Chapter I: The uniformization theorem

1.1. Two statements of the thm.

Thm 1.1.1. (The uniformization thm) A simply connected Riemann surface is isomorphic to either $\mathbb{P}^1$, $\mathbb{C}$, $\mathbb{D}$

cpt. Liouvill's thm

AIM (1.2–1.7) Thm 1.1.2 If a Riemann surface is connected and non-cpt, $H^1(X,\mathbb{R})=0$, then it is isomorphic to $\mathbb{C}$ or $\mathbb{D}$

$$\text{Thm } 1.1.2 \xrightarrow{\text{stronger}} \text{Thm } 1.1.1.$$

$$H^1(X,\mathbb{R}) = \mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{R})$$

$$H_1 \cong \pi_1(X,x)^{ab} = \cancel{H_1(X)}$$

$$\mathrm{Hom}(\pi_1(X,x)^{ab},\mathbb{R}) = \mathrm{Hom}(\pi_1(X,x),\mathbb{R})$$

$$\underset{0}{H^1(X,\mathbb{R})} = \mathrm{Hom}(\underset{0}{\pi_1(X)},\mathbb{R})$$

$$[g,h]=g^{-1}h^{-1}gh$$

$$X \text{ simply connected} \Rightarrow H^1(X,\mathbb{R})=0$$

$$\not\Leftarrow$$

$$\text{Thm } 1.1.2 \xleftarrow{\text{weaker}} \text{Thm } 1.1.1 \quad X \text{ non-cpt.}$$

Thm 1.1.3. Let X ~~is~~ be a connected cpt Riemann surface s.t. $H^1(X,\mathbb{R})=0$. Then X is iso to $\mathbb{P}^1$.

$$\left.\begin{array}{l} 1.2-1.7 \Rightarrow \text{Thm } 1.1.2 \\ \text{"Thm } 1.1.2 \Rightarrow \text{Thm } 1.1.3\text{"} \end{array}\right] \Rightarrow \text{Thm } 1.1.2 + \text{Thm } 1.1.3 \Rightarrow \text{Thm } 1.1.1.$$

Pf (From thm 1.1.2 ⇒ thm 1.1.3) It's enough to prove $x \in X$,

$X' := X - \{x\}$ is iso to $\mathbb{C}$

[let iso $f: X' \to \mathbb{C}$, $F: X \to \underset{\overset{\shortparallel}{P^1}}{\mathbb{C}\cup\{\infty\}}$ $F(y) = f(y)$ on X'

$F(x) = \infty$ local chart z: $z(x) = 0$ $h = F \circ z^{-1}$

$g = \frac{1}{h}$ $|g(z)| \to 0$ when $z \to 0$.

By removable singularity thm, g extends to $z = 0$, $g(0) = 0$

$\Rightarrow F$ biholomorphic $\therefore X \cong P^1$].

From $X' \cong \mathbb{C}$ we get Thm 1.1.3.

AIM: $X' \cong \mathbb{C}$

First: Show $H^1(X', R) = 0$

Lemma 1.1.4 If a cpt connected surface X s.t. $H^1(X, R) = 0$ then $\forall x \in X$, X' satisfies $H^1(X', R) = 0$

Pf: Let U be a neighborhood of x homeomorphic to a disc.

Mayer-Vietoris exact sequence of $(X; A, B)$ coefficient group G:

$$\cdots \to H^n(X; G) \to H^n(A, G) \oplus H^n(B, G) \to H^n(A \cap B, G) \to H^{n+1}(X; G) \to \cdots$$

$\therefore$ $(X; X', U)$ gives:

$$\cdots \to \underset{\overset{\shortparallel}{0}}{H^1(X, R)} \to H^1(X', R) \oplus \underset{\overset{\shortparallel}{0}}{H^1(U, R)} \to \underset{\downarrow\ R}{H^1(\underbrace{X' \cap U}_{\underset{S^1}{\simeq}}, R)} \to \underset{\underset{R}{\wr\shortparallel}}{H^2(X, R)} \to \cdots$$

Thm 1.1.2

$H^1(S^1, R) \cong R$

U open disc $\Rightarrow H^k(U; R) = 0$

$\Downarrow$

$$0 \to H^1(X', R) \xrightarrow{\alpha} R \xrightarrow{\beta} R \to 0$$

$\mathrm{Im}\,\beta = \ker(R \to 0) = R$

Now, suppose X' is not iso to $\mathbb{C}$, X' connected, non-cpt

$H^1(X', \mathbb{R}) = 0$; From 1.1.2. $X' \cong D$, X is one-point compactification of D.

But:

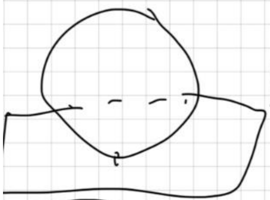
Lemma 1.1.5 $\bar{D} := D \cup \{\infty\}$ doesn't carry Riemann surface structure coinciding with the structure of D

Pf: If exists, $z: D \to \mathbb{C}$, near ∞, in $U_r = \{\infty\} \cup \{z \in D; |z| > r\}$ $r \in (0,1)$.
a neighborhood of ∞. On $U_r \setminus \{\infty\}$, z is bounded.

$\exists$ holomorphic $\tilde{z}: \bar{D} \to \mathbb{C}$ $\tilde{z}|_D = z$ $\tilde{z}$ continuous on ∞.

$\tilde{z}(\infty)$ doesn't exist. $\Rightarrow\Leftarrow$.

$X' \not\cong D \Rightarrow X' \cong \mathbb{C} \Rightarrow$ Thm 1.1.3



$P^1 \cong \mathbb{C} \cup \{\infty\}$.

Thm 1.1.2 $\Rightarrow$ 1.1.3

1.2–1.7. $\Rightarrow$ Thm 1.1.1

1.2. Subharmonic and harmonic funs.

Def 1.2.1 (Harmonic, subharmonic, superharmonic) Let X is a Riemann surface. A continuous fun $f: X \to \mathbb{R}$ is harmonic if $\forall\ \varphi: U \to X$ with $U \subset \mathbb{C}$ open, circle $|\zeta - \zeta_0| = r$ in U.

$$\left(\frac{1}{2\pi}\int_0^{2\pi} f(\varphi(\zeta_0 + re^{i\theta}))\,d\theta\right) - f(\varphi(\zeta_0)) = 0$$

$\geq$ (subharmonic)

$\leq$ (superharmonic).

$$\underline{u(z_0)} = \frac{1}{2\pi}\int_0^{2\pi} \underline{u(e^{i\psi})}\frac{1-|z_0|^2}{|z_0 - e^{i\psi}|^2}\,d\psi$$

Def 1.2.2. (Bounded Perron family) let M ~~is~~ be a real number. A set of subharmonic funs $\mathcal{F}$ on X. is called a ~~fee~~ Perron family bounded by M if:

1. if $f \in \mathcal{F}$, then $|f| \leq M$ 2. If $f_1, f_2 \in \mathcal{F}$, then $\sup(f_1, f_2) \in \mathcal{F}$
3. let $f \in \mathcal{F}$ be a function and let D be a disc in the image of a chart of X. If f_1 is a continuous fun that is f outside D, harmonic in D, then $f_1 \in \mathcal{F}$

Pro 1.2.3 (Perron thm). If $\mathcal{F}$ is a nonempty bounded Perron Family on a Riemann surface X, then $F := \sup \mathcal{F}$ is harmonic

Pf: Choose $z_0 \in X$, U is a neighborhood of z_0, $\zeta: U \to \mathbb{C}$ is a chart s.t. $\bar{\Delta} := \{|\zeta| \leq 1\}$ is a cpt disc in U. $\{x \mid |\zeta(x)| \leq 1\}$

$F = \sup \mathcal{F}$ $\quad \exists \{f_n\} \in \mathcal{F}$ s.t. $\sup f_n(z_0) = F(z_0)$.

Use $\sup(f_1, \cdots, f_n)$ replace $f_n \Rightarrow \{f_n\} \uparrow$ $\sup f_n(z_0) \to F(z_0)$

Let $\hat{f}$ be a continuous fun

let $\tilde{f}_n$ be a continuous fun, $\tilde{f}_n = f_n$ outside Δ, harmonic in Δ

$\Leftarrow \tilde{f}_n \in \mathcal{F}$, $\tilde{f}_n$ harmonic f_n subharmonic

$\tilde{f}_n \geq f_n$ $F(z_0) = \sup f_n(z_0) \Rightarrow F(z_0) = \sup \underline{\tilde{f}_n}(z_0)$.

$\sup \tilde{f}_n$ is harmonic in Δ.

AIM: Prove $F = \sup \mathcal{F} = \sup \tilde{f}_n$ in Δ

$\geq \sup \tilde{f}_n$

Let $z_1 \in \Delta$, similarly, construct $\{g_n\} \subset \mathcal{F}$, s.t. $\sup g_n(z_1) = F(z_1)$

Let $h_n = \sup(f_n, g_n)$. $h_n \in \mathcal{F}$, $\tilde{h}_n = h_n$ outside Δ, harmonic in Δ

$\tilde{h}_n \geq h_n \geq f_n$ $\tilde{h}_n \geq g_n$ $\sup \tilde{h}_n$ is harmonic.

Let $d(z) = \sup \tilde{h}_n(z) - \sup \tilde{f}_n(z) \geq 0$ in Δ.

At z_0. $\sup \tilde{f}_n(z_0) = F(z_0)$ $\sup \tilde{h}_n(z_0) \leq F(z_0)$

$$\overset{\shortparallel}{\sup \mathcal{F}(z_0)}.$$

$\therefore d(z_0) \leq 0 \Rightarrow d(z) = 0 \big|_{z_0}$.

~~min~~ minimum at z_0. $d \equiv 0$ $\sup \tilde{h}_n = \sup \tilde{f}_n$ in Δ.

$F(z_1) = \sup g_n(z_1) \leq \sup h_n(z_1) \leq \sup \tilde{h}_n(z_1) = \sup \tilde{f}_n(z_1) \leq F(z_1)$

$\therefore F(z_1) = \sup \tilde{f}_n(z_1)$.

z_1 arbitrary. $\forall z \in \Delta$, $F(z) = \sup \tilde{f}_n(z)$.

Since $\sup \tilde{f}_n(z)$ is harmonic in Δ, F is harmonic in Δ.

By arbitrary of z_0, F is harmonic in X $\boxed{//}$

Pro 1.2.4. (Existence of harmonic functions) Let $m \le M$ be two real numbers and X is a subsurface of a Riemann surface Y $\partial X \neq \emptyset$. Let $f: \partial X \to [m, M]$ be a bounded continuous fun. $\exists$ continuous fun $\tilde{f}: X \to [m, M]$ that is harmonic on $\mathrm{int}X$, equals f on the boundary of X.

Pf: $\mathcal{F} = \{g: X \to [m, M]$, subharmonic on the interior, $g \le f$ on $\partial X\}$.

$g \equiv m \in \mathcal{F}$ $\quad \mathcal{F} \neq \emptyset$. $\quad \sup = \tilde{f}$ harmonic on $\mathrm{int}X$.

AIM: $\tilde{f}$ continuous on X, $\tilde{f} = f|_{\partial X}$.

Let $x \in \partial X$, U is a neighborhood of x, $x \in U \subset Y$. $\zeta: U \to \mathbb{C}$ $\zeta(x) = 0$. $\{x_n\} \in U - X$, tending to X on the line orthogonal to ∂X.

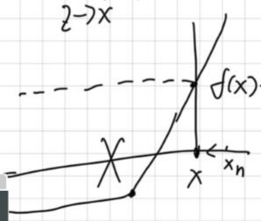
$$\forall \varepsilon > 0. \quad h_{n,\varepsilon}(z) = \sup\Big(m, \underbrace{\ln\Big|\frac{\zeta(x_n)}{\zeta(z) - \zeta(x_n)}\Big|}_{\downarrow \atop -\infty} + f(x) - \varepsilon\Big) \in \mathcal{F} \quad (n \to \infty)$$

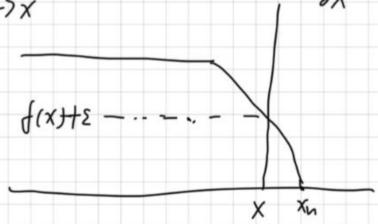
$$K_{n,\varepsilon}(z) = \inf\Big(M, \ln\Big|\frac{\zeta(z) - \zeta(x_n)}{\zeta(x_n)}\Big| + f(x) - \varepsilon\Big) \quad \text{superharmonic} \quad \ge f \ (n \to \infty)$$

$g \in \mathcal{F}$, $g < K_{n,\varepsilon}$. $\quad \forall x \in \partial X$, $\lim\limits_{z \to x} \tilde{f}(z) = f(x)$

$$\liminf_{z \to x} \tilde{f}(z) \ge \liminf h_{n,\varepsilon} = \sup(m, f(x) - \varepsilon) \ge f(x) - \varepsilon. \ \forall \varepsilon.$$

$\liminf\limits_{z \to x} \tilde{f} \ge f \qquad \limsup\limits_{z \to x} \tilde{f} \le f \Rightarrow \tilde{f} = f|_{\partial X}$ ▨



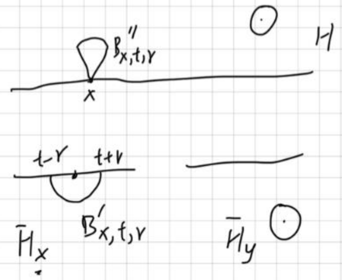


1.3 Rado's thm.

AIM: Every connected Riemann surface is A_2. second-countable.

Example (A surface that is not A_2)

$X := H \cup \bigsqcup_{x\in\mathbb{R}} \bar{H}_x$ $\quad H := \{z \mid \mathrm{Im}\, z > 0\}$. $\quad \bar{H}_x$: closed lower half-plane.



$t-r < \cot\arg(z-x) < t+r$

Pro 1.3.2. 1. If the universal covering space of a Riemann surface X is A_2, then so is X.

2. If X is connected Riemann surface, $\exists$ non-const analytic fun. $f: X \to \mathbb{C}$, then X is A_2.

Pf: 1. universal covering is local homoemorphism.

2. Let $\mathcal{B}$ be a countable top. basis of $\mathbb{C}$, $\forall U \in \mathcal{B}$ consider components of $f^{-1}(U)$ that are finite cover of their images. i.e. component V, $f: V \to f(V)$ 分支点 ≤ 1.

Let those V be $\mathcal{B}'$. AIM1: Show $\mathcal{B}'$ is top basis.

$\forall x \in X$, W includes x, U_x is a neighborhood of x, s.t. $f: U_x \to f(U_x)$ is finite sheeted covering map. $G = W \cap U_x$.

$\exists U'$ s.t. $f(x) \in U' \subset\subset \mathbb{C}$, $f^{-1}(U') \subset G$.

$\exists U \in \mathcal{B}$ s.t. $f(x) \in U \subset U' \cap f(G)$ $\quad U \subset f(G)$.

$f^{-1}(U) \subset f^{-1}(U') \subset G$.

$) \subset f(U_x)$ $\therefore f^{-1}(U) \cap U_x = f^{-1}(U)$ has finite component.

$\because$ V is the component including x, $V \in B'$

$V \subset f^{-1}(U) \subset G \subset W$ $\therefore x \in V \subset W$

$\therefore B'$ is top basis.

AIM2: Show B' is countable.

Firstly, If $V \in B'$ intersects with uncountable $W \in B'$, then

B countable $\Rightarrow \exists U \in B$, s.t. $f^{-1}(U)$ has uncountable components

intersects V., let them be W_i $i \in I$

$\therefore \{W_i \cap V\}$ is disjoint nonempty open set in V.

$V \cong$ open disc $\Rightarrow$ seperable. it won't have non countable disjoint nonempty

open sets.

$\therefore$ Every V only intersects with countable elements in B'.

Secondly, Def "$\sim$" : $x \sim y$ iff $\exists$ finite $V_1, \cdots, V_n \in B'$, $x \in V_1$,

$y \in V_n$, $V_i \cap V_{i+1} \neq \emptyset$

$x \sim y$.

Conside $[x]$, $\forall y \in [x]$, $\exists W \in B'$, including y, $\forall z \in W$, $z \sim y$

$\therefore z \sim x$ $\therefore z \in [x]$ $\therefore \forall y$, $\exists W$, s.t. $y \in W \subset [x]$.

$\Rightarrow [x]$ open.

$\therefore [x]$ is closed

$\therefore [x] = X$

$\therefore$ For one $V_0 \in B'$, let $S_0 = V_0$, $S_1 = \{W \in B' \mid W \cap V_0 \neq \emptyset\}$ countable

$\cdots$ S_n countable.

$S = \bigcup_{n=0}^{\infty} S_n$

$\forall W \in B'$, $p \in V_0$, $q \in W$, $p \sim q$ $\exists U_1, \cdots, U_m \in B'$ chain.

$U_1 \cap V_0 \neq \emptyset$ $\Rightarrow U_1 \in S_1$, $\because U_2 \cap U_1 \neq \emptyset$ $\therefore U_2 \in S_2$

$\cdots$ $\therefore U_m \in S_m$ $\because q \in U_m \cap W$ $\therefore W \in S_{m+1} \subset S$.

$B' = S$ countable.

$\therefore B' = S$. countable

$\therefore B'$ is the countable basis of X, induced by B in $\mathbb{C}$

Thm 1.3.3 (Rado's thm). Every connected Riemann surface X is A_2.

Pf: let $\zeta: U \to \mathbb{C}$ be a local coordinate and $Y := X - \zeta^{-1}(\bar{D}_1 \cup \bar{D}_2)$

D_1, D_2 are disjoint discs in the image of ζ

Y is a subsurface, $\partial Y = \zeta(\partial D_1 \cup \partial D_2)$

Let $\mathcal{F} = \{h: \bar{Y} \to [0,1] \mid h$ continuous, subharmonic on Y, $h|_{\partial D_1} = 0$, $h|_{\partial D_2} \le 1\}$.

$\Rightarrow g = \sup \mathcal{F}$ harmonic on Y

$\exists$ analytic h $\quad g = Re(h) = \frac{1}{2}(h + \bar{h})$

$\partial g = \frac{\partial g}{\partial z} dz \quad \frac{\partial g}{\partial z} = \frac{1}{2}\left(\frac{\partial h}{\partial z} + \underbrace{\frac{\partial \bar{h}}{\partial z}}_{0}\right) = \frac{1}{2} h'(z) \quad \partial g = \frac{1}{2} h'(z) dz$.

$y_0 \in Y$, $\tilde{Y}$: universal cover $\{[\gamma] \mid \gamma: [0,1] \to Y, \gamma(0) = y_0\}$.

def: $f([\gamma]) := \int_\gamma \partial g \qquad \int_{\gamma_1} \partial g = \int_{\gamma_2} \partial g \to$ well-define

$\tilde{Y}$ is connected, $\exists f: \tilde{Y} \to \mathbb{C}$ non const

From Pro 1.3.2.2. $\tilde{Y}$ is A_2

From Pro 1.3.2.1 Y is A_2.

$X = Y \cup \zeta^{-1}(\bar{D}_1 \cup \bar{D}_2)$ is A_2.

$\underbrace{U_1, \cdots, U_m}_{\uparrow \text{ countable basis}} \quad U_i \cong \mathbb{C} \quad U_i$ is A_2

$\square$

Note: Dim > 1 Rado's thm $\times$