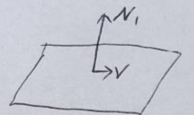
$\mathbb{R}^2$ and $C=\{(x,y,z)\in\mathbb{R}^3: x^2+z^2=1, y\in\mathbb{R}\}$ are locally isometric.

$(\psi:\mathbb{R}^2\to C: (x,y,0)\to(\sin x, y, \cos x)$ is locally isometric covering.

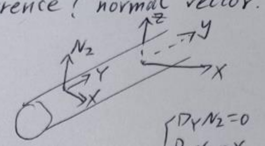
But they are different. How to describe difference? normal vector.

$\mathbb{R}^2$:



N_1: constant.

$D_V N_1 = 0$



$\begin{cases} D_Y N_2 = 0 \\ D_X N_2 = X \end{cases}$

S^2:

$D_W N_3 = W$.

C is similar to S^2 in one direction (X) but different in ~~the~~ the other one (Y)

$\Rightarrow$ To describe the shape of surface in $\mathbb{R}^3$, we should consider $D_X N$ $\quad$ N normal, X tangental.

Notice that $D_X N$ tangents to surface, $\langle D_X N, N\rangle = \frac{1}{2}X\langle N,N\rangle = 0$ $\quad D_X N \perp N$.

Since a vector is determined by its inner product to other vectors, so investigating $D_X N$ is equal to investigating $\langle D_X N, Y\rangle$, $X, Y\in TS$, this is a way to describe the shape of Riemannian manifold.

Def: Let $(\bar{M},\bar{g})$ be Riemannian manifold, (M,g) be its submanifold, g is induced by $\bar{g}$.

The inner product on $T_p\bar{M}$ splits into $T_pM\oplus(T_pM)^\perp$, $((T_pM)^\perp$ is the orthogonal complemen[t]

$V\in T_p\bar{M}$, $p\in M$, then $V=V^T+V^N$, $V^T\in T_pM$, $V^N\in(T_pM)^\perp$

Let $\bar{\nabla}$ be the Levi-Civita connection in $(\bar{M},\bar{g})$, let X,Y be tangental vector field of M,

Let $\nabla_X Y$ be tangental projection of $\bar{\nabla}_{\bar{X}}\bar{Y}$ to M ($\nabla_X Y=(\bar{\nabla}_{\bar{X}}\bar{Y})^T$, $\bar{X},\bar{Y}$ are local extensions to M),

then $\nabla_X Y$ defines Levi-Civita connection on M.

Check:

① $\nabla_{fX}Y=(\bar{\nabla}_{f\bar{X}}\bar{Y})^T=(f\bar{\nabla}_{\bar{X}}\bar{Y})^T=f(\bar{\nabla}_{\bar{X}}\bar{Y})^T=f\nabla_X Y$ ② $\nabla_X(aY+bZ)=(\bar{\nabla}_{\bar{X}}(a\bar{Y}+b\bar{Z}))^T=(a\bar{\nabla}_{\bar{X}}\bar{Y}+b\bar{\nabla}_{\bar{X}}\bar{Z})^T$ $=a\nabla_X Y+b\nabla_X Z$ ③ $\nabla_X(fY)=(\bar{\nabla}_{\bar{X}}(f\bar{Y}))^T=(f\bar{\nabla}_{\bar{X}}\bar{Y}+\bar{X}(f)\bar{Y})^T=f\nabla_X Y+X(f)Y$

Now, define $\mathrm{II}(X,Y)=\bar{\nabla}_{\bar{X}}\bar{Y}-\nabla_X Y$ normal to M, called second fundmental form.

Check that it doesn't depend ~~of~~ on extensions of X,Y: If $\bar{X}_1,\bar{X}_2$ are extensions of X.

$(\bar{\nabla}_{\bar{X}_1}\bar{Y}-\nabla_X Y)-(\bar{\nabla}_{\bar{X}_2}\bar{Y}-\nabla_X Y)=\bar{\nabla}_{(\bar{X}_1-\bar{X}_2)}\bar{Y}=0$ $\quad$ ($\bar{X}_1-\bar{X}_2$ vanishes on M)

Similarly, $\bar{\nabla}_{\bar{X}}(\bar{Y}_1-\bar{Y}_2)=0$ on M.

It is symmetric and bilinear: $\mathrm{II}(fX,Y)=f\mathrm{II}(X,Y)$, $\mathrm{II}(X,fY)=f\bar{\nabla}_{\bar{X}}\bar{Y}-f\nabla_X Y+\underbrace{\bar{X}(f)\bar{Y}-X(f)Y}_{0 \text{ on } M}=f\,\mathrm{II}(X,Y)$

$\mathrm{II}(X,Y)-\mathrm{II}(Y,X)=(\bar{\nabla}_{\bar{X}}\bar{Y}-\bar{\nabla}_{\bar{Y}}\bar{X})-(\nabla_X Y-\nabla_Y X)=[\bar{X},\bar{Y}]-[X,Y]=0$ on M.

Example1: $\bar{M}=R^3$, M is a surface with dim 2. $M=r(u,v)$

$X=\partial_u, Y=\partial_v$ $\because \mathrm{II}(X,Y)=(\bar{\nabla}_X Y)^{\perp}$ In R^3, $\Gamma_{ij}^k=0$ $\bar{\nabla}_X Y=X(Y)$

$\therefore \bar{\nabla}_{\partial_u}\partial_v=\partial_u(\partial_v)=\dfrac{\partial^2 r}{\partial u\partial v}$, $\bar{\nabla}_{\partial_u}\partial_u=\dfrac{\partial^2 r}{\partial u^2}$, $\bar{\nabla}_{\partial_v}\partial_v=\dfrac{\partial^2 r}{\partial v^2}$

$\mathrm{II}(\partial_u,\partial_v)=\langle\dfrac{\partial^2 r}{\partial u\partial v},\vec{n}\rangle\vec{n}$, $\mathrm{II}(\partial_u,\partial_u)=\langle\dfrac{\partial^2 r}{\partial u^2},\vec{n}\rangle\vec{n}$, $\mathrm{II}(\partial_v,\partial_v)=\langle\dfrac{\partial^2 r}{\partial v^2},\vec{n}\rangle=\vec{n}$

Let $\langle\dfrac{\partial^2 r}{\partial u\partial v},\vec{n}\rangle=\langle\dfrac{\partial^2 r}{\partial v\partial u},\vec{n}\rangle=M$, $\langle\dfrac{\partial^2 r}{\partial u^2},\vec{n}\rangle=L$, $\langle\dfrac{\partial^2 r}{\partial v^2},\vec{n}\rangle=N$.

$\mathrm{II}=Ldu^2+2Mdudv+Ndv^2$

Def: Let ν be normal vector of M, let $\mathrm{II}_\nu(X,Y)=\langle\nu,\mathrm{II}(X,Y)\rangle$ quadratic form.
called second fundamental form of f along ν.

Since $\mathrm{II}(X,Y)=\bar{\nabla}_{\bar{X}}\bar{Y}-\nabla_X Y$, $\mathrm{II}_\nu(X,Y)=\langle\nu,\bar{\nabla}_{\bar{X}}\bar{Y}\rangle=-\langle\bar{\nabla}_{\bar{X}}\nu,\bar{Y}\rangle$ (Weingarton equation)

Example2: Let $f:\bar{M}\to R$ be smooth function, c is regular value of f, let $M=f^{-1}(c)$ be a hypersurface of $\bar{M}$, let $\nu=\mathrm{grad}\, f$, ν is normal vector of M, for any tangental vector field X,Y, there are $\mathrm{II}_\nu(X,Y)=\langle\nu,\bar{\nabla}_X Y\rangle=\langle\mathrm{grad}\, f,\bar{\nabla}_X Y\rangle=(\bar{\nabla}_X Y)(f)=\underbrace{-YXf}_{\overset{\shortparallel}{0}\text{ on isosurface.}}+\bar{\nabla}_X Yf=-\bar{\nabla}^2 f(X,Y)$

(Hessian: $\nabla^2 f(X,Y)=\nabla_Y(df)(X)=Y(df(X))-df(\nabla_Y X)=YXf-\nabla_Y Xf$)
f is a function, $df=\nabla f$

(等值面的第二基本形式)

Def: $x\in T_pM$, $\eta\in(T_pM)^{\perp}$, ν is a local extension of η normal to M. Then, let $S_\eta(x)=-(\bar{\nabla}_x\nu)^T$
T_pM

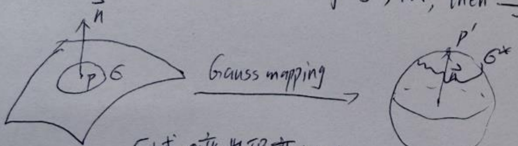
$\langle S_\eta(x),y\rangle=\langle-\bar{\nabla}_x\nu,y\rangle=-\langle Y,\bar{\nabla}_X\nu\rangle(p)=\langle\bar{\nabla}_X Y,N\rangle(p)=\langle\bar{\nabla}_X Y-\nabla_X Y,N\rangle(p)=\langle\mathrm{II}(X,Y)(p),N\rangle$

X,Y are local extensions of x,y. (S_η 称为关于法向η的形状算子)

If $\dim M=\dim\bar{M}-1$, $|\eta|=1$, the eigenvalues of S_η are called principal curvature, $\det S_\eta$ is called Gauss-Kronecker curvature.

Example3: Let surface $M\subset R^3$, orientable, $\exists$ idential normal vector field on M, Let it be η, let Γ: $M\to S^2$, $\Gamma(x)=\eta(x)$, called Gauss mapping, $d\Gamma(X)=S_\eta(X)$ $\forall X\in T_xM$, $x\in M$.

Let Ω_0,Ω be valume element of S^2, M, then $\dfrac{\Gamma^*\Omega_0}{\Omega}=\det d\Gamma=$ Gauss curvature.



$|$高斯曲率$|=\lim\limits_{\sigma\to p}\dfrac{\sigma^*\text{面积}}{\sigma\text{面积}}$

区域σ弯曲程度越大, σ^*也越大

Relation ~~to~~ among curvature of $M, \bar{M}$ and $\mathrm{I\!I}$

If $x, y \in T_pM \subset T_p\bar{M}$ are linearly independent, let sectional curvatures of $M, \bar{M}$ be $K(x,y), \bar{K}(x,y)$ in plane generated by x, y.

Thm (Gauss): $p \in M$, x, y are orthonormal vectors in T_pM, then:

$$K(x,y) - \bar{K}(x,y) = \langle \mathrm{I\!I}(x,x), \mathrm{I\!I}(y,y) \rangle - |\mathrm{I\!I}(x,y)|^2$$

Pf: Let X, Y be local extensions of x, y, tangant to M; $\bar{X}, \bar{Y}$ are local extensions to $\bar{M}$ of X, Y.

Then,

$$K(x,y) - \bar{K}(x,y) = \langle \nabla_Y \nabla_X X - \nabla_X \nabla_Y X - (\bar{\nabla}_{\bar{Y}} \bar{\nabla}_{\bar{X}} \bar{X} - \bar{\nabla}_{\bar{X}} \bar{\nabla}_{\bar{Y}} \bar{X}), Y \rangle (p) + \underbrace{\langle \nabla_{[X,Y]} X - \bar{\nabla}_{[\bar{X},\bar{Y}]} \bar{X}, Y \rangle (p)}_{\parallel}$$

$$- \langle (\bar{\nabla}_{[\bar{X},\bar{Y}]} \bar{X})^N, Y \rangle (p) = 0.$$

Denote $E_1, \cdots, E_m$ ($m = \dim \bar{M} - \dim M$) be local orthonormal fields which are normal to M.

Then, $\mathrm{I\!I}(X,Y) = \sum_i H_i(X,Y) E_i$, $H_i = \langle E_i, \mathrm{I\!I} \rangle$

At p, $\bar{\nabla}_{\bar{Y}} \bar{\nabla}_{\bar{X}} \bar{X} = \bar{\nabla}_{\bar{Y}} (\sum_i H_i(X,X) E_i + \nabla_X X) = \sum_i \{ H_i(X,X) \bar{\nabla}_{\bar{Y}} E_i + \bar{Y} H_i(X,X) E_i \} + \bar{\nabla}_{\bar{Y}} \nabla_X X$

Then $\langle \bar{\nabla}_{\bar{Y}} \bar{\nabla}_{\bar{X}} \bar{X}, Y \rangle = \sum_i [H_i(X,X) \langle \bar{\nabla}_{\bar{Y}} E_i, Y \rangle + \bar{Y}(H_i(X,X)) \underset{0}{\underset{\parallel}{\langle E_i, Y \rangle}}] + \langle \bar{\nabla}_{\bar{Y}} \nabla_X X, Y \rangle$

$$-\langle \bar{\nabla}_{\bar{Y}} Y, E_i \rangle = -\langle \bar{\nabla}_{\bar{Y}} Y - \nabla_Y Y, E_i \rangle = -H_i(Y,Y)$$

$$= -\sum_i H_i(X,X) H_i(Y,Y) + \langle \nabla_Y \nabla_X X, Y \rangle$$

$$\langle \bar{\nabla}_{\bar{X}} \bar{\nabla}_{\bar{Y}} \bar{X}, Y \rangle = -\sum_i H_i(X,Y) H_i(X,Y) + \langle \nabla_X \nabla_Y X, Y \rangle$$

$\therefore K(x,y) - \bar{K}(x,y) = \sum_i H_i(X,X) H_i(Y,Y) - \sum_i H_i(X,Y) H_i(X,Y) = \langle \mathrm{I\!I}(x,x), \mathrm{I\!I}(y,y) \rangle - |\mathrm{I\!I}(x,y)|^2$

Def: (Totally geodesic submanifold) An immersion $f: M \to \bar{M}$ is called geodesic at $p \in M$, if for every $\eta \in (T_pM)^\perp$, the second fundamental form is identially zero at p. An immersion f is called totally geodesic if it is geodesic for all $p \in M$. And we call M a totally geodesic submanifold of $\bar{M}$

Pro: An immersion $f: M \to \bar{M}$ is geodesic at $p \in M$ iff every geodesic γ of M starting from p is geodesic at p.

Pf: Let $\gamma(0) = p$, $\gamma'(0) = x$, N be a local extension of η normal to M at p. X is a local extension of $\gamma'(t)$ to a tangent field on M. $\langle X, N \rangle = 0$

$\mathrm{I\!I}_\eta(x,x) = -\langle \bar{\nabla}_X N, X \rangle = -X \underset{0}{\underset{\parallel}{\langle N, X \rangle}} + \langle N, \bar{\nabla}_X X \rangle = \langle N, \bar{\nabla}_X X \rangle$

$\therefore$ If $\bar{\nabla}_X X$ has no normal part at p, than f is geodesic at p.

$\therefore$ f is ~~geo desic~~ geodesic at p iff $\forall \gamma$ geodesic of M starting from p is geodesic of $\bar{M}$ at p.

$\square$

Geodesic cure is 1-dim totally geodesic submanifold. In most cases, totally geodesic submanifolds do not exist in general dimensions.

However: Let $\varphi: \bar{M} \to \bar{M}$ be isometric isomorphism, $M = \{p \in \bar{M} \mid \varphi(p) = p\}$. (不动点.)

Then every connected component is totally geodesic.

Def: Let M^n be submanifold of $\bar{M}^{n+p}$, $\{e_i\}^n$ is local orthnormal frame of M, $\{e_\alpha\}_{n+1}^{n+p}$ ~~be~~ is local orthrnormal frame of normal bundle.

$\langle \mathrm{II}(e_j, e_i), e_\alpha \rangle = \mathrm{II}_{e_\alpha}(e_j, e_i) \doteq h_{ji}^{\alpha}$. Let $H = \frac{1}{n} tr \mathrm{II} = \frac{1}{n} \sum_\alpha (\sum_{i=1}^n h_{ii}^\alpha) e_\alpha$, called average curvature vector. $H_\nu = \frac{1}{n} tr \mathrm{II}_\nu$ called average curvature of ν-direction.

If $H \equiv 0$, $\forall p \in M$, we call M minimal submanifold.

Example: $S^n(c) = \{(x^1)^2 + \cdots + (x^{n+1})^2 = c^2\}$ $\quad \nu = c^{-1} \vec{x}$.

X tangental, $\bar{\nabla}_X \nu = c^{-1} \bar{\nabla}_X x = c^{-1} X$. $\quad \mathrm{II}_\nu(X, Y) = -\langle \bar{\nabla}_X \nu, Y \rangle = -c^{-1} \langle X, Y \rangle$

$H_\nu = -c^{-1}$

(取单位内法向量, $c^{-1} = H_\nu$)

Example.

Let $f(x^1,\cdots,x^n)$ smooth. $x^{n+1}=f(x^1,\cdots,x^n)$ is a hypersurface in R^{n+1}

第一基本形式：

$$g=\sum_{i=1}^{n} dx^i\otimes dx^i+(\partial_i f dx^i)\otimes(\partial_j f dx^j)=g_{ij}\,dx^i\otimes dx^j \qquad g_{ij}=\delta_{ij}+\partial_i f\,\partial_j f$$

对应逆矩阵：$g^{ij}=\delta_{ij}-\dfrac{\partial_i f\,\partial_j f}{A^2}$ $\qquad A=\sqrt{1+|\nabla f|^2}$

法向量方向：$(\partial_1 f,\cdots,\partial_n f,-1)$，~~单位化~~ 单位化：$\nu=\frac{1}{A}\left(\sum_{i=1}^{n}\partial_i f\frac{\partial}{\partial x^i}-\frac{\partial}{\partial x^{n+1}}\right)$

在 R^{n+1} 中：

$$\overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu=\frac{\partial \nu}{\partial x^i}$$

$$\boxed{\frac{\partial}{\partial x^i}\left(\frac{1}{A}\right)=-\frac{\partial_i A}{A^2} \qquad \partial_i A=\frac{1}{2}\frac{2\sum_j\partial_j f\,\partial_i\partial_j f}{A}=\frac{\sum_j \partial_j f\,\partial_i\partial_j f}{A}}$$

$$=\frac{\partial}{\partial x^i}\left(A^{-1}\cdot\left(\partial_j f\frac{\partial}{\partial x^j}-\frac{\partial}{\partial x^{n+1}}\right)\right)$$

$$=\frac{\partial A^{-1}}{\partial x^i}\left(\partial_j f\frac{\partial}{\partial x^j}-\frac{\partial}{\partial x^{n+1}}\right)+A^{-1}\frac{\partial}{\partial x_i}\left(\partial_j f\frac{\partial}{\partial x^j}-\frac{\partial}{\partial x^{n+1}}\right)$$

$$=\underbrace{-\frac{\partial_i A}{A}\nu}_{\text{法向}}+\underbrace{A^{-1}\partial_i\partial_j f\frac{\partial}{\partial x^j}}_{\text{切向}}$$

$\therefore$ From Weingarten equation, $\mathrm{II}_\nu\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)=-\left\langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu,\frac{\partial}{\partial x^j}\right\rangle=-A^{-1}\partial_i\partial_j f$

$\therefore \mathrm{II}_\nu=-\frac{1}{A}\partial_i\partial_j f\,dx^i\otimes dx^j$

From $H_\nu=\frac{1}{n}\mathrm{tr}_g\mathrm{II}_\nu=\frac{1}{n}g^{ij}\mathrm{II}_{\nu ij}=-\frac{1}{n}g^{ij}\frac{1}{A}\partial_i\partial_j f=-\frac{1}{n}\partial_i(A^{-1}\partial_i f)$

when $n=2$ $\quad \partial_1\left(\frac{\partial_1 f}{A}\right)+\partial_2\left(\frac{\partial_2 f}{A}\right)=\frac{\partial_1^2 f+\partial_2^2 f}{A}-\frac{(\partial_1 f)^2\partial_1^2 f+(\partial_2 f)^2\partial_2^2 f+2\partial_1 f\,\partial_2 f\,\partial_1\partial_2 f}{A^3}.$

$A^2=1+(\partial_1 f)^2+(\partial_2 f)^2$

$$-\frac{1}{2}\left[\partial_1\left(\frac{\partial_1 f}{A}\right)+\partial_2\left(\frac{\partial_2 f}{A}\right)\right]=\frac{1}{2A}\left[\partial_1^2 f(1+(\partial_2 f)^2)+\partial_2^2 f(1+(\partial_1 f)^2)-2\partial_1\partial_2 f(\partial_1 f\,\partial_2 f)\right]$$

$\underset{E}{\underset{\parallel}{g_{11}}}=1+(\partial_1 f)^2 \quad \underset{F}{\underset{\parallel}{g_{12}}}=\partial_1 f\,\partial_2 f \quad \underset{G}{\underset{\parallel}{g_{22}}}=1+(\partial_2 f)^2$

$r(x^1,x^2)=(x^1,x^2,f(x^1,x^2))$. $r_1=(1,0,\partial_1 f)$, $r_2=(0,1,\partial_2 f)$ $\quad \nu=\frac{(\partial_1 f,\partial_2 f,1)}{A}.$

$\therefore L=\langle r_{11},\nu\rangle=\frac{\partial_1^2 f}{A}$, $\quad M=\frac{\partial_1\partial_2 f}{A}$ $\quad N=\frac{\partial_2^2 f}{A}$

$\therefore H_\nu=\dfrac{LG-2MF+NE}{2(EG-F^2)}$

二、The fundamental equations.

Give an isometric immersion $f: M^n \to \bar{M}^{n+m}$, $\forall p$, $T_p\bar{M} = T_pM \oplus (T_pM)^\perp$

Let X, Y, Z be vector fields tangent to M, ξ, η, ζ be vector fields normal to M.

$T\bar{M}$ can split into $TM \oplus TM^\perp$ (normal bundle)

$$\nabla_X^\perp \eta = (\bar{\nabla}_X \eta)^N = \bar{\nabla}_X \eta - (\bar{\nabla}_X \eta)^T = \bar{\nabla}_X \eta + S_\eta(X)$$

$$R^\perp(X,Y)\eta = \nabla_Y^\perp \nabla_X^\perp \eta - \nabla_X^\perp \nabla_Y^\perp \eta + \nabla_{[X,Y]}^\perp \eta \quad \text{(normal curvature)}$$

$$\begin{aligned}
\bar{R}(X,Y)Z &= \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_X \bar{\nabla}_Y Z + \bar{\nabla}_{[X,Y]} Z \\
&= \bar{\nabla}_Y(\nabla_X Z + \mathrm{II}(X,Z)) - \bar{\nabla}_X(\nabla_Y Z + \mathrm{II}(Y,Z)) + \nabla_{[X,Y]} Z + \mathrm{II}([X,Y],Z) \\
&= \nabla_Y \nabla_X Z + \mathrm{II}(Y,\nabla_X Z) + \bar{\nabla}_Y \mathrm{II}(X,Z) - \nabla_X \nabla_Y Z - \mathrm{II}(X,\nabla_Y Z) - \bar{\nabla}_X(\mathrm{II}(Y,Z)) + \nabla_{[X,Y]} Z + \mathrm{II}([X,Y],Z) \\
&= R(X,Y)Z + \mathrm{II}(Y,\nabla_X Z) + \nabla_Y^\perp \mathrm{II}(X,Z) - S_{\mathrm{II}(X,Z)} Y - \mathrm{II}(X,\nabla_Y Z) - \nabla_X^\perp \mathrm{II}(Y,Z) + S_{\mathrm{II}(Y,Z)} X + \mathrm{II}([X,Y],Z)
\end{aligned}$$

Consider tangent direction:

For any W tangent to M:

$$\begin{aligned}
\langle \bar{R}(X,Y)Z, W \rangle &= \langle R(X,Y)Z, W \rangle - \langle S_{\mathrm{II}(X,Z)} Y, W \rangle + \langle S_{\mathrm{II}(Y,Z)} X, W \rangle \\
&= \langle R(X,Y)Z, W \rangle - \langle \mathrm{II}(Y,W), \mathrm{II}(X,Z) \rangle + \langle \mathrm{II}(X,W), \mathrm{II}(Y,Z) \rangle \qquad (\langle S_\eta(X), Y \rangle = \langle \mathrm{II}(X,Y), \eta \rangle)
\end{aligned}$$

We call $\bar{R}(X,Y,Z,W) = R(X,Y,Z,W) - \langle \mathrm{II}(X,Z), \mathrm{II}(Y,W) \rangle + \langle \mathrm{II}(X,W), \mathrm{II}(Y,Z) \rangle$ Gauss equation.

Consider normal direction:

when M is hypersurface.

$$\langle \bar{R}(X,Y)Z, \eta \rangle = \langle \mathrm{II}(Y,\nabla_X Z), \eta \rangle + \langle \nabla_Y^\perp \mathrm{II}(X,Z), \eta \rangle - \langle \mathrm{II}(X,\nabla_Y Z), \eta \rangle - \langle \nabla_X^\perp \mathrm{II}(Y,Z), \eta \rangle + \underbrace{\langle \mathrm{II}([X,Y],Z), \eta \rangle}_{= \langle \mathrm{II}(\nabla_X Y, Z), \eta \rangle - \langle \mathrm{II}(\nabla_Y X, Z), \eta \rangle}$$

Def: $\mathrm{II}(X,Y,\eta) = \langle \mathrm{II}(X,Y), \eta \rangle$

Def:
$$\begin{aligned}
(\bar{\nabla}_X \mathrm{II})(Y,Z,\eta) &= X(\mathrm{II}(Y,Z,\eta)) - \mathrm{II}(\nabla_X Y, Z, \eta) - \mathrm{II}(Y, \nabla_X Z, \eta) - \mathrm{II}(Y,Z,\nabla_X^\perp \eta) \\
&= X\langle \mathrm{II}(Y,Z), \eta \rangle - \langle \mathrm{II}(\nabla_X Y, Z), \eta \rangle - \langle \mathrm{II}(Y,\nabla_X Z), \eta \rangle - \langle \mathrm{II}(Y,Z), \nabla_X^\perp \eta \rangle \\
&= \langle \nabla_X^\perp \mathrm{II}(Y,Z), \eta \rangle + \langle \mathrm{II}(Y,Z), \nabla_X^\perp \eta \rangle - \langle \mathrm{II}(\nabla_X Y, Z), \eta \rangle - \langle \mathrm{II}(Y,\nabla_X Z), \eta \rangle - \langle \mathrm{II}(Y,Z), \nabla_X^\perp \eta \rangle \\
&= \langle \nabla_X^\perp \mathrm{II}(Y,Z), \eta \rangle - \langle \mathrm{II}(\nabla_X Y, Z), \eta \rangle - \langle \mathrm{II}(Y,\nabla_X Z), \eta \rangle
\end{aligned}$$

$\therefore$ $\bar{R}(X,Y,Z,\eta) = (\bar{\nabla}_Y \mathrm{II})(X,Z,\eta) - (\bar{\nabla}_X \mathrm{II})(Y,Z,\eta)$

We call it Codazzi's equation.

When $\bar{M}$ is a manifold with constant curvature, $(\bar{\nabla}_Y \mathrm{II})(X,Z,\eta) = (\bar{\nabla}_X \mathrm{II})(Y,Z,\eta)$

$$\begin{aligned}
\bar{R}(X,Y)\eta &= \bar{\nabla}_Y \bar{\nabla}_X \eta - \bar{\nabla}_X \bar{\nabla}_Y \eta + \bar{\nabla}_{[X,Y]} \eta = \underbrace{\bar{\nabla}_Y(\nabla_X^\perp \eta - S_\eta X)}_{\textcircled{1}} - \underbrace{\bar{\nabla}_X(\nabla_Y^\perp \eta - S_\eta Y)}_{\textcircled{2}} + \nabla_{[X,Y]}^\perp \eta - S_\eta[X,Y] \\
&= \underbrace{\nabla_Y^\perp \nabla_X^\perp \eta - S_{\nabla_X^\perp \eta} Y - (\nabla_Y S_\eta X + \mathrm{II}(S_\eta X, Y))}_{\textcircled{1}} - \underbrace{\left[\nabla_X^\perp \nabla_Y^\perp \eta - S_{\nabla_Y^\perp \eta} X - (\nabla_X S_\eta Y + \mathrm{II}(S_\eta Y, X))\right]}_{\textcircled{2}} + \nabla_{[X,Y]}^\perp \eta - S_\eta[X,Y] \\
&= R^\perp(X,Y)\eta - S_{\nabla_X^\perp \eta} Y - \nabla_Y S_\eta X - \mathrm{II}(S_\eta X, Y) + S_{\nabla_Y^\perp \eta} X + \nabla_X S_\eta Y + \mathrm{II}(S_\eta Y, X) - S_\eta[X,Y]
\end{aligned}$$

$\langle \bar{R}(X,Y)\eta, \xi \rangle = \langle R^\perp(X,Y)\eta, \xi \rangle - \langle \mathrm{II}(S_\eta X, Y), \xi \rangle + \langle \mathrm{II}(S_\eta Y, X), \xi \rangle$ (切丛上的部分与ξ内积为0)

$(\langle S_\eta(X), Y \rangle = \langle \mathrm{II}(X,Y)(p), \eta \rangle)$

$$\begin{aligned}
&= \langle R^\perp(X,Y)\eta, \xi \rangle \cancel{- \langle S_\xi S_\eta X, Y \rangle + \langle S_\eta S_\xi X, Y \rangle} + \langle (S_\eta S_\xi - S_\xi S_\eta) X, Y \rangle \\
&= \langle R^\perp(X,Y)\eta, \xi \rangle + \langle [S_\eta, S_\xi] X, Y \rangle
\end{aligned}$$

$\bar{R}(X,Y,\eta,\xi) = R^\perp(X,Y,\eta,\xi) + \langle [S_\eta, S_\xi] X, Y \rangle$ called Ricci's equation