6.4. Teichmüller Spaces

Def: A quasiconformal surface S is a topological surface with a Rieman surface structure (mod $\sim$) : Two Riemann surface structures on S define the same quasiconformal structure if the id map is quasiconformal.
(U_α, ϕ_α), $\phi_\beta \circ \phi_\alpha^{-1}$ quasiformal.
If two cpt quasiconformal surfaces S_1, S_2 are homeomorphic, then they are isomorphic as quasiconformal surfaces.

Def: 6.4.1 (Teichmüller equivalence) Let X_1 and X_2 be Riemann surfaces, S a hyperbolic quasiconformal surface, $\varphi_1 : S \to X_1$, $\varphi_2 : S \to X_2$ quasi conformal mappings. The pairs (X_1, φ_1), (X_2, φ_2) are Teichmüller equivalent if $\exists$ an analytic isomorphism $\alpha : X_1 \to X_2$ s.t. $\varphi_2 = \alpha \circ \varphi_1$ on $I(S)$ and φ_2 is homotopic to $\alpha \circ \varphi_1$ rel the ideal boundary $I(S)$.

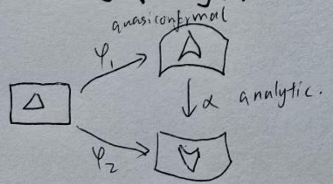
(Ideal boundary: X : hyperbolic Riemann surface D/Γ, $\bar{X} := (\bar{D} - \Lambda_\Gamma)/\Gamma$ has boundary $I(X) := (S^1 - \Lambda_\Gamma)/\Gamma$, which is 1-dim manifold. $I(X)$: ideal boundary $\simeq S^1$ or R
For Γ, $\bar{\Gamma_z} \cap S^1 = \Lambda_\Gamma(z)$, or Γ acts on $S^1 = \partial D$, $\Lambda_\Gamma(z_1) = \Lambda_\Gamma(z_2)$ $\therefore$ use Λ_Γ, called limit set.)

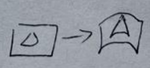
φ_2 homotopic to $\alpha \circ \varphi_1$ rel $I(S)$: $H(\cdot, 0) = \varphi_2$, $H(\cdot, 1) = \alpha \circ \varphi_1$, $\forall s \in I(S)$, $t \in [0,1]$, $H(s,t) = \varphi_2(s) = \alpha \circ \varphi_1(s)$
i.e. ~~unchanged~~ fixed on $I(S)$,

$$\begin{tikzcd} & X_1 \arrow[d, "\alpha"] \\ S \arrow[ur, "\varphi_1"] \arrow[r, "\varphi_2"'] & X_2 \end{tikzcd}$$

commutes on $I(S)$, but only commutes up to homotopy rel $I(S)$.

Def: 6.4.3. (Teichmüller space, marking) Let S be a hyperbolic quasiconformal surface. The Teichmüller space T_S modeled on S is the set of Teichmüller equivallence classes of pairs (X, φ), X is a Riemann surface, $\varphi : S \to X$ is a quasiconformal mapping. The mapping φ is a marking of X by S.



same up to α.
~~don't~~ only care about  "how it changes."

Pro/Def 6.4.4 (Teichmüller metric) Define:

$$d((X_1,\varphi_1),(X_2,\varphi_2)) := \inf_f \ln K(f)$$

$K(f)$ is the quasiconformal constant of f, f is quasiformal homeomorphism and satisfies $\varphi_2 = f\circ\varphi_1$ on $I(S)$, $\varphi_2 \simeq f\circ\varphi_1$ rel $I(S)$.

Then, d defines a metric on T_S, T_S is a complete metric space.

Pf: By Cor 4.5.10. $K(f_1\circ f_2) \le K(f_1)K(f_2)$, so take $\ln$, $\Rightarrow$ ~~tring~~ triangle inequality.

For $d=0 \Leftrightarrow$ equivalence:

Lemma 6.4.5. Let X, Y be hyperbolic Riemann surfaces, and $g: X\to Y$ a quasiconformal homeomorphism inducing $\hat{g}: I(X)\to I(Y)$ on the ideal boundary. $\forall K\geq 1$, let $F_K(X,g)$ be the set of K-quasiconformal homeomorphisms $f: X\to Y$ that coincide with $\hat{g}$ on $I(X)$ and are homotopic to g among maps that coincide with $\hat{g}$ on $I(X)$. Then $F_K(X,g)$ is cpt

Pf: Consider $f\circ g^{-1}: X\to X$, K-quasiconformal, $(f\circ g^{-1})|_{I(X)} = \hat{g}\circ\hat{g}^{-1} = id$

$f\circ g^{-1} \simeq g\circ g^{-1} = id$ rel $I(X)$.

$\therefore$ Just assume $X=Y$, g is id map.

$\forall f$, choose $f_t: X\times[0,1]\to X$ s.t. $f_0 = id_X$ $f_1 = f$ $\forall t$, on $I(X)$, $f_t = id$.

Universal covering $\pi: D\to X$, lift f on D, let $\tilde{f}_0 = id_D$, $\tilde{f}_t: D\to D$ inducing the identity on S^1, $\tilde{f} = \tilde{f}_1$

$\because$ {K-quasiconformal map : $D\to D$} is cpt under locally uniformly converge topology.

id on S^1: closed condition, $\tilde{f}\circ\gamma = \gamma\circ\tilde{f}$: closed condition. ($\tilde{f}_n \to \tilde{f}$, if every $\tilde{f}_n$ commutes γ, hence the limit f)

$\therefore$ $\tilde{F}_K(X,g)$ is cpt. ($\tilde{F}_K(X,g) := \{\tilde{f} \mid f\in F_K(X,g)\}$)

hence $F_K(X,g)$ is cpt

$\boxed{//}$

Now, if $d = \inf_f \ln K(f) = 0$, $\exists \{f_i\}$ s.t. $K(f_i)\to 1$, every f_i is quasiconformal homeomorphism $\varphi_2 = f_i\circ\varphi_1$ on $I(S)$, $\varphi_2 \simeq f_i\circ\varphi_1$ rel $I(S)$.

$\therefore$ Choose f_{i_k} has limit f. $K(f)=1$, so f is analytic isomorphism, as limit, $\varphi_2 = f\circ\varphi_1$ on $I(S)$, $\varphi_2 \simeq f\circ\varphi_1$ rel $I(S)$ as well. $\Rightarrow$ By def 6.4.1, $(X_1,\varphi_1), (X_2,\varphi_2)$ are equal.

For completeness: let $T_i = (X_i, \varphi_i)$ be a Cauchy sequence.

Choose T_{n_j} s.t. $d(T_{n_j}, T_{n_{j+1}}) < \frac{1}{2^j}$, choose $f_j : X_j \to X_{j+1}$ s.t. $k(f_j) < e^{\frac{1}{2^j}}$ and satisfies other boundary condition.

Def $g_i = f_{i-1} \circ \cdots \circ f_1 : X_1 \to X_i$, g_i induces a Beltrami form μ_i.

$\therefore T_i$ can be written as $(X_1)_{\mu_i}$

Beltrami form space

$d_{\mathcal{M}}(\mu_i, \mu_{i+1}) \leq \|\nu_i\|_\infty \leq \frac{1}{2^i}$ $\quad$ ν_i: Beltrami form corresponding f_i, $\frac{\partial f_i}{\partial \bar z} = \nu_i \frac{\partial f_i}{\partial z}$

($\left|\frac{\partial f}{\partial \bar z}\right| \leq k \left|\frac{\partial f}{\partial z}\right|$. $k = \frac{K-1}{K+1}$ $\Rightarrow$ $K(f) = \sup_z \frac{1+|\mu_f(z)|}{1-|\mu_f(z)|} = \frac{1+\|\mu_f\|_\infty}{1-\|\mu_f\|_\infty}$

here, $\|\nu_i\|_\infty = \frac{K(f_i)-1}{K(f_i)+1} < \frac{e^{\frac{1}{2^i}}-1}{e^{\frac{1}{2^i}}+1}$, use Tylar series, $\frac{e^x-1}{e^x+1} < x$, so $\|\nu_i\|_\infty \leq \frac{1}{2^i}$)

$\therefore \mu_i \xrightarrow{L^\infty} \mu_\infty$. so def $T_\infty = (id \circ \varphi_1 : S \to (X_1)_{\mu_\infty})$, it is the limit of T_i

(Teichmüller space Cauchy sequence $\Leftrightarrow$ Beltrami form space Cauchy sequence.

complete.

solve $\frac{\partial h_\infty}{\partial \bar z} = \mu_\infty \frac{\partial h_\infty}{\partial z}$, get h_∞. id: identity on top space, but change Riemann structure)

$\square$

Teichmüller space as a quotient of $\mathcal{M}(S)$:

Def: Beltrami form on a quasiconformal surface S: $((\varphi : S \xrightarrow{\text{isomorphism}} X), \mu)$.

Def 6.4.6: Let $m \in \mathcal{M}(S)$ be $((\varphi : S \to X), \mu)$. Then $\Phi_S(m) \in \mathcal{T}_S$

$$\Phi_S(m) = (\varphi : S \to X_\mu)$$

(X_μ: Riemann surface X, μ, cover $\{U_i\}$, atlas $\varphi_i : U_i \to V_i \subset \mathbb{C}$, give μ_i s.t. $\mu|_{U_i} = \varphi_i^*(\mu_i \frac{d\bar z}{dz})$

Then $\exists \psi_i(\mu) : V_i \to \mathbb{C}$ s.t. $\frac{\partial \psi_i(\mu)}{\partial \bar z} = \mu_i \frac{\partial \psi_i(\mu)}{\partial z}$. Then $\psi_i \circ \varphi_i : U_i \to \mathbb{C}$ is another atlas.

So we get another Riemann surface structure. $\Rightarrow X_\mu$.)

Note: If $((\varphi_1 : S \to X_1), \mu_1)$, $((\varphi_2 : S \to X_2), \mu_2)$ represent same element of $\mathcal{M}(S)$, i.e. $(\varphi_1 \circ \varphi_2^{-1})^* \mu_2 = \mu_1$

$\alpha := \varphi_2 \circ \varphi_1^{-1} : (X_1)_{\mu_1} \to (X_2)_{\mu_2}$ is an isomorphism

check that on $I(S)$, $\varphi_2 = \alpha \circ \varphi_1$, they are homotopic obviously.

$\therefore$ Teichmüller equivalent.

$\therefore \Phi_S$ is well-defined

Def 6.4.7. Let S be a quasiconformal surface. $QC(S)$ is the group of quasiconformal homeomorphisms of S. $QC^0(S) \subset QC(S)$ is quasiconformal homeomorphisms of S that fix $I(S)$ and $\simeq id$ rel $I(X)$

Let $X := H/\Gamma$ be a Riemann surface, Γ is a Fuchsian group, $\pi: H \to X$ is the universal covering. ~~Def~~ Let $f: X \to X$ be a quasiconformal homeomorphism $\simeq id$, f_t, $t \in [0,1]$, is the homotopy $f_0 = id$, $f_1 = f$, $\tilde{f}_t : H \to H$ is the lift
$$\begin{tikzcd} H \arrow[r, "\tilde{f}_t"] \arrow[d, "\pi"'] & H \arrow[d, "\pi"] \\ X \arrow[r, "f_t"'] & X \end{tikzcd}$$
, $\tilde{f}_0 = id$, $\tilde{f} := \tilde{f}_1$.

Pro 6.4.9. TFAE

1. f induces id on $I(X)$ and is isotopic to id rel $I(X)$;
2. f induces id on $I(X)$ and is homotopic to id rel $I(X)$;
3. $\tilde{f}$ extends to the id on $\bar{R}$

Pf: $1 \Rightarrow 2$. isotopy: f_t is homeomorphism, it is homotopy.

$2 \Rightarrow 3$. $\forall t \in [0,1]$, $\gamma \in \Gamma$, $\exists \gamma_t \in \Gamma$ s.t. $\gamma_t \circ \tilde{f}_t = \tilde{f}_t \circ \gamma$ $\quad \Gamma$ discrete. γ_t is continuous on t.

$\therefore t \to \gamma_t$ is constant. $\therefore \gamma \circ \tilde{f} = \tilde{f} \circ \gamma$.

Take $x \in R$ fixed by γ $\quad \tilde{f}(x) = \tilde{f}(\gamma(x)) = \gamma(\tilde{f}(x))$ $\quad \therefore \tilde{f}(x)$ is also a fixed point

If γ is parabolic, there is only one fixed point $\therefore \tilde{f}(x) = x$

If γ is hyperbolic, $x = \lim_{n\to\infty} \gamma^n(z)$ is attract fixed point, then: $\tilde{f}(x) = \tilde{f}(\lim \gamma^n(z)) = \lim \tilde{f}(\gamma^n(z)) = \lim \gamma^n(\tilde{f}(z)) = x$ $(\forall z \in H, \gamma^n(z) = x)$ $\quad \therefore \tilde{f}(x) = x$.

$\because \Gamma$ has dense fixed points set in Λ_Γ, so $\tilde{f}|_{\Lambda_\Gamma} = id|_{\Lambda_\Gamma}$ ($\tilde{f}$ is continuous)

$f = id$ on $I(X)$, after lifting, $\tilde{f} = id$ on $R - \Lambda_\Gamma$ $\quad (I(X) = \cancel{\phi}(S^1 - \Lambda_\Gamma)/\Gamma)$

$\therefore \tilde{f} = id$ on $\bar{R}$

$3 \Rightarrow 1$. Def $\mu = \frac{\partial \tilde{f}}{\partial \bar{z}} \Big/ \frac{\partial \tilde{f}}{\partial z}$ $\quad$ for $Im z < 0$, let $\mu(z) = \overline{\mu(\bar{z})}$

$\frac{\partial g_t}{\partial \bar{z}} = t\mu \frac{\partial g_t}{\partial z}$ $\quad 0 \le t \le 1$ $\quad$ let $g_t(0) = 0$, $g_t(1) = 1$, $g_t(\infty) = \infty$.

$\therefore g_t = id$ on R, g_1 has same Beltrami equation with $\tilde{f}$, $g_1 = \tilde{f}$

$t\mu$ is Γ-invariant, $g_t \circ \Gamma \circ g_t^{-1}$ is Fuchsian. For $t \neq 0,1$, g_t quasi-symmetric on $\mathbb{R}$.
$``\Gamma_t"$.

$\therefore g_t^{-1}|_{\mathbb{R}}$ has Douady-Earle extension $h_t: H \to H$ ~~h_t extend~~ $\frac{|f(x)-f(y)|}{|f(x)-f(z)|} \leq \eta(|\frac{x-y}{x-z}|)$

On $\mathbb{R}$: $g_t^{-1} \circ \Gamma_t \circ g_t = \Gamma$, extend to H, $h_t \circ \Gamma_t \circ h_t^{-1} = \Gamma$ $h_0 = id$ $h_1 = id$. $g_0 = id$ $g_1 = \tilde{f}$

Def $h_t \circ g_t$, is Γ-equivariant.

$$h_t \circ g_t \circ \Gamma \circ g_t^{-1} \circ h_t^{-1} = h_t \circ \Gamma_t \circ h_t^{-1} = \Gamma$$

$$F(z) = \frac{\int_{S^1} f(\xi) \frac{1-|z|^2}{|z-\xi|^2} d\xi}{\int_{S^1} \frac{1-|z|^2}{|z-\xi|^2} d\xi}$$

E extension
$(M \circ f \circ N) = M \circ E(f) \circ N$
$F \circ \gamma = \gamma \circ F$

And $h_t \circ g_t = id$ on $\mathbb{R}$. (h_t extended on $g_t^{-1}|_{\mathbb{R}}$)

$\therefore [h_t \circ g_t]: X \to X$ satisfy $[h_0 \circ g_0] = id$, $[h_1 \circ g_1] = [g_1] = [\tilde{f}] = f$

On $I(X)$, $[h_t \circ g_t]$ induces id.

$\therefore f$ induces id on $I(X)$ and isotopic to id rel $I(X)$

$\square$

$QC(S)$ acts on $\mathcal{M}(S)$: $f \in QC(S)$, $m \in \mathcal{M}(S)$, $m = ((\varphi: S \to X), \mu)$ then:

$$f^* m = ((\varphi \circ f: S \to X), \mu).$$

Prob. 4.11: Let m_1, m_2 be Beltrami forms on S, so that $\Phi_S(m_1)$ and $\Phi_S(m_2)$ are points in the Teichmüller space T_S. Then $\Phi_S(m_1) = \Phi_S(m_2)$ iff $\exists f \in QC^0(S)$ s.t. $m_1 = f^* m_2$

$T_S = \mathcal{M}(S) / QC^0(S)$.

Pf: "$\Rightarrow$" $m_1 = ((\varphi_1: S \to X_1), \mu_1)$, $m_2 = ((\varphi_2: S \to X_2), \mu_2)$

$\Phi_S(m_1) = [(\varphi_1: S \to (X_1)_{\mu_1})]$, $\Phi_S(m_2) = [(\varphi_2: S \to (X_2)_{\mu_2})]$.

$\because \Phi_S(m_1) = \Phi_S(m_2)$, $\therefore \exists$ analytic isomorphism $\alpha: (X_1)_{\mu_1} \to (X_2)_{\mu_2}$ s.t. On $I(S)$, $\alpha \circ \varphi_1 = \varphi_2$

$\alpha \circ \varphi_1$ homotopic φ_2 rel $I(S)$

Then let $f = \varphi_2^{-1} \circ \alpha \circ \varphi_1: S \to S$. f is quasiconformal, on $I(S)$, $f = \varphi_2^{-1} \circ \varphi_2 = id$.

f homotopic to id rel $I(S)$ $\therefore f \in QC^0(S)$

$f^* m_2 = f^*((\varphi_2: S \to X_2), \mu_2) = ((\varphi_2 \circ f: S \to X_2), \mu_2)$

$\because \varphi_2 \circ f = \alpha \circ \varphi_1$, so $f^* m_2 = ((\alpha \circ \varphi_1: S \to X_2), \mu_2)$ $\alpha: (X_1)_{\mu_1} \to (X_2)_{\mu_2}$ is isomorphism

so $\alpha^* \mu_2 = \mu_1$ $\therefore ((\alpha \circ \varphi_1: S \to X_2), \mu_2)$ and $((\varphi_1: S \to X_1), \mu_1)$ are same Beltrami form.

"$\Leftarrow$" $\exists f \in QC^0(S)$, s.t. $m_1 = f^* m_2$.
$m_2 = ((\varphi_2 : S \to X_2), \mu_2)$, $m_1 = f^* m_2 = ((\varphi_2 \circ f : S \to X_2), \mu_2)$

$\Phi_S(m_1) = [(\varphi_2 \circ f : S \to (X_2)_{\mu_2})]$

$\Phi_S(m_2) = [(\varphi_2 : S \to (X_2)_{\mu_2})]$.

Let $\alpha = id : (X_2)_{\mu_2} \to (X_2)_{\mu_2}$, On $I(S)$, $\alpha \circ (\varphi_2 \circ f) = \varphi_2 \circ f$. $\because f \in QC^0(S)$ fix $I(S)$,
So $\varphi_2 \circ f = \varphi_2$ on $I(S)$

$\alpha \circ (\varphi_2 \circ f) = \varphi_2 \circ f \simeq \varphi_2$ rel $I(S)$ $(f \in QC^0(S))$

$\therefore (\varphi_2 \circ f : S \to (X_2)_{\mu_2})$ are equivalent with $(\varphi_2 : S \to (X_2)_{\mu_2})$. ∎

Now, take a universal covering $\pi : H \to X$, with covering group Γ. Then $\mu \to \pi^* \mu$ maps $M(X)$ to $M^\Gamma(H)$. Use Bers' embedding: $\hat{\mu} \in M^\Gamma(C)$: $\hat{\mu}(z) = \begin{cases} \pi^*\mu(z), & \text{if } z \in H \\ 0, & \text{if } z \in H^* \end{cases}$

$f^{\hat{\mu}} : C \to C$ is the soluation of $\bar{\partial} f = \partial f \circ \hat{\mu}$, fixing $0, 1, \infty$.

So in H, $f^{\hat{\mu}}$ is quasi conformal, in H^*, $f^{\hat{\mu}}$ is conformal. (compatible with Γ).

Pro 6.4.12. $\Phi_S(m_1) = \Phi_S(m_2)$ iff $f^{\hat{\mu}_1} = f^{\hat{\mu}_2}$ on H^*

Pf: Let $m_1 = ((\varphi_1 : X \to X_1), \nu_1)$; $m_2 = ((\varphi_2 : X \to X_2), \nu_2)$

We have known that $\Phi_S(m_1) = \Phi_S(m_2)$ iff $\exists$ analytic iso $\alpha : (X_1)_{\nu_1} \to (X_2)_{\nu_2}$
s.t. $\alpha \circ \varphi_1 = \varphi_2$ on $I(X)$, $\alpha \circ \varphi_1 \simeq \varphi_2$ rel $I(X)$, $\alpha^* \nu_2 = \nu_1$

Def $g = \varphi_2^{-1} \circ \alpha \circ \varphi_1 : X \to X$. g is quasi conformal, id on $I(X)$, $g \simeq id$ rel $I(X)$

By 6.4.9. $\exists \tilde{g} : H \to H$ s.t. $\tilde{g}$ commutes Γ, $\tilde{g} = id$ on R.

$((f^{\hat{\mu}_1})^{-1} \circ f^{\hat{\mu}_2})^* \mu_2 = \mu_1$ $\quad g^* \mu_2 = \mu_1$ $\Rightarrow$ $(f^{\hat{\mu}_1})^{-1} \circ f^{\hat{\mu}_2}$ differ with g by an element in $Aut(H)$

They all fix $0, 1, \infty$, so the element is identity.

$\therefore$ Since $g = id$ on $\bar{R}$, $(f^{\hat{\mu}_1})^{-1} \circ f^{\hat{\mu}_2} = id$ on $\bar{R}$, $f^{\hat{\mu}_1} = f^{\hat{\mu}_2}$ on $\bar{R}$

$\because$ They are analytic in H^*

$\therefore f^{\hat{\mu}_1} = f^{\hat{\mu}_2}$ in H^* ∎

(We get $T_S \hookrightarrow Hol(H^*)/PSL(2,R)$ $\quad H^*$: simpler, has enough information for T_S)

Def 6.4.13. (Teichmüller Modular Group) $MCG(S) = QC(S)/QC^0(S)$.

quasiconformal ~~auto~~ homeomorphism ↺

Def 6.4.14. (Moduli Space) $Moduli(S) = T_S / MCG(S)$.

T_2: $T_1 \cong H$ $MCG(T^2) \cong SL(2,\mathbb{Z})$, $Moduli(T^2) \cong H/SL(2,\mathbb{Z})$