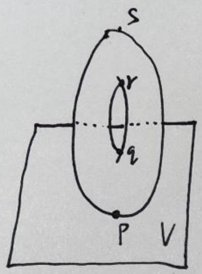
Chapter I Introduction



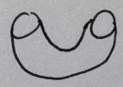
$f: M \to R$ height above V $\quad M^a = \{x \in M \mid f(x) < a\}$

(1) $a < 0 = f(p)$ $\quad M^a = \emptyset$

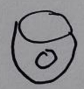
(2) $f(p) < a < f(q)$ $\quad M^a$ is homeomorphic to a 2-cell

Def: (n-cell) $e^n = \{x \in R^n : \|x\| \le 1\}$ R^n中闭单位球

(3) $f(q) < a < f(r)$, M^a is homeomorphic to a cylinder



(4) $f(r) < a < f(s)$, M^a is homeomorphic to a compact manifold of genus one, boundary: a circle
(亏格)

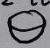


(5) $f(s) < a$, M^a is the full torus.

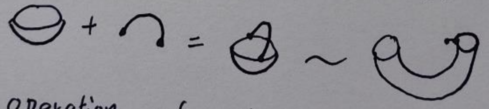
homeomorphism type ↑

homotopy type ↓

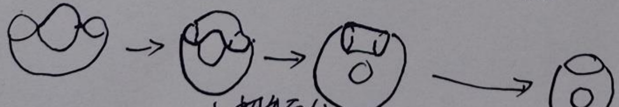
(1) → (2) $\quad \emptyset \to$ 2-cell ~~#~~ $\sim$ • ← 0-cell



(2) → (3) operation of attaching a 1-cell

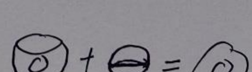


(3) → (4) operation of attaching a 1-cell

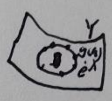


(把外面的部分拉上去)

(4) → (5) operation of attaching a 2-cell



Def (Attach a k-cell): Let Y be a top space. $e^k=\{x\in R^k : \|x\|\le 1\}$ be a k-cell
$\dot{e}^k=\{x\in R^k : \|x\|=1\}$ is the boundary (S^{k-1}). If $g: S^{k-1}\to Y$ continuous,
then $Y\cup_g e^k$ (Y with a k-cell attached by g). is obtained by:
first taking the topological sum (= disjoint union) of Y and e^λ.
then identifying each $x\in S^{k-1}$ with $g(x)\in Y$



If $k=0$, e^0 is a point, $\dot{e}^0=S^{-1}=\emptyset$, so Y with a 0-cell attached is just the union of Y and a disjoint point.

At p.q.r.s. ~~homotype~~ homotopy type changes, they are critical points of f.
Choose any (x,y), $\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0$. At p choose (x,y) s.t. $f=x^2+y^2$, then $f=c-x^2-y^2$ at s,
$f=c+x^2-y^2$ at q.r. Notice that "−" in f at each point = dim of cell attached
(0, 1, 1, 2) (0, 1, 1, 2).

(Chapter II + III ⇒ 证明)

Chapter II Definitions and Lemmas.

Def: $P\in M$ called critical point of f if $f_*: T_pM\to T_{f(p)}R$ is zero. Choose $(x^1,\cdots,x^n)$ in neighbor U of P, then $\frac{\partial f}{\partial x^1}=\cdots=\frac{\partial f}{\partial x^n}=0$. $f(P)\in R$ is called a critical value of f.
what's more: a critical point is called non-degenerate iff matrix $\left(\frac{\partial^2 f}{\partial x^i\partial x^j}(P)\right)$ is non-singular

Def: We define a symmetric bilinear functional f_{**} on T_pM (P is a critical point) called the Hessian of f at P. If $v,w\in T_pM$, then v,w have extensions $\tilde{v}$ and $\tilde{w}$ to vector fields ($\tilde{v}_p=v$, $\tilde{w}_p=w$).
Let $f_{**}(v,w)=\tilde{v}_p(\tilde{w}(f))$ ($\tilde{w}(f)$ is the directional derivate of f in the direction $\tilde{w}$)

③

Symmetric: $\tilde{v}_p(\tilde{w}(f)) - \tilde{w}_p(\tilde{v}(f)) = [\tilde{v},\tilde{w}]_p(f) = 0$ (p is a critical point of f)

well-defined: $\tilde{v}_p(\tilde{w}(f)) = v_p(\tilde{w}(f))$ is independent of the extension $\tilde{v}$ of v, $\tilde{w}_p(\tilde{v}(f))$ is also independent of $\tilde{w}$.

With a local coordinate system $(x_1,\cdots,x_n)$, $v=\sum a_i \frac{\partial}{\partial x^i}\Big|_p$, $w=\sum b_j \frac{\partial}{\partial x^j}\Big|_p$, take $\tilde{w}=\sum b_j \frac{\partial}{\partial x^j}\Big|_p$ where b_j = const function, then $f_{**}(v,w) = v(\tilde{w}(f))(p) = v(\sum b_j \frac{\partial f}{\partial x^j}) = \sum_{i,j} a_i b_j \frac{\partial^2 f}{\partial x^i \partial x^j}$

$\therefore$ Matrix $\left(\frac{\partial^2 f}{\partial x^i \partial x^j}(p)\right)$ represents the bilinear fun f_{**} with basis $\frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p$

Def: The index of a bilinear functional H on a vector space V is defined to be the maximal dimension of a subspace of V on which H is negative definite.

The nullity (零化度) is the dim of the null-space $\left(\begin{matrix}\text{All } v \in V \text{ s.t.}\\ H(v,w)=0 \ \forall w \in V\end{matrix}\right)$

Claim: A point p is a non-degenerate critical point of f iff f_{**} has nullity equal to 0 $\left(\left(\frac{\partial^2 f}{\partial x^i \partial x^j}(p)\right) \text{ non-singular}\right)$

接下来: behavior of f at $p \Leftarrow$ index

Lemma 2.1: Let f be a C^∞ function in a convex neighborhood V of 0 in R^n, $f(0)=0$. Then $f(x_1,\cdots,x_n) = \sum_{i=1}^n x_i g_i(x_1,\cdots,x_n)$ for some C^∞ fun g_i, with $g_i(0) = \frac{\partial f}{\partial x_i}(0)$

Proof: $f(x_1,\cdots,x_n) = \int_0^1 \frac{df(tx_1,\cdots,tx_n)}{dt} dt = \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(tx_1,\cdots,tx_n)\cdot x_i\, dt = \sum_{i=1}^n x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx_1,\cdots,tx_n)dt$

$\therefore g_i(x_1,\cdots,x_n) = \int_0^1 \frac{\partial f}{\partial x_i}(tx_1,\cdots,tx_n)dt$. $g_i(0) = \int_0^1 \frac{\partial f}{\partial x_i}(0) = \frac{\partial f}{\partial x_i}(0)$.

Lemma 2.2. (Lemma of Morse). Let p be a non-degenerate critical point for f. Then there is a local coordinate system $(y^1,\cdots,y^n)$ in a neighborhood U of p with $y^i(p)=0, \forall i$ s.t. $f = f(p)-(y^1)^2-\cdots-(y^\lambda)^2+(y^{\lambda+1})^2+\cdots+(y^n)^2$ holds throughout U, where λ is the index of f at p

(Index 完全描述了f在p的行为).

Proof: Firstly, if such expression for f exists, then λ must be the index of f at p.

Proof: For any $(z^1,\cdots,z^n)$, if $f(q)=f(p)-(z^1(q))^2-\cdots-(z^\lambda(q))^2+(z^{\lambda+1}(q))^2+\cdots+(z^n(q))^2$

then
$$\frac{\partial^2 f}{\partial z^i \partial z^j}(p)=\begin{cases} i=j\le\lambda & [-2(z^i)]'=-2. \\ i=j>\lambda & [2z^i]'=2 \\ 0 & \dfrac{\partial[\pm 2z^i]}{\partial z^j}=0 \end{cases}$$

∴ matrix representing f_{**} with respect to the basis $\frac{\partial}{\partial z^1}\Big|_p,\cdots,\frac{\partial}{\partial z^n}\Big|_p$ is

$$\begin{bmatrix} \overbrace{-2 & & }^{\lambda\text{个}} & & \\ & -2 & & 0 & \\ & & \ddots\ 2 & & \\ & 0 & & \ddots & 2 \end{bmatrix}$$
标准型

∴ ∃ subspace of T_pM of dim λ where f_{**} is negative definite, and a subspace of dim $n-\lambda$ where f_{**} is positive definite, call it V. If there were a subspace of dim larger than λ on which f_{**} were negative definite, then it must intersect V, hence a contradiction.

$\Rightarrow \lambda$ is the index of f_{**}

Secondly, a suitable coordinate system $(y^1,\cdots,y^n)$ exists.

Proof: WLOG, let ~~f(p)~~ p be origin, $f(p)=f(0)=0$. By lemma 2.1, $f(x_1,\cdots,x_n)=\sum_{j=1}^n x_j g_j(x_1,\cdots,x_n)$ $(x_1,\cdots,x_n)$ in some neighborhood of 0. Since 0 is p, (critical point), $g_j(0)=\frac{\partial f}{\partial x_j}(0)=0$.

∴ Use Lemma 2.1 again, $g_j(x_1,\cdots,x_n)=\sum_{i=1}^n x_i h_{ij}(x_1,\cdots,x_n)$ for certain smooth h_{ij}.

∴ $f(x_1,\cdots,x_n)=\sum_{i,j=1}^n x_i x_j h_{ij}(x_1,\cdots,x_n)$.

We can assume $h_{ij}=h_{ji}$, if not, use $\bar{h}_{ij}=\frac{1}{2}(h_{ij}+h_{ji})$ replace it. $(\bar{h}_{ij}(0))=\left(\frac{1}{2}\frac{\partial^2 f(0)}{\partial x_i x_j}\right)$

To show there exists coordinates $u_1, \cdots, u_n$ in neighborhood of 0 s.t. f is in desired expression.

~~#~~ Use ~~induction~~ induction: $f = \pm(u_1)^2 + \cdots \pm (u_{r-1})^2 + \sum_{i,j \geq r} u_i u_j H_{ij}(u_1, \cdots, u_n)$ in U_1 $\quad (H_{ij}(u_1, \cdots, u_n))_{n \times n}$ is symmetric

Assume $H_{rr} \neq 0$. if not, use linear change in last $n-r+1$ coordinates.

Let $g(u_1, \cdots, u_n) = \sqrt{|H_{rr}(u_1, \cdots, u_n)|}$ smooth in $U_2 \subset U_1$

Introduce new coordinates $v_1, \cdots, v_n$ by $v_i = u_i$ for $i \neq r$.

$$v_r(u_1, \cdots, u_n) = g(u_1, \cdots, u_n)\left(u_r + \sum_{i>r} u_i \frac{H_{ir}(u_1, \cdots, u_n)}{H_{rr}(u_1, \cdots, u_n)}\right)$$

($v_1, \cdots, v_n$ is coordinate funs in U_3 (small) by inverse function thm.)

$\therefore f = \sum_{i \leq r} \pm (v_i)^2 + \sum_{i,j \geq r} v_i v_j H'_{ij}(v_1, \cdots, v_n)$. Induction ✓.

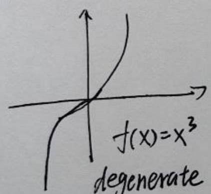
(把 v_i 乘进去, ~~换~~ 用 H'_{ij} 取代 H_{ij} ($H_{rr} = \frac{H_{rr}}{H_{rr}} g^2$ 不变)).

Cor 2.3. Non-degenerate critical points are isolated

(在 $f = f(p) - (y^1)^2 - \cdots - (y^\lambda)^2 + (y^{\lambda+1})^2 + \cdots + (y^n)^2$ 成立的 U 中, ~~若~~ $Df = 0 \Rightarrow y_i = 0 \Rightarrow U$ 中无~~更多~~临界点)

且 $\left|\frac{\partial^2 f}{\partial y_i \partial y_j}\right| \neq 0$

Examples:



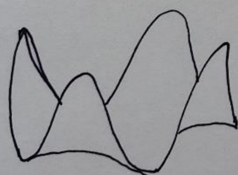
$f(x) = x^3$

degenerate critical point $(0,0)$.

$f(x,y) = x^3 - 3xy^2$ monkey saddle

猴鞍面

degenerate critical point $(0,0)$.

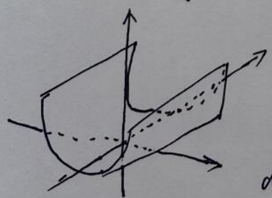


$f(x,y) = x^2y^2$

x-~~axis~~ axis $\cup$ y-axis

degenerate critical points

not even a submanifold of $\mathbb{R}^2$.



$f(x,y) = x^2$

x-axis

degenerate critical points

Review: 1-parameter group of diffeomorphisms: $\varphi: R\times M \to M$ s.t.

(1) $\forall t\in R$, $\varphi_t: R\times M \to M$ $\varphi_t(q)=\varphi(t,q)$ is a diffeomorphism of M onto M.

(2) $\forall t,s\in R$, $\varphi_{t+s}=\varphi_t\circ\varphi_s$

Given a 1-parameter group φ of diffeomorphisms of M, define a vector field of M:

$\forall f\in C^{\infty}(R)$, $X_q(f)=\lim\limits_{h\to 0}\dfrac{f(\varphi_h(q))-f(q)}{h}$. X is said to generate the group φ.

Lemma 2.4. A smooth vector field on M which vanishes outside of a compact set $K\subset M$ generates a unique 1-parameter group of diffeomorphisms of M.

Proof: $\forall$ smooth curve $t\to c(t)\in M$. define velocity vector $\frac{dc}{dt}\in T_{c(t)}M$ by

$$\frac{dc}{dt}(f)=\lim_{h\to 0}\frac{f_c(t+h)-f_c(t)}{h}$$

Now, let φ be a 1-parameter group of diffeomorphisms, generated by the vector field X. Then for fixed q, curve $t\to\varphi_t(q)$ satisfies $\dfrac{d\varphi_t(q)}{dt}=X_{\varphi_t}(q)$ with $\varphi_0(q)=q$, since

$$\frac{d\varphi_t(q)}{dt}(f)=\lim_{h\to 0}\frac{f(\varphi_{t+h}(q))-f(\varphi_t(q))}{h}=\lim_{h\to 0}\frac{f(\varphi_h(P))-f(P)}{h}=X_P(f)$$

$(P=\varphi_t(q))$

This ODE has unique soluation locally, depends smoothly on the initial ~~value~~ condition.

$\Rightarrow$. For each point of M, $\exists$ neighborhoods U, $\varepsilon>0$ s.t. $\dfrac{d\varphi_t(q)}{dt}=X_{\varphi_t}(q)$, $\varphi_0(q)=q$ has a unique smooth soluation, for $q\in U$, $|t|<\varepsilon$.

Set K is cpt $\Rightarrow$. Use finite neighborhoods U cover it. let ε_0 be ~~one~~ the smallest ε.

~~X vanishes~~ Setting $\varphi_t(q)=q$ for $q\notin K$ (X vanishes outside K), we get unique soluation $\varphi_t(q)$ of that ODE for $|t|<\varepsilon_0$ and for all $q\in M$. $\varphi_t(q)$ is smooth for t,q.

令 $\alpha(t)=\varphi_{t+s}(q)$ $\beta_t=\varphi_t(\varphi_s(q))$ $\alpha(0)=\beta(0)=\varphi_s(q)$. $\dfrac{d\alpha(t)}{dt}=\dfrac{d}{dt}\varphi_{t+s}(q)=X_{\varphi_{t+s}(q)}=X_{\alpha(t)}$

$\dfrac{d\beta(t)}{dt}=\dfrac{d}{dt}\varphi_t(\varphi_s(q))=X_{\varphi_t(p)}=X_{\beta(t)}$ $\Rightarrow\alpha(t)=\beta(t)$ $\Rightarrow$. $\varphi_{t+s}=\varphi_t\circ\varphi_s$.

$\Rightarrow$ (φ_{-t} exists, smooth). φ_t is a diffeomorphism

For $|t|\geq\varepsilon_0$ $t=k\cdot\frac{\varepsilon_0}{2}+r$, $k\geq 0$, $|r|<\frac{1}{2}\varepsilon_0$, let. $\varphi_t=\varphi_{\frac{\varepsilon_0}{2}}\circ\cdots\circ\varphi_{\frac{\varepsilon_0}{2}}\circ\varphi_r$

$k<0$ use $\varphi_{-\frac{1}{2}\varepsilon_0}$

$\square$

Note: "X vanishes outside of a cpt set" cannot be omitted.

Example: $M: (0,1) \subset \mathbb{R}$. $X = \frac{d}{dt}$.

$\cancel{\frac{d\varphi_t(q)}{dt} \quad X_q f = \frac{df}{dt} = \lim_{h\to 0} \frac{f(\varphi_h(q)) - f(q)}{h} \quad \varphi_h(q) = q + h}$

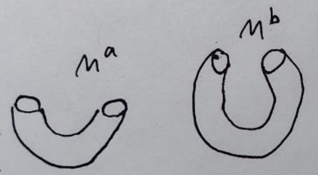
$\cancel{q \in (0,1) \quad \varphi_h(q)}$ ~~会~~

$X_q f = \frac{df}{dt} = \lim_{t\to 0} \frac{f(\varphi_t(q)) - f(q)}{t}$ $\varphi_t(q) = t + q$ $q \in (0,1)$ $\varphi_t(q)$会将M中点映到M外.

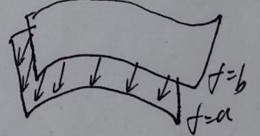
Chapter III: Homotopy Type in Terms of Critical Values.

f is real valued function on a manifold M, let $M^a = f^{-1}(-\infty, a] = \{p \in M : f(p) \le a\}$

Thm 3.1. Let f smooth real valued on M, Let $a < b$ and suppose $f^{-1}[a,b]$ (consisting of all $p \in M$, $a \le f(p) \le b$) is cpt, and contains no critical points of f. Then: M^a is diffeomorphic to M^b. Furthermore, M^a is a deformation retract of M^b, so that the inclusion map $M^a \to M^b$ is a homotopy equivalence
同伦等价



Push M^b down to M^a along the orthogonal trajectories of the hypersurfaces $f = \text{const}$.



Choose Riemannian metric on M; let $\langle X, Y \rangle$ denote inner product, $\operatorname{grad} f$ (vectorfield) is defined by $\langle X, \operatorname{grad} f \rangle = X(f)$ (directional derivative) $\Rightarrow$ $\operatorname{grad} f$ vanishes precisely at the critical points of f. For curve c, $\langle \frac{dc}{dt}, \operatorname{grad} f \rangle = \frac{d(f \circ c)}{dt}$.

Let $\rho: M \to \mathbb{R}$ be $\frac{1}{\langle \operatorname{grad} f, \operatorname{grad} f \rangle}$ throughout the cpt set $f^{-1}[a,b]$, and vanishes outside the cpt neighborhood of the set, define X: $X_q = \rho(q)(\operatorname{grad} f)_q$. $\Rightarrow$ From lemma 2.4. X generates a 1-parameter group of diffeomorphisms $\varphi_t: M \to M$.

For fixed $q \in M$, consider $\overset{\text{fun}}{\vee}$ $t \to f(\varphi_t(q))$, If $\varphi_t(q) \in f^{-1}[a,b]$, then $\frac{df(\varphi_t(q))}{dt} = \langle \frac{d\varphi_t(q)}{dt}, \operatorname{grad} f \rangle = \langle \overset{\rho \operatorname{grad} f}{X}, \operatorname{grad} f \rangle$
$= +1$

Thus, $t \to f(\varphi_t(q))$ is linear with derivative $+1$ if $f(\varphi_t(q)) \in [a,b]$

∴ For $p \in M^a$, $f(p)<a$, $f(\varphi_{b-a}(p)) < a+b-a < b$ ⇒ $\varphi_{b-a}(M^a) \subset M^b$

同理 $t \to f(\varphi_{-t}(q))$ derivative -1 ⇒ $\left\{ \begin{array}{l} \varphi_{a-b}(M^b) \subset M^a \\ \varphi_{b-a}(M^a) = M^b \end{array} \right.$ ⇒ diff. ($M^a \cong M^b$)

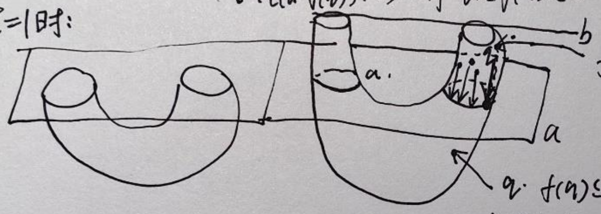
φ_{b-a} diff ⇒ φ_{b-a} smooth, φ_{b-a}^{-1} smooth

Furthurmore)

Next: Define 1-parameter family of maps $r_t : M^b \to M^b$. by

$$r_t(q) = \begin{cases} q & \text{if } f(q) \le a \\ \varphi_{t(a-f(q))}(q) & \text{if } a \le f(q) \le b \end{cases}$$

$t=1$时:



$f(q) \in [a,b]$ $r_t(q) = \varphi_{t[-(f(q)-a)]}(q)$ ⇓ 向下移动到 $f=a$ 平面上 ($t=1$)

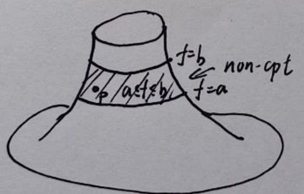
q. $f(q) \le a$, $r_t(q) = q$ 不变

$r_0 = id$ r_1 is a retraction from M^b to M^a.

∴ M^a is a ~~def~~ deformation retract of M^b □

Note: The condition $f^{-1}[a,b]$ is cpt can not be omitted.

Example:

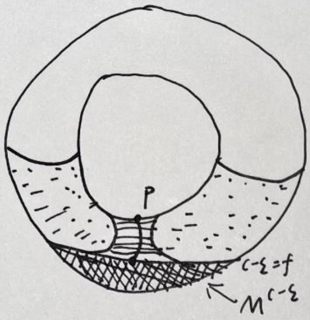


M^b can not ~~be~~ deformation retract ~~to~~ to M^a.

有 critical point 的情况:

Thm 3.2. Let $f: M \to \mathbb{R}$ be a smooth fun, let p be a non-degenerate critical point with index λ. Setting $f(p)=c$, suppose that $f^{-1}[c-\varepsilon, c+\varepsilon]$ is cpt, and contains no critical point of f other than p for some $\varepsilon > 0$. Then, for all sufficiently small ε, the set $M^{c+\varepsilon}$ has homotopy type of $M^{c-\varepsilon}$ with a λ-cell attached

⇓看图



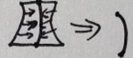
introduce $F: M \to \mathbb{R}$. $F = f$ others

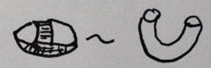
$F < f$ in a small neighbor of P

s.t. $F^{-1}(-\infty, c-\varepsilon] = M^{c-\varepsilon} \cup H$

H:  "把手"

Choosing a cell $e^\lambda \subset H$, push in along the horizontal lines

 $\therefore M^{c-\varepsilon} \cup e^\lambda$ is a deformation retract of $M^{c-\varepsilon} \cup H$

AIM: Then, ~~proof~~ prove $M^{c-\varepsilon} \cup H$ is a deformation retract of $M^{c+\varepsilon}$, then proof completed (Thm3.2). 

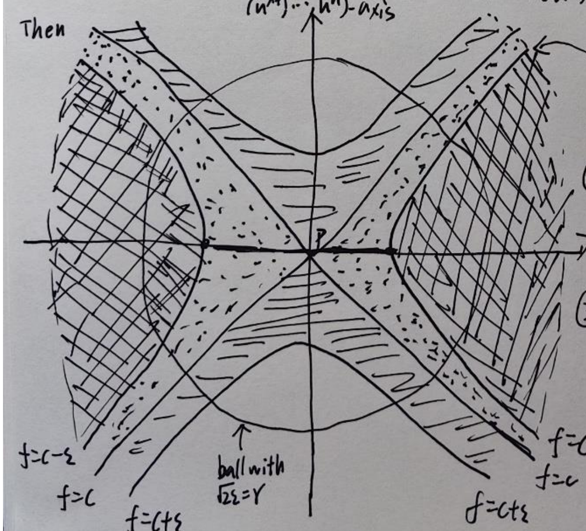
Pf: Choose a coordinate system $u^1, \cdots, u^n$ in a neighborhood of U of P s.t.

$f = c - (u^1)^2 - \cdots - (u^\lambda)^2 + (u^{\lambda+1})^2 + \cdots + (u^n)^2$ holds throughout U. Then $u^1(p) = \cdots = u^n(p) = 0$

Choose $\varepsilon > 0$ small enough s.t. (1) $f^{-1}[c-\varepsilon, c+\varepsilon]$ is cpt and contains no critical points other than P. (2) The image of U under the diffeomorphic imbedding $(u^1, \cdots, u^n): U \to \mathbb{R}^n$ contains the closed ball $\{(u^1, \cdots, u^n) \mid \sum (u^i)^2 \le 2\varepsilon\}$.

Now, define e^λ: $\{P(u^1, \cdots, u^n) \in U \mid (u^1)^2 + \cdots + (u^\lambda)^2 \le \varepsilon$ and $u^{\lambda+1} = \cdots = u^n = 0\}$

Then



① 当 $(u^1)^2 + \cdots + (u^\lambda)^2 = \varepsilon$, $u^{\lambda+1} = \cdots = u^n = 0$

此时 $f = c - \varepsilon + 0 = c - \varepsilon$.

且 $f = c - \varepsilon \iff (u^1)^2 + \cdots + (u^\lambda)^2 - (u^{\lambda+1})^2 - \cdots - (u^n)^2 = \varepsilon$

② $f = c \iff (u^1)^2 + \cdots + (u^\lambda)^2 = (u^{\lambda+1})^2 + \cdots + (u^n)^2$

③ $f = c + \varepsilon \iff (u^{\lambda+1})^2 + \cdots + (u^n)^2 - (u^1)^2 - \cdots - (u^\lambda)^2 = \varepsilon$

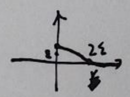
图例: (cross-hatched): $M^{c-\varepsilon}$ (lined): $f^{-1}[c, c+\varepsilon]$

(dotted): $f^{-1}[c-\varepsilon, c]$ (segment through P): e^λ

Since $e^\lambda \cap M^{c-\varepsilon}$ is $\dot{e}^\lambda$, e^λ is attached to $M^{c-\varepsilon}$ ✓. (10)

Then, prove $M^{c-\varepsilon} \cup e^\lambda$ is a deformation retract of $M^{c+\varepsilon}$

Let smooth fun $F: M \to R$. Let $\mu: R \to R$ C^∞ s.t. $\mu(0) > \varepsilon$, $\mu(r) = 0$ for $r \geq 2\varepsilon$, $-1 < \mu'(r) \leq 0$. $\forall r$



Let F coinside with f outside U.

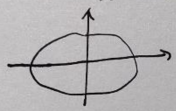
Let $F = f - \mu\left((u^1)^2 + \cdots + (u^\lambda)^2 + 2(u^{\lambda+1})^2 + \cdots + 2(u^n)^2\right)$ in U. Since $\mu(2\varepsilon) = 0$, μ is well-def smooth fun on M.

Def (ksi) (eita) $\xi, \eta: U \to [0, +\infty)$ $\xi = (u^1)^2 + \cdots + (u^\lambda)^2$, $\eta = (u^{\lambda+1})^2 + \cdots + (u^n)^2$ Then $f = c - \xi + \eta$

$F(q) = c - \xi(q) + \eta(q) - \mu(\xi(q) + 2\eta(q))$ $\forall q \in U$

Assertion 3.1. The region $F^{-1}(-\infty, c+\varepsilon]$ coincides with $M^{c+\varepsilon} = f^{-1}(-\infty, c+\varepsilon]$

Pf: Ellipsoid $\xi + 2\eta = 2\varepsilon$



Outside: $f = F$

Inside: $\mu \geq 0$. $F \leq f = c - \xi + \eta \leq c + \frac{1}{2}\xi + \eta \leq c + \varepsilon$. ▨

~~Assertion 3.2. The critical points of F are same as those of f.~~ 接下来关注 $F^{-1}[c-\varepsilon, c+\varepsilon]$ 的 critical point

$\frac{\partial F}{\partial \xi} = -1 - \mu'(\xi + 2\eta) < 0$ $(-1 < \mu')$ $\frac{\partial F}{\partial \eta} = 1 - 2\mu'(\xi + 2\eta) \geq 1$ $(\mu' \leq 0)$

$\therefore dF = 0 \Leftrightarrow d\xi, d\eta = 0 \Leftrightarrow$ origin. Thus: F has no critical points in U other than the origin.

Now, consider $F^{-1}[c-\varepsilon, c+\varepsilon]$, "Assertion 3.1" + "$F \leq f$" $\Rightarrow$. $F^{-1}[c-\varepsilon, c+\varepsilon] \subset f^{-1}[c-\varepsilon, c+\varepsilon]$.

$\Rightarrow$. This reigon is cpt, and the only possible critical point is p.

But: $F(p) = c - \mu(0) < c - \varepsilon$. Hence $F^{-1}[c-\varepsilon, c+\varepsilon]$ has no critical points.

Thm 3.1 ($F^{-1}[c-\varepsilon, c+\varepsilon]$ cpt + contains no critical point of f $\Rightarrow F^{-1}(-\infty, c-\varepsilon]$ is a deformation retract of $F^{-1}(-\infty, c+\varepsilon]$) + Assertion 3.1 ($F^{-1}(-\infty, c+\varepsilon] = M^{c+\varepsilon}$)

$\Rightarrow F^{-1}(-\infty, c-\varepsilon]$ deformation retract of $M^{c+\varepsilon}$ (Assertion 3.3). ✓

Let H denote the closure of $F^{-1}(-\infty, c-\varepsilon] - M^{c-\varepsilon}$, then $F^{-1}(-\infty, c-\varepsilon] = M^{c-\varepsilon} \cup H$ (Smale) ↑ handle

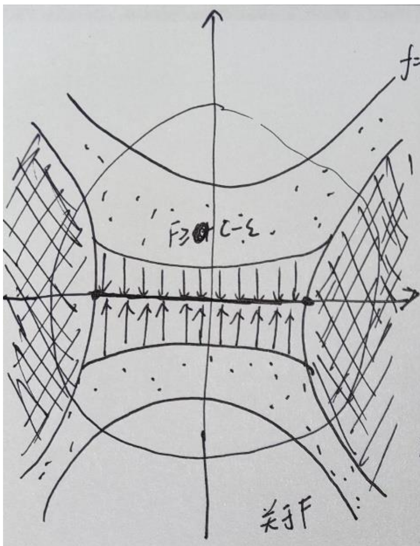
Now:

handle $\to$ e^λ (cell) (如何变?)

Consider the cell e^λ consisting of all points q with $\xi(q) \leq \varepsilon$, $\eta(q) = 0$ $\Rightarrow e^\lambda$ is contained in H.

Since $\frac{\partial F}{\partial \xi} < 0$, $F(q) \leq F(p) < c - \varepsilon$, but $f(q) \geq c - \varepsilon$ for $q \in e^\lambda$

⇓ 看图



: $M^{c-\varepsilon}$

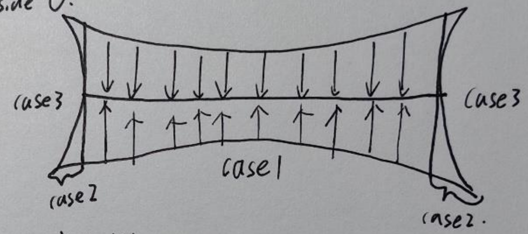
: $F^{-1}[c-\varepsilon, c+\varepsilon]$

: H.

Assertion 3.4. The region $M^{c-\varepsilon} \cup e^{\lambda}$ is a deformation retract of $M^{c-\varepsilon} \cup H$.

Pf: $r_t: M^{c-\varepsilon} \cup H \to M^{c-\varepsilon} \cup H$ $\downarrow\downarrow$ $\uparrow\uparrow$ Outside U: id

Inside U:



Case1: $\xi \le \varepsilon$. r_t: $(u^1, \cdots, u^n) \mapsto (u^1, \cdots, u^{\lambda}, tu^{\lambda+1}, \cdots, tu^n)$.

$\therefore r_1 = id$ r_0 maps to e^{λ}

Case2: $\varepsilon \le \xi \le \eta + \varepsilon$ $r_t: (u^1, \cdots, u^n) \longrightarrow (u^1, \cdots, u^{\lambda}, s_t u^{\lambda+1}, \cdots, s_t u^n)$ $s_t = t + (1-t)\left(\frac{\xi - \varepsilon}{\eta}\right)^{\frac{1}{2}}$.

$\therefore r_1 = id$ r_0 maps into $f^{-1}(c-\varepsilon)$

Case3: $\eta + \varepsilon \le \xi$ (in $M^{c-\varepsilon}$) Let $r_t = id$.

$\therefore M^{c-\varepsilon} \cup e^{\lambda}$ is a deformation retract of $\cancel{F^{-1}(-\infty, c+\varepsilon]}$ $M^{c-\varepsilon} \cup H$ ▨ 结论

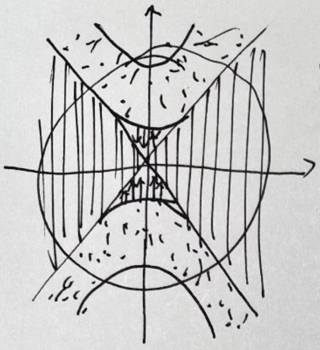
~~With Assertion3.3 ($F^{-1}(-\infty, c+\varepsilon]$)~~

$\therefore M^{c-\varepsilon} \cup e^{\lambda} \sim M^{c-\varepsilon} \cup H = F^{-1}(-\infty, c-\varepsilon] \overset{\text{Assertion3.3}}{\sim} M^{c+\varepsilon}$

$\therefore M^{c+\varepsilon}$ has ~~homotype~~ homotopic type of $M^{c-\varepsilon}$ with a λ-cell attached (Thm3.2) Proof completed ▨

Remark: k non-degenerate critical points: $p^1, \cdots, p^k$, index $\lambda_1, \cdots, \lambda_k$ in $f^{-1}(c)$, then $M^{c+\varepsilon}$ has the homotype of $M^{c-\varepsilon} \cup e^{\lambda_1} \cup \cdots \cup e^{\lambda_k}$.

Remark: M^c is also a deformation retract of $M^{c+\varepsilon}$. In fact M^c is a deformation retract of $F^{-1}(-\infty, c]$, $F^{-1}(-\infty, c] \sim M^{c+\varepsilon}$. $M^{c-\varepsilon} \cup e^\lambda \sim M^c$ (By thm 3-2)



$:M^c$ $: F^{-1}[c, c+\varepsilon]$

Thm 3.3. If f is a differentiable fun on M. with no degenerate critical points, 对 M^a is cpt. then M has the homotopy type of a CW-complex, with one cell of dim λ for each critical point of index λ.

Def: CW-complex.（单纯复形的推广） 胞腔粘合在一处（将空间X进行胞腔划分）

构造：Def 0-skeleton (X_0): points （由一组 0-cell 组成）

1-skeleton (X_1): attach 1-cells on 0-skeletons

...

$X_k = X_{k-1} \cup_\alpha e^k_\alpha$ （k-cells : $(e^k_\alpha)_\alpha$）

生成空间 $X = \cup_k X_k$ is a weak-topology : $U \subset X$ is open $\Leftrightarrow$. $U \cap X_k$ is open $\forall k$.

C: closure-finite : 每个cell的闭包只与有限多cell相交 W: weak-topo.

Example: 多面体: 点 → 线 → 面
X_0 X_1 X_2.

图: 1维 CW复形

S^n: 0-cell + n-cell $\Rightarrow$ "包包子"

T^2: 0-cell, 2个 1-cell + 2-cell (Thm 3.3)

$RP^n \sim D^n / \partial D^n$ 对径.

$RP^n = RP^{n-1} \cup_f D^n$

f: 二开一映射

D^n 自然粘到 S^{n-1}
$RP^{n-1} = S^{n-1}/\sim$ $\Rightarrow D^n + \sim = RP^n$

$\Leftarrow$ RP^2 是 CW复形，但不是单纯复形（不能被三角剖分）因为不可定向

$\Leftarrow$ 另一个三角形，全部三个边都粘在一个边上

一个三角形

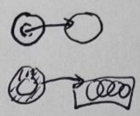
2维 CW复形 ✓
单纯复形 ✗

Lemma 3.4 (Whitehead) Let φ_0, φ_1 be homotopic maps from $\dot{e}^\lambda$ to X. Then the identity map of X extends to a homotopy equivalence $K: X \cup_{\varphi_0} e^\lambda \to X \cup_{\varphi_1} e^\lambda$

Pf: Define K: $K(x) = x \quad x \in X$ (K 是X上 id 的 extension)

$K(tu) = 2tu \quad 0 \le t \le \frac{1}{2}, u \in \dot{e}^\lambda$

$K(tu) = \varphi_{2-2t}(u) \quad \frac{1}{2} \le t \le 1, u \in \dot{e}^\lambda$



φ_t denotes the homotopy between φ_0 and φ_1

Def $L: X \cup_{\varphi_1} e^\lambda \to X \cup_{\varphi_0} e^\lambda$ $\quad L(x) = x \quad x \in X \quad L(tv) = 2tv \quad 0 \le t \le \frac{1}{2} \quad v \in \dot{e}^\lambda$

$L(tv) = \varphi_{2t-1}(v) \quad \frac{1}{2} \le t \le 1, v \in \dot{e}^\lambda$

$K \circ L \underset{同伦}{\sim} id$

$\therefore$ K is a homotopy equivalence

Lemma 3.5 Let $\varphi: \dot{e}^\lambda \to X$ be an attaching map. Any homotopy equivalence $f: X \to Y$ extends to a homotopy equivalence $F: X \cup_\varphi e^\lambda \to Y \cup_{f \circ \varphi} e^\lambda$ (Lemma 3.4: id 扩行 ... 延)

Pf: Def F: $\begin{cases} F|_X = f \\ F|_{e^\lambda} = id \end{cases}$ Let $g: Y \to X$ be a homotopy inverse to f, and define

$G: Y \cup_{f \circ \varphi} e^\lambda \to X \cup_{g \circ f \circ \varphi} e^\lambda \quad G|_Y = g \quad G|_{e^\lambda} = id$

Since $g \circ f \circ \varphi$ is homotopic to φ, by Lemma 3.4, $\exists K: X \cup_{g \circ f \circ \varphi} e^\lambda \to X \cup_\varphi e^\lambda$ (extension of id).

AIM: Firstly, prove $K \circ G \circ F: X \cup_\varphi e^\lambda \to X \cup_\varphi e^\lambda$ is homotopic to id map

Let h_t be a homotopy between $g \circ f$ and id. (因为 g 是 f 的 homotopy inverse)

$K \circ G \circ F(x) = gf(x)$ for $x \in X$, $K \circ G \circ F(tu) = 2tu \quad 0 \le t \le \frac{1}{2}, u \in \dot{e}^\lambda$, $K \circ G \circ F(tu) = h_{2-2t}\varphi(u) \quad \frac{1}{2} \le t \le 1, u \in \dot{e}^\lambda$

homotopy $q_T: X \cup_\varphi e^\lambda \to X \cup_\varphi e^\lambda$ defined by $q_T(x) = h_T(x)$ for $x \in X$

$q_T(tu) = \frac{2}{1+T} tu$ for $0 \le t \le \frac{1+T}{2}$, $u \in \dot{e}^\lambda$

$q_T(tu) = h_{2-2t+T}\varphi(u)$ for $\frac{1+T}{2} \le t \le 1$, $u \in \dot{e}^\lambda$

($T=0 \quad q_T = K \circ G \circ F$ ($h_0 = gf$, $h_1 = id$; $T = 1$, $q_T = id$)

$\therefore K \circ G \circ F$ is homotopic to id (q_T), F has a left homotopy inverse, similarly, G has a left homotopy inverse

Since F has left homotopy inverse L, right ~ R. then F is a homotopy equivalence

$L \circ F \simeq id \quad F \circ R \simeq id \quad L \simeq L(F \circ R) = (L \circ F) R \simeq R \quad \therefore R \circ F \simeq L \circ F \simeq id$

Therefore: ~~$k\circ F\circ G\simeq id$~~ $k\circ G\circ F\simeq id$, From Lemma 3.4 k has a left inverse (14)

$\Rightarrow (G\circ F)\circ k\simeq id$

$G\circ(F\circ k)\simeq id$ 且 Since G has a left inverse $\Rightarrow (F\circ k)\circ G\simeq id$

$F\circ(k\circ G)\simeq id$ + since F also has $(k\circ G)$ as left inverse $\Rightarrow$ F is a homotopy equivalence

(证明了 $f: X\to Y \Rightarrow F: X\cup_{\varphi} e^{\lambda}\longrightarrow Y\cup_{f\circ\varphi} e^{\lambda}$ homotopy equivalence). ▨

Proof of thm 3.3:

Let $c_1<c_2<\cdots$ be the critical values of $f: M\to\mathbb{R}$, $\{c_i\}$ has no cluster point (non-degenerate critical points are isolated). $M^a=\emptyset$ if $a<c_1$. Suppose $a>c_1, c_2, c_3\cdots$, and M^a is of the homotopy ~~tp~~ type of a CW-complex. let $c=c_i$ be smallest $c_i>a$. By Thm 3.1, 3.2. + remark $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}\cup_{\varphi_1} e^{\lambda_1}\cup\cdots\cup_{\varphi_{j(c)}} e^{\lambda_{j(c)}}$ for certain $\varphi_1,\cdots,\varphi_{j(c)}$, ε is small enough, and there is a homotopy equivalence $h: M^{c-\varepsilon}\longrightarrow M^a$. We have assumed $\exists$ a homotopy equivalence $h': M^a\to K$, where K is a CW-complex.

Then $h'\circ h\circ\varphi_j$ is homotopic (by cellular approximation) to a map:

(Psi) $\psi_j: \dot{e}^{\lambda_j}\longrightarrow (\lambda_j-1)$-skeleton of K (生成 CW复形的过程)

Then $K\cup_{\psi_1} e^{\lambda_1}\cup\cdots\cup_{\psi_{j(c)}} e^{\lambda_{j(c)}}$ is a CW-complex

Use lemma 3.4 to $h'\circ h$, it has the same homotopic type of $M^{c-\varepsilon}\cup_{\varphi_1} e^{\lambda_1}\cup\cdots\cup_{\varphi_{j(c)}} e^{\lambda_{j(c)}}$, so as $M^{c+\varepsilon}$

$\therefore$ By induction, each $M^{a'}$ has the homotopy type of a CW-complex

(加了一个 c_i, 依旧$\simeq$新的 CW-complex)

If M is cpt, then prove complete. If not, all critical points lie in one of the cpt sets M^a, then similar to proof of Thm 3.1, M^a is a deformation retract of M, then proof is complete.

If critical points are infinite, then we get $M^{a_1}\subset M^{a_2}\subset\cdots$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\downarrow\qquad\ \downarrow$

Let K denote union of K_i in the direct limit top $\quad K_1\subset K_2\subset\cdots$ i.e. the finest possible compatible top

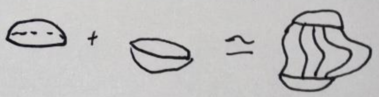
Let $g: M\to K$ be the limit map. Then g introduces isomorphisms of homotopy groups in all dims. Apply Whitehead's Thm: If M, K are CW-complex $\forall$ map $M\to K$ who induces iso$_*$ is a homotopy equivalence. K is dominated of itself

M: a retract of $\boxed{\text{dominated by}}$ tubular neighborhood in some Euclidean space ▨

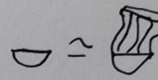
Note: we have proved M^a has the homotopy type of a finite CW-complex, with one cell of dim λ for each critical point of index λ in M^a. (That's true even if a is a critical value, compare to remark before)

Chapter VI Examples.

Thm 4.1 (Reeb). If M is cpt, f differentiable with only 2 critical points, both non-degenerate, then M is homeomorphic to a sphere.



Pf: From Morse's lemma, the critical points must be the minimum and maximum. Say $f(p)=0$ is the minimum and $f(q)=1$ is the maximum. By thm 3.2, ε small enough, then $M^{\varepsilon}=f^{-1}[0,\varepsilon]$ $f^{-1}[1-\varepsilon,1]$ are closed n-cells. By thm 3.1, $M^{\varepsilon}\simeq M^{1-\varepsilon}$



$\therefore M = M^{1-\varepsilon} + f^{-1}[1-\varepsilon,1] \simeq 2\times n\text{-cell}$ $\therefore M\simeq S^n$

Remark: diffeomorphic X

Ex 2. CP^n $(\sum|z_j|^2=1,\ (z_0:z_1:\cdots:z_n))$. Let $f(z_0:z_1:\cdots:z_n)=\sum C_j|z_j|^2$ $\quad C_i\in\mathbb{R}$.

Let U_0 be a set of $(z_0:z_1:\cdots:z_n)$, $z_0\neq 0$. $|z_0|\frac{z_j}{z_0}=x_j+iy_j$

Then $x_1,y_1,\cdots,x_n,y_n : U_0\to\mathbb{R}$ are the required coordinate funs. mapping U_0 diffeomorphically onto the open unit ball in $\mathbb{R}^{2n}$. $|z_j|^2=x_j^2+y_j^2$ $\quad |z_0|^2=1-\sum(x_j^2+y_j^2)$

$f=C_0+\sum_{j=1}^{n}(C_j-C_0)(x_j^2+y_j^2)$ the only critical point in U_0: $P_0=(1:0:\cdots:0)$

At P_0, f is non-degenerate, has index equal to twice the number of j s.t. $C_j<C_0$. (根据表达式)

Similar, we can consider other coordinate systems centered at the points:

$P_1=(0:1:0:\cdots:0),\cdots,P_n=(0:\cdots:0:1)$ $\Leftarrow$ critical points of f. index of f at P_k is equal to twice the number of j with $C_j<C_k$. $\Rightarrow$ Thus every possible even index between 0 and $2n$ occurs exactly once.

Thm 3.3 $\Rightarrow$ CP_n has the homotopy type of a CW-complex of the form $e^0\cup e^2\cup\cdots\cup e^{2n}$