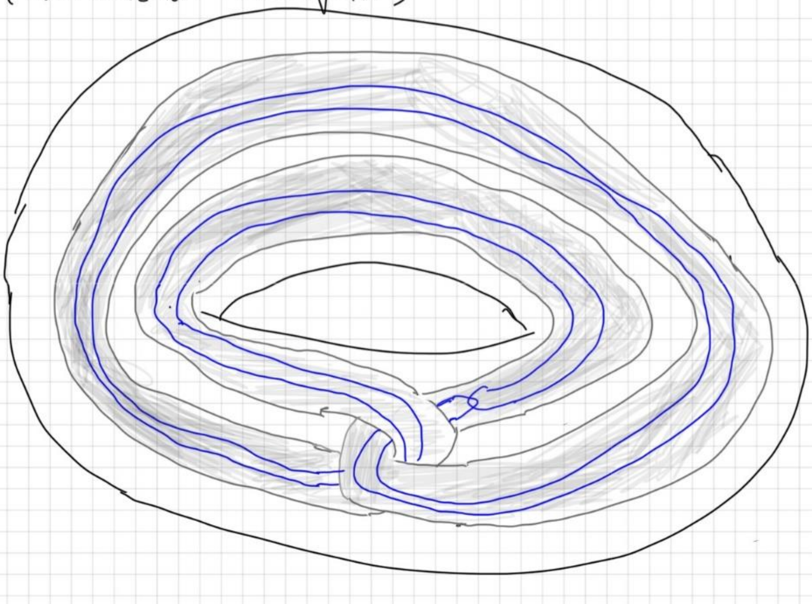
1.4. An exhaustion of X.

C^∞-piece of X: 2-dim cpt subsurface of X with C^∞-boundary. $X_0 \subset X_1 \subset \cdots$ s.t. $\bigcup_{n=0}^{\infty} X_n = X$

Pro 1.4.1 (A nice exhaustion of X) Let X be a Riemann surface, connected, non-cpt, $H^1(X,\mathbb{R})=0$, Let $x_0 \in X$ be a "base point", $\exists X_0 \subset X_1 \subset \cdots \subset X$ connected, cpt C^∞-pieces. s.t. 1. $x_0 \in X_0$ 2. $X_n \subset \operatorname{int} X_{n+1}$ 3. $\bigcup X_n = X$. 4. $H^1(X_n, \mathbb{R}) = 0$.

Note: 3-dim manifold it is false.

Example: (Whitehead manifold)



$Y = \mathbb{R}^3 \setminus W$

$T_{n+1} \subset T_n$ $\quad \bigcap T_n = W$. $\quad \therefore Y_0 \subset Y_1 \subset \cdots$ $\quad \bigcup Y_n = Y$.

$\{Y_n\}$ is an exhaustion.

$Y = \mathbb{R}^3 - W$

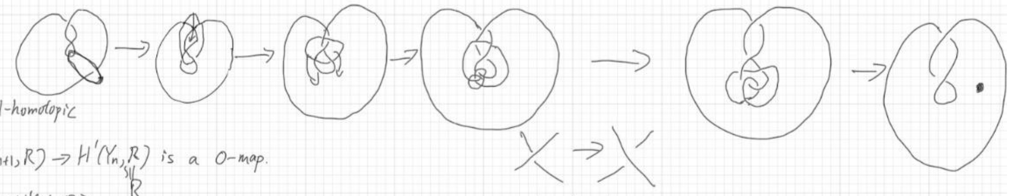
$T_{n+1} \subset T_n \quad \bigcap T_n = W. \quad \therefore Y_0 \subset Y_1 \subset \cdots \quad \bigcup Y_n = Y.$

$\{Y_n\}$ is an exhaustion.

① $H^1(Y_n, \mathbb{R}) \cong \mathbb{R}$ ② $\varprojlim H^1(Y_n, \mathbb{R}) = H^1(Y, \mathbb{R}) = 0$

$\pi_1(Y_n) \cong \mathbb{Z}$

$Hom(\mathbb{Z}, \mathbb{R})$ $\quad H^1(Y_{n+1}, \mathbb{R}) \to H^1(Y_n, \mathbb{R})$ is 0 map



$\Rightarrow$ null-homotopic

homotopy ✓

isotopy X

$H^1(Y_{n+1}, \mathbb{R}) \to H^1(Y_n, \mathbb{R})$ is a 0-map.
$\qquad\qquad\qquad\qquad\qquad\quad \cong \mathbb{R}$

$\varprojlim H^1(Y_n, \mathbb{R}) = 0$

Pf(1.4.1): Thm 1.3.3 $\Rightarrow X$ is A_2.

Choose a local finite cover of X by relatively cpt open sets U_n, choose partition of unity φ_n subordinate to this cover. Then $g(x) := \sum_{n=0}^{\infty} n\varphi_n(x)$ well-defined.

$\therefore \forall c \in \mathbb{R}$, $\{x \in X \mid g(x) \le c\}$ is cpt.

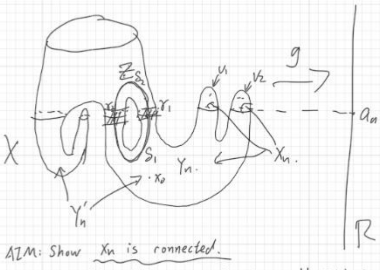
By Sard thm, $\exists a_0, a_1, \cdots$ increase regular value $\lim a_i = \infty$

Assume $a_0 > g(x_0)$. set $Y_n' = \{x \in X \mid g(x) \le a_n\}$, Y_n' cpt.

Let Y_n is connected component of Y_n', including x_0.

Since X is connected $a_n \to \infty$, $\bigcup Y_n = X$. $\{Y_n\} \uparrow$ cpt.

Consider connected component of $X - Y_n$, X_n is the union of Y_n and those component with cpt closure, let them be $V_1, \cdots, V_m$. i.e. $X_n = Y_n \cup \bar{V}_1 \cup \cdots \cup \bar{V}_m$



AIM: Show $\underline{X_n \text{ is connected.}}$

Let $X_n := U_1 \sqcup U_2$. open sets U_i. $\quad Y_n \subset U_1$ (WLOG)

if $V_i \subset U_2$, $\bar{V}_i \subset U_2 \quad \bar{V}_i \cap Y_n = \emptyset$

V_i is open / closed $\Rightarrow$ contridiction.

AIM: lemma 1.4.3 $H^1(X_n, \mathbb{R}) = 0$ $n \geq 0$.

Pf: let Z be the closure of a component of $X - X_n$.

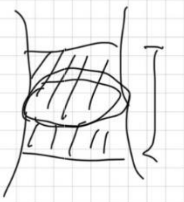
∂Z If γ_1, γ_2 are 2 connected components of ∂Z.

take a point from each, construct δ_1, δ_2, δ_1 in X_n,

δ_2 in Z $\delta = \delta_1 \cup \delta_2$ is closed ~~co~~ curve.

γ_1 is a component of $g^{-1}(a_n)$ $\therefore \gamma_1 \cong S^1$

find a neighborhood of γ_1 homeomorphic to an annulus



in which we can choose (x, y)

$x \in S^1$, $y \in (-\varepsilon, \varepsilon) \subset U \subset \mathbb{R}$

Let η be a positive fun on $\mathbb{R}$ with support in U.

s.t. $\int_{-\infty}^{\infty} \eta(y)\,dy = 1$

1-form $\varphi := \eta(y)\,dy$ $\Rightarrow$ closed.

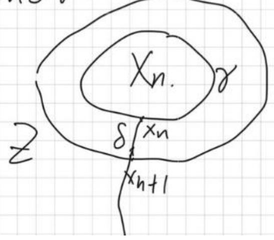
$\because \int_\delta \varphi = \pm 1$ $\therefore \varphi$ is not exact $\Rightarrow H^1(X, \mathbb{R}) \neq 0$.

$\Rightarrow$ There is no more then 1 connected components in ∂Z

let γ be the unique component of ∂Z.

AIM: $\rho_Z : Z \to \gamma$.

Let $x_n \in \gamma$.

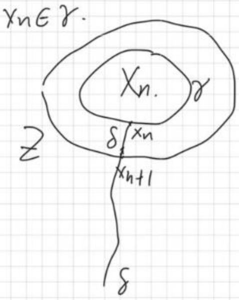


$\delta_{n,n+1}$ is arc connecting x_n, x_{n+1}

$\delta = \bigcup \delta_{n,n+1}$ connecting x_n to ∞

AIM: $\rho_Z : Z \to \gamma$.

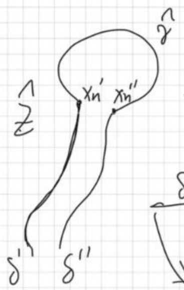
Let $x_n \in \gamma$.



$\delta_{n,n+1}$ is arc connecting x_n, x_{n+1}

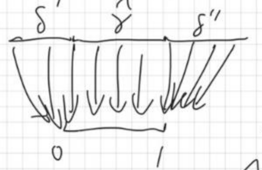
$\delta = \bigcup \delta_{n,n+1}$ connecting x_n to ∞.

$\delta' \cup \delta'' \cup [x_n', x_n'']$



$Z \sim \hat{Z}$ connected $\partial \hat{Z} \cong R$

Let $f : \partial \hat{Z} \to [0,1]$ be a continuous homeomorphism of $\hat{\gamma}$, send δ' to 0, send δ'' to 1.



By Tietze extension thm.
f can be extended to $\hat{f} : \hat{Z} \to [0,1]$

$\hat{f}$ induce a map $: Z \to \gamma$.
$\Downarrow$
ρ_Z.

Let $\rho_n : X \to X_n$ $\rho_n = id$ on X_n.

$\rho_Z : Z \to \gamma$. on component Z of $X - X_n$.

Then $X_n \xhookrightarrow{i} X \xrightarrow{\rho_n} X_n$ is id.

induces $H^1(X_n, R) \xrightarrow{i^*} H^1(X, R) \xrightarrow{\rho_n^*} H^1(X_n, R)$ $\quad (\rho_n \circ i)^* = i^* \circ \rho_n^* = id$

$H^1(X,R) \Downarrow 0$. $\quad i^* = 0 \quad \Rightarrow \underline{id = 0}$
$\Downarrow$
$H^1(X_n, R) = 0$

$\therefore$ Lemma 1.4.3 $(H^1(X_n, R) = 0)$

We find X_n, $H^1(X_n, R) = 0$ $X_n \subset int(X_{n+1})$ $\bigcup X_n = X$
connected, cpt

$\therefore$ Pro 1.4.1

$\boxed{W}$

1.5. Green functions

Def/Pro 1.5.1 (Green functions) Let X_n be a cpt C^∞ piece of Riemann surface X, let ξ be a local coordinate centered at $x_0 \in X_n$. $\forall n$. $\exists$ unique $G: X_n - \{x_0\} \to \mathbb{R}_+$ s.t.

1. continuous 2. harmonic on $\mathrm{int}(X_n - \{x_0\})$. 3. vanishes on ∂X_n.
4. $G + \ln(|\xi|)$ extends to a continuous fun on a neighborhood of x_0.

G is called Green's function of X_n with a pole at x_0.

Pf: Scale the local coordinate ξ so that its image contains the unit disc, consider the family g

$\{g$: positive on $X_n - \{x_0\}$, subharmonic on the interior, $g + \ln|\xi|$ is bounded near $x_0\} = \mathcal{F}$

$g_0 = \sup(0, -\ln|\xi|) \in \mathcal{F} \neq \emptyset$

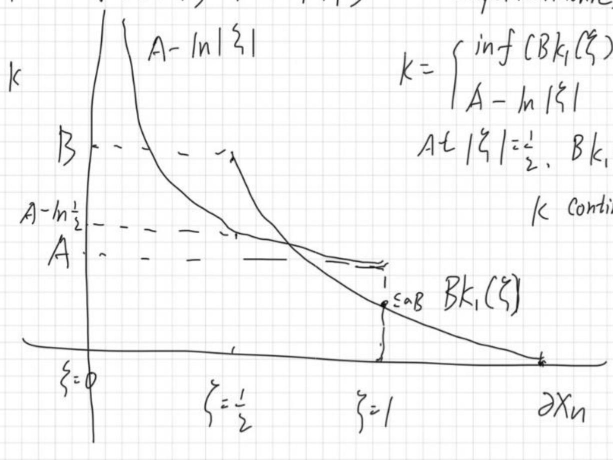
Consider $X_n \setminus \{|\xi| < \frac{1}{2}\}$ By Pro 1.2.4. 0 at ∂X_n, 1 at $|\xi| = \frac{1}{2}$

$\exists k_1$ harmonic s.t. $k_1|_{\partial X_n} = 0$ $k_1|_{|\xi| = \frac{1}{2}} = 1$ $k_1 \not\equiv$ const.

Let $a = \max_{|\xi|=1} k_1$

Next, find A, B. s.t. $A > aB$, $B > A - \ln\frac{1}{2}$

$k := \inf(Bk_1, A - \ln|\xi|)$ is superharmonic.



$$k = \begin{cases} \inf(Bk_1(\xi), A - \ln|\xi|), & |\xi| \geq \frac{1}{2} \\ A - \ln|\xi| & 0 < |\xi| < \frac{1}{2} \end{cases}$$

At $|\xi| = \frac{1}{2}$. $Bk_1 = B$. $A - \ln|\xi| = A - \ln\frac{1}{2}$

k continuous

$|\xi|<\frac{1}{2}$: $A-\ln|\xi|$ superharmonic ; $|\xi|\geq\frac{1}{2}$ $B k_1$ (super) harmonic

$A-\ln|\xi|$ superharmonic At ∂X_n, $k_1=0$ $\Rightarrow k=0$

$\xi\Rightarrow x_0$ $k\sim A-\ln|\xi|\sim -\ln|\xi|$

$\therefore k$ is what we want "$\mathcal{F}$"

$\therefore \mathcal{F}$ is Perron family with bounded k.

By Pro 1.2.3 $\underbrace{G=\sup\mathcal{F}}$ fix all requirements $\boxed{//}$

Remark : $\mathbb{C}$ $G=-\ln|x-x_0|-h(x)$ $\int_{\partial X_n} P(x,\xi)\cdot(-\ln|\xi-x_0|)\,d\xi$.

$\Downarrow$

$\Phi(x,x_0)=-\ln|x-x_0|$

1.6. Simply connected cpt pieces.

Pro 1.6.1 $\forall n\geq 0$, $\exists$ a homeomorphism $\varphi_n: X_n\to\overline{D}$ analytic on int X_n.

Pf: $X_n^*=X_n-\{x_0\}$, U is a neighborhood of x_0 $\cong$ unit disc.

Mayer - Vietoris $H^1(X_n,R)\to H^1(X_n^*,R)\oplus H^1(U,R)\to H^1(U-\{x_0\},R)\to H^2(X_n,R)$

$H^1(X_n,R)\to 0$ (Pro 1.4.1) $\cong$ $H^2(X_n,R)\to 0$

$H^1(U,R)=0$

$H^1(U-\{x_0\},R)\cong R$ $\Rightarrow$ $H^1(X_n^*,R)\cong R$.

$\Downarrow$

S^1 $d\theta$

$H^1(X_n^*,R)=\underline{Hom(H_1(X_n^*,Z),R)\cong R}$.

$H_1(X_n^*,Z)\cong Z^r\oplus T$

$Hom(H_1(X_n^*,Z),R)=Hom(Z^r\oplus T,R)\cong\underbrace{Hom(Z^r,R)}_{\substack{\cong\\ R^r=R}}\oplus \underbrace{Hom(T,R)}_{\substack{\shortparallel\\ 0}}$

$r=1$

$H_1(X_n^*,Z)=Z\oplus T$. proj $Z\oplus T\to Z$. $\underline{\ker proj=T}$.

$H_1(X_n^*, \mathbb{Z}) = \mathbb{Z} \oplus T$. $\quad$ proj $\mathbb{Z} \oplus T \to \mathbb{Z}$. $\quad$ ker proj $= T$.

$\phi : \pi_1(X_n^*) \to H_1(X_n^*, \mathbb{Z}) \to \mathbb{Z}$ $\quad$ Galois correspondence

$\ker\phi = H$

$p : \widetilde{X_n^*} \to X_n^*$

$\underbrace{\mathrm{Aut}}_{\parallel} = \pi_1(X_n^*)/H = \pi_1(X_n^*)/\ker\phi \cong \mathrm{Im}\,\phi \cong \mathbb{Z}$.

$\mathbb{Z} = \langle \alpha \rangle$

X_n^* find γ induce α in $\widetilde{X_n^*}$

$H_1(\widetilde{X_n^*}, \mathbb{Z}) \cong \pi_1(\widetilde{X_n})^{ab} = (\ker\phi)^{ab} \cong T$.

$$H^1(\widetilde{X_n^*}, \mathbb{R}) \cong \mathrm{Hom}(H_1(\widetilde{X_n^*}, \mathbb{Z}), \mathbb{R}) \oplus \underbrace{\mathrm{Ext}(H_0(\widetilde{X_n^*}, \mathbb{Z}), \mathbb{R})}_{\substack{\mathbb{Z} \\ \shortparallel \\ 0}}$$

$\mathrm{S}\!\parallel$

$\mathrm{Hom}(T, \mathbb{R}) = 0$.

$\therefore$ On $\widetilde{X_n^*}$ all closed 1-form are exact

$w := -\partial G$ on X_n^* analytic in int X_n^*

$G(z) = -\ln|z| + h(z)$ near x_0.

$w = (\frac{1}{z} + H(z))\,dz$. $\quad$ H holomorphic.

$\exists F$ on $\widetilde{X_n^*}$ s.t. $dF = p^* w$

$d(\mathrm{Re}F) = \mathrm{Re}(dF) = \mathrm{Re}(p^* w)$ $\quad$ $\mathrm{Re}F = -p^* G$

$\int_\gamma w = 2\pi i \,\mathrm{Res}_{x_0} w = 2\pi i$. $\quad$ $\alpha^* F = F + 2\pi i$

loop of x_0 $\quad$ $\alpha^* e^F = e^F$ $\quad$ $e^F \alpha = e^F$

$\exists f$ analytic on int X_n^* $\quad$ $p^* f = e^F$

$(\forall y \in X_n^*,\ x \in p^{-1}(y)$ $\ f(y) = e^F(x)$ $\quad$ $f \circ p(x) = f(y) = e^F(x)$, $\alpha^* e^F = e^F$ $\therefore$ f is well-defined)

$|f| = |e^F| = e^{\mathrm{Re}F} = e^{-p^* G}$, $\quad \ln|f| = -G$.

$|f| = e^{-G}$ $\therefore f(z) \to 0$ $\quad$ $f : X_n^* \to \overline{D}$

$(z \to x_0)$ $\quad$ extends to $f : X_n \to \overline{D}$ has

$|f| = e^{-G}$ $\therefore f(z) \to 0$ $(z \to x_0)$ $f: X_n^* \to \bar{D}$ extends to $f: X_n \to \bar{D}$ has simple zero at x_0.

$|f| = 1$ on ∂X_n. $(G = 0)$ $\Rightarrow$ maps boundary to boundary

$\Rightarrow f$ proper

$\ker f = \{x_0\}$ $f(z) \sim z$ at x_0 $\deg_{local} f = 1$

0 is regular value, $f^{-1}(0) = \{x_0\}$ $\deg f = 1$ $\Rightarrow$ homeomorphism.

$\therefore f$ is φ_n

$\boxed{//}$

1.7. Proof of thm 1.1.2.

Review: "If a Riemann surface X is connected, non-cpt. and $H^1(X, \mathbb{R}) \cong 0$, then it is isomorphic to $\mathbb{C}$ or $\mathbb{D}$".

Let $D_r = \{z \in \mathbb{C} \mid |z| < r\}$, choose $v \in T_{x_0}X$, by Pro 1.6.1, there exists isomorphism $\varphi_n : X_n \to D_{r_n}$. $(\forall X_n)$ s.t. $[D\varphi_n(x_0)]v = 1$

(If $\varphi_n(x_0)_* v = c_0$, then use $\frac{1}{c_0}\varphi_n$ replace φ_n)

Assume $r_0 = 1$.

Pro 1.7.1 The φ_n form a normal family.

(every sequence has a subsequence converges normally)

$\downarrow$

cpt

Pf: Firstly, prove $r_n \uparrow$

For $m > n$ $f(z) = \frac{1}{r_m}\varphi_m \circ \varphi_n^{-1}(r_n z)$

$f: \mathbb{D} \to \mathbb{D}$.

$f'(0) = r_n / r_m$ If $r_m \leq r_n$ $|f'(0)| \geq 1$

But, by Schwarz lemma: $|f'(0)|\leq 1$, $|f'(0)|=1$ f is isomorphism, $X_n \subsetneq X_m \Rightarrow$ contridiction $\therefore |f'(0)|<1$

$\Rightarrow r_m > r_n$.

If $\sup r_n$ is finite $\underline{\sup r_n = R}$ $|\varphi_n| \leq R$

By Montel Thm $\Rightarrow \{\varphi_n\}$ is a normal family

If $\sup r_n = \infty$, let $\psi_m = \varphi_m \circ \varphi_0^{-1} : D \Rightarrow \mathbb{C}$, $\psi_m'(0)=1$

By Koebe quarter thm $\psi_m(D)$ contains $D_{\frac{1}{4}|\psi_m'(0)|} = D_{\frac{1}{4}}$

$\forall n$, in $X_n \setminus X_0$, $|\varphi_m| \geq \frac{1}{4}$, $|\frac{1}{\varphi_m}| \leq 4$.

By Montel thm, $\{\frac{1}{\varphi_m}\}$ are normal family $\Rightarrow \{\varphi_m\}$ are normal family as well.

$\boxed{VII}$

$R := \sup r_n$, choose subsequence of φ_n converge uniformly on any cpt set. to a map $\varphi : X \to D_R$ $(R<\infty)$

$X \to \mathbb{C}$ $(R=\infty)$

φ is limit of analytic funs, hence analytic.

$\forall w \in D_R$, $\exists k$ s.t. $|w| < r_{n_k}$ $\because \varphi_{n_k}(X_{n_k}) = D_{r_{n_k}}$ $\exists x_{n_k} \in X_{n_k}$ s.t. $\varphi_{n_k}(X_k) = w$

$\{x_k\}$ has subsequence limit to x, $\varphi(x) = w \Rightarrow \varphi$ surjective.

φ is limit of inj analytic funs, $\varphi'(x_0) \neq 0 \Rightarrow \varphi$ injective.

$\therefore \varphi : X \to D_R$ is isomorphic, if $\sup r_n = R < \infty$, $\varphi : X \to D$

if $\sup r_n = \infty$, $\varphi : X \to \mathbb{C}$

$\therefore$ Thm 1.1.2

$\boxed{VIII}$

Thm 1.1.2 $\to$ Thm 1.1.3

$+$

Thm 1.1.1