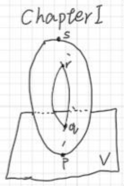
Chapter I Introduction



$f: M \to R$ height above V

$M^a = \{x \in M \mid f(x) < a\}$

(1) $a < 0 = f(p)$ $M^a = \emptyset$

(2) $f(p) < a < f(q)$ M^a is homeomorphic to a 2-cell 

Def: (n-cell) $e^n = \{x \in R^n : \|x\| \leq 1\}$

(3) $f(q) < a < f(r)$, M^a is homeomorphic to a cylinder

(4) $f(r) < a < f(s)$, M^a is homeomorphic to a cpt manifold of genus one, boundary: a circle

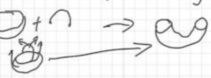
(5) $f(s) < a$, M^a is the full torus

homeomorphism type ↑

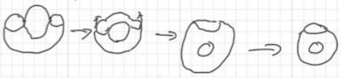
homotopy type ↓

$1 \to 2$ $\emptyset \to$ 2-cell $\simeq \cdot$ 0-cell

$2 \to 3$ operation of attaching a 1-cell



$3 \to 4$ operation of attaching a 1-cell



$4 \to 5$ operation of attaching a 2-cell

Def: (Attach a k-cell) Let Y be a topological space let e^k be the k-cell. $\dot{e}^k$ is the boundary (S^{k-1})
If $g: S^{k-1} \to Y$ continuous, then $Y \cup_g e^k$ is obtained by first taking the topological sum of Y and e^k then identifying each $x \in S^{k-1}$ with $g(x) \in Y$ 

$k=0$ e^0 is a point, $\dot{e}^0 = S^{-1} = \emptyset$ $Y \cup \{\text{disjoint point}\}$

critical point of f (x,y) $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$

At p choose (x,y) s.t. $f = x^2 + y^2$, then $f = c - x^2 - y^2$ at s.
At q, r. $f = c + x^2 - y^2$

Number of "$-$" $\sim$ dim of cell attached

$\Downarrow$

Chapter II Definitions and lemmas.

Def: A point $p \in M$ is called a critical point of f if $f_*: T_pM \to T_{f(p)}R$ is zero, if we choose a local coordinate system $(x^1, \cdots, x^n)$ in neighborhood U of p, then

$$\frac{\partial f}{\partial x^1} = \cdots = \frac{\partial f}{\partial x^n} = 0$$

$f(p) \in R$ is called a critical value of f.

what's more: A critical point is non-degenerate iff matrix $\left(\frac{\partial^2 f}{\partial x^i \partial x^j}(p)\right)$ is non-singular

Def: We define a symmetric bilinear functional f_{**} on T_pM (p is a critical point) called the Hessian of f at p.
If $v, w \in T_pM$, then they have extensions $\tilde{v}$ and $\tilde{w}$ to vector fields ($\tilde{v}_p = v$, $\tilde{w}_p = w$). Then let $f_{**}(v,w) = \tilde{v}_p(\tilde{w}(f))$
($\tilde{w}(f)$ directional derivate)

Symmetric: $\tilde{v}_p(\tilde{w}(f)) - \tilde{w}_p(\tilde{v}(f)) = [\tilde{v}, \tilde{w}]_p(f) = 0$

$\tilde{v}_p(\tilde{w}(f)) = v_p(\tilde{w}(f)) \to$ well-defined.

With a local coordinate system $(x_1, \cdots, x_n)$

$v = \sum a_i \frac{\partial}{\partial x^i}\Big|_p$, $w = \sum b_j \frac{\partial}{\partial x^j}\Big|_p$, take $\tilde{w} = \sum b_j \frac{\partial}{\partial x^j}\Big|_p$ $b_j \equiv$ const.

then $f_{**}(v,w) = v(\tilde{w}(f))(p) = v\left(\sum b_j \frac{\partial}{\partial x^j}\Big|_p\right) = \sum_{i,j} a_i b_j \frac{\partial^2 f}{\partial x^i \partial x^j}$

Matrix $\left(\frac{\partial^2 f}{\partial x^i \partial x^j}(p)\right) \Rightarrow f_{**}$ basis $\left\{\frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p\right\}$

Def: The index of a bilinear functional H on a vector space V is defined to be the maximal dim of a subspace of V s.t. H is ~~non~~ negative definite on it.

The nullity is dim of the null-space
($\text{All } v \in V \text{ s.t. } w \in V \cdot H(v,w)=0$)

Claim: A point p is a non-degenerate critical point of f iff f_{**} has nullity $= 0$

$$\left(\left(\frac{\partial^2 f}{\partial x^i \partial x^j}(p) \right) \text{ non-singular} \right)$$

index $\Rightarrow$ behavior of f at p.

Lemma 2.1: Let f be a C^∞ function in a convex neighborhood V of 0 in $\mathbb{R}^n$, $f(0)=0$. Then:

$$f(x_1,\cdots,x_n) = \sum_{i=1}^{n} x_i g_i(x_1,\cdots,x_n) \text{ for some } C^\infty\text{-fun } g_i$$

$$\left(g_i(0) = \frac{\partial f}{\partial x_i}\Big|_0 \right)$$

Pf: $f(x_1,\cdots,x_n) = \int_0^1 \frac{df(tx_1,\cdots,tx_n)}{dt} dt = \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(tx_1,\cdots,tx_n) x_i \, dt$

$= \sum_{i=1}^{n} x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx_1,\cdots,tx_n) dt.$

$\therefore g_i(x_1,\cdots,x_n) = \int_0^1 \frac{\partial f}{\partial x_i}(tx_1,\cdots,tx_n) dt$

$g_i(0) = \int_0^1 \frac{\partial f}{\partial x_i}(0) = \frac{\partial f}{\partial x_i}(0)$ ✓

Lemma 2.2. (Lemma of Morse) Let p be a non-degenerate critical point for f. Then there is a local coordinate system $(y^1,\cdots,y^n)$ in a neighborhood U of p with $y^i(p)=0 \ \forall i$ s.t.

$$f = f(p) - (y^1)^2 - (y^2)^2 - \cdots - (y^\lambda)^2 + (y^{\lambda+1})^2 + \cdots + (y^n)^2$$

holds throughout U. λ is the index of f at p.

Pf: Firstly, if such expression for f exists, then λ must be the index of f at p.

Pf: For any $(z^1,\cdots,z^n)$, if $f(q) = f(p) - (z^1(q))^2 - \cdots - (z^\lambda(q))^2 + (z^{\lambda+1}(q))^2 + \cdots + (z^n(q))^2$

then $\frac{\partial^2 f}{\partial z^i \partial z^j}(p) = \begin{cases} i=j\leq\lambda & (-2z^i)' = -2 \\ i=j>\lambda & (2z^i)' = 2 \\ 0 & \end{cases}$

$\therefore$ matrix f_{**}: basis $\left\{ \frac{\partial}{\partial z^1}\Big|_p, \cdots, \frac{\partial}{\partial z^n}\Big|_p \right\}$

$$\begin{bmatrix} \overbrace{-2 \quad \ddots \quad -2}^{\lambda} & & & 0 \\ & & & \\ 0 & & 2 & \ddots \quad 2 \end{bmatrix}$$

$\therefore \exists$ subspace of T_pM of dim λ where f_{**} is negative definite

And a subspace of dim $n-\lambda$ f_{**} is positive definite, call it V.

If there were a subspace of dim $>\lambda$, on which f_{**} is negative, then it must intersect V, $\times$

$\Rightarrow \lambda$ is the index of f_{**}

Secondly, a suitable coordinate system $(y^1,\cdots,y^n)$ exists:

Pf: WLOG, let p be the origin. $f(p)=f(0)=0$.

By lemma 2.1. $f(x_1,\cdots,x_n) = \sum_{j=1}^{n} x_j g_j(x_1,\cdots,x_n)$

$g_j(0) = \frac{\partial f}{\partial x_j}(0) = 0$

Use lemma 2.1 again, $g_j(x_1,\cdots,x_n) = \sum_{i=1}^{n} x_i h_{ij}(x_1,\cdots,x_n)$.

for certain smooth h_{ij}.

$\therefore f(x_1,\cdots,x_n) = \sum_{i,j=1}^{n} x_i x_j h_{ij}(x_1,\cdots,x_n)$

Assume $h_{ij} = h_{ji}$, if not, $\overline{h_{ij}} = \frac{1}{2}(h_{ij}+h_{ji})$ replace it.

To show exists coordinates $u_1,\cdots,u_n$ in a neighorhood of 0 s.t. f is in desired expression

Use induction: $\underbrace{f = \pm(u_1)^2 \pm \cdots \pm (u_{r-1})^2 + \sum_{i,j\geq r} u_i u_j H_{ij}(u_1,\cdots,u_n)}_{\text{in } U_1}$

Assume $H_{rr} \neq 0$, let $g(u_1,\cdots,u_n) = \sqrt{|H_{rr}|}$ in $U_2 \subset U_1$

Introduce $v_1,\cdots,v_n$ by $v_i = u_i \quad i \neq r$

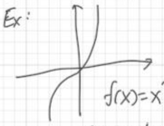
$$v_r(u_1,\cdots,u_n) = g(u_1,\cdots,u_n)\left(u_r + \sum_{i>r} u_i \frac{H_{ir}}{H_{rr}}\right)$$

$$f = \sum_{i\leq r} \pm(v_i)^2 + \sum_{i,j>r} v_i v_j H'_{ij}(v_1,\cdots,v_n)$$

$$H_{rr} = \frac{H_{rr}}{H_{rr}} \cdot g^2$$

Cor 2.3. Non-degenerate critical points are isolated.

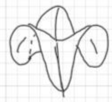
Ex:



$f(x)=x^3$

$(0,0)$: degenerate critical point

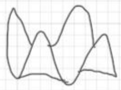
$f(x,y)=x^3-3xy^2$



monkey saddle

$(0,0)$: de — critical point

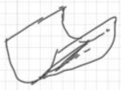
$f(x,y)=x^2y^2$



x-axis $\cup$ y-axis

de — critical point

$f(x,y)=x^2$



x-axis

de — critical point

Review: 1-parameter group of diffeomorphism $\varphi: \mathbb{R}\times M \to M$

s.t. (1) $\forall t \in \mathbb{R}$, $\varphi_t: \mathbb{R}\times M \to M$ $\varphi_t(q)=\varphi(t,q)$ is a diffeomorphism of M onto M.

(2) $\forall t,s \in \mathbb{R}$, $\varphi_{t+s}=\varphi_t\circ\varphi_s$

Give a 1-parameter group φ of diffeomorphisms of M, define a vector field of M: $\forall f \in C^\infty(\mathbb{R})$.

$X_q(f)=\lim_{h\to 0}\frac{f(\varphi_h(q))-f(q)}{h}$ X is said to generate the group φ.

Lemma 2.4. A smooth v.f. on M which vanishes outside of a cpt set $K\subset M$ generates a unique 1-parameter group of diffeomorphisms of M.

Pf: $\forall$ smooth curve $t\to c(t)\in M$ velocity vector $\frac{dc}{dt}\in T_{c(t)}M$

by $\frac{dc}{dt}(f)=\lim_{h\to 0}\frac{f_c(t+h)-f_c(t)}{h}$

Now, let φ be a 1-parameter group of diff— generated by v.f. X

Then for fixed q. curve $t\to\varphi_t(q)$ satisfies

$\frac{d\varphi_t(q)}{dt}=X_{\varphi_t(q)}$ with $\varphi_0(q)=q$

$$\frac{d\varphi_t(q)}{dt}(f)=\lim_{h\to 0}\frac{f(\varphi_{t+h}(q))-f(\varphi_t(q))}{h}=\lim_{\substack{h\to 0\\ p=\varphi_t(q)}}\frac{f(\varphi_h(p))-f(p)}{h}=X_p(f)$$

This ODE, for each point of M, $\exists$ neighborhood U, $\varepsilon>0$ s.t. $\frac{d\varphi_t(q)}{dt}=X_{\varphi_t(q)}$ $\varphi_0(q)=q$ has a unique smooth solnation for $q\in U$, $|t|<\varepsilon$.

Set K is cpt $\Rightarrow$ finite neighorhoods U cover it. Let ε_0 be the smallest ε.

(Setting $\varphi_t(q)=q$ for $q\notin K$, we get an unique soluation $\varphi_t(q)$ of that ODE for $|t|<\varepsilon_0$, and for all $q\in M$.

$\varphi_t(q)$ is smooth for t,q.

($\alpha(t)=\varphi_{t+s}(q)$, $\beta_t=\varphi_t(\varphi_s(q))$ $\alpha(0)=\beta(0)=\varphi_s(q)$.

$$\left.\begin{aligned}\frac{d\alpha(t)}{dt}&=\frac{d}{dt}\varphi_{t+s}(q)=X_{\varphi_{t+s}(q)}=X_{\alpha(t)}\\ \frac{d\beta(t)}{dt}&=\frac{d}{dt}\varphi_t(\varphi_s(q))=X_{\varphi_t(p)}=X_{\beta(t)}\end{aligned}\right]\to \alpha(t)=\beta(t)$$

$\Rightarrow \varphi_{t+s}=\varphi_t\circ\varphi_s$

φ_t smooth φ_t is diff—

$|t| \geq \varepsilon_0$, $t = k \cdot \frac{\varepsilon_0}{2} + r$, $k \geq 0$, $|r| < \frac{1}{2}\varepsilon_0$

Let $\varphi_t = \varphi_{\frac{\varepsilon_0}{2}} \circ \cdots \circ \varphi_{\frac{\varepsilon_0}{2}} \circ \varphi_r$.

$k < 0$, use $\varphi_{-\frac{1}{2}\varepsilon_0}$ ▨

Note: "X vanishes outside of a cpt set" cannot be omitted.

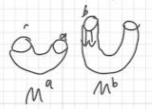
Ex: $M: (0,1) \subset \mathbb{R}$ $X = \frac{d}{dt}$

$$X_q f = \frac{df}{dt} = \lim_{t \to 0} \frac{f(\varphi_t(q)) - f(q)}{t} \Rightarrow \varphi_t(q) = t + q. \quad q \in (0,1)$$

Chapter III: Homotopy Type in Terms of Critical Values.

f is real fun on a manifold M, $M^a = f^{-1}(-\infty, a] = \{p \in M : f(p) \leq a\}$

Thm 3.1. Let f be a smooth, real valued fun on M, $a < b$, suppose $f^{-1}[a,b]$ is cpt, contains no critical points of f. Then: M^a is diffeomorphic to M^b. Further more, M^a is a deformation retract of M^b, so that the inclusion map $M^a \to M^b$ is a homotopy equivalence.



$\langle X, Y \rangle$ denotes inner product. gradf (v.f.) is defined by $\langle X, \text{grad} f \rangle = X(f) \Rightarrow \text{grad} f$ vanishes precisely at critical points of f. For curve c $\langle \frac{dc}{dt}, \text{grad} f \rangle = \frac{d(f \circ c)}{dt}$.

Let $\rho: M \to \mathbb{R}$ be $\frac{1}{\langle \text{grad} f, \text{grad} f \rangle}$ throughout the cpt set $f^{-1}[a,b]$, and vanishes outside the cpt neighborhood of the set, define X: $X_q = \rho(q)(\text{grad} f)_q$

Lemma 2.4 $\Rightarrow$ X generates a 1-parameter group of diff___ $\varphi_t: M \to M$.

For a fixed $q \in M$, consider fun $t \to f(\varphi_t(q))$. $\rho \, \text{grad} f$ $\frac{1}{\langle \text{grad} f, \text{grad} f \rangle}$

If $\varphi_t(q) \in f^{-1}[a,b]$, then $\frac{df(\varphi_t(q))}{dt} = \langle \frac{d\varphi_t(q)}{dt}, \text{grad} f \rangle = \langle X, \text{grad} f \rangle = +1$

Thus $t \to f(\varphi_t(q))$ is linear with derivative $+1$ if $f(\varphi_t(q)) \in [a,b]$.

$\therefore$ For $p \in M^a$, $f(p) < a$, $f(\varphi_{b-a}(p)) < a + (b-a) < b$. $\Rightarrow \varphi_{b-a}(M^a) \subset M^b$

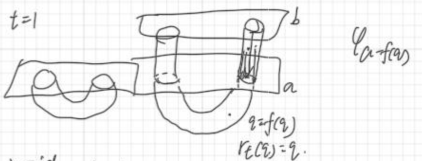
$$\left.\begin{array}{l} t \to f(\varphi_{-t}(q)) \quad \varphi_{a-b}(M^b) \subset M^a \Rightarrow \varphi_{b-a}(M^a) = M^b \\ \varphi_{b-a} \text{ diff} \end{array}\right\} \Rightarrow M^a \cong M^b$$

("Further more")

Define 1-parameter family of maps $r_t: M^b \to M^b$

$$r_t = \begin{cases} q & \text{if } f(q) \leq a \\ \varphi_{t(a-f(q))}(q) & \text{if } a \leq f(q) \leq b \end{cases}$$

$t = 1$

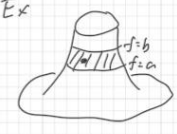


$r_0 = id$ r_1 is a retraction form M^b to M^a

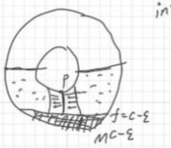
$\therefore M^a$ is a ~~retraction~~ deformation retract of M^b ▨

Note: The condition $f^{-1}[a,b]$ is cpt can not be omitted

Ex



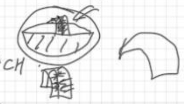
Thm 32. Let $f: M \to \mathbb{R}$ be a smooth fun, let p be a non-degenerate critical point with index λ. Set $f(p)=c$. Suppose that $f^{-1}[c-\varepsilon, c+\varepsilon]$ is cpt, and contains no other critical point. $(\varepsilon>0)$. Then, for small ε, the set $M^{c+\varepsilon}$ has homotopy type of $M^{c-\varepsilon}$ with a λ-cell attached



introduce $F: M \to \mathbb{R}$. $F=f$ others

$F<f$ in a small neighbor of p

s.t. $F^{-1}(-\infty, c-\varepsilon] = M^{c-\varepsilon} \cup H$



Choosing a cell $e^\lambda \subset H$

$M^{c-\varepsilon} \cup e^\lambda$ is a deformation retract of $M^{c-\varepsilon} \cup H$.

Then: prove $M^{c-\varepsilon} \cup H$ is a deformation retract of $M^{c+\varepsilon}$, then the pf complete

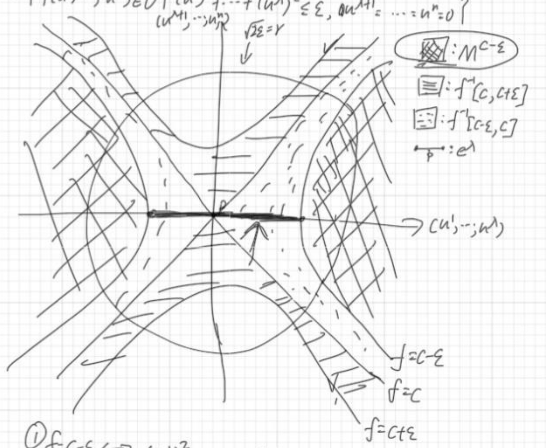
Pf: Choose a coordinate system $u^1, \cdots, u^n$ in a neighborhood U of p s.t.

$f = c - (u^1)^2 - \cdots - (u^\lambda)^2 + (u^{\lambda+1})^2 + \cdots + (u^n)^2$ holds throughout U, then $u^1(p) = \cdots = u^n(p) = 0$.

Choose $\varepsilon>0$ small enough s.t. (1) $f^{-1}[c-\varepsilon, c+\varepsilon]$ is cpt and contains no other critical point

(2) The image of U under the diffeomorphic embedding $(u^1, \cdots, u^n): U \to \mathbb{R}^n$ contains the ~~cot~~ closed ball $\{(u^1, \cdots, u^n) \mid \sum (u^i)^2 \leq 2\varepsilon\}$.

$e^\lambda: \{p(u^1, \cdots, u^n) \in U \mid (u^1)^2 + \cdots + (u^\lambda)^2 \leq \varepsilon,\ u^{\lambda+1} = \cdots = u^n = 0\}$



① $f = c-\varepsilon \iff (u^1)^2 + \cdots + (u^\lambda)^2 - (u^{\lambda+1})^2 - \cdots - (u^n)^2 = \varepsilon$

② $f = c \iff (u^1)^2 + \cdots + (u^\lambda)^2 = (u^{\lambda+1})^2 + \cdots + (u^n)^2$

③ $f = c+\varepsilon \iff (u^{\lambda+1})^2 + \cdots + (u^n)^2 - (u^1)^2 - \cdots - (u^\lambda)^2 = \varepsilon$

Since $e^\lambda \cap M^{c-\varepsilon}$ is $\dot{e}^\lambda$, e^λ is attached to $M^{c-\varepsilon}$

Then prove $M^{c-\varepsilon} \cup e^\lambda$ is a deformation retract of $M^{c+\varepsilon}$

Let a smooth fun $F: M \to \mathbb{R}$ Let $\mu: \mathbb{R} \to \mathbb{R}$ s.t. $\mu(0) > \varepsilon$

$\mu(r) = 0 \ \forall r \geq 2\varepsilon$

$-1 < \mu'(r) \leq 0 \ \forall r$

Let F conside with f outside U ✓

$F = f - \mu((u^1)^2 + \cdots + (u^\lambda)^2 + 2(u^{\lambda+1})^2 + \cdots + 2(u^n)^2)$ in U

$\mu(2\varepsilon) = 0$ f is a well-defined smooth fun on M

Def $\xi, \eta: U \to [0, +\infty)$ $\xi = (u^1)^2 + \cdots + (u^\lambda)^2$

$\eta = (u^{\lambda+1})^2 + \cdots + (u^n)^2$

$f = c - \xi + \eta$

Def $\xi, \eta : U \to [0, +\infty)$ $\quad \xi = (u^1)^2 + \cdots + (u^\lambda)^2$

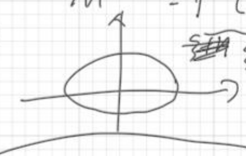
$\eta = (u^{\lambda+1})^2 + \cdots + (u^n)^2$

$f = c - \xi + \eta$

$F(q) = c - \xi(q) + \eta(q) - \mu(\xi(q) + 2\eta(q)) \quad \forall q \in U$

Assertion 3.1. The region $F^{-1}(-\infty, c+\varepsilon]$ coinsides with $M^{c+\varepsilon} = f^{-1}(-\infty, c+\varepsilon]$

Pf



$\xi + 2\eta = 2\varepsilon$

Outside $f = F$

Inside $\mu \geq 0$, $F \leq f = c - \xi + \eta \leq c + \frac{1}{2}\xi + \eta \leq c+\varepsilon$. ☐

Critical point $F^{-1}[c-\varepsilon, c+\varepsilon]$.

$\frac{\partial F}{\partial \xi} = -1 - \mu'(\xi + 2\eta) < 0 \qquad \frac{\partial F}{\partial \eta} = 1 - 2\mu'(\xi + 2\eta) \geq 1$

$\therefore dF = 0 \Leftrightarrow d\xi \; d\eta = 0 \Leftrightarrow$ origin.

$\therefore$ F has no critical point in U other than origin.

"Assertion 3.1" + "$F \leq f$" $\Rightarrow F^{-1}[c-\varepsilon, c+\varepsilon] \subset f^{-1}[c-\varepsilon, c+\varepsilon]$

$\Rightarrow$ cpt, possible critical point is p only.

$F(p) = c - \mu(0) < c - \varepsilon \Rightarrow F^{-1}[c-\varepsilon, c+\varepsilon]$ has no critical point.

Thm 3.1. ($F^{-1}[c-\varepsilon, c+\varepsilon]$ cpt + contains no critical point)

$\Rightarrow F^{-1}(-\infty, c-\varepsilon]$ is a deformation retract of $F^{-1}(-\infty, c+\varepsilon]$

+

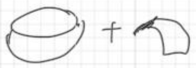
Assertion 3.1 ($F^{-1}(-\infty, c+\varepsilon] = M^{c+\varepsilon}$)

$\Downarrow$

$F^{-1}(-\infty, c-\varepsilon]$ is deformation retract of $M^{c+\varepsilon}$ (Assertion 3.3)

Let H denote the closure of $F^{-1}(-\infty, c-\varepsilon] - M^{c-\varepsilon}$.

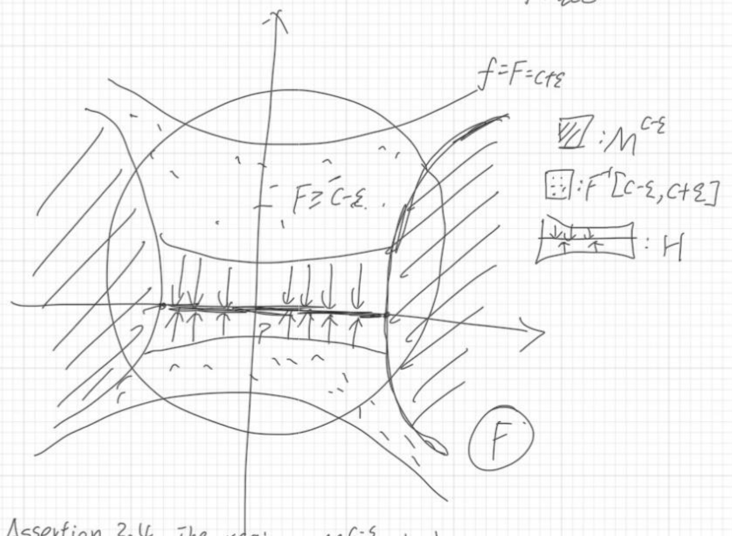
Then $F^{-1}(-\infty, c-\varepsilon] = M^{c-\varepsilon} \cup H$



Small handle

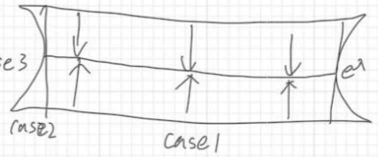
Now: handle $\xrightarrow{?}$ e^λ cell

Consider the cell e^λ ~~cost~~ consisting of all points q with $\xi(q)\le\varepsilon$, $\eta(q)=0 \Rightarrow e^\lambda$ is contained in H.
Since $\frac{\partial F}{\partial \xi}<0$, $F(q)\le F(p)<c-\varepsilon$. but $f(q)\ge c-\varepsilon$ for $q\in e^\lambda$



Assertion 3.4. The region $M^{c-\varepsilon}\cup e^\lambda$ is a deformation retract of $M^{c-\varepsilon}\cup H$

Pf: $r_t: M^{c-\varepsilon}\cup H \to M^{c-\varepsilon}\cup H$ $\downarrow\downarrow$ $\uparrow\uparrow$ Outside U: id
Inside U.



case1: $\xi\le\varepsilon$ $r_t:(u^1,\dots,u^n)\to(u^1,\dots,u^\lambda, tu^{\lambda+1},\dots,tu^n)$

$r_1=id$ r_0 maps to e^λ

case 2: $\varepsilon\le\xi\le\eta+\varepsilon$ $r_t:(u^1,\dots,u^n)\to(u^1,\dots,u^\lambda, s_tu^{\lambda+1},\dots,s_tu^n)$

$$s_t=t+(1-t)\left(\frac{\xi-\varepsilon}{\eta}\right)^{\frac{1}{2}}$$

$r_1=id$ r_0 maps to $f^{-1}(c-\varepsilon)$

case3: $\eta+\varepsilon\le\xi$ (in $M^{c-\varepsilon}$) let $r_t=id$

∴ $M^{c-\varepsilon}\cup e^\lambda$ is a deformation retract of $M^{c-\varepsilon}\cup H$. ▨

∴ $M^{c-\varepsilon}\cup e^\lambda\sim M^{c-\varepsilon}\cup H = F^{-1}(-\infty,c-\varepsilon]\sim M^{c+\varepsilon}$
↑ Assertion 3.3.

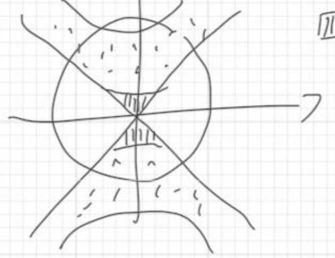
∴ $M^{c+\varepsilon}$ has homotopic type of $M^{c-\varepsilon}$ with a λ-cell attached Thm 3.2 ▨

Remark: k non-degenerate critical point $p^1,\cdots,p^k$ index $\lambda_1,\cdots,\lambda_k$ in $f^{-1}(c)$. then $M^{c+\varepsilon}$ has the homotype of $M^{c-\varepsilon}\cup e^{\lambda_1}\cup\cdots\cup e^{\lambda_k}$

Remark: M^c is also a deformation retract of $M^{c+\varepsilon}$.

$M^c \simeq F^{-1}[-\infty,c]$, $F^{-1}[-\infty,c]\sim M^{c+\varepsilon}$.

$M^{c-\varepsilon}\cup e^\lambda \sim M^c$.



▥ M^c

⊡ $F^{-1}[c,c+\varepsilon]$

Thm 3.3. If f is a differentiable fun on M with no degenerate critical points, if M^a is cpt, then M has the homotopy type of a CW-complew, with one cell of dim λ for each critical point of index λ.

Def: CW-complex

X_0: 0-skeleton points 0-cell

X_1: 1-skeleton attach 1-cells on X_0

...

$X_k = X_{k-1}\cup_\alpha e^k_\alpha$ (k-cells: (e^k_α))

$X=\cup_k X_k$ weak-topo $U\subset X$ is open $\Leftrightarrow U\cap X_k$ is open $\forall k$.

Closure-finite

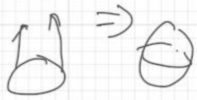
Ex: 点 → 线 → 面

X_0 X_1 X_2

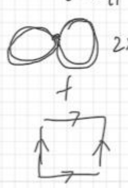
$\bigcup_{i=0}^{2} X_i$

图 1维 CW complex

S^n 0-cell + n-cell



T^2: 0-cell 2×1-cell +



$RP^n \sim D^n / \partial D^n/\sim$ $RP^n = RP^{n-1}\cup_f D^n$

$S^{n-1}/\sim$

$D^n+\sim$

$= RP^n$

RP^2 CW-complex ✓

单纯 ✗

Lemma 3.4. (Whitehead) Let φ_0, φ_1 be homotopic maps from $\dot{e}^\lambda$ to X. Then the id map of X can be extended to a homotopy equivalence $K: X \underset{\varphi_0}{\cup} e^\lambda \to X \underset{\varphi_1}{\cup} e^\lambda$

Pf: Define K: $K(x) = x \quad x \in X$

$K(tu) = 2tu \quad 0 \le t \le \frac{1}{2} \quad u \in \dot{e}^\lambda$

$K(tu) = \varphi_{2-2t}(u) \quad \frac{1}{2} \le t \le 1 \quad u \in \dot{e}^\lambda$

φ_t denotes the homotopy between φ_0 and φ_1.

Def $l: X \underset{\varphi_1}{\cup} e^\lambda \to X \underset{\varphi_0}{\cup} e^\lambda$

$l(x) = x \quad x \in X \qquad l(tv) = 2tv \quad 0 \le t \le \frac{1}{2} \quad v \in \dot{e}^\lambda$

$l(tv) = \varphi_{2t-1}(v) \quad \frac{1}{2} \le t \le 1, \ v \in \dot{e}^\lambda$

$k \circ l \sim id$.

$\therefore K$ is a homotopy equivalence.

Lemma 3.5. Let $\varphi: \dot{e}^\lambda \to X$ be an attaching map. Any homotopy equivalence $f: X \to Y$ extends to a homotopy equivalence

$F: X \underset{\varphi}{\cup} e^\lambda \to Y \underset{f\circ\varphi}{\cup} e^\lambda$

Pf: Def F: $\begin{cases} F|_X = f \\ F|_{e^\lambda} = id \end{cases}$

Let $g: Y \to X$ be a homotopy inverse to f. and define

$G: Y \underset{f\circ\varphi}{\cup} e^\lambda \to X \underset{g\circ f\circ\varphi}{\cup} e^\lambda$

$G|_Y = g \quad G|_{e^\lambda} = id$

Since $g\circ f\circ\varphi$ is homotopic to φ, by lemma 3.4. $\exists K: X \underset{g\circ f\circ\varphi}{\cup} e^\lambda \to X \underset{\varphi}{\cup} e^\lambda$

AIM: Firstly prove $K \circ G \circ F: X \underset{\varphi}{\cup} e^\lambda \to X \underset{\varphi}{\cup} e^\lambda$ is homotopic to id

Let h_t be a homotopy between $g\circ f$ and id

$K \circ G \circ F(x) = gf(x)$ for $x \in X$

$K \circ G \circ F(tu) = 2tu \quad 0 \le t \le \frac{1}{2}, u \in \dot{e}^\lambda$

$K \circ G \circ F(tu) = h_{2-2t}\varphi(u) \quad \frac{1}{2} \le t \le 1 \quad u \in \dot{e}^\lambda$

homotopy $q_T: X \underset{\varphi}{\cup} e^\lambda \to X \underset{\varphi}{\cup} e^\lambda$ defined by $q_T(x) = h_T(x)$ for $x \in X$

$q_T(tu) = \frac{2}{1+T} tu$ for $0 \le t \le \frac{1+T}{2}$, $u \in \dot{e}^\lambda$

$q_T(tu) = h_{2-2t+T}\varphi(u)$ for $\frac{1+T}{2} \le t \le 1$ $u \in \dot{e}^\lambda$

$T=0. \quad q_T = K \circ G \circ F$

$(h_0 = gf, h_1 = id)$; $T=1 \quad q_T = id$

$\frac{1+T}{2} = 1$

$K \circ G \circ F$ is homotopic to id, F has a left homotopy inverse, similar, G has a left homotopy inverse.

$K \circ G \circ F \simeq id$ + lemma 3.4 $\quad K$ has a left inverse

$\Rightarrow (G \circ F) \circ K \simeq id$

$G \circ (F \circ K) \simeq id$ + G has a left inverse

$\Rightarrow (F \circ K) \circ G \simeq id$.

$F \circ (K \circ G) \simeq id \quad F$ also has $(K \circ G)$ as left inverse

$\Downarrow$

F is a homotopy equivalence

$f: X \to Y$

$F: X \cup_{\varphi} e^{\lambda} \to Y \cup_{f \circ \varphi} e^{\lambda}$

$\boxed{///}$

Pf of thm 3.3:

Let $c_1 < c_2 < \cdots$ be critical values of $f: M \to \mathbb{R}$.

$M^a = \emptyset$ if $a < c_1$. Suppose $a > c_1, c_2, \cdots$ M^a is of the homotopy type of a CW-complex.

Let $c_i = c$ be the smallest $c_i > a$.

By thm 3.1 + 3.2 + Remark $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon} \cup_{\varphi_1} e^{\lambda_1} \cup \cdots \cup_{\varphi_{j(c)}} e^{\lambda_{j(c)}}$ for certain $\varphi_1, \cdots, \varphi_{j(c)}$

(ε is small), and there is a homotopy equivalence $h: M^{c-\varepsilon} \to M^a$, we assume $\exists$ homotopy equivalence $h': M^a \to K$, K is a CW-complex

Then $h' \circ h \circ \varphi_j$ is homotopic to a map:

$\psi_j : \dot{e}^{\lambda_j} \to (\lambda_j - 1)$ - Skeleton of K.

Then $K \cup_{\psi_1} e^{\lambda_1} \cup \cdots \cup_{\psi_{j(c)}} e^{\lambda_{j(c)}}$ is CW-complex

Use Lemma 3.4 to $h' \circ h$ it has same homotopic type of $M^{c-\varepsilon} \cup_{\varphi_1} e^{\lambda_1} \cup \cdots \cup_{\varphi_{j(c)}} e^{\lambda_{j(c)}}$, so as $M^{c+\varepsilon}$

By induction, each $M^{a'}$ has the homotopy type of a CW-complex.

If M cpt, then prove ✓.

If M not; all critical point lie in one of the cpt. sets M^a, then similar to proof of Thm 3.1, M^a is a deformation retract of M, then proof is also complete.

If critical points are infinite

(Whitehead's thm)

If M, K are dominated by CW-complex, $\forall\, M \xrightarrow{} K$ map induces isomorphisms of homotopy groups is a homotopy equivalence

$M^a \sim$ a finite CW-complex.

which one cell of dim λ for each critical point of index λ in M^a.