Discrete Mathematics

Dinessh Naik

Manjunath K Vanahalli

Department of Information Technology,
National Institute of Technology Karnataka, India

August 10, 2020

Propositional Logic

- The rules of logic give precise meaning to mathematical statements.
- These rules are used to distinguish between valid and invalid mathematical arguments.
- Logic has numerous applications to computer science. These rules are used in the design of computer circuits, the construction of computer programs, the verification of the correctness of programs, and in many other ways.

Propositional Logic

- A proposition is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.
- All the following declarative sentences are propositions.
  1. Washington, D.C., is the capital of the United States of America.
  2. Toronto is the capital of Canada.
  3. $1 + 1 = 2$.
  4. $2 + 2 = 3$.
- Propositions 1 and 3 are true, whereas 2 and 4 are false.

Propositional Logic

- Some sentences that are not propositions are as follows.
- Consider the following sentences.
  1. What time is it?
  2. Read this carefully.
  3. $x + 1 = 2$.
  4. $x + y = z$.
- Sentences 1 and 2 are not propositions because they are not declarative sentences.

Propositional Logic

- Sentences 3 and 4 are not propositions because they are neither true nor false.
- Note that each of sentences 3 and 4 can be turned into a proposition if we assign values to the variables.
- We use letters to denote propositional variables (or sentential variables), that is, variables that represent propositions, just as letters are used to denote numerical variables.
- The conventional letters used for propositional variables are p, q, r, s, . . .

Propositional Logic

- The truth value of a proposition is true, denoted by T, if it is a true proposition, and the truth value of a proposition is false, denoted by F, if it is a false proposition.
- Propositions that cannot be expressed in terms of simpler propositions are called atomic propositions.
- The area of logic that deals with propositions is called the propositional calculus or propositional logic.
- Many mathematical statements are constructed by combining one or more propositions. New propositions, called compound propositions, are formed from existing propositions using logical operators.

Propositional Logic

Let p be a proposition. The *negation of p*, denoted by $\neg p$ (also denoted by $\overline{p}$), is the statement

"It is not the case that p."

The proposition $\neg p$ is read "not p." The truth value of the negation of p, $\neg p$, is the opposite of the truth value of p.

Figure 1. Negation

Propositional Logic

TABLE 1 The Truth Table for the Negation of a Proposition.

p	$\neg p$
T	F
F	T

Figure 2. The Truth Table of Negation

Propositional Logic

- Find the negation of the proposition – “Michael’s PC runs Linux” – and express this in simple English.
- The negation is – “It is not the case that Michael’s PC runs Linux.”
- This negation can be more simply expressed as – “Michael’s PC does not run Linux.”

Propositional Logic

- Find the negation of the proposition – "Vandana's smartphone has at least 32 GB of memory" – and express this in simple English.
- The negation is – "It is not the case that Vandana's smartphone has at least 32 GB of memory."
- This negation can also be expressed as – "Vandana's smartphone does not have at least 32 GB of memory"
- Even more simply as – "Vandana's smartphone has less than 32 GB of memory."

Propositional Logic

- The negation of a proposition can also be considered the result of the operation of the negation operator on a proposition
- The negation operator constructs a new proposition from a single existing proposition.
- We will now introduce the logical operators that are used to form new propositions from two or more existing propositions.
- These logical operators are also called connectives.

Propositional Logic

Let p and q be propositions. The *conjunction* of p and q, denoted by $p \wedge q$, is the proposition "p and q." The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

Figure 3. Conjunction

Propositional Logic

TABLE 2 The Truth Table for the Conjunction of Two Propositions.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Figure 4. Conjunction Truth Table

Propositional Logic

- Find the conjunction of the propositions p and q where p is the proposition "Rebecca's PC has more than 16 GB free hard disk space" and q is the proposition "The processor in Rebecca's PC runs faster than 1 GHz."

- Solution: The conjunction of these propositions, $p \wedge q$, is the proposition "Rebecca's PC has more than 16 GB free hard disk space, and the processor in Rebecca's PC runs faster than 1 GHz." This conjunction can be expressed more simply as "Rebecca's PC has more than 16 GB free hard disk space, and its processor runs faster than 1 GHz." For this conjunction to be true, both conditions given must be true. It is false when one or both of these conditions are false.

Propositional Logic

Let p and q be propositions. The *disjunction* of p and q, denoted by $p \vee q$, is the proposition "p or q." The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

Figure 5. Disjunction

Propositional Logic

TABLE 3 The Truth Table for the Disjunction of Two Propositions.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Figure 6. Disjunction Truth Table

Propositional Logic

- Rebecca's PC has at least 16 GB free hard disk space, or the processor in Rebecca's PC runs faster than 1 GHz
- This proposition is true when Rebecca's PC has at least 16 GB free hard disk space, when the PC's processor runs faster than 1 GHz, and when both conditions are true. It is false when both of these conditions are false, that is, when Rebecca's PC has less than 16 GB free hard disk space and the processor in her PC runs at 1 GHz or slower.

Propositional Logic

Let p and q be propositions. The *exclusive or* of p and q, denoted by $p \oplus q$ (or p XOR q), is the proposition that is true when exactly one of p and q is true and is false otherwise.

Figure 7. Exclusive OR

Propositional Logic

TABLE 4 The Truth Table for the Exclusive Or of Two Propositions.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Figure 8. Exclusive OR Truth Table

Propositional Logic

- Let p and q be the propositions that state "A student can have a salad with dinner" and "A student can have soup with dinner," respectively. What is $p \oplus q$, the exclusive or of p and q?

- The exclusive or of p and q is the statement that is true when exactly one of p and q is true. That is, $p \oplus q$ is the statement "A student can have soup or salad, but not both, with dinner." Note that this is often stated as "A student can have soup or a salad with dinner," without explicitly stating that taking both is not permitted.

Propositional Logic

Let p and q be propositions. The *conditional statement* $p \to q$ is the proposition "if p, then q." The conditional statement $p \to q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \to q$, p is called the *hypothesis* (or *antecedent* or *premise*) and q is called the *conclusion* (or *consequence*).

Figure 9. Conditional Statement

Propositional Logic

TABLE 5 **The Truth Table for the Conditional Statement $p \rightarrow q$.**

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Figure 10. Conditional Statement OR Truth Table

Propositional Logic

- The statement p $\rightarrow$ q is called a conditional statement because p $\rightarrow$ q asserts that q is true Assessment on the condition that p holds. A conditional statement is also called an implication.
- Let p be the statement "Maria learns discrete mathematics" and q the statement "Maria will find a good job." Express the statement p $\rightarrow$ q as a statement in English.
- "If Maria learns discrete mathematics, then she will find a good job."
- "Maria will find a good job when she learns discrete mathematics."

Propositional Logic

- We can form some new conditional statements starting with a conditional statement $p \rightarrow q$.
- The proposition $q \rightarrow p$ is called the converse of $p \rightarrow q$.
- The contrapositive of $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$.
- The proposition $\neg p \rightarrow \neg q$ is called the inverse of $p \rightarrow q$.