General Linear Model

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1 Expression

General linear model is expressed as:

$$\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon}$$

The matrix form of the equation:

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1p} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2p} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{np} \end{bmatrix}_{n\times p} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_p \end{bmatrix}_{p\times 1} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \vdots \\ \epsilon_n \end{bmatrix}_{n\times 1}$$

The goal of general linear model is to find $\boldsymbol{\beta}$ that minimizes the distance between $\boldsymbol{Y}$ and $\boldsymbol{X\beta}$.

2 Least Squares Estimate of $\boldsymbol{\beta}$

$$\boldsymbol{\beta} = (\boldsymbol{X^T X})^{-1}\boldsymbol{XY}$$

3 Examples

Consider a system of equations:

$$
\begin{aligned}
p_{11} &= p + b_1 + \epsilon \\
p_{12} &= p + b_1 + \epsilon \\
p_{21} &= p + b_2 + \epsilon \\
p_{22} &= p + b_2 + \epsilon
\end{aligned}
$$

Express it in matrix form:

$$
\underbrace{\begin{bmatrix} p_{11} \\ p_{12} \\ p_{21} \\ p_{22} \end{bmatrix}}_{\text{observed values}} = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}}_{\text{a matrix with linear dependent columns}} \begin{bmatrix} p \\ b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \end{bmatrix}
$$

The matrix with 1 and 0 is used to form the X matrix in general linear model by removing redundant columns.

For example, remove the third column, X matrix is

$$
\boldsymbol{X} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ \underbrace{1}_{\text{intercept}} & 1 \end{bmatrix}
$$

The column with all 1 is called the intercept.

Or, remove the first column, X matrix is

$$
\boldsymbol{X} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}
$$

The two X matrices are row equivalent.

New X matrices can also be obtained using elementary row operations.

For example,

$$\begin{bmatrix}1 & 0\\1 & 0\\0 & 1\\0 & 1\end{bmatrix} \xrightarrow{R1 \to R1 - R3} \begin{bmatrix}1 & -1\\1 & 0\\0 & 1\\0 & 1\end{bmatrix} \xrightarrow{R2 \to R2 - R3} \begin{bmatrix}1 & -1\\1 & -1\\0 & 1\\0 & 1\end{bmatrix} \xrightarrow[R4 \to R4 \times 2]{R3 \to R3 \times 2} \begin{bmatrix}1 & -1\\1 & -1\\0 & 2\\0 & 2\end{bmatrix}$$

$$\xrightarrow[R4 \to R4 + R2]{R3 \to R3 + R1} \begin{bmatrix}1 & -1\\1 & -1\\1 & 1\\1 & 1\end{bmatrix}$$

The new X matrix is $\begin{bmatrix}1 & -1\\1 & -1\\1 & 1\\1 & 1\end{bmatrix}$.

Different X matrices give different equations:

$$\underbrace{\begin{bmatrix}1 & 0\\1 & 0\\1 & 1\\1 & 1\end{bmatrix}}_{\boldsymbol{X}} \underbrace{\begin{bmatrix}p_1\\p_2 - p_1\end{bmatrix}}_{\boldsymbol{\beta}} = \underbrace{\begin{bmatrix}p_1\\p_1\\p_2\\p_2\end{bmatrix}}_{\text{predicted values}}$$

$$\underbrace{\begin{bmatrix}1 & 0\\1 & 0\\0 & 1\\0 & 1\end{bmatrix}}_{\boldsymbol{X}} \underbrace{\begin{bmatrix}p_1\\p_2\end{bmatrix}}_{\boldsymbol{\beta}} = \underbrace{\begin{bmatrix}p_1\\p_1\\p_2\\p_2\end{bmatrix}}_{\text{predicted values}}$$

Note that different X matrices have different $\boldsymbol{\beta}$ vector. But they have the same predicted values.

Choosing an X matrix is called **contrast**.