

$$\max(c, p)$$

where c is the value of the call underlying the option and p is the value of the put underlying the option.

If the options underlying the chooser option are both European and have the same strike price, put–call parity can be used to provide a valuation formula. Suppose that S_1

[2] See R. Geske, "The Valuation of Compound Options," *Journal of Financial Economics*, 7 (1979): 63–81; M. Rubinstein, "Double Trouble," *Risk*, December 1991/January 1992: 53–56.

[3] See Technical Note 5 at www-2.rotman.utoronto.ca/~hull/TechnicalNotes for a numerical procedure for calculating M. A function for calculating M is also on the website.

is the asset price at time T_1, K is the strike price, T_2 is the maturity of the options, and r is the risk-free interest rate. Put–call parity implies that

$$\begin{aligned}\max(c, p) &= \max(c,\ c + Ke^{-r(T_2-T_1)} - S_1 e^{-q(T_2-T_1)}) \\ &= c + e^{-q(T_2-T_1)} \max(0,\ Ke^{-(r-q)(T_2-T_1)} - S_1)\end{aligned}$$

This shows that the chooser option is a package consisting of:

1. A call option with strike price K and maturity T_2
2. $e^{-q(T_2-T_1)}$ put options with strike price $Ke^{-(r-q)(T_2-T_1)}$ and maturity T_1

As such, it can readily be valued.

More complex chooser options can be defined where the call and the put do not have the same strike price and time to maturity. They are then not packages and have features that are somewhat similar to compound options.

26.9 BARRIER OPTIONS

Barrier options are options where the payoff depends on whether the underlying asset's price reaches a certain level during a certain period of time.

A number of different types of barrier options regularly trade in the over-the-counter market. They are attractive to some market participants because they are less expensive than the corresponding regular options. These barrier options can be classified as either *knock-out options* or *knock-in options*. A knock-out option ceases to exist when the underlying asset price reaches a certain barrier; a knock-in option comes into existence only when the underlying asset price reaches a barrier.

Equations (17.4) and (17.5) show that the values at time zero of a regular call and put option are

$$\begin{aligned}c &= S_0 e^{-qT} N(d_1) - Ke^{-rT} N(d_2) \\ p &= Ke^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1)\end{aligned}$$

where

$$\begin{aligned}d_1 &= \frac{\ln(S_0/K) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}} \\ d_2 &= \frac{\ln(S_0/K) + (r - q - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}\end{aligned}$$

A *down-and-out call* is one type of knock-out option. It is a regular call option that ceases to exist if the asset price reaches a certain barrier level H. The barrier level is below the initial asset price. The corresponding knock-in option is a *down-and-in call*. This is a regular call that comes into existence only if the asset price reaches the barrier level.

If H is less than or equal to the strike price, K, the value of a down-and-in call at time zero is

$$c_{\text{di}} = S_0 e^{-qT} (H/S_0)^{2\lambda} N(y) - Ke^{-rT} (H/S_0)^{2\lambda-2} N(y - \sigma\sqrt{T})$$

where

$$\lambda = \frac{r - q + \sigma^2/2}{\sigma^2}$$

$$y = \frac{\ln[H^2/(S_0K)]}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$$

Because the value of a regular call equals the value of a down-and-in call plus the value of a down-and-out call, the value of a down-and-out call is given by

$$c_{\text{do}} = c - c_{\text{di}}$$

If $H \geqslant K$, then

$$c_{\text{do}} = S_0N(x_1)e^{-qT} - Ke^{-rT}N(x_1 - \sigma\sqrt{T}) - S_0e^{-qT}(H/S_0)^{2\lambda}N(y_1) + Ke^{-rT}(H/S_0)^{2\lambda-2}N(y_1 - \sigma\sqrt{T})$$

and

$$c_{\text{di}} = c - c_{\text{do}}$$

where

$$x_1 = \frac{\ln(S_0/H)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}, \quad y_1 = \frac{\ln(H/S_0)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$$

An *up-and-out call* is a regular call option that ceases to exist if the asset price reaches a barrier level, H, that is higher than the current asset price. An *up-and-in call* is a regular call option that comes into existence only if the barrier is reached. When H is less than or equal to K, the value of the up-and-out call, c_{uo}, is zero and the value of the up-and-in call, c_{ui}, is c. When H is greater than K,

$$c_{\text{ui}} = S_0N(x_1)e^{-qT} - Ke^{-rT}N(x_1 - \sigma\sqrt{T}) - S_0e^{-qT}(H/S_0)^{2\lambda}[N(-y) - N(-y_1)] + Ke^{-rT}(H/S_0)^{2\lambda-2}[N(-y + \sigma\sqrt{T}) - N(-y_1 + \sigma\sqrt{T})]$$

and

$$c_{\text{uo}} = c - c_{\text{ui}}$$

Put barrier options are defined similarly to call barrier options. An *up-and-out put* is a put option that ceases to exist when a barrier, H, that is greater than the current asset price is reached. An *up-and-in put* is a put that comes into existence only if the barrier is reached. When the barrier, H, is greater than or equal to the strike price, K, their prices are

$$p_{\text{ui}} = -S_0e^{-qT}(H/S_0)^{2\lambda}N(-y) + Ke^{-rT}(H/S_0)^{2\lambda-2}N(-y + \sigma\sqrt{T})$$

and

$$p_{\text{uo}} = p - p_{\text{ui}}$$

When H is less than or equal to K,

$$p_{\text{uo}} = -S_0N(-x_1)e^{-qT} + Ke^{-rT}N(-x_1 + \sigma\sqrt{T}) + S_0e^{-qT}(H/S_0)^{2\lambda}N(-y_1) - Ke^{-rT}(H/S_0)^{2\lambda-2}N(-y_1 + \sigma\sqrt{T})$$

and

$$p_{\text{ui}} = p - p_{\text{uo}}$$

A *down-and-out put* is a put option that ceases to exist when a barrier less than the current asset price is reached. A *down-and-in put* is a put option that comes into existence only when the barrier is reached. When the barrier is greater than the strike price, $p_{\mathrm{do}} = 0$ and $p_{\mathrm{di}} = p$. When the barrier is less than the strike price,

$$p_{\mathrm{di}} = -S_0 N(-x_1)e^{-qT} + Ke^{-rT}N(-x_1 + \sigma\sqrt{T}) + S_0 e^{-qT}(H/S_0)^{2\lambda}[N(y) - N(y_1)] - Ke^{-rT}(H/S_0)^{2\lambda-2}[N(y - \sigma\sqrt{T}) - N(y_1 - \sigma\sqrt{T})]$$

and

$$p_{\mathrm{do}} = p - p_{\mathrm{di}}$$

All of these valuations make the usual assumption that the probability distribution for the asset price at a future time is lognormal. An important issue for barrier options is the frequency with which the asset price, S, is observed for purposes of determining whether the barrier has been reached. The analytic formulas given in this section assume that S is observed continuously and sometimes this is the case.[4] Often, the terms of a contract state that S is observed periodically; for example, once a day at 3 p.m. Broadie, Glasserman, and Kou provide a way of adjusting the formulas we have just given for the situation where the price of the underlying is observed discretely.[5] The barrier level H is replaced by $He^{0.5826\sigma\sqrt{T/m}}$ for an up-and-in or up-and-out option and by $He^{-0.5826\sigma\sqrt{T/m}}$ for a down-and-in or down-and-out option, where m is the number of times the asset price is observed (so that T/m is the time interval between observations).

Barrier options often have quite different properties from regular options. For example, sometimes vega is negative. Consider an up-and-out call option when the asset price is close to the barrier level. As volatility increases, the probability that the barrier will be hit increases. As a result, a volatility increase can cause the price of the barrier option to decrease in these circumstances.

One disadvantage of the barrier options we have considered so far is that a "spike" in the asset price can cause the option to be knocked in or out. An alternative structure is a *Parisian option*, where the asset price has to be above or below the barrier for a period of time for the option to be knocked in or out. For example, a down-and-out Parisian put option with a strike price equal to 90% of the initial asset price and a barrier at 75% of the initial asset price might specify that the option is knocked out if the asset price is below the barrier for 50 days. The confirmation might specify that the 50 days are a "continuous period of 50 days" or "any 50 days during the option's life." Parisian options are more difficult to value than regular barrier options.[6] Monte Carlo simulation and binomial trees can be used with the enhancements discussed in Sections 27.5 and 27.6.

26.10 BINARY OPTIONS

Binary or digital options are options with discontinuous payoffs. A simple example of a binary option is a *cash-or-nothing call*. This pays off nothing if the asset price ends up

[4] One way to track whether a barrier has been reached from below (above) is to send a limit order to the exchange to sell (buy) the asset at the barrier price and see whether the order is filled.

[5] M. Broadie, P. Glasserman, and S. G. Kou, "A Continuity Correction for Discrete Barrier Options," *Mathematical Finance* 7, 4 (October 1997): 325–49.

[6] See, for example, M. Chesney, J. Cornwall, M. Jeanblanc-Picqué, G. Kentwell, and M. Yor, "Parisian pricing," *Risk*, 10, 1 (1997), 77–79.

below the strike price at time T and pays a fixed amount, Q, if it ends up above the strike price. In a risk-neutral world, the probability of the asset price being above the strike price at the maturity of an option is, with our usual notation, $N(d_2)$. The value of a cash-or-nothing call is therefore $Qe^{-rT}N(d_2)$. A *cash-or-nothing put* is defined analogously to a cash-or-nothing call. It pays off Q if the asset price is below the strike price and nothing if it is above the strike price. The value of a cash-or-nothing put is $Qe^{-rT}N(-d_2)$.

Another type of binary option is an *asset-or-nothing call*. This pays off nothing if the underlying asset price ends up below the strike price and pays the asset price if it ends up above the strike price. With our usual notation, the value of an asset-or-nothing call is $S_0e^{-qT}N(d_1)$. An *asset-or-nothing put* pays off nothing if the underlying asset price ends up above the strike price and the asset price if it ends up below the strike price. The value of an asset-or-nothing put is $S_0e^{-qT}N(-d_1)$.

Binary options have discontinuous payoffs. This can create problems if the underlying asset is thinly traded so that relatively small buy or sell trades move the price. Consider a cash-or-nothing call with a strike price of \$20 and a payoff of \$1 million. If the final asset price is \$19.99, there is no payoff; if it is \$20 or more, the payoff is \$1 million. If there is one day left in the life of the option and the price a little below \$20, it would be tempting for the holder of the cash-or-nothing call to place buy orders for the underlying asset to tip the price over \$20. A similar issue can arise with barrier options on thinly traded assets. If the price of the underlying asset is close to the barrier, one side is likely to be tempted to place buy or sell orders to ensure that the barrier is reached.

A regular European call option is equivalent to a long position in an asset-or-nothing call and a short position in a cash-or-nothing call where the cash payoff in the cash-or-nothing call equals the strike price. Similarly, a regular European put option is equivalent to a long position in a cash-or-nothing put and a short position in an asset-or-nothing put where the cash payoff on the cash-or-nothing put equals the strike price.

26.11 LOOKBACK OPTIONS

The payoffs from lookback options depend on the maximum or minimum asset price reached during the life of the option. The payoff from a *floating lookback call* is the amount that the final asset price exceeds the minimum asset price achieved during the life of the option. The payoff from a *floating lookback put* is the amount by which the maximum asset price achieved during the life of the option exceeds the final asset price.

Valuation formulas have been produced for floating lookbacks.[7] The value of a floating lookback call at time zero is

$$c_{\text{fl}} = S_0e^{-qT}N(a_1) - S_0e^{-qT}\frac{\sigma^2}{2(r-q)}N(-a_1) - S_{\min}e^{-rT}\left[N(a_2) - \frac{\sigma^2}{2(r-q)}e^{Y_1}N(-a_3)\right]$$

[7] See B. Goldman, H. Sosin, and M. A. Gatto, "Path-Dependent Options: Buy at the Low, Sell at the High," *Journal of Finance*, 34 (December 1979): 1111–27.; M. Garman, "Recollection in Tranquility," *Risk*, March (1989): 16–19.

where

$$a_1 = \frac{\ln(S_0/S_{\min}) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$a_2 = a_1 - \sigma\sqrt{T},$$

$$a_3 = \frac{\ln(S_0/S_{\min}) + (-r + q + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$Y_1 = -\frac{2(r - q - \sigma^2/2)\ln(S_0/S_{\min})}{\sigma^2}$$

and $S_{\min}$ is the minimum asset price achieved to date. (If the lookback has just been originated, $S_{\min} = S_0$.) See Problem 26.23 for the $r = q$ case.

The value of a floating lookback put is

$$p_{\text{fl}} = S_{\max}e^{-rT}\left[N(b_1) - \frac{\sigma^2}{2(r-q)}e^{Y_2}N(-b_3)\right] + S_0e^{-qT}\frac{\sigma^2}{2(r-q)}N(-b_2) - S_0e^{-qT}N(b_2)$$

where

$$b_1 = \frac{\ln(S_{\max}/S_0) + (-r + q + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$b_2 = b_1 - \sigma\sqrt{T}$$

$$b_3 = \frac{\ln(S_{\max}/S_0) + (r - q - \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$Y_2 = \frac{2(r - q - \sigma^2/2)\ln(S_{\max}/S_0)}{\sigma^2}$$

and $S_{\max}$ is the maximum asset price achieved to date. (If the lookback has just been originated, then $S_{\max} = S_0$.)

A floating lookback call is a way that the holder can buy the underlying asset at the lowest price achieved during the life of the option. Similarly, a floating lookback put is a way that the holder can sell the underlying asset at the highest price achieved during the life of the option.

Example 26.2

Consider a newly issued floating lookback put on a non-dividend-paying stock where the stock price is 50, the stock price volatility is 40% per annum, the risk-free rate is 10% per annum, and the time to maturity is 3 months. In this case, $S_{\max} = 50$, $S_0 = 50$, $r = 0.1$, $q = 0$, $\sigma = 0.4$, and $T = 0.25$, $b_1 = -0.025$, $b_2 = -0.225$, $b_3 = 0.025$, and $Y_2 = 0$, so that the value of the lookback put is 7.79. A newly issued floating lookback call on the same stock is worth 8.04.

In a fixed lookback option, a strike price is specified. For a *fixed lookback call option*, the payoff is the same as a regular European call option except that the final asset price is replaced by the maximum asset price achieved during the life of the option. For a *fixed lookback put option*, the payoff is the same as a regular European put option except that the the final asset price is replaced by the minimum asset price achieved during the life of the option. Define $S^*_{\max} = \max(S_{\max}, K)$, where as before $S_{\max}$ is the maximum asset price achieved to date and K is the strike price. Also, define p^*_{fl} as the value of a floating lookback put which lasts for the same period as the fixed lookback call when the actual maximum asset price so far, $S_{\max}$, is replaced by $S^*_{\max}$. A put–call parity type of argument shows that the value of the fixed lookback call option, c_{fix} is given by[8]

$$c_{\text{fix}} = p^*_{\text{fl}} + S_0 e^{-qT} - K e^{-rT}$$

Similarly, if $S^*_{\min} = \min(S_{\min}, K)$, then the value of a fixed lookback put option, p_{fix}, is given by

$$p_{\text{fix}} = c^*_{\text{fl}} + K e^{-rT} - S_0 e^{-qT}$$

where c^*_{fl} is the value of a floating lookback call that lasts for the same period as the fixed lookback put when the actual minimum asset price so far, $S_{\min}$, is replaced by $S^*_{\min}$. This shows that the equations given above for floating lookbacks can be modified to price fixed lookbacks.

Lookbacks are appealing to investors, but very expensive when compared with regular options. As with barrier options, the value of a lookback option is liable to be sensitive to the frequency with which the asset price is observed for the purposes of computing the maximum or minimum. The formulas above assume that the asset price is observed continuously. Broadie, Glasserman, and Kou provide a way of adjusting the formulas we have just given for the situation where the asset price is observed discretely.[9]

26.12 SHOUT OPTIONS

A *shout option* is a European option where the holder can "shout" to the writer at one time during its life. At the end of the life of the option, the option holder receives either the usual payoff from a European option or the intrinsic value at the time of the shout, whichever is greater. Suppose the strike price is \$50 and the holder of a call shouts when the price of the underlying asset is \$60. If the final asset price is less than \$60, the holder receives a payoff of \$10. If it is greater than \$60, the holder receives the excess of the asset price over \$50.

A shout option has some of the same features as a lookback option, but is considerably less expensive. It can be valued by noting that if the holder shouts at a time τ when the asset price is S_τ the payoff from the option is

$$\max(0,\ S_T - S_\tau) + (S_\tau - K)$$

where, as usual, K is the strike price and S_T is the asset price at time T. The value at time τ if the holder shouts is therefore the present value of $S_\tau - K$ (received at time T) plus the value of a European option with strike price S_τ. The latter can be calculated using Black–Scholes–Merton formulas.

A shout option is valued by constructing a binomial or trinomial tree for the underlying asset in the usual way. Working back through the tree, the value of the option if the holder shouts and the value if the holder does not shout can be calculated at each node.

[8] The argument was proposed by H. Y. Wong and Y. K. Kwok, "Sub-replication and Replenishing Premium: Efficient Pricing of Multi-state Lookbacks," *Review of Derivatives Research*, 6 (2003), 83–106.

[9] M. Broadie, P. Glasserman, and S. G. Kou, "Connecting Discrete and Continuous Path-Dependent Options," *Finance and Stochastics*, 2 (1998): 1–28.

The option's price at the node is the greater of the two. The procedure for valuing a shout option is therefore similar to the procedure for valuing a regular American option.

26.13 ASIAN OPTIONS

Asian options are options where the payoff depends on the arithmetic average of the price of the underlying asset during the life of the option. The payoff from an *average price call* is $\max(0, S_{ave} - K)$ and that from an *average price put* is $\max(0, K - S_{ave})$, where S_{ave} is the average price of the underlying asset. Average price options tend to be less expensive than regular options and are arguably more appropriate than regular options for meeting some of the needs of corporate treasurers. Suppose that a U.S. corporate treasurer expects to receive a cash flow of 100 million Australian dollars spread evenly over the next year from the company's Australian subsidiary. The treasurer is likely to be interested in an option that guarantees that the average exchange rate realized during the year is above some level. An average price put option can achieve this more effectively than regular put options.

Average price options can be valued using similar formulas to those used for regular options if it is assumed that S_{ave} is lognomal. As it happens, when the usual assumption is made for the process followed by the asset price, this is a reasonable assumption.[10] A popular approach is to fit a lognormal distribution to the first two moments of S_{ave} and use Black's model.[11] Suppose that M_1 and M_2 are the first two moments of S_{ave}. The value of average price calls and puts are given by equations (18.9) and (18.10), with

$$F_0 = M_1 \tag{26.3}$$

and

$$\sigma^2 = \frac{1}{T}\ln\left(\frac{M_2}{M_1^2}\right) \tag{26.4}$$

When the average is calculated continuously, and r, q, and σ are constant (as in DerivaGem):

$$M_1 = \frac{e^{(r-q)T} - 1}{(r-q)T} S_0$$

and

$$M_2 = \frac{2e^{[2(r-q)+\sigma^2]T} S_0^2}{(r-q+\sigma^2)(2r-2q+\sigma^2)T^2} + \frac{2S_0^2}{(r-q)T^2}\left(\frac{1}{2(r-q)+\sigma^2} - \frac{e^{(r-q)T}}{r-q+\sigma^2}\right)$$

More generally, when the average is calculated from observations at times T_i ($1 \leqslant i \leqslant m$),

$$M_1 = \frac{1}{m}\sum_{i=1}^{m} F_i \quad \text{and} \quad M_2 = \frac{1}{m^2}\left(\sum_{i=1}^{m} F_i^2 e^{\sigma_i^2 T_i} + 2\sum_{j=1}^{m}\sum_{i=1}^{j-1} F_i F_j e^{\sigma_i^2 T_i}\right)$$

where F_i and σ_i are the forward price and implied volatility for maturity T_i. See Technical Note 27 on `www-2.rotman.utoronto.ca/~hull/TechnicalNotes` for a proof of this.

[10] When the asset price follows geometric Brownian motion, the geometric average of the price is exactly lognormal and the arithmetic average is approximately lognormal.

[11] See S. M. Turnbull and L. M. Wakeman, "A Quick Algorithm for Pricing European Average Options," *Journal of Financial and Quantitative Analysis*, 26 (September 1991): 377–89.

Example 26.3

Consider a newly issued average price call option on a non-dividend-paying stock where the stock price is 50, the strike price is 50, the stock price volatility is 40% per annum, the risk-free rate is 10% per annum, and the time to maturity is 1 year. In this case, $S_0 = 50$, $K = 50$, $r = 0.1$, $q = 0$, $\sigma = 0.4$, and $T = 1$. If the average is calculated continuously, $M_1 = 52.59$ and $M_2 = 2{,}922.76$. From equations (26.3) and (26.4), $F_0 = 52.59$ and $\sigma = 23.54\%$. Equation (18.9), with $K = 50$, $T = 1$, and $r = 0.1$, gives the value of the option as 5.62. When 12, 52, and 250 observations are used for the average, the price is 6.00, 5.70, and 5.63, respectively.

We can modify the analysis to accommodate the situation where the option is not newly issued and some prices used to determine the average have already been observed. Suppose that the averaging period is composed of a period of length t_1 over which prices have already been observed and a future period of length t_2 (the remaining life of the option). Suppose that the average asset price during the first time period is $\bar{S}$. The payoff from an average price call is

$$\max\left(\frac{\bar{S}t_1 + S_{\text{ave}}t_2}{t_1 + t_2} - K,\ 0\right)$$

where S_{ave} is the average asset price during the remaining part of the averaging period. This is the same as

$$\frac{t_2}{t_1 + t_2}\max(S_{\text{ave}} - K^*,\ 0)$$

where

$$K^* = \frac{t_1 + t_2}{t_2}K - \frac{t_1}{t_2}\bar{S}$$

When $K^* > 0$, the option can be valued in the same way as a newly issued Asian option provided that we change the strike price from K to K^* and multiply the result by $t_2/(t_1 + t_2)$. When $K^* < 0$ the option is certain to be exercised and can be valued as a forward contract. The value is

$$\frac{t_2}{t_1 + t_2}[M_1e^{-rt_2} - K^*e^{-rt_2}]$$

Another type of Asian option is an average strike option. An *average strike call* pays off $\max(0,\ S_T - S_{\text{ave}})$ and an *average strike put* pays off $\max(0,\ S_{\text{ave}} - S_T)$. Average strike options can guarantee that the average price paid for an asset in frequent trading over a period of time is not greater than the final price. Alternatively, it can guarantee that the average price received for an asset in frequent trading over a period of time is not less than the final price. It can be valued as an option to exchange one asset for another when S_{ave} is assumed to be lognormal.

26.14 OPTIONS TO EXCHANGE ONE ASSET FOR ANOTHER

Options to exchange one asset for another (sometimes referred to as *exchange options*) arise in various contexts. An option to buy yen with Australian dollars is, from the point of view of a U.S. investor, an option to exchange one foreign currency asset for another foreign currency asset. A stock tender offer is an option to exchange shares in one stock for shares in another stock.

Consider a European option to give up an asset worth U_T at time T and receive in return an asset worth V_T. The payoff from the option is

$$\max(V_T - U_T,\ 0)$$

A formula for valuing this option was first produced by Margrabe.[12] Suppose that the asset prices, U and V, both follow geometric Brownian motion with volatilities σ_U and σ_V. Suppose further that the instantaneous correlation between U and V is ρ, and the yields provided by U and V are q_U and q_V, respectively. The value of the option at time zero is

$$V_0 e^{-q_V T} N(d_1) - U_0 e^{-q_U T} N(d_2) \tag{26.5}$$

where

$$d_1 = \frac{\ln(V_0/U_0) + (q_U - q_V + \hat{\sigma}^2/2)T}{\hat{\sigma}\sqrt{T}}, \quad d_2 = d_1 - \hat{\sigma}\sqrt{T}$$

and

$$\hat{\sigma} = \sqrt{\sigma_U^2 + \sigma_V^2 - 2\rho\sigma_U\sigma_V}$$

and U_0 and V_0 are the values of U and V at times zero.

This result will be proved in Chapter 28. It is interesting to note that equation (26.5) is independent of the risk-free rate r. This is because, as r increases, the growth rate of both asset prices in a risk-neutral world increases, but this is exactly offset by an increase in the discount rate. The variable $\hat{\sigma}$ is the volatility of V/U. Comparisons with equation (17.4) show that the option price is the same as the price of U_0 European call options on an asset worth V/U when the strike price is 1.0, the risk-free interest rate is q_U, and the dividend yield on the asset is q_V. Mark Rubinstein shows that the American version of this option can be characterized similarly for valuation purposes.[13] It can be regarded as U_0 American options to buy an asset worth V/U for 1.0 when the risk-free interest rate is q_U and the dividend yield on the asset is q_V. The option can therefore be valued as described in Chapter 21 using a binomial tree.

An option to obtain the better or worse of two assets can be regarded as a position in one of the assets combined with an option to exchange it for the other asset:

$$\min(U_T, V_T) = V_T - \max(V_T - U_T,\ 0)$$
$$\max(U_T, V_T) = U_T + \max(V_T - U_T,\ 0)$$

26.15 OPTIONS INVOLVING SEVERAL ASSETS

Options involving two or more risky assets are sometimes referred to as *rainbow options*. One example is the bond futures contract traded on the CBOT described in Chapter 6. The party with the short position is allowed to choose between a large number of different bonds when making delivery.

[12] See W. Margrabe, "The Value of an Option to Exchange One Asset for Another," *Journal of Finance*, 33 (March 1978): 177–86.

[13] See M. Rubinstein, "One for Another," *Risk*, July/August 1991: 30–32

Probably the most popular option involving several assets is a European *basket option*. This is an option where the payoff is dependent on the value of a portfolio (or basket) of assets. The assets are usually either individual stocks or stock indices or currencies. A European basket option can be valued with Monte Carlo simulation, by assuming that the assets follow correlated geometric Brownian motion processes. A much faster approach is to calculate the first two moments of the basket at the maturity of the option in a risk-neutral world, and then assume that value of the basket is lognormally distributed at that time. The option can then be valued using Black's model with the parameters shown in equations (26.3) and (26.4). In this case,

$$M_1 = \sum_{i=1}^{n} F_i \quad \text{and} \quad M_2 = \sum_{i=1}^{n} \sum_{j=1}^{n} F_i F_j e^{\rho_{ij}\sigma_i\sigma_j T}$$

where n is the number of assets, T is the option maturity, F_i and σ_i are the forward price and volatility of the ith asset, and ρ_{ij} is the correlation between the ith and jth asset. See Technical Note 28 at www-2.rotman.utoronto.ca/~hull/TechnicalNotes.

26.16 VOLATILITY AND VARIANCE SWAPS

A volatility swap is an agreement to exchange the realized volatility of an asset between time 0 and time T for a prespecifed fixed volatility. The realized volatility is usually calculated as described in Section 15.4 but with the assumption that the mean daily return is zero. Suppose that there are n daily observations on the asset price during the period between time 0 and time T. The realized volatility is

$$\bar{\sigma} = \sqrt{\frac{252}{n-2} \sum_{i=1}^{n-1} \left[\ln\left(\frac{S_{i+1}}{S_i} \right) \right]^2}$$

where S_i is the ith observation on the asset price. (Sometimes $n-1$ might replace $n-2$ in this formula.)

The payoff from the volatility swap at time T to the payer of the fixed volatility is $L_{\text{vol}}(\bar{\sigma} - \sigma_K)$, where L_{vol} is the notional principal and σ_K is the fixed volatility. Whereas an option provides a complex exposure to the asset price and volatility, a volatility swap is simpler in that it has exposure only to volatility.

A variance swap is an agreement to exchange the realized variance rate $\bar{V}$ between time 0 and time T for a prespecified variance rate. The variance rate is the square of the volatility ($\bar{V} = \bar{\sigma}^2$). Variance swaps are easier to value than volatility swaps. This is because the variance rate between time 0 and time T can be replicated using a portfolio of put and call options. The payoff from a variance swap at time T to the payer of the fixed variance rate is $L_{\text{var}}(\bar{V} - V_K)$, where L_{var} is the notional principal and V_K is the fixed variance rate. Often the notional principal for a variance swap is expressed in terms of the corresponding notional principal for a volatility swap using $L_{\text{var}} = L_{\text{vol}}/(2\sigma_K)$.

Valuation of Variance Swap

Technical Note 22 at www-2.rotman.utoronto.ca/~hull/TechnicalNotes shows that, for any value S^* of the asset price, the expected average variance between times 0 and T is

$$\hat{E}(\bar{V}) = \frac{2}{T}\ln\frac{F_0}{S^*} - \frac{2}{T}\left[\frac{F_0}{S^*} - 1\right] + \frac{2}{T}\left[\int_{K=0}^{S^*}\frac{1}{K^2}e^{rT}p(K)\,dK + \int_{K=S^*}^{\infty}\frac{1}{K^2}e^{rT}c(K)\,dK\right] \tag{26.6}$$

where F_0 is the forward price of the asset for a contract maturing at time T, $c(K)$ is the price of a European call option with strike price K and time to maturity T, and $p(K)$ is the price of a European put option with strike price K and time to maturity T.

This provides a way of valuing a variance swap.[14] The value of an agreement to receive the realized variance between time 0 and time T and pay a variance rate of V_K, with both being applied to a principal of L_{var}, is

$$L_{\text{var}}[\hat{E}(\bar{V}) - V_K]e^{-rT} \tag{26.7}$$

Suppose that the prices of European options with strike prices K_i $(1 \leqslant i \leqslant n)$ are known, where $K_1 < K_2 < \cdots < K_n$. A standard approach for implementing equation (26.6) is to set S^* equal to the first strike price below F_0 and then approximate the integrals as

$$\int_{K=0}^{S^*}\frac{1}{K^2}e^{rT}p(K)dK + \int_{K=S^*}^{\infty}\frac{1}{K^2}e^{rT}c(K)dK = \sum_{i=1}^{n}\frac{\Delta K_i}{K_i^2}e^{rT}Q(K_i) \tag{26.8}$$

where $\Delta K_i = 0.5(K_{i+1} - K_{i-1})$ for $2 \leqslant i \leqslant n-1$, $\Delta K_1 = K_2 - K_1$, $\Delta K_n = K_n - K_{n-1}$. The function $Q(K_i)$ is the price of a European put option with strike price K_i if $K_i < S^*$ and the price of a European call option with strike price K_i if $K_i > S^*$. When $K_i = S^*$, the function $Q(K_i)$ is equal to the average of the prices of a European call and a European put with strike price K_i.

Example 26.4

Consider a 3-month contract to receive the realized variance rate of an index over the 3 months and pay a variance rate of 0.045 on a principal of \$100 million. The risk-free rate is 4% and the dividend yield on the index is 1%. The current level of the index is 1020. Suppose that, for strike prices of 800, 850, 900, 950, 1,000, 1,050, 1,100, 1,150, 1,200, the 3-month implied volatilities of the index are 29%, 28%, 27%, 26%, 25%, 24%, 23%, 22%, 21%, respectively. In this case, $n = 9$, $K_1 = 800$, $K_2 = 850, \ldots, K_9 = 1{,}200$, $F_0 = 1{,}020e^{(0.04-0.01)\times 0.25} = 1{,}027.68$, and $S^* = 1{,}000$. DerivaGem shows that $Q(K_1) = 2.22$, $Q(K_2) = 5.22$, $Q(K_3) = 11.05$, $Q(K_4) = 21.27$, $Q(K_5) = 51.21$, $Q(K_6) = 38.94$, $Q(K_7) = 20.69$, $Q(K_8) = 9.44$, $Q(K_9) = 3.57$. Also, $\Delta K_i = 50$ for all i. Hence,

$$\sum_{i}^{n}\frac{\Delta K_i}{K_i^2}e^{rT}Q(K_i) = 0.008139$$

From equations (26.6) and (26.8), it follows that

$$\hat{E}(\bar{V}) = \frac{2}{0.25}\ln\left(\frac{1027.68}{1{,}000}\right) - \frac{2}{0.25}\left(\frac{1027.68}{1{,}000} - 1\right) + \frac{2}{0.25}\times 0.008139 = 0.0621$$

[14] See also K. Demeterfi, E. Derman, M. Kamal, and J. Zou, "A Guide to Volatility and Variance Swaps," *The Journal of Derivatives*, 6, 4 (Summer 1999), 9–32. For options on variance and volatility, see P. Carr and R. Lee, "Realized Volatility and Variance: Options via Swaps," *Risk*, May 2007, 76–83.

From equation (26.7), the value of the variance swap (in millions of dollars) is $100 \times (0.0621 - 0.045)e^{-0.04\times 0.25} = 1.69$.

Valuation of a Volatility Swap

To value a volatility swap, we require $\hat{E}(\bar{\sigma})$, where $\bar{\sigma}$ is the average value of volatility between time 0 and time T. We can write

$$\bar{\sigma} = \sqrt{\hat{E}(\bar{V})}\sqrt{1 + \frac{\bar{V} - \hat{E}(\bar{V})}{\hat{E}(\bar{V})}}$$

Expanding the second term on the right-hand side in a series gives the approximation:

$$\bar{\sigma} = \sqrt{\hat{E}(\bar{V})}\left\{1 + \frac{\bar{V} - \hat{E}(\bar{V})}{2\hat{E}(\bar{V})} - \tfrac{1}{8}\left[\frac{\bar{V} - \hat{E}(\bar{V})}{\hat{E}(\bar{V})}\right]^2\right\}$$

Taking expectations,

$$\hat{E}(\bar{\sigma}) = \sqrt{\hat{E}(\bar{V})}\left\{1 - \tfrac{1}{8}\left[\frac{\mathrm{var}(\bar{V})}{\hat{E}(\bar{V})^2}\right]\right\} \qquad \textbf{(26.9)}$$

where $\mathrm{var}(\bar{V})$ is the variance of $\bar{V}$. The valuation of a volatility swap therefore requires an estimate of the variance of the average variance rate during the life of the contract. The value of an agreement to receive the realized volatility between time 0 and time T and pay a volatility of σ_K, with both being applied to a principal of L_{vol}, is

$$L_{\mathrm{vol}}[\hat{E}(\bar{\sigma}) - \sigma_K]e^{-rT}$$

Example 26.5

For the situation in Example 26.4, consider a volatility swap where the realized volatility is received and a volatility of 23% is paid on a principal of $100 million. In this case $\hat{E}(\bar{V}) = 0.0621$. Suppose that the standard deviation of the average variance over 3 months has been estimated as 0.01. This means that $\mathrm{var}(\bar{V}) = 0.0001$. Equation (26.9) gives

$$\hat{E}(\bar{\sigma}) = \sqrt{0.0621}\left(1 - \tfrac{1}{8} \times \frac{0.0001}{0.0621^2}\right) = 0.2484$$

The value of the swap in (millions of dollars) is

$$100 \times (0.2484 - 0.23)e^{-0.04\times 0.25} = 1.82$$

The VIX Index

In equation (26.6), the ln function can be approximated by the first two terms in a series expansion:

$$\ln\left(\frac{F_0}{S^*}\right) = \left(\frac{F_0}{S^*} - 1\right) - \tfrac{1}{2}\left(\frac{F_0}{S^*} - 1\right)^2$$

This means that the risk-neutral expected cumulative variance is calculated as

$$\hat{E}(\bar{V})T = -\left(\frac{F_0}{S^*} - 1\right)^2 + 2\sum_{i=1}^{n}\frac{\Delta K_i}{K_i^2}e^{rT}Q(K_i) \qquad \textbf{(26.10)}$$

Since 2004 the VIX volatility index (see Section 15.11) has been based on equation (26.10). The procedure used on any given day is to calculate $\hat{E}(\bar{V})T$ for options that trade in the market and have maturities immediately above and below 30 days. The 30-day risk-neutral expected cumulative variance is calculated from these two numbers using interpolation. This is then multiplied by 365/30 and the index is set equal to the square root of the result. More details on the calculation can be found on the CBOE website.

26.17 STATIC OPTIONS REPLICATION

If the procedures described in Chapter 19 are used for hedging exotic options, some are easy to handle, but others are very difficult because of discontinuities (see Business Snapshot 26.1). For the difficult cases, a technique known as static options replication is sometimes useful.[15] This involves searching for a portfolio of actively traded options that approximately replicates the exotic option. Shorting this position provides the hedge.[16]

The basic principle underlying static options replication is as follows. If two portfolios are worth the same on a certain boundary, they are also worth the same at all interior points of the boundary. Consider as an example a 9-month up-and-out call option on a non-dividend-paying stock where the stock price is 50, the strike price is 50, the barrier is 60, the risk-free interest rate is 10% per annum, and the volatility is 30% per annum. Suppose that $f(S,t)$ is the value of the option at time t for a stock price of S. Any boundary in (S,t) space can be used for the purposes of producing the replicating portfolio. A convenient one to choose is shown in Figure 26.1. It is defined by $S = 60$ and $t = 0.75$. The values of the up-and-out option on the boundary are given by

$$f(S, 0.75) = \max(S - 50,\ 0) \quad \text{when } S < 60$$
$$f(60, t) = 0 \quad \text{when } 0 \leqslant t \leqslant 0.75$$

There are many ways that these boundary values can be approximately matched using regular options. The natural option to match the first boundary is a 9-month European call with a strike price of 50. The first component of the replicating portfolio is therefore one unit of this option. (We refer to this option as option A.)

One way of matching the $f(60,t)$ boundary is to proceed as follows:

1. Divide the life of the option into N steps of length Δt
2. Choose a European call option with a strike price of 60 and maturity at time $N\Delta t$ (= 9 months) to match the boundary at the $\{60, (N-1)\Delta t\}$ point
3. Choose a European call option with a strike price of 60 and maturity at time $(N-1)\Delta t$ to match the boundary at the $\{60, (N-2)\Delta t\}$ point

and so on. Note that the options are chosen in sequence so that they have zero value on the parts of the boundary matched by earlier options.[17] The option with a strike price

[15] See E. Derman, D. Ergener, and I. Kani, "Static Options Replication," *Journal of Derivatives* 2, 4 (Summer 1995): 78–95.

[16] Technical Note 22 at www-2.rotman.utoronto.ca/~hull/TechnicalNotes provides an example of static replication. It shows that the variance rate of an asset can be replicated by a position in the asset and out-of-the-money options on the asset. This result, which leads to equation (26.6), can be used to hedge variance swaps.

[17] This is not a requirement. If K points on the boundary are to be matched, we can choose K options and solve a set of K linear equations to determine required positions in the options.

Business Snapshot 26.1 Is Delta Hedging Easier or More Difficult for Exotics?

As described in Chapter 19, we can approach the hedging of exotic options by creating a delta neutral position and rebalancing frequently to maintain delta neutrality. When we do this we find some exotic options are easier to hedge than plain vanilla options and some are more difficult.

An example of an exotic option that is relatively easy to hedge is an average price option where the averaging period is the whole life of the option. As time passes, we observe more of the asset prices that will be used in calculating the final average. This means that our uncertainty about the payoff decreases with the passage of time. As a result, the option becomes progressively easier to hedge. In the final few days, the delta of the option always approaches zero because price movements during this time have very little impact on the payoff.

By contrast barrier options are relatively difficult to hedge. Consider a down-and-out call option on a currency when the exchange rate is 0.0005 above the barrier. If the barrier is hit, the option is worth nothing. If the barrier is not hit, the option may prove to be quite valuable. The delta of the option is discontinuous at the barrier making conventional hedging very difficult.

of 60 that matures in 9 months has zero value on the vertical boundary that is matched by option A. The option maturing at time $i\,\Delta t$ has zero value at the point $\{60, i\,\Delta t\}$ that is matched by the option maturing at time $(i+1)\Delta t$ for $1 \leqslant i \leqslant N-1$.

Suppose that $\Delta t = 0.25$. In addition to option A, the replicating portfolio consists of positions in European options with strike price 60 that mature in 9, 6, and 3 months. We will refer to these as options B, C, and D, respectively. Given our assumptions

Figure 26.1 Boundary points used for static options replication example.

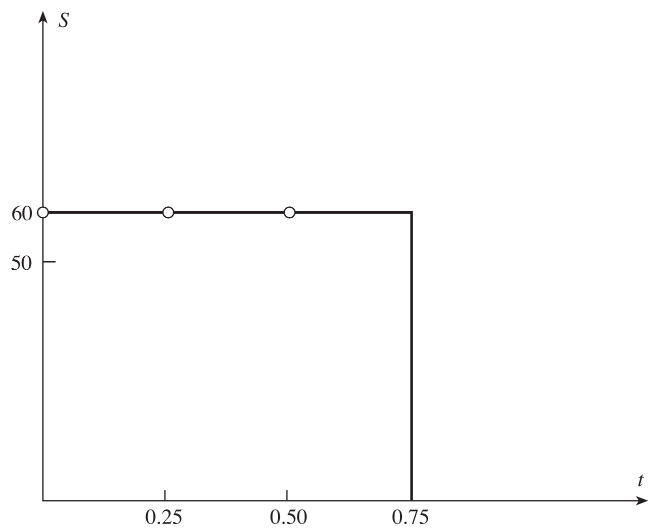

Table 26.1 The portfolio of European call options used to replicate an up-and-out option.

Option	*Strike price*	*Maturity (years)*	*Position*	*Initial value*
A	50	0.75	1.00	+6.99
B	60	0.75	−2.66	−8.21
C	60	0.50	0.97	+1.78
D	60	0.25	0.28	+0.17

about volatility and interest rates, option B is worth 4.33 at the $\{60, 0.5\}$ point. Option A is worth 11.54 at this point. The position in option B necessary to match the boundary at the $\{60, 0.5\}$ point is therefore $-11.54/4.33 = -2.66$. Option C is worth 4.33 at the $\{60, 0.25\}$ point. The position taken in options A and B is worth -4.21 at this point. The position in option C necessary to match the boundary at the $\{60, 0.25\}$ point is therefore $4.21/4.33 = 0.97$. Similar calculations show that the position in option D necessary to match the boundary at the $\{60, 0\}$ point is 0.28.

The portfolio chosen is summarized in Table 26.1. (See also Sample Application F of the DerivaGem Applications.) It is worth 0.73 initially (i.e., at time zero when the stock price is 50). This compares with 0.31 given by the analytic formula for the up-and-out call earlier in this chapter. The replicating portfolio is not exactly the same as the up-and-out option because it matches the latter at only three points on the second boundary. If we use the same procedure, but match at 18 points on the second boundary (using options that mature every half month), the value of the replicating portfolio reduces to 0.38. If 100 points are matched, the value reduces further to 0.32.

To hedge a derivative, the portfolio that replicates its boundary conditions must be shorted. The hedge lasts until the end of the life of the derivative or until the boundary is reached, whichever happens first. If the boundary is reached, the hedge portfolio must be unwound and a new hedge portfolio set up.

Static options replication has the advantage over delta hedging that it does not require frequent rebalancing. It can be used for a wide range of derivatives. The user has a great deal of flexibility in choosing the boundary that is to be matched and the options that are to be used.

SUMMARY

Exotic options are options with rules governing the payoff that are more complicated than standard options. We have discussed 15 different types of exotic options: packages, perpetual American options, nonstandard American options, gap options, forward start options, cliquet options, compound options, chooser options, barrier options, binary options, lookback options, shout options, Asian options, options to exchange one asset for another, and options involving several assets. We have discussed how these can be valued using the same assumptions as those used to derive the Black–Scholes–Merton model in Chapter 15. Some can be valued analytically, but using much more complicated formulas than those for regular European calls and puts, some can be handled using analytic approximations, and some can be valued using extensions of the numerical procedures in Chapter 21. We will present more numerical procedures for valuing exotic options in Chapter 27.

Some exotic options are easier to hedge than the corresponding regular options; others are more difficult. In general, Asian options are easier to hedge because the payoff becomes progressively more certain as we approach maturity. Barrier options can be more difficult to hedge because delta is discontinuous at the barrier. One approach to hedging an exotic option, known as static options replication, is to find a portfolio of regular options whose value matches the value of the exotic option on some boundary. The exotic option is hedged by shorting this portfolio.

FURTHER READING

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Practice Questions (Answers in Solutions Manual)

26.1. Explain the difference between a forward start option and a chooser option.

26.2. Describe the payoff from a portfolio consisting of a floating lookback call and a floating lookback put with the same maturity.

26.3. Consider a chooser option where the holder has the right to choose between a European call and a European put at any time during a 2-year period. The maturity dates and strike prices for the calls and puts are the same regardless of when the choice is made. Is it ever optimal to make the choice before the end of the 2-year period? Explain your answer.

26.4. Suppose that c_1 and p_1 are the prices of a European average price call and a European average price put with strike price K and maturity T, c_2 and p_2 are the prices of a European average strike call and European average strike put with maturity T, and c_3 and p_3 are the prices of a regular European call and a regular European put with strike price K and maturity T. Show that $c_1 + c_2 - c_3 = p_1 + p_2 - p_3$.

26.5. The text derives a decomposition of a particular type of chooser option into a call maturing at time T_2 and a put maturing at time T_1. Derive an alternative decomposition into a call maturing at time T_1 and a put maturing at time T_2.

26.6. Section 26.9 gives two formulas for a down-and-out call. The first applies to the situation where the barrier, H, is less than or equal to the strike price, K. The second applies to the situation where $H \geqslant K$. Show that the two formulas are the same when $H = K$.

26.7. Explain why a down-and-out put is worth zero when the barrier is greater than the strike price.

26.8. Suppose that the strike price of an American call option on a non-dividend-paying stock grows at rate g. Show that if g is less than the risk-free rate, r, it is never optimal to exercise the call early.

26.9. How can the value of a forward start put option on a non-dividend-paying stock be calculated if it is agreed that the strike price will be 10% greater than the stock price at the time the option starts?

26.10. If a stock price follows geometric Brownian motion, what process does $A(t)$ follow where $A(t)$ is the arithmetic average stock price between time zero and time t?

26.11. Explain why delta hedging is easier for Asian options than for regular options.

26.12. Calculate the price of a 1-year European option to give up 100 ounces of silver in exchange for 1 ounce of gold. The current prices of gold and silver are \$1,520 and \$16, respectively; the risk-free interest rate is 10% per annum; the volatility of each commodity price is 20%; and the correlation between the two prices is 0.7. Ignore storage costs.

26.13. Is a European down-and-out option on an asset worth the same as a European down-and-out option on the asset's futures price for a futures contract maturing at the same time as the option?

26.14. Answer the following questions about compound options:
(a) What put–call parity relationship exists between the price of a European call on a call and a European put on a call? Show that the formulas given in the text satisfy the relationship.
(b) What put–call parity relationship exists between the price of a European call on a put and a European put on a put? Show that the formulas given in the text satisfy the relationship.

26.15. Does a floating lookback call become more valuable or less valuable as we increase the frequency with which we observe the asset price in calculating the minimum?

26.16. Does a down-and-out call become more valuable or less valuable as we increase the frequency with which we observe the asset price in determining whether the barrier has been crossed? What is the answer to the same question for a down-and-in call?

26.17. Explain why a regular European call option is the sum of a down-and-out European call and a down-and-in European call. Is the same true for American call options?

26.18. What is the value of a derivative that pays off \$100 in 6 months if an index is greater than 1,000 and zero otherwise? Assume that the current level of the index is 960, the risk-free rate is 8% per annum, the dividend yield on the index is 3% per annum, and the volatility of the index is 20%.

26.19. In a 3-month down-and-out call option on silver futures the strike price is \$20 per ounce and the barrier is \$18. The current futures price is \$19, the risk-free interest rate is 5%, and the volatility of silver futures is 40% per annum. Explain how the option works and calculate its value. What is the value of a regular call option on silver futures with the same terms? What is the value of a down-and-in call option on silver futures with the same terms?

26.20. A new European-style floating lookback call option on a stock index has a maturity of 9 months. The current level of the index is 400, the risk-free rate is 6% per annum, the dividend yield on the index is 4% per annum, and the volatility of the index is 20%. Use DerivaGem to value the option.

26.21. Estimate the value of a new 6-month European-style average price call option on a non-dividend-paying stock. The initial stock price is \$30, the strike price is \$30, the risk-free interest rate is 5%, and the stock price volatility is 30%.

26.22. Use DerivaGem to calculate the value of:
(a) A regular European call option on a non-dividend-paying stock where the stock price is \$50, the strike price is \$50, the risk-free rate is 5% per annum, the volatility is 30%, and the time to maturity is one year
(b) A down-and-out European call which is as in (a) with the barrier at \$45
(c) A down-and-in European call which is as in (a) with the barrier at \$45.

Show that the option in (a) is worth the sum of the values of the options in (b) and (c).

26.23. Explain adjustments that have to be made when $r = q$ for (a) the valuation formulas for floating lookback call options in Section 26.11 and (b) the formulas for M_1 and M_2 in Section 26.13.

26.24. Value the variance swap in Example 26.4 of Section 26.16 assuming that the implied volatilities for options with strike prices 800, 850, 900, 950, 1,000, 1,050, 1,100, 1,150, 1,200 are 20%, 20.5%, 21%, 21.5%, 22%, 22.5%, 23%, 23.5%, 24%, respectively.

26.25. Verify that the results in Section 26.2 for the value of a derivative that pays Q when $S = H$ are consistent with those in Section 15.6.

Further Questions

26.26. What is the value in dollars of a derivative that pays off £10,000 in 1 year provided that the dollar/sterling exchange rate is greater than 1.5000 at that time? The current exchange rate is 1.4800. The dollar and sterling interest rates are 4% and 8% per annum, respectively. The volatility of the exchange rate is 12% per annum.

26.27. Consider an up-and-out barrier call option on a non-dividend-paying stock when the stock price is 50, the strike price is 50, the volatility is 30%, the risk-free rate is 5%, the time to maturity is 1 year, and the barrier at \$80. Use the DerivaGem software to value the option and graph the relationship between (a) the option price and the stock price, (b) the delta and the stock price, (c) the option price and the time to maturity, and

(d) the option price and the volatility. Provide an intuitive explanation for the results you get. Show that the delta, gamma, theta, and vega for an up-and-out barrier call option can be either positive or negative.

26.28. Sample Application F in the DerivaGem Application Builder Software considers the static options replication example in Section 26.17. It shows the way a hedge can be constructed using four options (as in Section 26.17) and two ways a hedge can be constructed using 16 options.
(a) Explain the difference between the two ways a hedge can be constructed using 16 options. Explain intuitively why the second method works better.
(b) Improve on the four-option hedge by changing Tmat for the third and fourth options.
(c) Check how well the 16-option portfolios match the delta, gamma, and vega of the barrier option.

26.29. Consider a down-and-out call option on a foreign currency. The initial exchange rate is 0.90, the time to maturity is 2 years, the strike price is 1.00, the barrier is 0.80, the domestic risk-free interest rate is 5%, the foreign risk-free interest rate is 6%, and the volatility is 25% per annum. Use DerivaGem to develop a static option replication strategy involving five options.

26.30. Suppose that a stock index is currently 900. The dividend yield is 2%, the risk-free rate is 5%, and the volatility is 40%. Use the results in Technical Note 27 on the author's website to calculate the value of a 1-year average price call where the strike price is 900 and the index level is observed at the end of each quarter for the purposes of the averaging. Compare this with the price calculated by DerivaGem for a 1-year average price option where the price is observed continuously. Provide an intuitive explanation for any differences between the prices.

26.31. Use the DerivaGem Application Builder software to compare the effectiveness of daily delta hedging for (a) the option considered in Tables 19.2 and 19.3 and (b) an average price call with the same parameters. Use Sample Application C. For the average price option you will find it necessary to change the calculation of the option price in cell C16, the payoffs in cells H15 and H16, and the deltas (cells G46 to G186 and N46 to N186). Carry out 20 Monte Carlo simulation runs for each option by repeatedly pressing F9. On each run record the cost of writing and hedging the option, the volume of trading over the whole 20 weeks and the volume of trading between weeks 11 and 20. Comment on the results.

26.32. In the DerivaGem Application Builder Software modify Sample Application D to test the effectiveness of delta and gamma hedging for a call on call compound option on a 100,000 units of a foreign currency where the exchange rate is 0.67, the domestic risk-free rate is 5%, the foreign risk-free rate is 6%, the volatility is 12%. The time to maturity of the first option is 20 weeks, and the strike price of the first option is 0.015. The second option matures 40 weeks from today and has a strike price of 0.68. Explain how you modified the cells. Comment on hedge effectiveness.

26.33. Outperformance certificates (also called "sprint certificates," "accelerator certificates," or "speeders") are offered to investors by many European banks as a way of investing in a company's stock. The initial investment equals the stock price, S_0. If the stock price goes up between time 0 and time T, the investor gains k times the increase at time T, where k is a constant greater than 1.0. However, the stock price used to calculate the gain at time T is capped at some maximum level M. If the stock price goes down, the investor's loss is equal to the decrease. The investor does not receive dividends.

(a) Show that an outperformance certificate is a package.

(b) Calculate using DerivaGem the value of a one-year outperformance certificate when the stock price is 50 euros, $k = 1.5$, $M = 70$ euros, the risk-free rate is 5%, and the stock price volatility is 25%. Dividends equal to 0.5 euros are expected in 2 months, 5 months, 8 months, and 11 months.

26.34. Carry out the analysis in Example 26.4 of Section 26.16 to value the variance swap on the assumption that the life of the swap is 1 month rather than 3 months.

26.35. What is the relationship between a regular call option, a binary call option, and a gap call option?

26.36. Produce a formula for valuing a cliquet option where an amount Q is invested to produce a payoff at the end of n periods. The return earned each period is the greater of the return on an index (excluding dividends) and zero.