1. Let $X=[-1,1]\times[-1,1]$, $(-1,y)\sim(1,y)$, $(x,-1)\sim(x,1)$.

Let $p: X\to T^2$ be the quotient map. WLOG, delete $p:(0,0)$

Define deformation retraction $g:(X\setminus\{(0,0)\})\times[0,1]\to X$ by

$$g((x,y),t)=\left(\frac{x}{\phi(t)},\frac{y}{\phi(t)}\right),\quad \phi(t)=(1-t)+t\cdot\max\{|x|,|y|\}.$$

It is continuous (ϕ is continuous, $\phi(t)>0$ for $(x,y)\neq(0,0)$)

At $t=0$, $\phi(0)=1$, $g((x,y),0)=(x,y)$, which is identity.

At $t=1$, $\phi(1)=\max\{|x|,|y|\}$, so $g((x,y),1)=\left(\frac{x}{\max\{|x|,|y|\}},\frac{y}{\max\{|x|,|y|\}}\right)$. one of them must be 1, so it lies on ∂X, where $|x|=1$ or $|y|=1$

Now, for $p\circ g:(X\setminus\{(0,0)\})\times[0,1]\to T^2$, it is continuous ($p$, g are continuous)

If two points are identified by p:

① if they are not on the boundary, preimage of p is just one point

② if points on the boundary, $\max\{|x|,|y|\}=1$, so $\phi(t)=1$, g leaves these points fixed.

$\therefore$ p maps them to same point in torus

$\therefore$ $p\circ g$ is well-defined.

At $t=1$, $Img=\partial X=\{|x|=1 \text{ or } |y|=1\}$. under p, it becomes union of two circles:

$p(\{(x,y)\in X \mid x=1 \text{ or } -1\})$ is the longitude circle.

$p(\{(x,y)\in X \mid y=1 \text{ or } -1\})$ is the meridian circle.

$\therefore$ The map $p\circ g$ induces a deformation retraction of $T^2\setminus\{p(0,0)\}$.

2. Def $\pi: \mathbb{R}^n-\{0\}\to S^{n-1}$ by $\pi(x)=\frac{x}{\|x\|}$

And let $H:(\mathbb{R}^n-\{0\})\times[0,1]\to\mathbb{R}^n-\{0\}$ by $H(x,t)=(1-t)x+t\frac{x}{\|x\|}$, it is continuous.

$H(x,0)=x$ $\forall x\in\mathbb{R}^n-\{0\}$, $H(x,1)=\pi(x)\in S^{n-1}$, and take all $x\in\mathbb{R}^n-\{0\}$, it is onto.

If $x\in S^{n-1}$, $\frac{x}{\|x\|}=\frac{x}{1}=x$ so $H(x,1)|_{S^{n-1}}=x$

(Besides, $H(x,t)=\left[(1-t)+t\frac{1}{\|x\|}\right]x$, $x\neq0$, $(1-t)+t\frac{1}{\|x\|}>0$, so $H(x,t)\neq0$)

$\therefore$ H is a deformation retraction of $\mathbb{R}^n\setminus\{0\}$ onto S^{n-1}

3. (a) $\exists f: X\to Y, f': Y\to X$ s.t. $f\circ f' \simeq id_X$, $f'\circ f \simeq id_Y$

$\exists g: Y\to Z, g': Z\to Y$ s.t. $g\circ g' \simeq id_Y$, $g'\circ g \simeq id_Z$.

Now, let $h = g\circ f: X\to Z$, $h' = f'\circ g': Z\to X$.

$h'\circ h = (f'\circ g')\circ(g\circ f) = f'\circ(g'\circ g)\circ f$

$\because g'\circ g \simeq id_Y$ $\therefore \exists K: Y\times[0,1]\to Y$ s.t. $K(y,0) = (g'\circ g)(y)$, $K(y,1) = y$.

$f'\circ K\circ(f\times id_{[0,1]})$ gives homotopy from $f'\circ(g'\circ g)\circ f$ to $f'\circ id\circ f = f'\circ f \simeq id_X$

$\therefore f'\circ(g'\circ g)\circ f \simeq id_X$

$\therefore h'\circ h \simeq id_X$

Totally same, $h\circ h' \simeq id_Z$

$\therefore h, h'$ define a homotopy equivalence between X, Z. (Transitivity)

Besides, $\forall$ space X, $id_X: X\to X$ is a homotopy equivalence ($id_X\circ id_X = id_X \simeq id_X$)

$\therefore X$ is homotopy equivalent to itself. (Reflexivity)

And if X is homotopy equivalent to Y, $\exists f: X\to Y$, $g: Y\to X$ s.t. $g\circ f \simeq id_X$, $f\circ g \simeq id_Y$

$\therefore Y$ is homotopy equivalent to X obviously. (Symmetry)

$\therefore$ It is an equivalence relation.

(b) Reflexity: for $f: X\to Y$, $H(x,t) = f(x)$ $\forall t\in[0,1]$, $H(x,0) = f(x)$, $H(x,1) = f(x)$

$\therefore f\simeq f$.

Symmetry: If $f\simeq g$. $\exists H$ s.t. $H(x,0) = f(x)$, $H(x,1) = g(x)$

Def $G(x,t) = H(x,1-t)$, $G(x,0) = H(x,1) = g(x)$, $G(x,1) = f(x)$. $\therefore g\simeq f$

Transitivity: If $f\simeq g$, $g\simeq h$, $\exists H_1, H_2$ s.t. $H_1(x,0) = f(x)$, $H_1(x,1) = g(x)$, $H_2(x,0) = g(x)$

$H_2(x,1) = h(x)$. Def $H(x,t) = \begin{cases} H_1(x,2t) & 0\le t\le \frac{1}{2} \\ H_2(x,2t-1) & \frac{1}{2}\le t\le 1 \end{cases}$ at $t=\frac{1}{2}$, $H(x,\frac{1}{2}) = H_1(x,1) = g(x) = H_2(x,0)$

$\therefore H$ is continuous. $H(x,0) = H_1(x,0) = f(x)$, $H(x,1) = H_2(x,1) = h(x)$

$\therefore f\simeq h$.

$\therefore$ Homotopy is an equivalence relation

(C) let $f: X \to Y$ be a homotopy equivalence, $\exists g: Y \to X$ s.t. $g \circ f \simeq id_X$, $f \circ g \simeq id_Y$

let f' be a map s.t. $f' \simeq f$.

$\exists H: X \times [0,1] \to Y$, with $H(x,0) = f(x)$, $H(x,1) = f'(x)$

for $g \circ H: X \times [0,1] \to X$, this gives a homotopy from $g \circ f$ to $g \circ f'$

$\therefore g \circ f \simeq g \circ f'$ since $g \circ f \simeq id_X$ $\therefore g \circ f' \simeq id_X$

for $K: Y \times [0,1] \to Y$ by $K(y,t) = H(g(y), t)$, induces homotopy from $f \circ g$ to $f' \circ g$

$\therefore f' \circ g \simeq f \circ g \simeq id_Y$

$\therefore g$ is homotopy inverse of f', so f' is a homotopy equivalence.

4. $\exists f_t: X \to X$ s.t. $f_0 = id_X$ $f_t(X) \subseteq A$ $\forall t \in [0,1]$, $f_t(A) \subseteq A$ $\forall t \in [0,1]$.

Def $j: X \to A$ $j(x) = f_1(x)$ $\because f_1(X) \subseteq A$, j is well defined. f_t continuous, j continuous

$i \circ j: X \to X$. $i \circ j(x) = i(j(x)) =$ ~~[illegible]~~ $f_1(x)$

$\therefore f_0 = id_X$, $f_1 = i \circ j$ $\therefore i \circ j \simeq id_X$.

$j \circ i: A \to A$. $j \circ i(a) = j(a) = f_1(a)$

$\because f_t(A) \subseteq A$, $\therefore f_t|_A = g_t: A \to A$, ~~$g_t(a) = f$~~

$g_0(a) = f_0|_A(a) = a = id_A(a)$. $g_1(a) = f_1(a) = j \circ i(a)$.

$\therefore g_t$ ~~induces~~ is homotopy of id_A to $j \circ i$ $\therefore j \circ i \simeq id_A$

$\therefore i$ is a homotopy equivalence.

9. Let X be contractible, $A \subseteq X$ is a retract of X. $\exists H: X \times [0,1] \to X$ s.t. $H(x,0) = x$, $H(x,1) = c$ for some $c \in X$. $\exists r: X \to A$, continuous, $r(a) = a$ $\forall a \in A$.

Def $G: A \times [0,1] \to A$, $G(a,t) = r(H(a,t))$, it is continuous.

At $t=0$, $G(a,0) = r(H(a,0)) = r(a) = a$

At $t=1$, $G(a,1) = r(H(a,1)) = r(c)$. Let $a_0 = r(c) \in A$, Then $G(a,1) = a_0$ $\forall a \in A$.

$\therefore G$ is a homotopy from id on A to a constant map at a_0.

$\therefore A$ is contractible.

10. ① If X is contractible. ~~$\forall f: X \to Y$~~ $\exists H: X \times [0,1] \to X$ s.t. $H(x,0)=x$. $H(x,1)=x_0$.

($x_0 \in X$) $\forall f: X \to Y$. Def $G: X \times [0,1] \to Y$ by $G(x,t) = f(H(x,t))$.

$\therefore G(x,0) = f(x)$, $G(x,1) = f(x_0)$, constant.

$\therefore f$ is nullhomotopic.

If $\forall f: X \to Y$ is nullhomotopic. Take $Y=X$, $id_X: X \to X$, it is nullhomotopic.

So $\exists H: X \times [0,1] \to X$ s.t. $H(x,0)=x$, $H(x,1)=c$. $c \in X$. $\Rightarrow X$ is contractible.

② If X is contractible, $\exists H: X \times [0,1] \to X$, s.t. $H(x,0)=x$, $H(x,1)=x_0$, $x_0 \in X$

$\forall f: Y \to X$ Def $G: Y \times [0,1] \to X$, $G(y,t) = H(f(y),t)$. Then $G(y,0)=f(y)$

$G(y,1)=x_0$, it is constant, so f is nullhomotopic.

If $\forall f: Y \to X$. is nullhomotopic, take $Y=X$, $id_X: X \to X$. it is nullhomotopic,

so $\exists H: X \times [0,1] \to X$ s.t. $H(x,0)=x$, $H(x,1)=c$ $\Rightarrow X$ is contractible.

11. If $fg \simeq 1$, $hf \simeq 1$.

Def: $F: Y \times I \to Y$ s.t. $F(y,0) = fg(y)$, $F(y,1) = y$.

$G: X \times I \to X$ s.t. $G(x,0) = hf(x)$, $G(x,1) = x$

Let $H: Y \times I \to X$

on $t \in [0, \frac{1}{2}]$ $\quad H(y,t) = G(g(y), 1-2t)$

$H(y,0) = G(g(y), 1) = g(y)$

$H(y, \frac{1}{2}) = G(g(y), 0) = hfg(y)$

on $t \in [\frac{1}{2}, 1]$ $\quad H(y,t) = h(F(y, 2t-1))$

$H(y, \frac{1}{2}) = h(F(y,0)) = hfg(y)$

$H(y,1) = h(F(y,1)) = h(y)$

on $t = \frac{1}{2}$, $H(y, \frac{1}{2}) = hfg(y)$ $\Rightarrow$ H is continuous, $H(y,0) = g(y)$, $H(y,1) = h(y)$

$\therefore g \simeq h$

$\therefore g \circ f \simeq h \circ f \simeq id_X$ (by problem 3.(c)).

$f \circ g \simeq id_Y$

$\therefore f$ is homotopy equivalence

More generally.

If $f \circ g$, $h \circ f$ are homotopy equivalence.

$\exists\, p: Y \to Y$, $p \circ (f \circ g) \simeq 1_Y$, $(f \circ g) \circ p \simeq 1_Y$

$\exists\, q: X \to X$, $q \circ (h \circ f) \simeq 1_X$, $(h \circ f) \circ q \simeq 1_X$.

Let $m = g \circ p : Y \to X$.

$f \circ m = f \circ (g \circ p) = (f \circ g) \circ p \simeq 1_Y$

$m \circ f = (g \circ p) \circ f = g \circ (p \circ f)$

$\because p \circ (f \circ g) \simeq 1_Y$ $\therefore h \circ f \circ g \circ p \simeq h$ $\quad \therefore q \circ h \circ f \circ g \circ p \simeq q \circ h$

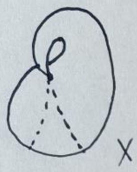
$q \circ h \circ f \circ g \circ p = (q \circ h \circ f) \circ (g \circ p) \simeq g \circ p$.

$\therefore g \circ p \simeq q \circ h$. $\quad \therefore g \circ p \circ f \simeq q \circ h \circ f$

$\therefore g \circ (p \circ f) \simeq 1_X \simeq m \circ f$ $\quad \therefore m = g \circ p$ is inverse of f, f is homotopy equivalence

20. The klein bottle intersects itself in a circle C. The intersection place is a ~~de~~ disk D s.t. $\partial D = C$. Since $D \simeq$ a point, X ~~can~~ has a deformation of X_1:

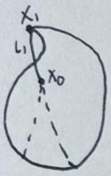
"......": inside the bottle.



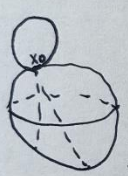
(let D become a point.)

X_1.

Then, let this point stretch ~~to~~ into a segment L_1, $X_1 \simeq X_2$, the two end points of L_1 is x_0, x_1



Then, move x_1 to x_0, X_2 becomes X_3 ~~homotop~~, $X_2 \simeq X_3$

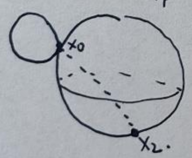


X_3

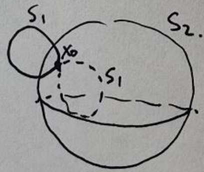
Use the similar~~ly~~ way to transform the inside part: $X_3 \simeq X_4$, $X_4 \simeq X_5$

X_4:

$X_3 \rightarrow$



move x_2 to x_0 $\longrightarrow$



X_5

$\therefore X_5 \simeq X$, i.e. $X \simeq S_1 \vee S_1 \vee S_2$

23. $A, B, A\cap B$ are all contractible. ($A\vee B = X$, $A\cap B$ is also a subcomplex)

$(X, A\cap B)$ is a CW pair, $A\cap B$ are contractible

$\therefore X \simeq X/(A\cap B)$

$X/(A\cap B) = A/(A\cap B) \vee B/(A\cap B)$

$\because A\cap B$ is contractible, $(A, A\cap B)$ is a CW pair

$\therefore A \simeq A/(A\cap B)$, $A/(A\cap B)$ is also contractible

Similarly, $B/(A\cap B)$ is contractible.

$\therefore \exists H_A : A/(A\cap B) \times I \to A/(A\cap B)$; $H_B : B/(A\cap B) \times I \to B/(A\cap B)$

contract each quotient space to one point.

$\therefore$ Combine them, get $H : (A/(A\cap B) \vee B/(A\cap B)) \times I \to A/(A\cup B) \vee B/(A\cap B)$

contracts the wedge to ~~the wedge of two points~~, a common basepoint.

$\therefore X/(A\cap B)$ is contractible

$\because X \simeq X/(A\cap B)$

$\therefore X$ is contractible

28. Given (X_1, A) has HEP. Let $Y = X_0 \sqcup_f X_1$, $q: X_0 \sqcup X \to Y$ be the quotient map
($f(a) \sim a \ \forall a \in A$)

Then $Y \times I \simeq X_0 \times I \sqcup X_1 \times I / \sim$

Let $Z = Y \times \{0\} \cup X_0 \times I$

Since (X_1, A) has HEP, $\exists\, r: X_1 \times I \to X_1 \times \{0\} \cup A \times I$ is a retraction

Def $R': X_0 \times I \sqcup X_1 \times I \to Z = Y \times \{0\} \cup X_0 \times I$

On $X_0 \times I$, $R'(x_0, t) = (x_0, t)$

On $X_1 \times I$, for (x,t), $r(x,t) = (a, b) \in X_1 \times \{0\} \cup A \times I$

if $b = 0$, $a \in X_1$, def $R'(x,t) = (q(a), 0) \in Y \times \{0\}$.

if $b > 0$, $a \in A$, def $R'(x,t) = (f(a), b) \in X_0 \times I$ (Notice that $f: A \to X_0$)

For $(a,t) \in X_1 \times I$, $a \in A$, $r(a,t) = (a,t) \in A \times I$ Then $R'(a,t) = (f(a), t)$.

For $(f(a), t) \in X_0 \times I$, $R'(f(a), t) = (f(a), t)$

$\therefore R'$ obeys the equivalence relation.

Since R' is continuous, obeys the equivalence relation, it induces R, which is continuous.

$$R: Y \times I \to Z.$$

$\forall z \in Z$, if $z = (x_0, t) \in X_0 \times I$, $R(z) = R'(x_0, t) = (x_0, t) = z$.

if $z = (y, 0) \in Y \times \{0\}$, $y = q(x_1)$ for some $x_1 \in X_1$, or $y = q(x_0)$ for ~~sme~~ some $x_0 \in X_0$.

if $y = q(x_1)$, $z = (x_1, 0) \in X_1 \times \{0\}$, $r(x_1, 0) = (x_1, 0)$

$\therefore R'(x_1, 0) = (q(x_1), 0) = (y, 0) = z$.

if $y = q(x_0)$, $z = (x_0, 0)$, $R'(x_0, 0) = (x_0, 0) = (q(x_0), 0) = (y, 0) = z$.

$\Rightarrow R(z) = z$ on $Y \times \{0\}$.

$\therefore R$ is a retraction

$\therefore R: Y \times I \to Y \times \{0\} \cup X_0 \times I$ is a retraction, so (Y, X_0) satisfies HEP.