5.4. Space of Quadratic Differentials

Pro 5.4.9 (Reproducing formula for $Q^{\infty}(X)$)

Let $q \in (Q^{\infty})^{\Gamma}(H^*)$ $\quad q(w)\,dw^2 = \frac{12}{\pi}\left(\int_H \frac{q(\bar{z})\,y^2}{(z-w)^4}\,|dz|^2\right)dw^2$

Pro 5.4.11 (~ $Q^1(X)$)

Let $q \in (Q^1)^{\Gamma}(H)$ $\quad q(w)\,dw^2 = \frac{12}{\pi}\left(\int_{H^*} \frac{q(\bar{z})\,y^2}{(z-w)^4}\,|dz|^2\right)dw^2$

Pf: $|q(z)|\,y^2 = \left(|q(z)|\,|dz|^2\right)\left(\frac{y^2}{|dz|^2}\right)$: Γ-inv

measure: $\frac{|dz|^2\,|dw|^2}{|z-w|^4}$ inv under Aut H, acting diagonally $H \times H^*$

$\int_{\Omega^*}\left(\int_{H}\right.$ …

$= \int_{\Omega^*}\left(\sum\right.$ …

$= \sum_{\gamma\in\Gamma} \int_{\Omega}$ …

$z \to \gamma(z)$ …

$= \sum_{\gamma\in\Gamma}$ …

$= \sum_{\gamma\in\Gamma} \int$ …

$z|^2)dw^2$

$$\int_{\Omega^*}\left(\int_H \frac{|q(\bar z)|\,y^2}{(z-w)^4}|dz|^2\right)|dw|^2$$

$$=\int_{\Omega^*}\left(\sum_{\gamma\in\Gamma}\int_{\gamma^{-1}(\Omega)}\frac{|q(\bar z)|\,y^2}{(z-w)^4}|dz|^2\right)|dw|^2$$

$$=\sum_{\gamma\in\Gamma}\int_{\Omega^*}\int_{\gamma^{-1}(\Omega)}\_\_\_\_\_\_$$

$|z|^2)dw^2$

$z\to\gamma(z)$

$$=\sum_{\gamma\in\Gamma}\int_{\gamma(\Omega^*)}\int_{\Omega}\frac{|q(\bar z)|\,y^2}{(z-w)^4}|dz|^2|dw|^2$$

ly $\Gamma\backslash H\times H^*$

$$=\sum_{\gamma\in\Gamma}\int_{\Omega}|q(\bar z)|\,y^2\left(\int_{\gamma(\Omega^*)}\frac{|dw|^2}{|z-w|^4}\right)|dz|^2$$

$$=\frac{\pi}{4}\int_{\Omega}|q(\bar z)|\,|dz|^2=\frac{\pi}{4}\|q\|_1<\infty$$

$$(f\times f)^* F(z,w) = \bar{F}(z,w)$$

$$(f\times f)^* \bar{F}(z,w)\, dz^2 \otimes dw^2 = \bar{F}(f(z_1), f(w_1))\, (f'(z_1))^2 (f'(w_1))^2\, dz_1^2 \otimes dw_1^2$$

$w|^2$

$=$

$w|^2$

$dz|^2$

$$\int_{\Omega^*}\left(\int_H \frac{|q(\bar z)|y^2}{(z-w)^4}|dz|^2\right)|dw|^2$$
$$=\int_{\Omega^*}\left(\sum_{\gamma\in\Gamma}\int_{\gamma^{-1}(\Omega)}\frac{|q(\bar z)|y^2}{(z-w)^4}|dz|^2\right)|dw|^2$$
$$=\sum_{\gamma\in\Gamma}\int_{\Omega^*}\int_{\gamma^{-1}(\Omega)} \underline{\qquad\qquad} \quad \int_H$$
$z\to\gamma(z)$
$$=\sum_{\gamma\in\Gamma}\int_{\gamma(\Omega^*)}\int_{\Omega}\frac{|q(\bar z)|y^2}{(z-w)^4}|dz|^2|dw|^2$$
$$=\sum_{\gamma\in\Gamma}\int_{\Omega}|q(\bar z)|y^2\left(\int_{\gamma(\Omega^*)}\frac{|dw|^2}{|z-w|^4}\right)|dz|^2$$
$$=\frac{\pi}{4}\int_\Omega|q(\bar z)||dz|^2=\frac{\pi}{4}\|q\|_1<\infty$$

5.4.12 (Duality theorem) $\forall$ hyperbolic Riemann surface X,
$Q^1(X)\times Q^\infty(X^*)\longrightarrow\mathbb{C}:\langle q,p\rangle\longrightarrow\int_X\frac{qp}{\rho^2}$ induces an isomorphism $Q^\infty(X^*)\longrightarrow(Q^1(X))^*$, $p\longmapsto\langle\cdot,p\rangle$

Pf: Def $T: Q^\infty(X^*)\longrightarrow(Q^1(X))^*$, $T(p)(q)=\langle q,p\rangle$
$\forall\alpha\in(Q^1(X))^*$ α extends to $(LQ^1(X))^*$, $\tilde\alpha\in(LQ^1(X))^*$
$\exists p\in LQ^\infty(X^*)$ s.t. $\forall q\in LQ^1(X)$, $\tilde\alpha(q)=\langle q,p\rangle$

Lemma: $\forall q\in(LQ^1)^\Gamma(H)$, $p\in(LQ^\infty)^\Gamma(H^*)$
there is $\langle P^1_q,p\rangle=\langle q,P^\infty_p\rangle$
$\left(P^1(q)=\frac{12}{\pi}\left(\int_{H^*}\frac{q(\bar z)y^2}{(z-w)^4}\right)dw^2,\ P^\infty(p)\longrightarrow\right)$

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$w = u + iv$

Pf: $\langle P^1_P q, P\rangle = \int_\Omega \left(\int_{H^*} \frac{q(\bar{z})y^2}{(\bar{z}-w)^4}|dz|^2\right) p(\bar{w}) v^2 |dw|^2$

$= \int_\Omega \left(\sum_{\gamma\in\Gamma} \int_{\gamma^{-1}(\Omega^*)} \text{\_\_\_\_\_\_}\right)$

$= \sum_{\gamma\in\Gamma} \int_\Omega \int_{\gamma^{-1}(\Omega^*)} \text{\_\_\_\_\_\_}$

$z \to \gamma(z)$

$= \sum_{\gamma\in\Gamma} \int_{\gamma(\Omega)} \int_{\Omega^*} \text{\_\_\_\_\_\_}$

$= \int_H \int_{\Omega^*}$

$= \int_{\Omega^*} \int_H \text{\_\_\_\_\_\_}$

$= \langle q, P^\infty_P \rangle$

For

$q =$

$\therefore \exists$

tials $\quad w = u + iv$

$^2 (dw)^2$

$p^{\infty}{}_{p} \in Q^{\infty}(X^{*})$

$T(p^{\infty}{}_{p})(q) = \langle q, p^{\infty}{}_{p} \rangle = \langle p' q, p \rangle = \langle q, p \rangle = \alpha(q)$

$\therefore T(p^{\infty}{}_{p})(\cdot) = \alpha(\cdot)$

$\therefore T$ is surjective.

For $p \in Q^{\infty}(X^{*})$ $\quad p \neq 0$, $\Rightarrow \exists q' \in LQ'(X)$, s.t $\langle q', p \rangle \neq 0$

$q = P' q' \in Q'(X)$ $\quad \langle q, p \rangle = \langle P' q, p \rangle = \langle q', p^{\infty}{}_{p} \rangle$

5.4.9 $\Rightarrow p^{\infty}{}_{p} = p$

$\therefore \exists q$ $\quad \therefore \langle q, p \rangle = \langle q', p \rangle \neq 0$

s.t $\langle q, p \rangle \neq 0$

$\therefore p \neq 0 \Rightarrow T(p) \neq 0$

$\therefore$ injective

$\therefore$ bij + linear

$\therefore$ isomorphism

5.4.12 (1)

$Q'(X) \times Q$

n isomor

Def T

$(Q'$

LQ

mr

$(P' ia$

$\rangle, p\rangle = \alpha(q)$

s.t $\langle q', p\rangle \neq 0$

$q', p\,\hat{p}\rangle$

Let $\pi: Y \to X$ covering map, $\pi_*: Q^1(Y) \to Q^1(X)$ called Poincaré operator.

Def 5.4.15. (direct image operator) If $v \in T_x X$

Then $(\underline{\pi_*}\varphi)(v) = \sum_{y \in \pi^{-1}(x)} \varphi([D\pi(y)]^{-1}(v))$

for $U \subset X$, $\zeta: U \to \mathbb{C}$ is local coordinate, π maps U_i of $\pi^{-1}(U)$ iso to U. $\zeta_i = \zeta \circ \pi|_{U_i}$ $\varphi|_{U_i} = \varphi_i(\zeta_i)\, d\zeta_i^2$

$$(\pi_*\varphi)|_U = \left(\sum \varphi_i\right) d\zeta^2$$

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Pro 5.4.16 π_* is continuous, linear, $\|\pi_*\| \le 1$

Pf: $\int_U |\pi_* \varphi| = \int_U |\sum_i \varphi_i(\zeta)| \, |d\zeta|^2 \le \sum_i \int_{U_i} |\varphi(\zeta_i)| \, |d\zeta_i|^2 = \int_Y |\varphi|$

$\|\pi_* \varphi\|_1 \le \|\varphi\|_1 \qquad \|\pi_*\| \le 1$

Pro 5.4.17. X hyper. $\pi: Y \to X$ $\pi_*: Q^1(Y) \to Q^1(X)$ is surjective.

Pf. $Y = H$, $X = H/\Gamma$ Fuchsian group

$\Omega \subset H$

$\forall q \in Q^1(X)$, $p(w) dw^2 := \frac{12}{\pi} \left(\int_\Omega \frac{q(y) y^2}{(z-w)^4} |dz|^2 \right) dw^2$

$\|P\|_1 = \int_H |P(w)|$

$\therefore P$

$\pi_* P = \frac{12}{\pi}$

$= \frac{12}{\pi} \sum$

$= \frac{12}{\pi} \int$

$= \int_Y |\varphi|$

$)'(x))$

$\|P\|_1 = \int_H |P(w)|\,|dw|^2 \le \frac{12}{\pi} \int_H \int_{\Omega^*} \frac{|q(\bar z)|\, y^2}{|z-w|^4} |dz|^2 |dw|^2$

$= \frac{12}{\pi} \int_{\Omega^*} |q(\bar z)|\, y^2 \Big(\underbrace{\int_H \frac{|dw|^2}{|z-w|^4}}_{\frac{\pi}{4y^2}} \Big) |dz|^2$

$= 3 \int_{\Omega^*} |q(\bar z)|\, y^2$

$= 3\|q\|_1$

$\|P\|_1 \le 3\|q\|_1$

$\therefore P \in Q^1(Y)$

$\pi_* P = \frac{12}{\pi} \sum_{\gamma \in \Gamma} \gamma^* \left(\left(\int_{\Omega^*} \frac{q(\bar z)}{(\bar z - w)^4} |dz|^2 \right) dw^2 \right)$

$= \frac{12}{\pi} \sum_{\gamma \in \Gamma} \int_{\gamma(\Omega^*)} \underline{\qquad}$

$= \frac{12}{\pi} \int_{H^*} \underline{\qquad}$

$= q(w)\, dw^2 \Rightarrow$ surj

$t: Y \to X$

Poi

$(d$

$\pi_* Y$

$U \subset X$

$U.$

Summary:

Finite type X: $Q^1(X) = Q^\infty(X)$

D: $Q^1(D) \subset Q^\infty(D)$

general X: $Q^1(X) \subset Q^\infty(X)$

$$\begin{array}{ccc} \cong & & \cong \\ (Q^1)^\Gamma(H) & & (Q^\infty)^\Gamma(H) \end{array}$$

Reproducing formula: $(Q^1)^\Gamma(H) \quad (Q^\infty)^\Gamma(H)$

Duality thm: $(Q^1(X))^* \cong Q^\infty(X^*)$

cover:
$$\begin{tikzcd} Y \arrow[r, "\pi"] \arrow[d] & X \arrow[d] \\ Q^1(Y) \arrow[r, "\pi_*", "\text{surjective}"'] & Q^1(X) \end{tikzcd}$$