Chapter 4:

Conditional Entropy $H_\mu(\xi|\gamma) = -\sum_{C\in\xi, D\in\gamma} \mu(C\cap D)\log\frac{\mu(C\cap D)}{\mu(D)}$

theoretic entropy $\rightarrow$ $h_\mu(T,\xi) = \lim_{n\to\infty} H_\mu\left(\xi \,\middle|\, \bigvee_{i=1}^{n} T^{-i}\xi\right)$

$x\in \xi_n(x) \in \xi_n = \bigvee_{k=0}^{n-1} T^{-k}\xi$

Shannon-McMillan-Breiman: $T: X\to X$ perserves measure μ

Then the limit $h_\mu(T,\xi,x) := \lim_{n\to\infty} -\frac{1}{n}\log\mu(\xi_n(x))$ exists for μ-almost every $x\in X$. Moreover, $f(x) = h_\mu(T,\xi,x)$ is T-invariant almost everywhere, μ-integrable, and $h_\mu(T,\xi) = \int_X h_\mu(T,\xi,x)\,d\mu(x)$.

Chapter 5: Thermodynamic Formalism

Def: Topological pressure of $\varphi: X\to\mathbb{R}$ with respect to T is

$$P_T(\varphi) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\sup_E \sum_{x\in E}\exp\sum_{k=0}^{n-1}\varphi(T^k(x)) \quad (E: (n,\varepsilon)\text{-separated})$$

Ex: $\varphi = c$ $\quad \sum_{x\in E}\exp\sum_{k=0}^{n-1}\varphi(T^k(x)) = \sum_{x\in E} e^{nc} = e^{nc}\cdot \operatorname{card} E$

$\therefore \sup_E \sum_{x\in E}\exp\sum\varphi(T^k(x)) = e^{nc} N(d_n,\varepsilon)$

$P_T(c) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log(e^{nc}N(d_n,\varepsilon)) = c + h(T)$, $P_T(0) = h(T)$

Thm: Topological pressure for one-side shift, $\sigma: X_k^+\to X_k^+$, $\varphi: X_k^+\to\mathbb{R}$

$P_\sigma(\varphi) = \lim_{n\to\infty}\frac{1}{n}\log\sum_{i_1\cdots i_n}\exp\sup_{C_{i_1\cdots i_n}}\sum_{k=0}^{n-1}\varphi\circ\sigma^k$

$\Rightarrow$ Ex: given $\lambda_1,\cdots,\lambda_k > 0$, $\varphi: X_k^+\to\mathbb{R}$ $\varphi(i_1 i_2\cdots) = \log\lambda_{i_1}$

$P_\sigma(\varphi) = \lim_{n\to\infty}\frac{1}{n}\log\sum_{i_1\cdots i_n}\exp\sum_{k=1}^{n}\log\lambda_{i_k} = \lim_{n\to\infty}\frac{1}{n}\log\sum_{i_1\cdots i_n}\prod_{k=1}^{n}\lambda_{i_k}$

$= \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{j=1}^{k}\lambda_j\right)^n = \log\sum_{j=1}^{k}\lambda_j$

Thm: (Variational principle for the topological pressure) $T: X\to X$, $\varphi: X\to\mathbb{R}$ continuous then,

$P_T(\varphi) = \sup_\mu\{h_\mu(T) + \int_X\varphi\,d\mu\}$

$\uparrow$ all T-invariant probablity measure in X

Def: Equilibrium measure: $P_T(\varphi) = h_\mu(T) + \int_X\varphi\,d\mu$

Thm: If $T: X\to X$ is a one-sided expansive continuous transformation of a compact metric space, then any continuous $\varphi: X\to\mathbb{R}$ has at least one equilibrium measure.

(two-sided) ✓

if $\exists\,\varepsilon$ s.t. $d(T^n(x), T^n(y)) < \varepsilon$ then $x = y$

$\Downarrow$ set $\varphi = 0$ $\exists\mu$ s.t. $P_T(0) = h_\mu(T) + \int_X 0\,d\mu$
$= h_\mu T$

Thm: $T: X\to X$ one-sided expansive, $\exists$ T-invariant μ with $h_\mu(T) = h(T)$

$\varphi = c$ $\quad P_T(c) = c + h(T)$

$c = 0$. $\quad P_T(c) = h(T) = h_\mu(T)$

Part II Hyperbolic Dynamics

Chapter 6

Def: Hyperbolic set(Λ): If $\exists\,\tau\in(0,1)$, $C>0$, decomposition $T_xM = \overset{\text{stable}}{E^s(x)} \oplus \overset{\text{unstable}}{E^u(x)}$ $\forall x\in\Lambda$

s.t. $d_xf\,E^s(x) = E^s(f(x))$, $d_xf\,E^u(x) = E^u(f(x))$

$\|d_xf^n v\| \le C\tau^n\|v\|$, $\forall v\in E^s(x)$; $\|d_xf^{-n}v\| \le C\tau^n\|v\|$, $\forall v\in E^u(x)$

Def: Fixed point $x = f(x)$ called hyperbolic if $\{x\}$ is hyperbolic set

m-Periodic point $x = f^m(x)$ called hyperbolic if $O_f(x) = \{f^k(x): k=0,\cdots,m-1\}$ is hyperbolic set

Pro: A fixed point x of a diffeomorphism f is hyperbolic iff $Sp(d_x f) \cap S^1 = \phi$ (spectrum)

$T: T^2 \to T^2$ induced $B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ $\longrightarrow$ $\tau = \frac{3+\sqrt{5}}{2}$, $\tau^{-1} = \frac{3-\sqrt{5}}{2}$ $\notin S^1$

Def: Matrix B is hyperbolic if $Sp(B) \cap S^1 = \phi$

$\because d_x T = B$

$\|d_x T^n(v)\| = \tau^n \|v\|$ $v \in F^u$

$\|d_x T^n v\| = \tau^{-n} \|v\|$ $v \in F^s$

$F^u \oplus F^s = R^2 = T_x T^2$

$d_x T F^s = F^s$

$d_x T F^u = F^u$ $\Rightarrow T^2$ is hyperbolic

Pro: If Λ is a hyperbolic set, then 1. $E^s(x), E^u(x)$ vary continuously with $x \in \Lambda$

2. $\inf \{ \angle(E^s(x), E^u(x)) : x \in \Lambda \} > 0$

If Λ is hyperbolic for a diffeomorphism f, then $\exists$ a inner product $\langle \cdot, \cdot \rangle'$ in TM

And 1. $\angle'(E^s(x), E^u(x)) = \frac{1}{2}\pi$ $\forall x \in \Lambda$

2. $\exists \rho \in (0,1)$ s.t. $\forall x \in \Lambda$, $\|d_x f v\|' \le \rho \|v\|'$, $v \in E^s(x)$

$\|d_x f^{-1} v\|' \le \rho \|v\|'$, $v \in E^u(x)$

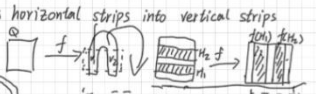
Horseshoe: $Q = [0,1]^2$ $H_1 = [0,1] \times [\frac{1}{9}, \frac{4}{9}]$ $H_2 = [0,1] \times [\frac{5}{9}, \frac{8}{9}]$ horizontal strips

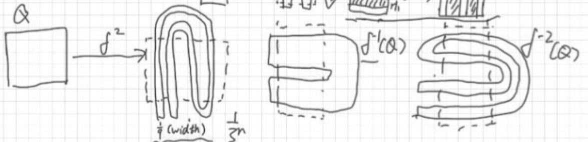
$V_1 = [\frac{1}{9}, \frac{4}{9}] \times [0,1]$ $V_2 = [\frac{5}{9}, \frac{8}{9}] \times [0,1]$ vertical strips

$$g(x,y) = \begin{cases} (\frac{1}{3}x, 3y) + (\frac{1}{9}, -\frac{1}{3}) & \text{if } (x,y) \in H_1 \\ (-\frac{1}{3}x, -3y) + (\frac{8}{9}, \frac{8}{3}) & \text{if } (x,y) \in H_2 \end{cases}$$

$f|_{(H_1 \cup H_2)} = g$ transforms horizontal strips into vertical strips

$f(H_1) = V_1$, $f(H_2) = V_2$



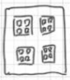


Def: $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(Q)$ is called Smale horseshoe

Pro: It's a hyperbolic set for diffeomorphism f

$d_x f: \begin{cases} B & x \in H_1 \\ -B & x \in H_2 \end{cases}$ $B = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 3 \end{pmatrix}$

Let $w = (\cdots i_{-1}(w) \, i_0(w) \, i_1(w) \cdots)$ $\forall w \in X, n \in N$, $Q_n(w) = \bigcap_{k=-n}^{n-1} f^{-k}(V_{i_k(w)})$



$Q, Q_1, Q_2 \cdots$

$H(w) = \bigcap_{n \in \mathbb{Z}} f^{-n}(V_{i_n(w)}) = \bigcap_{n \in N} Q_n(w)$ card $H(w) = 1$

$\Rightarrow h: X \to \Lambda$ $h(w) = \bigcap_{n = \mathbb{Z}} f^{-n}(V_{i_n(w)})$

Pro: 1. h is a homeomorphism

2. $h^{-1}(x) = (\cdots i_{-1} i_0 i_1 \cdots)$ $i_n = \begin{cases} 1, & f^n(x) \in V_1 \\ 2, & f^n(x) \in V_2 \end{cases}$

3. $h \circ \sigma = f \circ h$

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X \\ \downarrow h & \circlearrowleft & \downarrow h \\ \Lambda & \xrightarrow{f} & \Lambda \end{array}$$

Pro: 1. $\forall m \in N$ card $\{ w \in X : \sigma^m(w) = w \} = 2^m$ $f|_\Lambda$

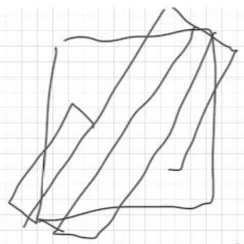
2. periodic points of σ is dense in X

$f|\Lambda$ ✓

Def: Let $T:\mathbb{T}^2\to\mathbb{T}^2$ be a hyperbolic automorphism, $\{R_i : i=1,\cdots,k\}$ is a Markov partition, $\sigma|X_A : X_A\to X_A$ is Markov chain, A is the transition matrix. Coding map $h: X_A\to\mathbb{T}^2$ by $h(\omega)=\bigcap_{n\in\mathbb{Z}} T^{-n}R_{i_n(\omega)}$

Measure μ : $\mu(h^{-1}C)=\lambda(C)$

Lebesgue measure



If $\mathrm{int}\,T(R_i)\cap\mathrm{int}\,R_j\neq\emptyset$

$T(R_i)$ intersect R_j

unstable dir

Pro: μ is a σ-invariant Markov measure in X_k

$(\mu(C_{i_{-m}\cdots i_m})=p_{i_{-m}}p_{i_{-m}i_{-m+1}}\cdots p_{i_{m-1}i_m}$, $C_{i_{-m}\cdots i_m}$ is cylinder set$)$

$f:U\to M$, C^1, U open in smooth manifold M

Def: Given a compact f-invariant set $J\subset U$, f is expanding on J and J is a repeller for f if $\exists\ c>0,\ \tau>1$ s.t. $\|d_xf^n v\|\geq c\tau^n\|v\|$ $\forall x\in J,\ v\in T_xM,\ n\in\mathbb{N}$

Ex: $E_q(z)=z^q$ on S^1 $(\|d_zE_q\|=q>1)$

Def: (Markov Partition of Repeller) Let J be a repeller for f, a collection of closed sets $R_1,\cdots,R_k\subset J$ is called a Markov partition of J if

1. $J=\bigcup_{i=1}^{k}R_i$, $R_i=\overline{\mathrm{int}R_i}$ $\forall i$ 2. $\mathrm{int}R_i\cap\mathrm{int}R_j=\emptyset$ when $i\neq j$
3. If $f(R_i)\cap\mathrm{int}R_j\neq\emptyset$, then $R_j\subset f(R_i)$

Thm: Any repeller has Markov partitions of arbitrarily small diameter.

Chapter 7.

Let $A:\mathbb{R}^n\to\mathbb{R}^n$ be invertible linear transformation: $A(x,y)=(\tau x,\rho y)$. $(x,y)\in\mathbb{R}^p\times\mathbb{R}^q$, $\tau,\rho^{-1}\in(0,1)$

horizontal axes $E^s=\mathbb{R}^p\times\{0\}=\{(x,y)\in\mathbb{R}^n : \|A^m(x,y)\|\to 0$ when $m\to+\infty\}$

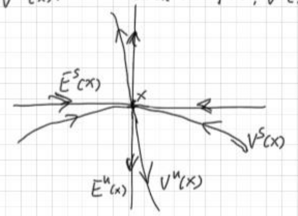
vertical axes $E^u=\{0\}\times\mathbb{R}^q=\{(x,y)\in\mathbb{R}^n : \|A^m(x,y)\|\to 0$ when $m\to-\infty\}$

$AE^s=E^s$, $AE^u=E^u\Rightarrow E^s,E^u$ are A-invariant sets.

Thm: (Hadamard-Perron) If x is a hyperbolic fixed point of a C^1 diffeomorphism f, Then there exist C^1 manifolds $V^s(x), V^u(x)$ containing x s.t.

$T_xV^s(x)=E^s(x)$, $T_xV^u(x)=E^u(x)$, $f(V^s(x))\subset V^s(x)$, $f^{-1}(V^u(x))\subset V^u(x)$

$V^u(x)$: local unstable manifold, $V^s(x)$: local stable manifold



$\Downarrow$ 推广到一般的Λ上:

Thm: If Λ is a hyperbolic set for a C^1 diffeomorphism f, then $\forall\varepsilon>0$

1. $\forall x\in\Lambda$, $\begin{cases} T_xV^s(x)=E^s(x) & T_xV^u(x)=E^u(x) \\ f(V^s(x))\subset V^s(f(x)), & f^{-1}(V^u(x))\subset V^u(f^{-1}(x)) \end{cases}$
2. There exists $\gamma=\gamma(\varepsilon)>0$ s.t.

$V^s(x)\supset B^s(x,\gamma)$, $V^u(x)\supset B^u(x,\gamma)$ $\forall x\in\Lambda$ ← 统一的γ, λ

Balls

3. $\forall\lambda>\tau$, $\exists C>0$ s.t. $d(f^n(x),f^n(y))\leq C\lambda^n d(x,y)$, $y\in V^s(x)$ $\forall x\in\Lambda$

$d(f^{-n}(x),f^{-n}(y))\leq C\lambda^n d(x,y)$, $y\in V^u(x)$, $n\in\mathbb{N}$

Thm: The Lebesgue measure is ergodic with respect to any hyperbolic toral automorphism

Birkhoff $f^* = \frac{1}{n}\sum_{k=0}^{n-1} f(T^k(x))$ $\int^* f$, $\int_{T^n} f$.

→ Hopf's argument

$\varphi : T^n \to R$ by Birkhoff

$\varphi^+(x) = \lim_{m\to\infty} \frac{1}{m}\sum_{k=0}^{m-1} \varphi(T^k(x))$

$\varphi^-(x) = \lim_{m\to\infty} \frac{1}{m}\sum_{k=0}^{m-1} \varphi(T^{-k}(x))$

lemma: $\varphi^+ = \varphi^-$ a.e.

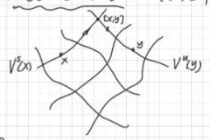
$Y = \{x \in T^n ; \varphi^+(x) > \varphi^-(x)\}$

$$\int_Y \varphi^+ d\lambda = \int_{T^n} \varphi^+ \chi_Y d\lambda = \underbrace{\int_{T^n} (\varphi \chi_Y)^+ d\lambda = \int_{T^n} (\varphi \chi_\varphi) d\lambda = \int_{T^n} (\varphi \chi_Y)^- d\lambda}$$
$$= \int_{T^n} (\varphi^- \chi_Y) d\lambda = \int_Y \varphi^- d\lambda$$

Def: A hyperbolic set Λ has a product structure if $\exists \varepsilon > 0, \delta > 0$ s.t.

$\mathrm{card}\,(V^s_\varepsilon(x) \cap V^u_\varepsilon(y)) = 1$ when $x, y \in \Lambda$, $d(x,y) \le \delta$

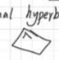
$[x,y] = V^s_\varepsilon(x) \cap V^u_\varepsilon(y)$ $\quad ([\cdot,\cdot] : \{(x,y) \in \Lambda \times \Lambda : d(x,y) \le \delta\} \to M$



Pro: A hyperbolic set has a product structure, and $[\cdot,\cdot]$ is continuous.

Def. A hyperbolic set Λ for a diffeomorphism f is said to be locally maximal if $\exists$ open set $U \supset \Lambda$ s.t. $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$

Pro. In a locally maximal hyperbolic set Λ, if $x, y \in \Lambda$ are sufficiently close, then $[x,y] \in \Lambda$



Def. $\alpha > 0$. $(x_n)_{a \le n \le b} \subset M$ $(a \in \mathbb{Z} \cup \{-\infty\}, b \in \mathbb{Z} \cup \{+\infty\})$ is called an α-orbit of f if $d(f(x_n), x_{n+1}) < \alpha$ $\forall n \in [a,b)$ $\quad f(x_n) = x_{n+1}$

$\beta > 0$, $x \in M$ is called β-shadow $(x_n)_{a\le n \le b} \subset M$ if

$d(f^n(x), x_n) \le \beta$ $\forall n \in [a,b]$ $\quad \Downarrow \quad f^n(x) = x_n$

(x_n) is β-shadowed by x

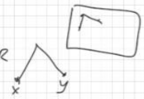
Thm. Let Λ be a locally maximal hyperbolic set for a diffeomorphism f. For each $\beta > 0$, $\exists \alpha > 0$ s.t. each α-orbit $(x_n)_{a\le n\le b} \subset \Lambda$ of f is β-shadowed by some point $x \in \Lambda$

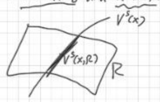
Thm. Let Λ be ____________________, $\forall \alpha > 0$, $\forall x \in \Lambda$ and $d(f^m(x), x) < \alpha$, then $\exists y \in \Lambda$ s.t. $f^m(y) = y$ and

$d(f^n(x), f^n(y)) < \varepsilon$ $\forall n \in [0,m]$

Def: A closed set $R \subset \Lambda$ is called rectangle if

1. $\mathrm{diam}\, R < \delta$ and $R = \overline{\mathrm{int}\, R}$ $\quad$ 2. $[x,y] \in R$ $\forall x, y \in R$





(Let $V^s(x,R) = V^s(x) \cap R$
$V^u(x,R) = V^u(x) \cap R$)

Def. A collection of rectangles $R_1, \cdots, R_k \subset \Lambda$ is called Markov partition of Λ if

一般的双曲集下

1. $\operatorname{int} R_i \cap \operatorname{int} R_j = \emptyset$ when $i \neq j$
2. If $x \in \operatorname{int} R_i$, $f(x) \in \operatorname{int} R_j$, then $\left.\begin{array}{l} f(V^u(x, R_i)) \supset V^u(f(x), R_j) \\ f^{-1}(V^s(f(x), R_j)) \supset V^s(x, R_i) \end{array}\right\}$

Thm. Any locally maximal hyperbolic set has Markov partitions of arbitrary small diameter.

Thm. $\forall$ Markov partition of a locally maximal hyperbolic set Λ and its coding map h $\left(h(\omega) = \bigcap_{n \in \mathbb{Z}} f^{-n}(R_{i_n(\omega)})\right)$, then:

1. h is continuous and onto
2. $h \circ \sigma = f \circ h$ in X_A
3. h is injective in $\left(\Lambda \setminus \bigcup_{n \in \mathbb{Z}} \bigcup_{i=1}^{k} f^n(\partial R_i) = \Lambda \setminus \bigcup_{n \in \mathbb{Z}} \bigcup_{i=1}^{k} f^n(\partial^s R_i \cup \partial^u R_i)\right)$

   $(\partial^s R_i = \{x \in R_i : x \notin \operatorname{int} V^u(x, R_i)\}, \partial^u R_i = \{x \in R_i : x \notin \operatorname{int} V^s(x, R_i)\})$
4. $\operatorname{card} h^{-1}x \leq k^2$, $\forall x \in \Lambda$

Part III

Chapter 8

Def. The diameter of collection $\mathcal{U}$ of subsets of X

$$\operatorname{diam} \mathcal{U} = \sup\{\operatorname{diam} U : U \in \mathcal{U}\}$$

$Z \subset X, \alpha \in \mathbb{R}$

α-dimensional Hausdorff measure of Z:

$$m(Z, \alpha) = \lim_{\varepsilon \to 0} \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} (\operatorname{diam} U)^\alpha$$

$\uparrow$ all finite or countable covers with $\operatorname{diam} \mathcal{U} \leq \varepsilon$

Def: Hausdorff dimension $\quad$ H-dim $\quad \alpha \uparrow\ m(Z, \alpha) \downarrow$.

$$\dim_H Z = \inf\{\alpha \in \mathbb{R} : m(Z, \alpha) = 0\} = \sup\{\alpha \in \mathbb{R} : m(Z, \alpha) = +\infty\}$$

lower box dimension : $\underline{\dim}_B Z = \liminf_{\varepsilon \to 0} \frac{\log N(Z, \varepsilon)}{-\log \varepsilon} \Leftarrow$ least number of B_ε balls needed to cover the set Z

upper box dimension : $\overline{\dim}_B Z = \limsup_{\varepsilon \to 0} \frac{\log N(Z, \varepsilon)}{-\log \varepsilon}$

Pro: $\dim_H Z \leq \underline{\dim}_B Z \leq \overline{\dim}_B Z$

Ex: $J \subset [0, 1]$ compose of the points with a base-3 representation without the digit 1. Give $n \in \mathbb{N}$, let $\varepsilon \in (3^{-(n+1)}, 3^{-n}]$, we have

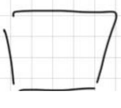
$$2^{n+1} > N(J, \varepsilon) \geq 2^n$$

$$\Rightarrow \frac{\log 2^n}{-\log(3^{-(n+1)})} < \frac{\log N(J, \varepsilon)}{-\log \varepsilon} < \frac{\log(2^{n+1})}{-\log(3^{-n})}$$

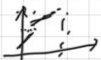
$$\Rightarrow \frac{n}{n+1} \cdot \frac{\log 2}{\log 3} < \frac{\log N(J, \varepsilon)}{-\log \varepsilon} < \frac{n+1}{n} \cdot \frac{\log 2}{\log 3} \quad \forall N$$

Let $\varepsilon \to 0$, $n \to \infty$

$$\underline{\dim}_B J = \overline{\dim}_B J = \frac{\log 2}{\log 3}$$



More elaborate: Given numbers $a_1, a_2 \in [0,1)$, $\lambda_1, \lambda_2 \in (0,1)$, consider functions $f_i(x) = \lambda_i x + a_i$, $i = 1,2$. Assume $f_i([0,1]) \subset [0,1]$ for $i=1,2$.

$f_1([0,1]) \cap f_2([0,1]) = \emptyset \Rightarrow \lambda_1 + a_1 < a_2,\ \lambda_2 + a_2 \leq 1$ 

Consider $J = \bigcap_{n=1}^{\infty} \bigcup_{i_1 \cdots i_n} \Delta_{i_1 \cdots i_n}$, $i_1, \cdots, i_n \in \{1,2\}$ where $\Delta_{i_1 \cdots i_n} = (f_{i_1} \circ \cdots \circ f_{i_n})([0,1])$

$\Delta_{i_1 \cdots i_n}$ is a closed interval of length $\lambda_{i_1} \cdots \lambda_{i_n}$,

$\Delta_{i_1 \cdots i_n} \cap \Delta_{j_1 \cdots j_n} = \emptyset$ whenever $(i_1 \cdots i_n) \neq (j_1 \cdots j_n)$

Pro: $\dim_H J = \underline{\dim}_B J = \overline{\dim}_B J = s$, where $s \in (0,1)$ is the unique root of the equation $\lambda_1^s + \lambda_2^s = 1$

$\Rightarrow k=2$, $f_1(x) = \frac{1}{3}x$, $f_2(x) = \frac{1}{3}x + \frac{2}{3}$ $\quad \lambda_1 = \lambda_2 = \frac{1}{3}$

$\dim_H J = \underline{\dim}_B J = \overline{\dim}_B J = s$

Since $2(\frac{1}{3})^s = 1$ $\quad s = \frac{\log 2}{\log 3}$

Def. Let μ be a finite measure of X, the Hausdorff dimension, lower, upper box dimensions of μ are defined by,

$$\dim_H \mu = \inf\{\dim_H Z : \mu(X \setminus Z) = 0\}$$

$$\underline{\dim}_B \mu = \lim_{\delta \to 0} \inf\{\underline{\dim}_B Z : \mu(Z) \geq \mu(X) - \delta\}$$

$$\overline{\dim}_B \mu = \lim_{\delta \to 0} \inf\{\overline{\dim}_B Z : \mu(Z) \geq \mu(X) - \delta\}$$

$$\Rightarrow \dim_H \mu = \lim_{\delta \to 0} \inf\{\dim_H Z : \mu(Z) \geq \mu(X) - \delta\}$$

There is: $\dim_H \mu \leq \underline{\dim}_B \mu \leq \overline{\dim}_B \mu$

Def. The lower and upper pointwise dimensions of the measure μ at the point $x \in X$ are defined by

$$\underline{d}_\mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \qquad \overline{d}_\mu(x) = \limsup_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}$$

Ex: μ is Lebesgue measure in R^n, then $\mu(B(x,r)) = C_n r^n \ \forall x \in R^n$

const depend on n

$$\underline{d}_\mu(x) = \overline{d}_\mu(x) = n \quad \forall x \in R^n$$

Thm: 1. If $\underline{d}_\mu(x) \geq \alpha$ for μ-almost every $x \in X$, then $\dim_H \mu \geq \alpha$

2. If $\underline{d}_\mu(x) \leq \alpha$ for every $x \in Z \subset X$, then $\dim_H Z \leq \alpha$

3. $\dim_H \mu = \operatorname{ess\,sup}\{\underline{d}_\mu(x) : x \in X\}$

Thm (Young). If μ is a finite measure in X and there exists $d \geq 0$ s.t.

$$\lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} = d \quad \text{for } \mu\text{-almost every } x \in X$$

then $\dim_H \mu = \underline{\dim}_B \mu = \overline{\dim}_B \mu = d$

$\uparrow$ pointwise dimension of measure μ at x, $d_\mu(x)$

Chapter 9

Let $f: U \to M$ be C^1, $J \subset U$ be a repeller for f

Def: f is conformal on J if $d_x f$ is a multiple of an isometry $\forall x \in J$

Thm. If J is a repeller for a $C^{1+\alpha}$ transformation f, for some $\alpha \in (0,1]$ s.t. f is conformal on J, then

$$\dim_H J = \underline{\dim}_B J = \overline{\dim}_B J = s$$

where s is the unique real number s.t. $P_{f|J}(s\varphi) = 0$

$\varphi: J \to R \quad \varphi(x) = -\log \|d_x f\|$

For $f: M \to M$ be a diffeomorphism, $\Lambda \subset M$ be a hyperbolic set for f.

Def. f is conformal on Λ if $d_x f|E^s(x)$, $d_x f|E^u(x)$ are multiples of isometries $\forall x \in \Lambda$

If M is a surface and $\dim E^s(x) = \dim E^u(x) = 1 \ \forall x \in \Lambda$ then f is conformal

Thm. Let Λ be a locally maximal hyperbolic set for a $C^{1+\alpha}$ diffeomorphism, for some $\alpha \in (0,1]$, s.t. f is conformal and topologically mixing on Λ. Then

$$\dim_H (V^s(x) \cap \Lambda) = \underline{\dim}_B (V^s(x) \cap \Lambda) = \overline{\dim}_B (V^s(x) \cap \Lambda) = t_s$$

and

$$\dim_H (V^u(x) \cap \Lambda) = \underline{\dim}_B (V^u(x) \cap \Lambda) = \overline{\dim}_B (V^u(x) \cap \Lambda) = t_u$$

where $t_s, t_u \in R$ s.t. $P_{f|\Lambda}(t_s \varphi_s) = P_{f|\Lambda}(t_u \varphi_u) = 0$

$(\varphi_s(x) = \log \|d_x f|E^s(x)\|, \ \varphi_u(x) = -\log \|d_x f|E^u(x)\|)$

Thm. $\dim_H \Lambda = \underline{\dim}_B \Lambda = \overline{\dim}_B \Lambda = t_s + t_u$