5.4 Space of Quadratic Differentials

Def. Beltrami Eq $\frac{\partial f}{\partial \bar{z}} = \mu(z)\frac{\partial f}{\partial z} \rightarrow$ qusi-conformal

C-R $\frac{\partial f}{\partial \bar{z}} = 0 \longrightarrow$ conformal

(p,q)-form: $a(z)\,dz_{i_1}\wedge\cdots\wedge dz_{i_p}\wedge d\bar{z}_{j_1}\wedge\cdots\wedge d\bar{z}_{j_q}$

$(-1,1)$-form: $v = v(\zeta)\frac{d\bar{\zeta}}{d\zeta}$ $\quad v(\zeta)\frac{d\bar{\zeta}}{d\zeta}\left(w(\zeta)\frac{\partial}{\partial \zeta}\right) = v(\zeta)\overline{w(\zeta)}\frac{\partial}{\partial \zeta}$

Infinitesimal Beltrami form: $\mu \in \mathcal{L}_*(TX, TX)$ $\quad \mu = \mu(z)\frac{d\bar{z}}{dz}$

$T(av) = \bar{a}\,T(v)$

$q = q(z)\,dz^2$

conformal

$\mu = (\bar{\partial} f)^{-1} \partial f$

$\mu : T_x X \to T_x X$

$) \overline{\mu(\zeta)} \frac{\partial}{\partial \bar{\zeta}}$

$\mu = \mu(z) \frac{d\bar{z}}{dz}$

$q\mu = q(z)\mu(z)|dz|^2 \quad \langle \mu, q \rangle = \int_X q\mu$

Def: pre-dual of E $(F : E = F^*)$

$L^\infty_x(TX, TX)$: esssup $|\theta(x)| < \infty$

Let X is a hyperbolic Riemannian surface ρ

$\|q\|_1 = \int_X |q| \quad \|q\|_\infty := \sup_{x \in X} \frac{|q|(x)}{\rho^2(x)}$

Integrable quardric differential

$Q^1(X) := \{q \in Q(X), \|q\|_1 < \infty\}$

Bounded

$Q^\infty(X) := \{q \in Q(X), \|q\|_\infty < \infty\}$

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Pro 5.4.3. X: Riemann surface, finite type, then $Q^1(X) = Q^\infty(X)$

Pf: $\forall q \in Q^\infty(X)$. $\|q\|_\infty < \infty$

$$\|q\|_1 = \int_X |q| = \int_X \frac{|q|}{\rho^2}\rho^2 \qquad \frac{|q|}{\rho^2} < \infty$$

$$\leq \|q\|_\infty \underline{\int_X \rho^2} = 2\pi(2g-2+n)\|q\|_\infty < \infty$$

$\therefore q \in Q^1(X)$, $Q^\infty(X) \subseteq Q^1(X)$

"$D - \{0\}$" if $\int |q| < \infty$, it has at most simple pole at origin $\sim \frac{dz^2}{z}$

$\rho = \frac{|dz|}{r|\ln r|}$ $\quad \rho^2 = \frac{|dz|^2}{r(\ln r)^2}$ $\quad \frac{|q|}{\rho^2} = r(\ln r)^2 \to 0$

$\|q\|_\infty < \infty$

$Q^1(X) \subseteq Q^\infty(X)$

$\square$

Pro 5.4.4. $Q^1(D) \subset Q^\infty(D)$, $\|q\|_\infty \leq \frac{1}{4\pi}\|q\|_1$

Pf: $\rho = \frac{2}{1-|z|^2}$, $\|q\|_\infty = \sup_{z\in D} |q(z)| \frac{(1-|z|^2)^2}{4}$,

$\|q\|_1 = \int_D |q(z)||dz|^2$

$\phi \in \mathrm{Aut}D$ $\|\phi^* q\|_{1,\infty} = \|q\|_{1,\infty}$

At origin $\frac{|q(0)|}{4} \leq \frac{1}{4\pi}\|q\|_1 \iff |q(0)| \leq \frac{\|q\|_1}{\pi}$

$q(0) \leq \frac{1}{\pi}\int_D |q(z)||dz|^2 = \frac{1}{\pi}\|q\|_1$

$\therefore \|q\|_\infty \leq \frac{1}{4\pi}\|q\|_1$

Def. 5.4.6. Th

$X \to$ Surface X^*

is a local

is a

(x)

$<\infty$

$\frac{dz^2}{z}$

$\square$

Def. 5.4.6. The conjugate Riemann

$X \rightsquigarrow$ Surface X^* def: if $U \subset X$, $\varphi: U \to \mathbb{C}$

is a local coordinate; then $\overline{\varphi}$

is a local coordinate of X^*

Ex: if $\Gamma \subset PSL_2\mathbb{R}$ $X = H/\Gamma$

$X^* = H^*/\Gamma$

$$|a(0)| \leq \frac{\|a\|_1}{\pi}$$

$= \sup$

$\frac{1}{\pi}\|a\|_1$

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Ex 5.4.8 $\dfrac{dz^2\otimes dw^2}{(z-w)^4} \in \Gamma(\Omega^{\otimes 2}P^1 \otimes \Omega^{\otimes 2}P^1) \xrightarrow{\pi} (P^1\times P^1)\setminus\Delta$

If $f: U \to P^1$ analytic $z = f(z_1)$, $w = f(w_1)$

$$(f\times f)^*[F(z,w)\,dz^2\otimes dw^2] = F(f(z_1), f(w_1))\,(f'(z_1))^2(f'(w_1))^2\,dz_1^2\otimes dw_1^2$$

Invariant under acting diagonally: $(f\times f)^*(F(z,w)) = F(z,w)$

$f(z)=\dfrac{az+b}{cz+d}$ $\quad f'(z) = \dfrac{ad-bc}{(cz+d)^2}$

$$\frac{(f'(z))^2(f'(w))^2}{(f(w)-f(z))^4} = \frac{1}{(w-z)^4} \Rightarrow \frac{dz^2\otimes dw^2}{(z-w)^4}$$ invariant under acting diagonally.

Pro 5.4.9 (Reproducing formula of Q^∞)

let $q \in (Q^\infty)^\Gamma(H^*)$, Then

$$q(w)dw^2 = \left(\frac{12}{\pi}\int_H \frac{q(\bar z)y^2}{(z-w)^4}|dz|^2\right)dw^2$$

Pf: $\frac{12}{\pi}\int_H \left(q(\bar z)\,d\bar z^2 \frac{dz^2 \otimes dw^2}{(z-w)^4}\frac{y^2}{|dz|^2}\right)$

$q \in Q^\infty(H^*)$ $\therefore y^2|q(\bar z)| < \infty$, $\mu_w := \frac{|dz|^2}{|z-w|^4}$

Lemma: let $\frac{\alpha}{\beta} = w$, $\Phi: \xi \mapsto \frac{\alpha\xi + \bar\alpha}{\beta\xi + \bar\beta}$ iso $D \to H$

$$\Phi^*\mu_w = \frac{1}{4(\mathrm{Im}\,w)^2}|d\xi|^2 \qquad \int_H \mu_w = \frac{\pi}{4(\mathrm{Im}\,w)^2}$$

$z = \frac{\alpha\xi + \bar\alpha}{\beta\xi + \bar\beta}$, $\frac{\alpha}{\beta} = w$

$$\frac{12}{\pi}\int_H \frac{q(\bar z)y^2}{(z-w)^4}|dz|^2$$

$$= \frac{3}{\pi}\int_D q\left(\frac{\overline{\alpha\xi}+\alpha}{\overline{\beta\xi}+\beta}\right)\frac{(1-|\xi|^2)^2}{(\overline{\beta\xi}+\beta)^4}\beta^4|d\xi|^2$$

Let $\xi = re^{i\theta}$, $|\xi| = r$

$$\frac{3\beta^4}{\pi}\int_0^1\left(\int_0^{2\pi} \underbrace{q\left(\frac{\overline{\alpha\xi}+\alpha}{\overline{\beta\xi}+\beta}\right)\frac{1}{(\overline{\beta\xi}+\beta)^4}}_{f(\xi)}d\theta\right)(1-r^2)^2\,r\,dr$$

$$\int_0^{2\pi} f(\xi)\,d\theta = 2\pi f(0) = \frac{2\pi}{\beta^4}q(w)$$

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$$\frac{3\beta^4}{\pi}\int_0^1 \frac{2\pi}{\beta^4} q(w)\,(1-r^2)^2\, r\,dr$$

$$= 6\,q(w) \underbrace{\int_0^1 (1-r^2)^2\, r\,dr}_{\frac{1}{6}}$$

$$= q(w)$$