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On a generalized modularization theorem

Theodosis Dimitrakos [a,*], Tom Maibaum [b,1]

[a] ISE-CLRC, Rutherford Appleton Laboratory, Oxfordshire, OX11 0QX, UK

[b] Department of Computer Science, King's College, London, WC2R 2LS, UK

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Abstract

The relation between a metalogical property of entailment (interpolation) and a structural property of categories of theory presentations (stability of faithful morphisms under pushouts) is studied in a general logical framework. The stability of faithful morphisms for any specific instance of the pushout construction is shown to be equivalent to the existence of interpolants over the corresponding locus of the underlying logic. The latter neither assumes nor requires interpolation to be a global property of the formalism. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we revise, generalize and re-establish, in a general logical framework [7,15], an equivalence between the pushout generalization of a "splitting" form of interpolation [11,20,21,26,5] and the stability of conservative extensions and, more generally, faithful morphisms under pushouts (*modularization*) [26, 25,5]. The latter provides the formal basis for the composability of a form of refinement [24,26,5], the correctness of (multi)parametric axiomatic specifications [25,4], the modularity of databases [14] and some operations of module algebras ([19] discussing an earlier version of [1]). We note that an analogous relation between ordinary Craig interpolation and amalgamation of models (which leads to the stability of conservativeness in the amalgamation of a span of conservative extensions) had been observed earlier in [13], where an equivalence was shown to hold for *every* (propositional) *super-intuitionistic* logic. (See also [9, 17].) The generalized form of the interpolation property we are interested in, called CRI, and an analogous generalization of Craig's interpolation, called CI, are analyzed and compared in Section 3. The equivalence between CRI and the stability of faithful morphisms under pushouts (which generalizes the "Modularization Theorem" of [24,26]) is re-established in Section 4 in a revised form in the general logical framework of "Entailment Systems" [15] (called π-Institutions earlier in [7]). Theorem 6, in particular, exhibits a deeper relation than originally envisioned, between the existence of CRI-interpolants and the stability of faithful theory interpretations under pushouts, for every *reflexive*, *transitive* and *monotonic* family of entailment relations. As Corollaries 10 and 11 illustrate, asserting that the faithfulness of a specific theory interpretation is stable under pushouts is equivalent to asserting that CRI holds locally on specific syntactical loci. This does not assume nor imply that CRI (or in-

* Corresponding author. Email: theo@inf.rl.ac.uk.

1 Email: tom@dcs.kcl.ac.uk.

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deed any other form of interpolation) is possessed as a global metalogical property. The essence of our generalization is emphasized in Section 3.1, where we explain when CRI, CI and ordinary Craig interpolation become equivalent in first order logics with equality and finite syntax, and in Example 12 which involves a non-trivial extension of first order logic which *possesses* modularization and CRI while it *lacks* ordinary Craig interpolation.

2. Preliminaries

Ordinary Craig interpolation states that if γ and φ are sentences in (the context of) languages L_1, L_2, respectively, and such that γ *entails* φ in $L_1 \cup L_2$, in symbols $\{\gamma\} \vdash_{L_1 \cup L_2} \varphi$, then there is an interpolant sentence ϑ in $L_1 \cap L_2$ such that $\{\gamma\} \vdash_{L_1} \vartheta$ and $\{\vartheta\} \vdash_{L_2} \varphi$. Craig–Robinson ("splitting") interpolation involves a primary assertion γ_1 in L_1 and a secondary assertion γ_2 in L_2 and states that if $\{\gamma_1, \gamma_2\} \vdash_{L_1 \cup L_2} \varphi$, then there is a ϑ in $L_1 \cap L_2$ such that $\{\gamma\} \vdash_{L_1} \vartheta$ and $\{\vartheta, \gamma_2\} \vdash_{L_2} \varphi$. The *Modularization property* [12,24] asserts that theorem conservation is stable under the amalgamation of theory interpretations.

In order to state the critical form of interpolation and modularization in a general logical framework, we use the concept of an entailment system (ES). An ES [15] (π-Institution [7,8]) is a triple $\langle \mathbf{Sign}, \mathbf{gram}, \vdash \rangle$ where **Sign** is a category of (objects called) signatures, $\mathbf{gram} : \mathbf{Sign} \to \mathbf{Set}$ is the grammar functor and $\vdash$ is a **Sign**-indexed family of binary relations $\vdash_\Sigma \subseteq 2^{\mathbf{gram}(\Sigma)} \times \mathbf{gram}(\Sigma)$ such that, for every Σ in **Sign**, $\vdash_\Sigma$ is *reflexive*, *monotonic*, *transitive* and the family $\vdash$ is *stable under translation*.[2] In addition, an ES is called *compact* iff, for every Σ and every $A \cup \{\varphi\} \in \mathbf{gram}(\Sigma)$ such that $A \vdash_\Sigma \varphi$, there is a *finite* $\Gamma \subseteq A$ such that $\Gamma \vdash_\Sigma \varphi$. The writing of δ^i denotes $\mathbf{gram}(i)(\delta)$, i.e., the image of δ via the translation induced by i, and $\Delta^i = \{\delta^i \colon \delta \in \Delta\}$. Finally, $\Gamma \vdash_\Sigma \Delta$ means $\Gamma \vdash_\Sigma \delta$ for each $\delta \in \Delta$.

The category of theory presentations over an ES is denoted by **Pres**. A theory interpretation $i : \mathsf{A} \to \mathsf{B}$ is called *faithful* iff it is theorem conserving, i.e., for every sentence a in the language of A, the sentence a is a theorem of A iff the translation a^i of a via i is a theorem of B. The morphism i is called a *conservative extension* iff i is faithful and the language of A is *included* in the language of B.

[2] I.e., $\{\varphi\} \vdash_\Sigma \varphi$; if $\Gamma \subseteq \Delta$ and $\Gamma \vdash_\Sigma \varphi$ then $\Delta \vdash_\Sigma \varphi$; $\Delta \vdash_\Sigma \varphi$ and $\Gamma \vdash_\Sigma \delta$, for all $\delta \in \Delta$, then $\Gamma \vdash_\Sigma \varphi$; and for each $i : \Sigma_1 \to \Sigma_2$ in **Sign**, $\Gamma \vdash_{\Sigma_1} \varphi$ implies $\mathbf{gram}(i)(\Gamma) \vdash_{\Sigma_2} \mathbf{gram}(i)(\varphi)$.

3. A generalized pushout version of interpolation

The following generalized *pushout version* of *Craig–Robinson Interpolation*, labeled CRI, provides, as we prove in Section 4, a *necessary and sufficient* condition for modularization over an arbitrary ES.

Definition 1. An ES possesses CRI iff for every pushout diagram D in **Sign** (depicted in Fig. 1) and every $A \subseteq \mathbf{gram}(\Sigma_A)$, $B \subseteq \mathbf{gram}(\Sigma_B)$, $\varphi \in \mathbf{gram}(\Sigma_B)$, such that $A^{i'} \cup B^{e'} \vdash_{\Sigma_{A+B}} \varphi^{e'}$, there is a set $I(A, B, \varphi, D) \subseteq \mathbf{gram}(\Sigma_R)$ of *interpolants* such that
(i) $A \vdash_{\Sigma_A} I^e(A, B, \varphi, D)$ and
(ii) $I^i(A, B, \varphi, D) \cup B \vdash_{\Sigma_B} \varphi$.

The writing of CRI$[A, B, D]$ denotes the specialization of the above stated CRI-condition to a specific diagram D presenting the pushout completion of a (specific) span $\Sigma_A \leftarrow e - \Sigma_R - i \rightarrow \Sigma_B$, and (specific) $A \subseteq \mathbf{gram}(\Sigma_A)$, $B \subseteq \mathbf{gram}(\Sigma_B)$.

A property similar to CRI first appeared explicitly in the literature in [11] for propositional calculus. Since then, the terms "Maehara" [21], "Splitting" [19], and "Craig–Robinson" Interpolation [20,25,5] have been used to refer to instances of CRI. The following weakening of CRI corresponds to a generalized *pushout version* of *Craig Interpolation*, labeled CI, that was introduced in [22]:

Definition 2. An ES possesses CI iff it possesses CRI$[A, \emptyset, D]$ for every D and every $A \subseteq \mathbf{gram}(\Sigma_A)$.

The distinction between CI, CRI and ordinary interpolation is delicate. On the one hand, both CI and CRI *weaken* ordinary interpolation by
(1) abstracting the implication to an entailment,
(2) generalizing the premise of an implication to the set of assertions in an entailment, and
(3) generalizing the interpolant sentence to a set of interpolants on the shared language (i.e., an interpolant presentation).

$$
\begin{tikzcd}
A,I^e(A,B,\varphi,D) \arrow[rrr, "\mathbf{gram}(i')"] & & & A^{i'}\cup B^{e'},\varphi \\
& \Sigma_A \arrow[ul, dash] \arrow[r, "i'"] & \Sigma_{A+B} \arrow[ur, dash] & \\
& \Sigma_R \arrow[u, "e"] \arrow[r, "i"'] \arrow[ur, phantom, "D"] & \Sigma_B \arrow[u, "e'"'] \arrow[dr, dash] & \\
I(A,B,\varphi,D) \arrow[uuu, "\mathbf{gram}(e)"] \arrow[rrr, "\mathbf{gram}(i)"'] \arrow[ur, dash] & & & B\cup I^i(A,B,\varphi,D),\varphi \arrow[uuu, "\mathbf{gram}(e')"']
\end{tikzcd}
$$

Fig. 1. A pictorial presentation of CRI$[A, B, D]$.

On the other hand, they *strengthen* ordinary interpolation by

(4) generalizing the (extra-logical) language sharing from a union of language inclusions to an arbitrary pushout in the corresponding category.

In addition, CRI *strengthens* CI by

(5) distinguishing a set A of *primary* and a set B of *secondary* assertions which reside in different and possibly overlapping languages.

See [5] for examples of CRI-interpolants and extensive lists of concrete calculi that possess CRI and calculi that do not possess CRI globally.

3.1. CI, CRI, and ordinary Craig interpolation in super-intuitionistic logics

It is often the case that the instances of different, in general non-equivalent, metalogical properties conflate over structurally rich calculi. CRI and CI, in particular, conflate in the presence of compactness and deduction detachment. First of all, in a compact ES, for each A, B and φ, one can find a *finite* subset $I_0(A, B, \varphi, D)$ of the interpolants $I(A, B, \varphi, D)$, and a finite subset $B_0(A, B, \varphi, D)$ of B such that

$$I_0^i(A, B, \varphi, D) \cup B_0(A, B, \varphi, D) \vdash_{\Sigma_B} \varphi.$$

If, in addition, the ES admits deduction detachment then the CRI-interpolants do not depend on B. That is, given a $\varphi \in \mathbf{gram}(\Sigma_B)$, one can construct [3] a $\varphi_{\mathsf{B}} \in \mathbf{gram}(\Sigma_B)$ such that $I(A, B, \varphi, D)$ and $I(A, \emptyset, \varphi_{\mathsf{B}}, D)$ are interchangeable:

$$A^{i'} \vdash_{\Sigma_{A+B}} \varphi_B^{e'} \quad \text{iff} \quad A^{i'} \cup B^{e'} \vdash_{\Sigma_{A+B}} \varphi^{e'}$$

and

$$I^i(A, \emptyset, \varphi_{\mathsf{B}}, D) \vdash_{\Sigma_B} \varphi_B \quad \text{iff}$$
$$I^i(A, B, \varphi, D) \cup B \vdash_{\Sigma_B} \varphi.$$

Proposition 3. *In the presence of compactness, there is a finite set $I(A, B, \varphi, D)$ of CRI-interpolants, for each φ, A, B and D. If, in addition, deduction detachment is admitted, then CRI and CI conflate with each other for each D and each $A \subseteq \mathbf{gram}(\Sigma_A)$.*

In the particular case of a super-intuitionistic propositional and first order logic with equality, CI and CRI conflate not only with each other, but also with ordinary Craig interpolation. As we elaborate in the sequel, this is caused by a delicate mixture of metalogical and structural properties of the corresponding ESs. (See also [25].)

First of all, we need some auxiliary definitions: Let $\sigma : \Sigma_1 \to \Sigma_2$ be a *retraction* in **Sign**. (That is, an epimorphism with a monomorphic right inverse.) A *kernel* for σ is a *consistent* set $\ker(\sigma) \subset \mathbf{gram}(\Sigma_1)$ such that, for every $\varphi \in \mathbf{gram}(\Sigma_2)$ and ρ, τ right inverses of σ, the following holds:

$$\ker(\sigma) \cup \varphi^\rho \vdash_{\Sigma_1} \varphi^\tau \quad \text{and} \quad \emptyset \vdash_{\Sigma_2} \ker(\sigma)^\sigma.$$

In first order logics with equality,

$$\begin{aligned}\ker(\sigma) = \big\{&\forall x.\, p(x) \leftrightarrow q(x) \mid \\ &\quad p, q \in \mathit{Pred}(\Sigma_1) \text{ and } \sigma(p) = \sigma(q)\big\} \cup \\ \big\{&\forall x.\, f(x) = g(x) \mid \\ &\quad f, g \in \mathit{Func}(\Sigma_1) \text{ and } \sigma(f) = \sigma(g)\big\},\end{aligned}$$

[3] Let $B_0(A, B, \varphi, D) = \{\beta_1, \ldots, \beta_n\}$ and let $\cdot \to \cdot$ be the connective such that $X \cup \{\chi\} \vdash \psi$ iff $X \vdash \chi \to \psi$. Then φ_{B} can be constructed by prefixing the assertions in $B_0(A, B, \varphi, D)$ as *premises* to the conclusion φ, i.e., $\varphi_{\mathsf{B}} \equiv \beta_1 \to (\cdots \to (\beta_n \to \varphi) \cdots)$.

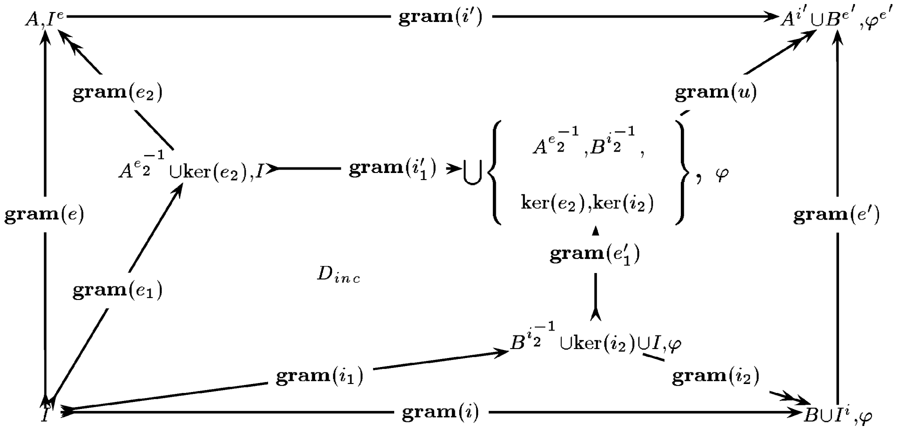

Fig. 2. The reduction of CRI$[A, B, D]$ to CRI$[A^{e_2^{-1}} \cup \ker(e_2), B^{i_2^{-1}} \cup \ker(i_2), D_{inc}]$.

and the pushout completion of an arbitrary span $\mathsf{A} \leftarrow e - \mathsf{R} - i \rightarrow \mathsf{B}$ reduces to a pushout D_{inc} of extensions as is depicted in Fig. 2 where **sign**(e_1), **sign**(e'_1), **sign**(i_1), **sign**(i'_1) are all signature inclusions, **sign**(e_2), **sign**(i_2), **sign**(u) are retractions and e_2, i_2, u are faithful. Hence, CRI$[A, B, D]$ reduces to CRI$[A^{e_2^{-1}} \cup \ker(e_2), B^{i_2^{-1}} \cup \ker(i_2), D_{inc}]$. Note that $\ker(e_2) \cup \ker(i_2) \dashv\vdash \ker(u)$ and that for every pair e_2^{-1}, i_2^{-1} there is a unique u^{-1} such that $i'_1 \circ e_2^{-1} = u^{-1} \circ i'$ and $e'_1 \circ i_2^{-1} = u^{-1} \circ e'$ and therefore $A^{e_2^{-1}} \cup B^{i_2^{-1}} \cup \ker(e_2) \cup \ker(i_2) \dashv\vdash \ker(u) \cup A^{u^{-1}} \cup B^{u^{-1}}$, where $e_2^{-1}, i_2^{-1}, u^{-1}$ are injective right inverses for, respectively, e_2, i_2, u. Also ordinary Craig interpolation and CI are equivalent over D_{inc} because the ES is compact and has finite conjunction. Moreover there is a deduction theorem for implication and so CRI and CI conflate with each other over D_{inc} by Proposition 3. Hence,

Proposition 4. *CRI, CI and ordinary Craig interpolation are equivalent in first order logic with equality.*

Note that, in addition to compactness and deduction detachment, we only used the substitutivity of equals/equivalents and the existence of a kernel together with a basic mono-epi factorization [4] of translations. The latter depend on the traditional view of formulae as inductively defined strings of symbols. Notably, (conditional) equational logic possesses CI for all pushout completions of spans $\cdot \leftarrow e - \cdot - i \rightarrow \cdot$ where i is injective [18] but does not possess CI globally. In contrast, classical and intuitionistic propositional and first order logics possess Craig interpolation [2,9] and hence, by Proposition 4, both CRI and CI.

[4] This can be generalized to a "*distributive inclusion system*" [3] which ensures that each translation has a unique factorization to an inclusion and a surjection, where inclusions have unions and finite intersections with the usual distributive laws.

4. CRI and Modularization

The *Modularization property* [12,24,26] is generalized within the employed framework by the following property MP which asserts the stability of the class of theorem conserving (viz., faithful) morphisms in **Pres**:

Definition 5. An ES possesses MP iff the pushout $e'_{\mathsf{B(A+B)}} : \mathsf{B} \rightarrow (\mathsf{A} + \mathsf{B})$ of any faithful morphism $e_{\mathsf{RA}} : \mathsf{R} \rightarrow \mathsf{A}$ along any morphism $i_{\mathsf{RB}} : \mathsf{R} \rightarrow \mathsf{B}$ is also faithful.

The axiomatization of the pushout object of the **Pres**-diagram depicted in Fig. 3 is $A^{i'} \cup R^{i' \circ e} \cup B^{e'}$ which is logically equivalent to $A^{i'} \cup B^{e'}$, by the transitivity and monotonicity of $\vdash$. Also, writing MP$[A, B, D]$ denotes the specialization of the above stated MP to the pushout completion of a (specific) span $\langle \Sigma_A, A \rangle \leftarrow e_{\mathsf{RA}} - \langle \Sigma_R, R \rangle - i_{\mathsf{RB}} \rightarrow \langle \Sigma_B, B \cup R^i \rangle$ in **Pres**, with D as the underlying square of **Sign**.

$$\begin{tikzcd}
\mathsf{A} = \langle \Sigma_A, A\rangle \arrow[rrr, "i'_{\mathsf{A(A+B)}}"] \arrow[dr, dash, gray] & & & \mathsf{A+B} = \langle \Sigma_{A+B}, A^{i'} \cup B^{e'}\rangle \arrow[dl, dash, gray] \\
& \Sigma_A \arrow[r, "i'"] & \Sigma_{A+B} & \\
& \Sigma_R \arrow[u, "e"] \arrow[r, "i"'] \arrow[ur, phantom, "D"] & \Sigma_B \arrow[u, "e'"'] & \\
\mathsf{R} = \langle \Sigma_R, R\rangle \arrow[uuu, "e_{\mathsf{RA}}"] \arrow[ur, dash, gray] \arrow[rrr, "i_{\mathsf{AB}}"'] & & & \mathsf{B} = \langle \Sigma_B, A \cup B^i\rangle \arrow[uuu, "e'_{\mathsf{B(A+B)}}"'] \arrow[ul, dash, gray]
\end{tikzcd}$$

Fig. 3. A pictorial presentation MP[A, B, D].

Note that for every $B \subseteq \mathbf{gram}(\Sigma_B)$, $\langle \Sigma_B, B \cup R^i\rangle$ interprets B by the stability under translation and the monotonicity of $\vdash$. Finally, the argument R is omitted in MP[A, B, D] because R is determined by A and D as the pre-image of A via e. A proof that CRI is *necessary and sufficient* for MP (Corollary 8) follows from our main contribution, Theorem 6, together with two novel, critically stronger results: Corollaries 10 and 11.

Theorem 6. CRI[A, B, D] *is necessary and sufficient for* MP[A, B, D] *for every diagram D depicting the pushout completion of the span $\Sigma_A \leftarrow e - \Sigma_R - i \rightarrow \Sigma_B$ and all $A \subseteq \mathbf{gram}(\Sigma_A)$, $B \subseteq \mathbf{gram}(\Sigma_B)$.*

Proof. ($\Rightarrow$) It suffices to prove that CRI[A, B, D] together with the assumption that $e_{\mathsf{RA}} : \langle \Sigma_R, R\rangle \rightarrow \langle \Sigma_A, A\rangle$ is faithful implies that $e'_{\mathsf{B(A+B)}}$ is faithful (Fig. 3). That is, *for every $\varphi \in$* $\mathbf{gram}(\Sigma_B)$,

if $A^{i'} \cup B^{e'} \vdash_{\Sigma_{A+B}} \varphi^{e'}$ then $B \vdash_{\Sigma_B} \varphi$.

Indeed, by CRI[A, B, D], *for every $\varphi \in$* $\mathbf{gram}(\Sigma_B)$ such that $A^{i'} \cup B^{e'} \vdash_{\Sigma_{A+B}} \varphi^{e'}$, there is some $I(A, B, \varphi, D) \subseteq \mathbf{gram}(\Sigma_R)$ such that:

(i) $A \vdash_{\Sigma_A} I^e(A, B, \varphi, D)$, and

(ii) $I^i(A, B, \varphi, D) \cup B \vdash_{\Sigma_B} \varphi$.

By the faithfulness of e, the above statement (i) implies $R \vdash_{\Sigma_R} I(A, B, \varphi, D)$, and, hence,

$$B \cup R^i \vdash_{\Sigma_B} I^i(A, B, \varphi, D)$$

by the stability under translation and the transitivity of $\vdash$. The latter, together with statement (ii) above and the transitivity of $\vdash$ imply that $B \cup R^i \vdash_{\Sigma_B} \varphi$. $\square$

Proof. ($\Leftarrow$) It suffices to establish the existence of an *interpolant presentation* on Σ_R for each $\varphi \in \mathbf{gram}(\Sigma_B)$. We claim that the axiomatization R of $\langle \Sigma_R, R\rangle$ is the strongest such, given that

(iii) $e_{\mathsf{RA}} : \langle \Sigma_R, R\rangle \rightarrow \langle \Sigma_A, A\rangle$ is faithful, and

(iv) the faithfulness of e_{RA} *implies* the faithfulness of its pushout along i_{RB}.

Indeed, $A \vdash_{\Sigma_A} R^e$ by (iii) above and, for all $\varphi \in \mathbf{gram}(\Sigma_B)$, if $A^{i'} \cup B^{e'} \vdash_{\Sigma_{A+B}} \varphi^{e'}$ then $B \cup R^i \vdash_{\Sigma_B} \varphi$ by (iv) above. Hence, for any specific φ there is some $I(A, B, \varphi, D) \subseteq R$ such that $A \vdash_{\Sigma_A} I^e(A, B, \varphi, D)$ and $B \cup I^i(A, B, \varphi, D) \vdash_{\Sigma_B} \varphi^{e'}$. $\square$

Remark 7. Note that even in the presence of compactness, R need not be finite. The theory axiomatized by R includes the $\{\varphi \mid A^{i'} \cup B^{e'} \vdash_{\Sigma_{A+B}} \varphi^{e'}\}$-indexed union of all CRI-interpolants for arbitrary but fixed A, B and D.

By shifting the (meta-)quantification on A, B and D inside the equivalence of Theorem 6, we obtain a generalization of the "Modularization Theorem":

Corollary 8. *An ES E possesses MP iff E possesses CRI.*

Moreover, if language inclusions are stable under pushouts, then the specialization to conservative extensions is immediate:

Corollary 9. *CRI guarantees the stability of conservative extensions under pushouts over ESs where language inclusions are stable under pushouts.*

The following two corollaries are easily derived from Theorem 6. The first, Corollary 10, states that the equivalence between CRI and MP holds in the locus of *all* theory presentations and theory presentation morphisms on an arbitrary but fixed **Sign**-diagram D. It is derived by shifting the universal (meta-)quantification on sets of sentences (theory presentations) inside the equivalence in the statement of Theorem 6. The second, Corollary 11, provides a necessary and sufficient condition for the pushout stability of faithfulness for any specific theory interpretation. It is derived by shifting the universal (meta-)quantification on the set of *secondary* assertions B and the underlying diagram D inside the equivalence in the statement of Theorem 6.

Corollary 10. *Let D be an arbitrary but fixed pushout diagram of* **Sign**. *Then, CRI holds* locally *in the* (*grammatical*) *locus of* all *sentences/theory presentations over D iff all faithful morphisms over e are stable under the pushout along any morphism over i.*

Corollary 11. *A faithful theory interpretation $e_{\mathsf{RA}}:\mathsf{R}\to\mathsf{A}$ over $e:\Sigma_R\to\Sigma_A$, is stable under pushouts in* **Pres** *iff for every theory interpretation $i_{\mathsf{RB}}:\mathsf{R}\to\mathsf{B}$ over $i:\Sigma_R\to\Sigma_B$, such that the pushout of e_{RA} along i_{RB} exists,* CRI$[A, B, D]$ *holds. Here A is the axiomatization of* A, *B is the axiomatization of* B *and D is the underlying* **Sign**-*diagram of the above mentioned pushout.*

Note that CI is in general weaker than CRI and, hence, in itself CI is necessary but not sufficient for modularization. Also note that the above Theorem 6 and Corollaries 10 and 11 neither assume nor imply that the underlying ES possesses CRI (and therefore MP) globally. Furthermore, as is illustrated in the following example, there are interesting, nontrivial, ESs that poses CRI/MP globally but lack ordinary interpolation. [5]

Example 12. The calculus of predicate extensions of first order arithmetic, possesses CRI, and therefore MP, but lacks ordinary interpolation. The standard interpolation theorem for first order logic does not apply because there can be sentences γ and φ such that $\gamma\to\varphi$ is established by induction, in which case the (logical) induction axiom that is used may depend on the union of the extra-logical predicate (extension) symbols that appear in γ and φ. Applying the standard deductive proof of the ordinary interpolation theorem using a cut free Gentzen-style proof system fails at the ω-rule:

$$\frac{\cdots\alpha\to\beta(n)\cdots}{\alpha\to\forall x\beta(x)}.$$

Having determined an interpolant ϑ_n for each premise $\alpha\to\beta(n)$, where $n\in\mathbb{N}$, one has to take an *infinite* conjunction $\bigwedge_n\vartheta_n$ as the overall interpolant and this is not allowed by the syntax. In contrast, one can consider the possibly infinite set $\{\vartheta_n\colon\ n\in\mathbb{N}\}$ as the CRI-interpolant presentation. See also [16] for a detailed proof that ordinary interpolation fails for predicate extensions of first order arithmetic. The proof is by contradiction using Tarski's undefinability theorem [23].

5. Conclusion

The first attempt to analyze this property, in the early 1980s, based on an observation of Sadler [12], indicated a strong interconnection between modularization and interpolation. The stability of conservative extensions under amalgamation with arbitrary theory interpretations was equivalent—in first order logic—to ordinary interpolation. [6] Also a group working with Bergstra [1] attributed the lack of certain modularity properties in some equational formalisms to the absence of an appropriate interpolation property. Further work on this, revealed the important observation, put forward by Rodenburg and van Glabbeek in [19], that in some logics, such as (conditional) equational logic, the various formulations of Craig interpolation are not equivalent and only some of them are relevant to modularization. As we elaborated in this paper, *the CRI* (*Definition* 1) *is the critical form of interpolation which is relevant to Modularization*. The proof

[5] The example is based on a note given to the authors by Professor G. Mints in the summer of 1998. (Hints that ordinary interpolation does not hold for the predicate extensions of first order arithmetic can also be found in [10].)

[6] An equivalence between Craig interpolation and amalgamation of models had been established earlier by Maksimova [13] and Pitts [17] for super-intuitionistic propositional logics.

of Theorem 6 illustrates that the equivalence between CRI and the stability of theorem conservation under pushouts can also be studied as a phenomenon over appropriate loci of an ES. This new observation lead to Corollaries 10 and 11. The latter state that the stability of theorem conservation under pushouts is equivalent to the existence of CRI-interpolants in the corresponding loci of the underlying logic. As we illustrated in this paper, these results appear to have the status of *general laws* of specification theory. This is of particular importance since both Theorem 6 and Corollaries 10 and 11 do not assume nor imply that CRI (or indeed any other form of interpolation) is possessed as a global metalogical property.

References

[1] J.A. Bergstra, J. Heering, P. Klint, Module algebra, J. ACM 37 (2) (1990) 335–372.

[2] W. Craig, Three uses of the Herbrand–Getzen theorem in relating model theory and proof theory, J. Symbolic Logic 22 (1957) 269–285.

[3] R. Diaconnescu, J. Goguen, P. Stefaneas, Logical support for modularisation, in: Logical Environments, Cambridge University Press, Cambridge, 1993, pp. 83–130.

[4] T. Dimitrakos, Parameterising (algebraic) specifications on diagrams, in: Automated Software Engineering—ASE'98, IEEE Press, New York, 1998, pp. 221–224.

[5] T. Dimitrakos, T.S.E. Maibaum, Notes on refinement, interpolation and uniformity, in: Automated Software Engineering—ASE'97, IEEE Press, New York, 1997.

[6] H.D. Ehrich, On the theory of specification, implementation and parametrization of abstract data types, J. ACM 29 (1) (1982).

[7] J.L. Fiadeiro, A. Sernadas, Structuring theories on consequence, in: Recent Trends in Data Type Specification, 1988, pp. 44–72.

[8] J.L. Fiadeiro, T.S.E. Maibaum, Generalising interpretations between Theories in the Context of (π-)institutions, in: Theory and Formal Methods, Springer, Berlin, 1993, pp. 126–147.

[9] D.M. Gabbay, Semantic proof of the Craig interpolation theorem for intuitionistic logic and extensions, in: Logic Colloquium'69, North-Holland, Amsterdam, 1971.

[10] G. Kreisel, Survey of proof theory, J. Symbolic Logic (1968).

[11] S. Maehara, Craig's interpolation theorem, Sugaku (1961) 235–237.

[12] T.S.E. Maibaum, M.R. Sadler, Axiomatising specification theory, in: Recent Trends in Data Type Specification, Springer, Bremen, 1985, pp. 171–177.

[13] L.L. Maksimova, Craig's interpolation theorem and amalgamable varieties, Soviet Math. Dokl. 18 (6) (1977) 1550–1553.

[14] M. Marx, Interpolation, modularization and knowledge representation, in: Proc. Dutch Conference on AI (NAIC'95), June 1995.

[15] J. Meseguer, General logics, in: Logic Colloquium'87, 1989, pp. 275–329.

[16] G. Mints, Modularisaton and interpolation, Kestrel Institute, Dec. 1998 (Available at `ftp://www.kestrel.edu/pub/papers/mints/mod.ps`).

[17] A.M. Pitts, Amalgamation and interpolation in the category of Heyting algebras, J. Pure Appl. Algebra 29 (1983) 155–165.

[18] G. Roşu, J. Goguen, On equational Craig interpolation, J. Univ. Comput. Sci. (2000), to appear.

[19] P.H. Rodenburg, R.J. van Glabbeek, An interpolation theorem in equational logic, Technical Report CS-R8838, Centre for Mathematics and Computer Science, Amsterdam, The Netherlands, 1988.

[20] J.R. Shoenfield, Mathematical Logic, Addison-Wesley, Reading, MA, 1967.

[21] G. Tăkeuti, Proof Theory, North-Holland, Amsterdam, 1976.

[22] A. Tarlecki, Bits and pieces of the theory of institutions, in: Workshop on Category Theory and Computer Science, Lecture Notes in Comput. Sci., Vol. 240, Springer, Berlin, 1986.

[23] A. Tarski, A. Mostowski, R.M. Robinson, Undecidable Theories, North-Holland, Amsterdam, 1953.

[24] W.M. Turski, T.S.E. Maibaum, The Specification of Computer Programs, Addison-Wesley, Reading, MA, 1987.

[25] P.A.S. Veloso, On pushout consistency, modularity and interpolation for logical specifications, Inform. Process. Lett. 60 (2) (1996) 59–66.

[26] P.A.S. Veloso, T.S.E. Maibaum, On the modularisation theorem for logical specifications, Inform. Process. Lett. 53 (1995) 287–293.