1.8 A first classification of Riemann surface

Aut of $\mathbb{P}^1$, $\mathbb{D}$, $\mathbb{C}$

Def. 1.8.1. (Möbius transformation)

$$\mathbb{P}^1 \to \mathbb{P}^1$$

$$z \to \frac{az+b}{cz+d} \quad ad-bc \neq 0$$

Thm 1.8.2 (Automorphism of simply connected Riemann surface)

1. $\mathrm{Aut}\,\mathbb{P}^1$ is the group of all Möbius trans
2. $\mathrm{Aut}\,\mathbb{C}$ $\quad z \to az+b \quad a \neq 0$
3. $\mathrm{Aut}\,\mathbb{D}$ $\quad z \to \dfrac{az+b}{\bar{b}z+\bar{a}}, \ |a|^2 > |b|^2$

Pf: 1. Möbius trans 3 distint points to 3 points.

$$z \to \frac{(z-a)(c-b)}{(z-b)(c-a)} : a, b, c \to 0, \infty, 1$$

Let α be $\forall\, \mathrm{Aut}\,\mathbb{P}^1$ $\quad \alpha(0), \alpha(1), \alpha(\infty)$

$\exists$ Möbius β s.t $\beta(\alpha(0))=0 \quad \beta(\alpha(1))=1 \quad \beta(\alpha(\infty))=\infty$

$\beta\circ\alpha=\gamma \quad \gamma(0)=0 \quad \gamma(1)=1 \quad \gamma(\infty)=\infty$

analytic on $\mathbb{C}$, ∞: simple pole

$w=\frac{1}{z}$, $w \to 0$ $\quad h(w)=\dfrac{1}{\gamma(\frac{1}{w})}$ $\quad h(w) \to 0$

$h(w)=c_1 w + c_2 w^2 + \cdots$, $c_1 \neq 0$

$$\gamma(z)=\frac{1}{h(\frac{1}{z})}=\frac{1}{\frac{c_1}{z}+\frac{c_2}{z^2}+\cdots}=\frac{z}{c_1}\left(1-\frac{c_2}{c_1}\frac{1}{z}+O\left(\frac{1}{z^2}\right)\right)$$

$$\frac{z}{c_1} \cdot \frac{1}{1-\bigcirc}$$

$$\frac{z}{c_1}+\frac{c_2}{c_1^2}+O\left(\frac{1}{z}\right)$$

$$\gamma(z)=az+b+O(1)$$

$\gamma(z)=az+O(1)$

$h(z)=\gamma(z)-az$ $\to \mathbb{C}$

$h(z)\equiv c$

$\gamma(z)=az+c$

$c=0$

$a=1$

$\gamma(z)=z=\beta\circ\alpha$

2 Let $f: \mathbb{C} \to \mathbb{C}$ auto

$z \to \infty$ $f(z) \to \infty$

$w = \frac{1}{z}$ $g(w) = \frac{1}{f(\frac{1}{w})}$ $w \to 0$ $g \to 0$

$g(w) = c_1 w + \cdots$

$f(z) = \frac{1}{g(w)} = \frac{1}{c_1 w + \cdots} \sim \frac{1}{c} z$

$f \to \tilde{f}$ on P^1 $\underline{\tilde{f}(\infty) = \infty}$

$z \to az + b$
$a \neq 0$

Scanned with CamScanner

3. let α be Aut on D $\quad \bar{D}$

$\tilde{\alpha}: P^1 \to P^1 \quad \tilde{\alpha}(z) = \begin{cases} \alpha(z) \\ \dfrac{1}{\overline{\alpha(\frac{1}{\bar{z}})}} & |z|>1 \end{cases}$

$\tilde{\alpha}(z)$ continuous

α maps $\partial D \to \partial D$

$|z| \to |f(z)|$

$z \to \dfrac{dz+b}{cz+a}$

$e^{i\theta} \dfrac{z-z_0}{1-\bar{z}_0 z}$

$z \to \dfrac{az+b}{\bar{b}z+\bar{a}} \quad |a|^2 > |b|^2$

$|a|^2 - |b|^2 = 1$

$|z|=1$

$|az+b| = |cz+d|$

$\bar{a} = d$

$c = \bar{b}$

$a = \sqrt{\dfrac{\overline{\alpha'(0)}}{1-|\alpha^{-1}(0)|^2}}$

$b = \alpha^{-1}(0)\bar{a}$

3: let α be Aut on D

$\tilde{\alpha}: P^1 \to P^1$ $\quad \tilde{\alpha}(z) = \begin{cases} \alpha(z) & D \\ \dfrac{1}{\overline{\alpha\left(\frac{1}{\bar z}\right)}} & |z|>1 \end{cases}$

$\tilde{\alpha}(z)$ continuous

α maps $\partial D \to \partial D$

$e^{i\theta}\dfrac{z-z_0}{1-\bar{z}_0 z}$

$|z| \mapsto |\alpha(z)|$

$z \to \dfrac{az+b}{cz+d}$

$|z|=1$

$|az+b| = |cz+d|$

$\bar{a} = d$

$c = \bar{b}$

$z \to \dfrac{az+b}{\bar{b}z+\bar{a}}$ $\quad |a|^2 > |b|^2$

$|a|^2 - |b|^2 = 1$

$a = \sqrt{\dfrac{\alpha'(0)}{1-|\alpha^{-1}(0)|^2}}$

$b = \alpha^{-1}(0)$

Ex1.8.4 (1) $\pi: GL(2,\mathbb{C}) \to \mathrm{Aut}(P^1)$

$PGL(2,\mathbb{C}) = \underbrace{GL(2,\mathbb{C})}_{4} \Big/ \underbrace{\ker\pi}_{\{\mathbb{C}^* I_2\}} \cong \mathrm{Im}\,\pi = \mathrm{Aut}\,P^1$

$\dim_{\mathbb{C}} \mathrm{Aut}\,P^1 = 4-1 = 3$

(2) $\mathrm{Aut}\,\mathbb{C}$ $\quad az+b$ $\quad 2$

$a \neq 0$

(3) $SU(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in GL(2,\mathbb{C}) : |\alpha|^2 - |\beta|^2 = 1 \right\}$

$\alpha = x+iy$

$\beta = u+iv$

$x^2+y^2-u^2-v^2=1$

$\mathrm{Aut}\,D \cong SU(1,1)/\{\pm I\}$

$4-1=3$

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If X a Riemann surface universal covering

P^1, $\mathbb{C}$, D

Thm 1.8.8. 1. The P^1 is the universal covering of itself only

2. $\mathbb{C}$ is ________ itself, $\mathbb{C}-\{a\}$, all cpt

Riemann surface $\cong T^2$

3. All other Riemann surface ________ analytically iso to D

(hyperbolic)

Pf: let $\tilde{X}$ be

$z = \frac{az+b}{cz+d}$

deck trans

$f(\tilde{x}) = \tilde{x}$

f trivial

P^1 is only

2. $z \to az$

$\Gamma \leq (\mathbb{C}, +)$

$\{0\}$ $\{n$

$\langle c_1, c_2 \rangle =$

surface
covering

of itself
only
all cpt

analytically

Pf: let X be the covering space of $\mathbb{P}^1$ $z \to \frac{az+b}{cz+d}$

$z = \frac{az+b}{cz+d}$ 2 fixed point

deck trans $f \neq id$ $\tilde{x} \in \tilde{X}$ $f(\tilde{x}) = \tilde{x}$ $p \circ f|_{\tilde{x}} = p$ $p \circ id|_{\tilde{x}} = p$

$f(\tilde{x}) = \tilde{x} = id(\tilde{x})$

f trivial

$\mathbb{P}^1$ is only universal covering of $\mathbb{P}^1$

2. $z \to az+b \Rightarrow a=1$ $b \neq 0$ $z \to z+b$

$\Gamma \leq (\mathbb{C}, +)$

$\{0\}$ $\{nc\} = \langle c \rangle$

$\langle c_1, c_2 \rangle = \{n_1c_1 + n_2c_2 \mid \frac{c_1}{c_2} \notin \mathbb{Q}\}$

$\{0\}: X = \mathbb{C}/\Gamma \cong \mathbb{C}$

$\langle c \rangle$: $\mathbb{C}/\langle c \rangle \cong \mathbb{C} - \{0\}$ $(\varphi: \mathbb{C} \to \mathbb{C} \setminus \{0\})$

$\varphi(z) = e^{2\pi z i/c}$ $\varphi(z+c) = \varphi(z)$ $\tilde{\varphi}: \mathbb{C}/\Gamma \to \mathbb{C} \setminus \{0\}$

$X \cong \mathbb{C} \setminus \{0\}$

$\Gamma: \langle c_1, c_2 \rangle$ $X \cong \mathbb{R}^2/\mathbb{Z}^2 = \underline{T}^2$

$\exists, D$

$X \cong T^2$ $\underline{\pi_1(X)} \cong \mathbb{Z} \times \mathbb{Z}$ free Abelian of rk2

$\Gamma \cong \mathbb{Z} \times \mathbb{Z}$

face
ring

t itself
only
cpt

alytically

Aut D non-trivial

f, g $\quad e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z}$

$\mathrm{Fix}f = \{p, q\}$ $\quad p\in D$ $\quad q\notin D$

$f\circ g|_p = g\circ f(p) = g(p)$

$f(p)\in D$ $\quad g f(p) = g(p)\in D$

$\therefore g(p) = p$

$g(q)\in \mathrm{fix}f$

$g(q) = q$

$\therefore \mathrm{Fix}f = \mathrm{Fix}g$

1. have one fixed point: generate a discrete subgroup iff have a common power

$f = e^{i\theta}z$ $\quad g = e^{i\psi}z$ $\quad k\theta - l\psi = 2\pi m$ $\quad \underline{f^k = g^l}$

2. two fixed point

$f = z$ $\quad g = \frac{1}{z}$