5.4 Space of Quadratic Differentials.

Def. Beltrami Equation

$$\frac{\partial f}{\partial \bar z} = \mu(z)\frac{\partial f}{\partial z} \qquad \left(\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),\ \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\right)$$

This equation shows how f strech space. C-R equation: $\frac{\partial f}{\partial \bar z} = 0$. $\to$ conformal

Beltrami Eq $\to$ quasi conformal $\quad \circ \to \bigcirc$

(P,q)-form : $a(z)\, dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar z_{j_1} \wedge \cdots \wedge d\bar z_{j_q}$.

$(-1,1)$-form

$$V = v(\zeta)\frac{d\bar\zeta}{d\zeta} \qquad v(\zeta)\frac{d\bar\zeta}{d\zeta}\Big(\underbrace{w(\zeta)\frac{\partial}{\partial\zeta}}_{TX}\Big) = \underbrace{v(\zeta)\overline{w(\zeta)}\frac{\partial}{\partial\zeta}}_{TX}$$

$\therefore$ $L^\infty_x(TX, TX)$. antilinearly map.

$$\frac{\partial f}{\partial \bar z} = \mu\frac{\partial f}{\partial z} \Rightarrow \mu = (\partial f)^{-1} \circ \bar\partial f \qquad \mu \text{ maps } T_xX \text{ to } T_xX \quad \text{(express it to tangent bundle)}$$

$\therefore$ $\mu \in L_x(TX, TX)$

$\therefore$ Infinitesimal Beltrami forms : $\mu \in L_x(TX, TX)$ $\quad \mu = \mu(z)\frac{d\bar z}{dz}$

antilinear : $T(av) = \bar a\, T(v)$

Quadratic differential $q = q(z)\,dz^2$

$\therefore$ $q\mu = q(z)\mu(z)\,|dz|^2$ (measure) $\qquad \langle \mu, q\rangle = \int_X q\mu$.

Note: For infinite dim Banach space, $E^{**} \neq E$, def pre-dual of E as F if $E = F^*$

Now, try to find dual / pre-dual of $L^\infty_x(TX, TX)$ ($\operatorname{ess\,sup}|v(x)| < \infty$, $v: TX \to TX$), it is natural to research $Q(x) = \{q = q(z)\,dz^2\}$.

Let X be a hyperbolic Riemann surface with hyperbolic metric ρ.

$$\|q\|_1 := \int_X |q| \qquad \|q\|_\infty := \sup_{x\in X} \frac{|q|(x)}{\rho^2(x)}$$

Def. 5.4.1. The Banach space of integrable quadratic differentials is

$$Q^1(X) := \{ q \in Q(X) \mid \|q\|_1 < \infty \}$$

The Banach space of bounded quadratic differentials is

$$Q^\infty(X) := \{ q \in Q(X) \mid \|q\|_\infty < \infty \}.$$

Prop. 5.4.3. let X be a Riemann surface of finite type (finite points removed), then $Q^1(X) = Q^\infty(X)$

Pf: let $q \in Q^\infty(X)$, $\|q\|_\infty < \infty$

hyperbolic area

$$\|q\|_1 = \int_X |q| = \int_X \frac{|q|}{\rho^2} \rho^2$$

$\because \frac{|q|}{\rho^2} \leq \|q\|_\infty \qquad \int_X \frac{|q|}{\rho^2} \rho^2 \leq \|q\|_\infty \int_X \rho^2 = 2\pi(2g-2+n) \|q\|_\infty < \infty$

$\therefore \|q\|_1 < \infty, \quad q \in Q^1(X)$

$\therefore Q^\infty(X) \subseteq Q^1(X)$

Conversely, when X is cpt, $Q(X)$ has finite dim (Riemann-Roch), $Q^1(X) = Q^\infty(X)$.

For puncture surfaces: Consider $D - \{0\}$.

if $\int |q| < \infty$, it has at most ~~one~~ a simple pole at the origin $\sim \frac{dz^2}{z}$ near 0.

$\rho = \frac{|dz|}{r|\ln r|} \quad r = |z| \quad \rho^2 = \frac{|dz|^2}{r^2 (\ln r)^2} \qquad \frac{|q|}{\rho^2} = \frac{\frac{|dz|^2}{r}}{\frac{|dz|^2}{r^2(\ln r)^2}} = r(\ln r)^2 \to 0 \quad (r \to 0)$

$\therefore \|q\|_\infty < \infty$

For general puncture surface, it is similar.

$\therefore Q^1(X) \subseteq Q^\infty(X)$ $\square$

Pro. 5.4.4. $Q^1(D) \subset Q^\infty(D)$, $\|q\|_\infty \leq \frac{1}{4\pi}\|q\|_1$

Pf: $\rho(z) = \frac{2}{1-|z|^2}$ $\qquad \|q\|_\infty = \sup_{z\in D} |q(z)| \cdot \frac{(1-|z|^2)^2}{4}$

$\|q\|_1 = \int_D |q(z)||dz|^2$

$\because \forall \phi \in Aut(D)$, $\|\phi^* q\|_1 = \|q\|_1$ isometry $\|\phi^* q\|_\infty = \|q\|_\infty$

It is sufficient to prove:

At origin $\frac{|q(0)|}{4} \leq \frac{1}{4\pi}\|q\|_1 \Leftrightarrow |q(0)| \leq \frac{\|q\|_1}{\pi}$

$q(0) = \frac{1}{\pi}\int_D q(z)|dz|^2$ $\quad$ $q(z)$ analytic (mean value)

$|q(0)| \leq \frac{1}{\pi}\int_D |q(z)||dz|^2 = \frac{1}{\pi}\|q\|_1$

$\therefore \|q\|_\infty \leq \frac{1}{4\pi}\|q\|_1$ ▨

(when integral of q limited, q can not be very large)

Def. 5.4.6. The conjugate Riemann surface X^* of Riemann surface X is ~~defind~~ defined: if $U \subset X$ is open, $\varphi: U \to \mathbb{C}$ is a local coordinate for X, then $\bar{\varphi}: U \to \mathbb{C}$ is a local coordinate of X^*

Ex: If $\Gamma \subset PSL_2\mathbb{R}$, $X = H/\Gamma$, $X^* = H^*/\Gamma$

Exercises 5.4.8. $\frac{dz^2 \otimes dw^2}{(w-z)^4} \in \Gamma(\Omega^{\otimes 2}P^1 \otimes \Omega^{\otimes 2}P^1)$ defined ~~on~~ over $(P^1\times P^1)\setminus\Delta$.

If $f: U \to P^1$ analytic $z = f(z_1)$, $w = f(w_1)$

$(f\times f)^*[F(z,w)dz^2\otimes dw^2] = F(f(z_1), f(w_1))(f'(z_1))^2(f'(w_1))^2 dz_1^2 \otimes dw_1^2$

Invariant under acting diagonally: $(f\times f)^*(F(z,w)) = F(z,w)$.

Now: $f(z) = \frac{az+b}{cz+d}$ $ad-bc\neq 0$, $\quad f'(z) = \frac{ad-bc}{(cz+d)^2}$

$$f(w)-f(z)=\frac{(ad-bc)(w-z)}{(cw+d)(cz+d)}$$

$$(f'(z))^2(f'(w))^2=\left(\frac{ad-bc}{(cz+d)^2}\right)^2\cdot\left(\frac{ad-bc}{(cw+d)^2}\right)^2=\frac{(ad-bc)^4}{(cz+d)^4(cw+d)^4}$$

$$(f(w)-f(z))^4=\frac{(ad-bc)^4(w-z)^4}{(cz+d)^4(cw+d)^4}$$

$$\therefore\ \frac{(f'(z))^2(f'(w))^2}{(f(w)-f(z))^4}=\frac{1}{(w-z)^4}\quad\Rightarrow\quad\frac{dz^2\otimes dw^2}{(w-z)^4}$$ invariant under acting diagonally.

(q is invariant differential under Γ, in H^*, $im<0$)

Pro 5.4.9. (Reproducing formula for Q^∞). let $q\in(Q^\infty)^\Gamma(H^*)$

Then: $$q(w)\,dw^2=\frac{12}{\pi}\underbrace{\left(\int_H\frac{q(\bar z)y^2}{(z-w)^4}|dz|^2\right)}_{\text{reproducing kernel}}dw^2$$

Pf: Rewrite $$\frac{12}{\pi}\int_H\Big(\underbrace{q(\bar z)\,d\bar z^2}_{\text{invariant.}}\ \underbrace{\frac{dz^2\otimes dw^2}{(z-w)^4}}_{\text{invariant}}\ \underbrace{\frac{y^2}{|d\bar z|^2}}_{\text{invariant}}\Big)$$

① $q\in(Q^\infty)^\Gamma(H^*)$ ② eversis before

③ invarse of hyperbolic metric in H

$q\in Q^\infty(H^*)$ $\quad\therefore y^2|q(\bar z)|$ is bounded in H.

$\mu_w:=\frac{|dz|^2}{|z-w|^4}$ is a smooth finite measure on H.

Lemma 5.4.10. if $\frac{\alpha}{\beta}=w$. $\Phi:\zeta\to\frac{\alpha\zeta+\bar\alpha}{\beta\zeta+\bar\beta}$ is an isomorphism $D\to H$ and

$$\Phi^*\mu_w=\frac{1}{4(\mathrm{Im}\,w)^2}|d\zeta|^2,\quad\int_H\mu_w=\frac{\pi}{4(\mathrm{Im}\,w)^2}$$

$\therefore$ Integral converge absulately.

Now, set $z:=\frac{\alpha\zeta+\bar\alpha}{\beta\zeta+\bar\beta}$, $\frac{\alpha}{\beta}=w$.

$$\frac{12}{\pi}\int_H\frac{q(\bar z)y^2}{(z-w)^4}|dz|^2=\frac{3}{\pi}\int_D q\left(\frac{\bar\alpha\bar\zeta+\alpha}{\bar\beta\bar\zeta+\beta}\right)\frac{(1-|\zeta|^2)^2}{(\bar\beta\bar\zeta+\beta)^4}\beta^4|d\zeta|^2$$

($\frac{y^2}{|dz|^2}\to\frac{(1-|\zeta|^2)^2}{4|d\zeta|^2}$; $\frac{1}{(z-w)^4}\to\frac{\beta^4(\beta\zeta+\bar\beta)^4}{(\underbrace{\bar\alpha\beta-\alpha\bar\beta}_{\text{imaginary}})^4}$; $|dz|^4\to\frac{|\alpha\bar\beta-\bar\alpha\beta|^4}{|\beta\zeta+\bar\beta|^8}|d\zeta|^4$ $(|\beta\zeta+\bar\beta|)^8=[(\bar\beta\bar\zeta+\beta)(\beta\zeta+\bar\beta)]^4$

multiply them)

Now, let $\zeta = re^{i\theta}$, it becomes

$$\frac{3\beta^4}{\pi}\int_0^1 \left(\int_0^{2\pi} q\left(\frac{\bar{\alpha}\bar{\zeta}+\alpha}{\bar{\beta}\bar{\zeta}+\beta}\right)\frac{1}{(\bar{\beta}\bar{\zeta}+\beta)^4}\,d\theta\right)(1-r^2)^2\, r\,dr$$

$f(\zeta) = q\left(\frac{\bar{\alpha}\bar{\zeta}+\alpha}{\bar{\beta}\bar{\zeta}+\beta}\right)\frac{1}{(\bar{\beta}\bar{\zeta}+\beta)^4}$ is anti-holomorphic. on $|z-z_0|\le r$

(composite, inverse of holomorphic fun)

$\frac{1}{2\pi}\int f(re^{i\theta})d\theta = f(0)$

$$\therefore \int_0^{2\pi} \underset{f(re^{i\theta})}{f(\zeta)}\, d\theta = 2\pi f(0) = \frac{2\pi}{\beta^4} q(w)$$

$$\frac{3\beta^4}{\pi}\int_0^1 \frac{2\pi}{\beta^4} q(w)(1-r^2)^2 r\,dr = 6\cdot q(w)\int_0^1 (1-r^2)^2 r\,dr = 6q(w)\cdot\frac{1}{6} = q(w)$$

$\boxtimes$

Pro 5.4.11 (Reproducing formula for Q') let $q \in (Q')^{\Gamma}(H)$, Then

$$q(w)\,dw^2 = \frac{12}{\pi}\left(\int_{H^*}\frac{q(\bar{z})y^2}{(\bar{z}-w)^4}|dz|^2\right)dw^2$$

Pf: $|q(z)|y^2 = (|q(z)||dz|^2)\left(\frac{y^2}{|dz|^2}\right)$. Γ-invariant.

measure: $\frac{|dz|^2|dw|^2}{|z-w|^4}$ is invariant under Aut H, acting diagonally on $H\times H^*$

$$\int_{\Omega^*}\left(\int_H \frac{|q(\bar{z})|y^2}{|z-w|^4}|dz|^2\right)|dw|^2$$

Ω^*: fundamental domain, contains exactly one point from each of these orbits.

$$= \int_{\Omega^*}\left(\sum_{\gamma\in\Gamma}\int_{\gamma^{-1}(\Omega)}\frac{|q(\bar{z})|y^2}{|z-w|^4}|dz|^2\right)|dw|^2$$

$$= \sum_{\gamma\in\Gamma}\int_{\Omega^*}\int_{\gamma^{-1}(\Omega)}\frac{|q(\bar{z})|y^2}{|z-w|^4}|dz|^2|dw|^2$$

$z\to\gamma(z)$

$$= \sum_{\gamma\in\Gamma}\int_{\gamma(\Omega^*)}\int_{\Omega}\frac{|q(\bar{z})|y^2}{|z-w|^4}|dz|^2|dw|^2$$

$$= \sum_{\gamma\in\Gamma}\int_{\Omega}|q(\bar{z})|y^2\left(\int_{\gamma(\Omega^*)}\frac{|dw|^2}{|z-w|^4}\right)|dz|^2$$

$\sum_{\gamma\in\Gamma}\int_{\gamma(\Omega^*)} = \int_{H^*}$

Lemma 5.4.10 $= \frac{\pi}{4\,\underbrace{\mathrm{Im}(z)}_{y}{}^2}$

$$= \frac{\pi}{4}\int_{\Omega}|q(\bar{z})||dz|^2 = \frac{\pi}{4}\|q\|_1 < \infty \Rightarrow \text{integral converges.}$$

Other proof is same as Pro 5.4.9.

$\boxtimes$

Thm. 5.4.12 (Duality theorem) $\forall$ hyperbolic Riemann surface X, the pairing ⑥

$Q^1(X) \times Q^\infty(X^*) \to \mathbb{C} : \langle q, p\rangle \longmapsto \int_X \frac{qp}{\rho^2}$ induces an isomorphism $Q^\infty(X^*) \to (Q^1(X))^T$.

($Q^\infty(X^*)$ is the dual of $Q^1(X)$, $Q^1(X)$ is the predual of $Q^\infty(X^*)$)

Pf: Def $T: Q^\infty(X^*) \to (Q^1(X))^*$ $\quad T(p)(q) = \langle q, p\rangle$.

measurable quadratic differentials (may not holomorphic)

$\forall \alpha \in (Q^1(X))^*$. By Hahn-Banach thm, α extends to $LQ^1(X)^*$, $\tilde{\alpha} \in (LQ^1(X))^*$

$\exists\, p \in LQ^\infty(X^*)$ s.t. $\forall q \in LQ^1(X)$, $\tilde{\alpha}(q) = \langle q, p\rangle$ (Riesz-rep)

Lemma 5.4.14: $\forall q \in (LQ^1)^\Gamma(H)$, $p \in (LQ^\infty)^\Gamma(H^*)$, there is $\langle P^1 q, p\rangle = \langle q, P^\infty p\rangle$

($P^\infty(p) = \frac{12}{\pi}\left(\int_H \frac{p(\bar{z})y^2}{(z-w)^4}|dz|^2\right)dw^2$; $P^1(q) = \frac{12}{\pi}\left(\int_{H^*} \frac{q(\bar{z})y^2}{(z-w)^4}|dz|^2\right)dw^2$, they are projections from $(LQ^\infty)^\Gamma(H^*) \to (Q^\infty)^\Gamma(H^*)$, $(LQ^1)^\Gamma(H) \to (Q^1)^\Gamma(H)$)

Pf: $\langle P^1 q, p\rangle = \int_\Omega \left(\int_{H^*} \frac{q(\bar{z})y^2}{(z-w)^4}|dz|^2\right) p(\bar{w}) v^2 |dw|^2 \quad w = u + iv$

$= \int_\Omega \left(\sum_{\gamma\in\Gamma} \int_{\gamma^{-1}(\Omega^*)} \frac{q(\bar{z})y^2}{(z-w)^4}|dz|^2\right) p(\bar{w}) v^2 |dw|^2 = \sum_{\gamma\in\Gamma} \int_\Omega \int_{\gamma^{-1}(\Omega^*)} \sim\sim$

$z \to \gamma(z)$ + Γ-invariance

$= \sum_{\gamma\in\Gamma} \int_{\gamma(\Omega)} \int_{\Omega^*} \frac{q(\bar{z})y^2}{(z-w)^4}|dz|^2 p(\bar{w}) v^2 |dw|^2$

$= \int_H \int_{\Omega^*} \sim\sim$

$= \int_{\Omega^*} \int_H \sim\sim = \langle q, P^\infty p\rangle$ ☒

$\therefore$ Consider $P^\infty p \in Q^\infty(X^*)$ $\quad T(P^\infty p)(q) = \langle q, P^\infty p\rangle = \langle P^1 q, p\rangle \underset{(5.4.11)}{=} \langle q, p\rangle = \alpha(q)$

$\therefore T(P^\infty p) = \alpha$, surjective

Then, for $p \in Q^\infty(X^*)$ $p \neq 0$, $\exists q' \in LQ^1(X)$, s.t. $\langle q', p\rangle \neq 0$.

Let $q = P^1 q' \in Q^1(X)$, $\langle q, p\rangle = \langle P^1 q', p\rangle = \langle q', P^\infty p\rangle$

$\because p \in Q^\infty(X^*)$ is holomorphic, by 5.4.9, $P^\infty p = p$ $\therefore \langle q, p\rangle = \langle q', p\rangle \neq 0$

$\therefore \exists q \in Q^1(X)$, s.t. $\langle q, p\rangle \neq 0$. $\therefore p \neq 0 \Rightarrow T(p) \neq 0$

$\therefore$ injective $\qquad \therefore$ bijective + linear $\Rightarrow$ isomorphism ☒

Let $\pi: Y \to X$ be a covering map, $\pi_*: Q^1(Y) \to Q^1(X)$ is called Poincaré operator

Def. 5.4.15. (The direct image operator) If $v \in T_x X$, then

$$(\pi_* \varphi)(v) = \sum_{y \in \pi^{-1}(x)} \varphi([D\pi(y)]^{-1}(v)) \qquad D\pi(y): T_y Y \to T_x X$$

For $U \subset X$, $\zeta: U \to \mathbb{C}$ is a local coordinate. π maps connected components U_i of $\pi^{-1}(U)$ isomorphically to U, $\zeta_i := \zeta \circ \pi|_{U_i}$ is a local coordinate in U_i, so $\forall \varphi \in Q^1(Y)$ $\varphi|_{U_i} = \varphi_i(\zeta_i) d\zeta_i^2$

$d\zeta_i = d(\zeta \circ \pi) = \pi^* d\zeta$

$$\pi_* \varphi|_U = (\sum \varphi_i) d\zeta^2$$

$\Big(v \in T_x X$, $v = a\frac{\partial}{\partial \zeta}$ $(D\pi(y))^{-1}(v) = a\frac{\partial}{\partial \zeta_i}$, $\varphi(a\frac{\partial}{\partial \zeta_i}) = \varphi_i(\zeta_i) d\zeta_i^2 (a\frac{\partial}{\partial \zeta_i}) = \varphi_i(\zeta_i) a^2$

$\therefore (\pi_* \varphi)(v) = \sum_i \varphi_i(\zeta_i) a^2$

$\sum \varphi_i d\zeta^2(v) = \sum_i \varphi_i \cdot a^2 \Big)$

Prop. 5.4.16. π_* is continuous linear operator from $Q^1(Y) \to Q^1(X)$, $\|\pi_*\| \le 1$.

Pf. $\int_U |\pi_* \varphi| = \int_U |\sum_i \varphi_i(\zeta)|\, |d\zeta|^2 \le \sum_i \int_{U_i} |\varphi(\zeta_i)|\, |d\zeta_i|^2 = \int_Y |\varphi|$

$\left(\int_U |\varphi_i(\zeta)|\, |d\zeta|^2 = \int_{U_i} |\varphi(\zeta_i)|\, |d\zeta_i|^2 \right)$

$\pi: U_i \to U$ bi-holomorphic

$\therefore \|\pi_* \varphi\|_1 \le \|\varphi\|_1$, $\|\pi_*\| \le 1$ ☐

norm ≤ 1 (derivatives are dirct images) these maps are contracting.

Pro 5.4.17. Let X be a hyperbolic Riemann surface, $\pi: Y \to X$ is a covering map. Then the operator $\pi_*: Q^1(Y) \to Q^1(X)$ is surjective

Pf: $Y = \mathbb{H}$, $X = \mathbb{H}/\Gamma$ (Γ: Fuchsian group) $\Omega \subset \mathbb{H}$ is fundmental domain.

$\forall q \in Q^1(X)$, consider $p(w) dw^2 := \frac{12}{\pi} \left(\int_{\Omega^*} \frac{q(\bar{y}) y^2}{(z - w)^4} |dz|^2 \right) dw^2$ $\quad \Omega^* \subset \mathbb{H}^*$

Firstly $\|p\|_1 = \int_{\mathbb{H}} |p(w)|\, |dw|^2 \le \frac{12}{\pi} \int_{\mathbb{H}} \int_{\Omega^*} \frac{|q(\bar{z})| y^2}{|z - w|^4} |dz|^2 |dw|^2 = \frac{12}{\pi} \int_{\Omega^*} |q(\bar{z})| y^2 \Big(\underbrace{\int_{\mathbb{H}} \frac{|dw|^2}{|z - w|^4}}_{\frac{\pi}{4y^2}.\ (5.4.10)} \Big) |dz|^2$ (Fubini)

$$= \frac{12}{\pi} \int_{\Omega^*} |q(\bar{z})| y^2 \frac{\pi}{4y^2} |dz|^2 = 3 \int_{\Omega^*} |q(\bar{z})|\, |dz|^2 = 3\|q\|_1 \quad (q \text{ is } \Gamma\text{-invariant})$$

$\therefore \|p\|_1 \le 3\|q\|_1$, $p \in Q^1(\mathbb{H})$.

Secondly

$$\pi_* p = \frac{12}{\pi} \sum_{\gamma \in \Gamma} \gamma^* \left(\left(\int_{\Omega^*} \frac{q(\bar{z}) y^2}{(z - w)^4} |dz|^2 \right) dw^2 \right)$$

On U_0, $\varphi = \varphi_0(\zeta_0) d\zeta_0^2$, On U_i, $\zeta_i = \zeta_0 \circ \gamma_i^{-1}$, $\pi \circ \gamma_i = \pi$

$\therefore \zeta_i = \zeta \circ \pi = \zeta \circ \pi \circ \gamma_i = \zeta_0 \circ \gamma_i$; $\varphi_i = \gamma_i^* \varphi_0$

$(\gamma^* \varphi)(w) = \varphi(\gamma(w)) \cdot (\gamma'(w))^2 dw^2$

$\pi_* \varphi = \sum_{\gamma \in \Gamma} \gamma^* \varphi$

$\sum \varphi_i d\zeta^2 = \sum_i \varphi_0(\zeta_0 \circ \gamma_i) \cdot (\gamma_i')^2 d\zeta^2$

$$= \frac{12}{\pi} \sum_{\gamma\in\Gamma} \int_{\gamma(\Omega^*)} \frac{q(\bar{z})\,y^2}{(z-w)^4} |dz|^2 dw^2$$ ⑧

$$= \frac{12}{\pi} \int_{H^*} \frac{q(\bar{z})y^2}{(z-w)^4} |dz|^2 dw^2 \;=\; q(w)\,dw^2 \text{ (by 5.4.11)}$$

$\therefore$ surjective $\boxed{}$

$$(\pi_* \varphi)(v) = \sum_{y\in\pi^{-1}(x)} \varphi([D\pi(y)]^{-1}(v)) \quad v \in T_x X$$

$\gamma: Y \to Y$, $(\gamma^*\varphi)_y(u) := \varphi_{\gamma(y)}(D\gamma(y)\cdot u)$ $u\in T_yY$

If $\varphi = f(z)dz^2$ $\quad \gamma^*\varphi = f(\gamma(w))(\gamma'(w))^2 dw^2$

$\pi: H \to H/\Gamma = X$ $\quad \pi\circ\gamma = \pi \;\; \forall \gamma\in\Gamma$ $\quad \pi^{-1}(x) = \{\gamma(y)\} \;\forall y \in \pi^{-1}(x)$,

$$(\pi_*\varphi)(v) = \sum_{y\in\pi^{-1}(x)} \varphi([D\pi(y)]^{-1}(v)) = \sum_{\gamma\in\Gamma} \varphi([D\pi(\gamma(y_0))]^{-1}(v))$$

$\pi\circ\gamma = \pi$ $\quad D\pi(\gamma(y_0))\circ D\gamma(y_0) = D\pi(y_0)$

$$\therefore (\pi_*\varphi)(v) = \sum_{\gamma\in\Gamma} \varphi(D\gamma(y_0)\circ[D\pi(y_0)]^{-1}(v)) = \sum_{\gamma\in\Gamma} (\gamma^*\varphi)_{y_0}([D\pi(y_0)]^{-1}(v))$$

Summary:

Finite type X : $Q^1(X) = Q^\infty(X)$

D : $Q^1(D) \subset Q^\infty(D)$

most of general X : $Q^1(X) \subset Q^\infty(X)$

Reproducing formula: $\underbrace{(Q^1)^\Gamma(H) \quad (Q^\infty)^\Gamma(H^*)}$

$\downarrow$

Duality thm: $(Q^1(X))^T \cong Q^\infty(X^*)$

Covering:

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & X \\ \downarrow & & \downarrow \\ Q^1(Y) & \xrightarrow{\pi_*} & Q^1(X) \end{array}$$

surjective