Discrete Mathematics

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n-ary Relations and Their Applications

n-ary Relations

Definition: Let $A_1, A_2, ..., A_n$ be sets. An *n-ary relation* on these sets is a subset of $A_1 \times A_2 \times ... \times A_n$. The sets $A_1, A_2, ..., A_n$ are called the *domains* of the relation, and n is called its *degree*.

Primary Key

Definition: A domain of an n-ary relation is called a *primary key* when the value of the n-tuple from this domain determines the n-tuple.

Composite Key

Definition: Combinations of domains can also uniquely identify n-tuples in an n-ary relation. When the values of a set of domains determine an n-tuple in a relation, the Cartesian product of these domains is called a *composite key*.

n-ary Relations and Their Applications

Selection

Definition: Let R be an n-ary relation and C a condition that elements in R may satisfy. Then the *selection operator* S_C maps the n-ary relation R to the n-ary relation of all n-tuples from R that satisfy the condition C.

Projection

Definition: The *projection* $P_{i_1 i_2,\ldots,i_m}$ where $i_1 < i_2 < \ldots < i_m$, maps the n-tuple $(a_1, a_2, \ldots, a_n)$ to the m-tuple $(a_{i_1}, a_{i_2}, \ldots, a_{i_m})$, where $m \leq n$.

Join

Definition: Let R be a relation of degree m and S a relation of degree n. The *join* $J_p(R, S)$, where $p \leq m$ and $p \leq n$, is a relation of degree $m + n - p$ that consists of all $(m + n - p)$-tuples $(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$, where the m-tuple $(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p)$ belongs to R and the n-tuple $(c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$ belongs to S.

n-ary Relations and Their Applications

We can have relation between more than just 2 sets

A binary relation involves 2 sets and can be described by a set of pairs
A ternary relation involves 3 sets and can be described by a set of triples
...

An n-ary relation involves n sets and can be described by a set of n-tuples

Relations are used to represent computer databases

n-ary Relations and Their Applications

Let A_1, A_2, Λ, A_n be sets

An $n-ary$ relation is a subset of the cartesian product $A_1 \times A_2 \times \Lambda \times A_n$

The sets A_1, A_2, Λ, A_n are the *domains* of the relation

The degree of the relation is n

n-ary Relations and Their Applications

Let R be the relation on $N \times Z \times N \times Z$ consisting of 4-tuples (a,b,c,d) such that $(a+b \neq c+d) \wedge (a+b+c+d=0)$

Note: N is the set of natural numbers {0,1,2,3,...}
Z is the set of integers {...,-2,-1,0,1,2,...}

$(0,-1,1,0) \in R$
$(5,-11,3,3) \in R$
$(6,6,3,9) \notin R$

The relation has degree 4

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Relational databases

Database is made up of records.
Typical operations on a database are

- find records that satisfy a given criteria
- delete records
- add records
- update records

Some everyday databases
- student records
- health records
- tax information
- telephone directories
- banking records
- ...

Databases *may* be represented using the relational model

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Relational databases

The relational data model

Database made up of *records*, they are *n-tuples*, made up of *fields*

Student record might look as follows

(name,metricNo,faculty,gpa)

gpa is an attribute

(Jones,200401986,Arts,4.9)
(Lee,200408972,Science,3.6)
(Kuhns,200501728,Humanities,5.0)
(Moore,200308327,Science,5.5)

relations (in relDB) also called *tables*

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Relational databases

The relational data model

Name	*metricNo*	*Dept*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.49
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

Attributes: name, metric No, Dept and GPA

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Relational databases

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primary key:
An attribute/domain/column is a primary key when
the value of this attribute uniquely defines tuples
i.e. no two tuples have the same value for that attribute

Name cannot be a primary key, neither can Dept or GPS
metricNo is a primary key

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Relational databases

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The current collection of n-tuples (records) in the relation (table) is called ***the extension of the relation***

The permanent aspects of the relation (table) such as the attribute names is called ***the intention of the relation***

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Relational databases

The relational data model

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A ***composite key*** is a combination of attributes
That uniquely define tuples

n-ary Relations and Their Applications

Relational databases Selection Operations on n-ary relations

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Let R be an $n-ary$ relation and C a condition that elements in R must satisfy.
The selection operator S_c maps R to the new $n-ary$ relation of all $n-tuples$ from R that satisfy the condition C

n-ary Relations and Their Applications

Relational databases

Selection

Operations on n-ary relations

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Let R be an $n-ary$ relation and C a condition that elements in R must satisfy. The selection operator S_c maps R to the new $n-ary$ relation of all $n-tuples$ from R that satisfy the condition C

Apply the selection operator S_c where C is the condition GPA > 3.45

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n-ary Relations and Their Applications

Relational databases   Projection   Operations on n-ary relations

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The *projection* $P_{i_1 i_2 \Lambda\, i_m}$ where $i_1 < i_2 < \Lambda < i_m$ maps the $n-tuple$ $(a_1, a_2, \Lambda\ , a_n)$ to the $m-tuple$ $(a_{i_1}, a_{i_2}, \Lambda\ , a_{i_m})$ where $m \le n$

It strips out specific columns

n-ary Relations and Their Applications

Relational databases Projection Operations on n-ary relations

The *projection* $P_{i_1 i_2 \Lambda\, i_m}$ where $i_1 < i_2 < \Lambda < i_m$ maps the $n-tuple(a_1, a_2, \Lambda\ , a_n)$ to the $m-tuple$ $(a_{i_1}, a_{i_2}, \Lambda\ , a_{i_m})$ where $m \le n$

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Apply the projection $P_{1,4}$

Name			*GPA*
Ackermann			3.88
Adams			3.45
Chou			3.49
Goodfriend			3.49
Rao			3.90
Stevens			2.99

n-ary Relations and Their Applications

Relational databases Join Operations on n-ary relations

Lecturer	*Dept*	*Course*
Cruz	Zoology	335
Cruz	Zoology	412
Faber	Psychology	501
Faber	Psychology	617
Grammer	Physics	544
Grammer	Physics	551
Rosen	Computer Science	518
Rosen	Mathematic s	575

Dept	*Course*	*Room*	*Time*
Computer Science	518	N521	14.00
Mathematic s	575	N502	15.00
Mathematic s	611	N521	16.00
Physics	544	B505	16.00
Psychology	501	A100	15.00
Psychology	617	A110	11.00
Zoology	335	A100	09.00
Zoology	412	A100	08.00

The join operator $J_p(R,S)$ where R and S are $m-ary$ and $n-ary$ relations respectively and $p \leq m$ and $p \leq n$ delivers a new relation of degree $m+n-p$ such that the first $m-p$ attributes come R and the last $n-p$ attributes come from S where the overlappin g p attributes match (see Rosen p.534 Defn 4)

Joins two tables/relations together, matching up on specific attributes

n-ary Relations and Their Applications

Relational databases Join Operations on n-ary relations

Lecturer	*Dept*	*Course*
Cruz	Zoology	335
Cruz	Zoology	412
Faber	Psychology	501
Faber	Psychology	617
Grammer	Physics	544
Grammer	Physics	551
Rosen	Computer Science	518
Rosen	Mathematic s	575

Relation R

Dept	*Course*	*Room*	*Time*
Computer Science	518	N521	14.00
Mathematic s	575	N502	15.00
Mathematic s	611	N521	16.00
Physics	544	B505	16.00
Psychology	501	A100	15.00
Psychology	617	A110	11.00
Zoology	335	A100	09.00
Zoology	412	A100	08.00

Relation S

$$J_2(R, S)$$

Lecturer	*Dept*	*Course*	*Room*	*Time*
Cruz	Zoology	335	A100	09.00
Cruz	Zoology	412	A100	08.00
Faber	Psychology	501	A100	15.00
Faber	Psychology	617	A110	11.00
Grammer	Physics	544	B505	16.00
Rosen	Computer Science	518	N521	14.00
Rosen	Mathematic s	575	N502	15.00

Closures of Relations

Recall: Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.

$$\begin{aligned} R \text{ is reflexive } &\Leftrightarrow \forall x \in A, (x, x) \in R. \\ R \text{ is symmetric } &\Leftrightarrow [\forall x, y \in A, ((x, y) \in R) \Rightarrow ((y, x) \in R)]. \\ R \text{ is transitive } &\Leftrightarrow [\forall x, y, z \in A, ((x, y) \in R \text{ and } (y, z) \in R) \Rightarrow (x, z) \in R]. \end{aligned}$$

In general, let R be a relation on a set A. R may or may not have some property $\mathbf{P}$, such as reflexivity, symmetry, or transitivity. If there is a relation S with property $\mathbf{P}$ containing R such that S is a subset of every relation with property $\mathbf{P}$ containing R, then S is called the closure of R with respect to $\mathbf{P}$.

Definition 1. *Let R be a relation on the set A. R may or may not have some property $\boldsymbol{P}$ (e.g. reflexive). If there is a relation R^p such that*

- *R^p has the property $\boldsymbol{P}$.*
- *$R \subseteq R^p$.*
- *If S is any other relation that contains R and has the property $\boldsymbol{P}$, then $R^p \subseteq S$.*

then R^p is the $\boldsymbol{P}$-closure of R.

Closures of Relations

Example 1. *The relation* $R = \{(1,3),(2,2),(3,4)\}$ *on the set* $\{1,2,3,4\}$ *is not reflexive. What is the reflexive closure of* R*?*

Solution. This can be done by adding $(1,1),(3,3)$, and $(4,4)$ to R, because these are the only pairs of the form (a,a) that are not in R. Clearly, this new relation contains R. Furthermore, any reflexive relation that contains R must also contain $(1,1),(3,3)$, and $(4,4)$. Because this relation contains R, is reflexive, and is contained within every reflexive relation that contains R, it is called the reflexive closure of R. Thus the reflexive closure of R is

$$R^r = \{(1,3),(2,2),(3,4),(1,1),(3,3),(4,4)\}. \qquad \square$$

In general, given a relation R on a set A, we may form the reflexive closure of R by adding:

$$R^r = R \cup \Delta,$$

where $\Delta = \{(a,a) \mid a \in A\}$ is the diagonal relation on A.

Similarly, in general, given a relation R on a set A, we may form the symmetric closure of R, R^s, by taking the union of R with R^{-1}:

$$R^s = R \cup R^{-1} = R \cup \{(b,a) \mid (a,b) \in R\}.$$

Closures of Relations

Example 2. *The relation* $R = \{(1,3),(2,2),(3,4)\}$ *on the set* $\{1,2,3,4\}$ *is not symmetric. What is the symmetric closure of* R*?*

Solution. The symmetric closure of R is:

$$R^s = R \cup R^{-1} = \{(1,3),(2,2),(3,4),(3,1),(4,3)\}.$$

□

Example 3. *What is the symmetric closure of* R*, where* $R = \{(a,b) \mid a \text{ divides } b\}$ *on the set* $\mathbb{Z}$*?*

Solution. $R^s = R \cup R^{-1} = \{(a,b) \mid a \text{ divides } b \text{ or } b \text{ divides } a\}$. □

Generally speaking, a relation fails to be transitive because it fails to contain certain ordered pairs. For example, if $(1,3)$ and $(3,4)$ are in a relation R, then the pair $(1,4)$ must be in R if R is to be transitive. To obtain a transitive relation from one that is not transitive, it is necessary to add ordered pairs. Roughly speaking, the relation obtained by adding the least number of ordered pairs to ensure transitivity is called the transitive closure of the relation. In a sense made precise by the formal definition, the transitive closure of a relation is the smallest transitive relation that contains the relation.

Closures of Relations

Definition 2. *Let A be a set and R a relation on A. The transitive closure of R is the relation R^t on A that satisfies the following three properties:*

1. *R^t is transitive.*
2. *$R \subseteq R^t$.*
3. *If S is any other transitive relation that contains R, then $R^t \subseteq S$.*

Example 4. *Let $A = \{0, 1, 2, 3\}$ and consider the relation R on A as follows:*

$$R = \{(0, 1), (1, 2), (2, 3)\}.$$

Find the transitive closure of R.

Solution. Every pair in R is in R^t, so

$$\{(0, 1), (1, 2), (2, 3)\} \subseteq R^t.$$

Thus the directed graph of R contains the arrows shown below.

$$\begin{tikzcd}
0 \arrow[r] & 1 \arrow[d] \\
3 & 2 \arrow[l]
\end{tikzcd}$$