Chapter 16. A Finite Dimensional Approximation to Ω^c

M a connected Riemann Manifold, p, q is two point

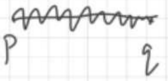
$\Omega = \Omega(M; p, q)$ piecewise C^∞ paths from p to q

$\uparrow$

Def topology: let ρ denote the topological metric on M by Riemann metric ($L(\gamma) = \int_a^b \sqrt{\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \rangle}\, dt \to ds \quad \inf L$)

$\forall$ path $\omega_1, \omega_2 \in \Omega$, arc-lengths $s_1(t), s_2(t)$

Def $d(\omega_1, \omega_2) = \underbrace{\max_{t \in [0,1]} \rho(\omega_1(t), \omega_2(t))} + \underbrace{\left(\int_0^1 \left(\frac{ds_1}{dt} - \frac{ds_2}{dt} \right)^2 dt \right)^{\frac{1}{2}}}_{L^2\text{-}}$



$E_a^b(\omega)$

$\therefore d$ induces a topology on Ω: $\{\omega_n\} \to \omega$ iff $\omega_n(t) \to \omega(t)$ uniformly and $\frac{ds_n}{dt} \xrightarrow{L^2} \frac{ds}{dt}$

Given $c > 0$, $\Omega^c := E^{-1}([0, c]) \subset \Omega$ ($E = E_0^1 : \Omega \to \mathbb{R}$)

$\mathrm{Int}\, \Omega^c := E^{-1}([0, c))$

Now: find topology of Ω^c by constructing a finite dim approximation

Choose $0 = t_0 < t_1 < \cdots < t_k = 1$, let $\Omega(t_0, \cdots, t_k)$ be paths: $\omega : [0,1] \to M$ s.t. $\omega(0) = p$, $\omega(1) = q$, $\omega|_{[t_i, t_{i+1}]}$ is a geodesic $\forall i = 0, \cdots, k-1$.

Def: $\Omega(t_0, \cdots, t_k)^c = \Omega^c \cap \Omega(t_0, \cdots, t_k)$; $\mathrm{Int}\, \Omega(t_0, \cdots, t_k)^c = \mathrm{Int}\, \Omega^c \cap \Omega(t_0, \cdots, t_k)$

Lemma 16.1, let M be a complete Riemannian manifold, $c>0$, $\Omega^c \neq \emptyset$, then for all sufficiently fine subdivisions $(t_0, \cdots, t_k)$ of $[0,1]$, set $\mathrm{Int}\,\Omega(t_0, \cdots, t_k)^c$ can be given the structure of a smooth finite dim manifold.

Pf: let $S: \{x \in M : \rho(x,p) \leq \sqrt{c}\} \Rightarrow \forall w \in \Omega^c, \because L^2 \leq E \leq c$

$\therefore w \in S \subset M$. M is complete $\Rightarrow S$ is compact.

Cor 10.11: $\forall$ cpt set $K \subset M$, $\exists \delta > 0$, s.t. $\forall$ two points of K with distance $< \delta$ are joined by a unique geodesic of length $< \delta$ depends differentiably on end points

$\Downarrow$

$\therefore \exists \varepsilon > 0$, s.t. $x, y \in S$, $\rho(x,y) < \varepsilon$, $\exists!$ geodesic from x to y of length $< \varepsilon$, and depends differentiably on x, y.

Choose $(t_0, t_1, \cdots, t_k)$ of $[0,1]$ s.t. $t_i - t_{i-1} < \frac{\varepsilon^2}{c}$, then $\forall$ broken geodesic $w \in \Omega(t_0, \cdots, t_k)^c$

$$\begin{aligned}(L_{t_{i-1}}^{t_i} w)^2 = (t_i - t_{i-1})(E_{t_{i-1}}^{t_i} w) &\leq (t_i - t_{i-1})(Ew) \\ &\leq (t_i - t_{i-1})c < \varepsilon^2\end{aligned}$$

$\rho < \varepsilon \Rightarrow$ geodesic $w|_{[t_{i-1}, t_i]}$ is unique, differentiably by end points

The broken geodesic w is uniquely determined by $(k-1)$-tuple

$$w(t_1), \cdots, w(t_{k-1}) \in M \times M \times \cdots \times M$$

$\therefore w \rightarrow (w(t_1), \cdots, w(t_{k-1}))$ defines a homeomorphism between $\mathrm{Int}\,\Omega(t_0, \cdots, t_k)^c$ and an open set of $\underbrace{M \times \cdots M}_{k-1 \text{ times}}$

Take over the structure of $M \times \cdots \times M$, $\mathrm{Int}\,\Omega(t_0, \cdots, t_k)^c$ get its smooth structure, it is a smooth manifold of $(k-1)\dim M$ $\blacksquare$

let $\operatorname{Int}\Omega(t_0,\cdots,t_k)^c$ be B, $E':B\to R$ denote the restriction of B of energy fun $E:\Omega\to R$.

Thm 16.2. $E':B\to R$ is smooth; For each $a<c$, $B^a=E'^{-1}[0,a]$ is cpt, and is a deformation retract of Ω^a; The critical points of E' are precisely the same as the critical points of E in $\operatorname{Int}\Omega^c$ — unbroken geodesics from p to q and length $<\sqrt{c}$; The index/nullity of Hessian E'_{**} at such critical point (geodesic named γ) = index/nullity of E_{**} at γ

Thus, the finite dim model $B(\subseteq M\times\cdots\times M)$ gives a faithful model for the infinite dim path space $\operatorname{Int}\Omega^c$

Pf: $E'(\omega)=\sum_{i=1}^{k}\frac{\rho(\omega(t_{i-1}),\omega(t_i))^2}{t_i-t_{i-1}}$ is smooth

For $a<c$, $B^a\cong\{(p_1,\cdots,p_{k-1})\in S\times\cdots\times S$, s.t. $\sum_{i=1}^{k}\frac{\rho(p_{i-1},p_i)^2}{t_i-t_{i-1}}\leq a\}$ $p_0=p$, $p_k=q$

It is a closed subset of cpt set, hence cpt.

Then, define the retraction $r:\operatorname{Int}\Omega^c\to B$

$r(\omega)$ denote the unique geodesic in B s.t.

$r(\omega)|_{[t_{i-1},t_i]}$ is geodesic of length $<\varepsilon$, from $\omega(t_{i-1})$ to $\omega(t_i)$

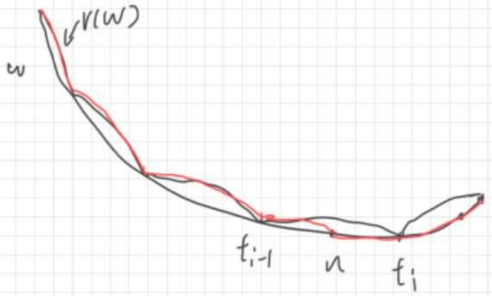
$\rho(p,\omega(t))^2\leq(L\omega)^2\leq E\omega<c\Rightarrow\omega\subset S$

$\rho(\omega(t_{i-1}),\omega(t_i))^2\leq(t_i-t_{i-1})(E_{t_{i-1}}^{t_i}\omega)<\frac{\varepsilon^2}{c}\cdot c=\varepsilon^2\Rightarrow$ By Cor 10.11, geodesics exists, unique, $L<\varepsilon$.

$\therefore r(\omega)$ is well-defined, $E(r(\omega))\leq E(\omega)<c$.

Def: $r_u : Int\,\Omega^c \to Int\,\Omega^c$ For $t_{i-1} \le u \le t_i$ let

$$\begin{cases} r_u(w)\big|_{[0,t_{i-1}]} = r(w)\big|_{[0,t_{i-1}]} \\ r_u(w)\big|_{[t_{i-1},u]} = \text{minimal geodesic from } w(t_{i-1}) \text{ to } w(u) \\ r_u(w)\big|_{[u,1]} = w\big|_{[u,1]} \end{cases}$$



$\therefore r_0$ is id map

$r_1 = r$

$r_u(w)$ is continuous

$\therefore B$ is a deformation retract of $Int\,\Omega^c$

$\because E(r_u(w)) \le E(w)$ $\therefore$ If $w \in \Omega^a$, $r_u(w) \in \Omega^a$

$\therefore B^a$ is also a deformation retract of Ω^a

Every geodesic is a broken geodesic, so critical point of E in $Int\,\Omega^c$ is critical point of B.

For critical point of B, it is unbroken geodesic, hence critical point of $Int\,\Omega^c$

At critical point: geodesic γ, $T_\gamma B$ Let $\bar{\alpha} : (-\varepsilon, \varepsilon) \to B$ $\bar{\alpha}(0) = \gamma$

$\forall u$, $\bar{\alpha}(u)$ is a broken geodesic.

$\therefore \bar{\alpha}$ is a variation, $W(t) = \frac{\partial \bar{\alpha}}{\partial u}(0,t)$ is a variation vector field along γ.

By lemma 14.5, $W(t)$ is Jacobi field on $[t_{i-1}, t_i]$

$W(t)$ is a broken Jacobi field.

$\{W(t)\} = T_\gamma \Omega(t_0, \cdots, t_n)$ space of broken Jacobi fields along γ

By lemma 15.4 (Index/nullity of E_{**} = index/nullity of E_{**} restricted to $T_\gamma\Omega(t_0, \cdots, t_n)$ of broken Jacobi field)

index/nullity of E_{**} at γ = index/nullity of E'_{**} at γ

Thm 16.3. Let M be a complete Riemannian manifold and let $p, q \in M$ be two points which are not conjugate along any geodesic of $L \leq \sqrt{a}$. Then Ω^a has the homotopy type of a finite CW-complex, with one cell of dim λ for each geodesic in Ω^a at which E_{**} has index λ.

Pf: Thm 3.3: If f is a differentiable fun on M with no degenerate critical point, and if each M^a is cpt, then M has homotopy type of a CW-complex, with one cell of dim λ for each critical point of index λ.

By thm 16.2. E' is smooth, $E'^{-1}([0,a])$ is cpt.

Since p, q are not conjugate $\Rightarrow$ E_{**} has 0-nullity at critical point.

By lemma 15.4. E_{**} and E'_{**} has index/nullity $\Rightarrow$

E'_{**} only has non-degenerate critical points.

$\therefore$ Thm 3.3 $\Rightarrow B^a$ has homotopy type of a CW-complex,
cell of dim $\lambda \Leftrightarrow$ critical point of E' of index $\lambda \Leftrightarrow$ geodesic
at which E_{**} has index λ.

By thm 16.2. B^a is a deformation retract of Ω^a.
$\therefore \Omega^a$ has homotopy type of a CW complex, cell of dim λ
$\Leftrightarrow$ geodesic, E_{**} has index λ. $\blacksquare$

$\Omega^a \leftrightarrow$ CW complex

Thm 16.3 $\Rightarrow \Omega^a$ contains only finite geodesics