Discrete Mathematics

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Propositional Logic

- Show that p $\vee$ (q $\wedge$ r) and (p $\vee$ q) $\wedge$ (p $\vee$ r) are logically equivalent
- This is the distributive law of disjunction over conjunction.

TABLE 5 **A Demonstration That $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ Are Logically Equivalent.**

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Propositional Logic

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Propositional Logic

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

Propositional Logic

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

$$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$$

$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Propositional Logic

- Use De Morgan's laws to express the negations of "Miguel has a cellphone and he has a laptop computer"
- Let p be "Miguel has a cellphone" and q be "Miguel has a laptop computer."
- Then Assessment "Miguel has a cellphone and he has a laptop computer" can be represented by $p \land q$.
- By the first of De Morgan's laws, $\neg(p \land q)$ is equivalent to $\neg p \lor \neg q$. Consequently, we can express the negation of our original statement as "Miguel does not have a cellphone or he does not have a laptop computer."

Propositional Logic

- Use De Morgan's laws to express the negations of "Heather will go to the concert or Steve will go to the concert."
- Let r be "Heather will go to the concert" and s be "Steve will go to the concert."
- Then "Heather will go to the concert or Steve will go to the concert" can be represented by $r \lor s$.
- By the second of De Morgan's laws, $\neg(r \lor s)$ is equivalent to $\neg r \land \neg s$. Consequently, we can express the negation of our original statement as "Heather will not go to the concert and Steve will not go to the concert."

Propositional Logic

- Show that ¬(p → q) and p ∧ ¬q are logically equivalent.

$$
\begin{aligned}
\neg(p \rightarrow q) &\equiv \neg(\neg p \vee q) && \text{by the conditional-disjunction equivalence} \\
&\equiv \neg(\neg p) \wedge \neg q && \text{by the second De Morgan law} \\
&\equiv p \wedge \neg q && \text{by the double negation law}
\end{aligned}
$$

Propositional Logic

- Show that ¬(p ∨ (¬p ∧ q)) and ¬p ∧ ¬q are logically equivalent by developing a series of logical equivalences.

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law} \\
&\equiv \mathbf{F} \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv \mathbf{F} \\
&\equiv (\neg p \wedge \neg q) \vee \mathbf{F} && \text{by the commutative law for disjunction} \\
&\equiv \neg p \wedge \neg q && \text{by the identity law for } \mathbf{F}
\end{aligned}
$$

Propositional Logic

- Show that $(p \wedge q) \to (p \vee q)$ is a tautology.

$$
\begin{aligned}
(p \wedge q) \to (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{by Example 3} \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{by the first De Morgan law} \\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{by the associative and commutative laws for disjunction} \\
&\equiv \mathbf{T} \vee \mathbf{T} && \text{by Example 1 and the commutative law for disjunction} \\
&\equiv \mathbf{T} && \text{by the domination law}
\end{aligned}
$$

Propositional Logic

- A compound proposition is satisfiable if there is an assignment of truth values to its variables that makes it true (that is, when it is a tautology or a contingency).
- When no such assignments exists, that is, when the compound proposition is false for all assignments of truth values to its variables, the compound proposition is unsatisfiable.
- Note that a compound proposition is unsatisfiable if and only if its negation is true for all assignments of truth values to the variables, that is, if and only if its negation is a tautology.
- When we find a particular assignment of truth values that makes a compound proposition true,we have shown that it is satisfiable; such an assignment is called a solution of this particular satisfiability problem.

Propositional Logic

- Determine whether the compound propositions $(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p)$ is satisfiable.
- Instead of using a truth table to solve this problem, we will reason about truth values.
- Note that $(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p)$ is true when the three variables p, q, and r have the same truth value.
- Hence, it is satisfiable as there is at least one assignment of truth values for p, q, and r that makes it true.

Propositional Logic

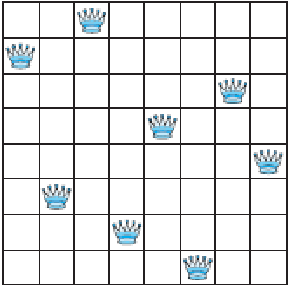
- Determine whether each of the compound propositions $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is satisfiable.
- Similarly, note that $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is true when at least one of p, q, and r is true and at least one is false.
- Hence, $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is satisfiable, as there is at least one assignment of truth values for p, q, and r that makes it true.

Propositional Logic

- Determine whether the compound propositions and $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is satisfiable.
- Note that for $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ to be true, $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ and $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ must both be true.
- For the first to be true, the three variables must have the same truth values, and for the second to be true, at least one of the three variables must be true and at least one must be false.
- However, these conditions are contradictory. From these observations we conclude that no assignment of truth values to p, q, and r makes $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ true. Hence, it is unsatisfiable.

The n-Queens Problem

- The n-queens problem asks for a placement of n queens on an n $\times$ n chessboard so that no queen can attack another queen. This means that no two queens can be placed in the same row, in the same column, or on the same diagonal. We display a solution to the eight-queens problem in following Figure.



The n-Queens Problem

p: You will join a MNC

q: You are expert in ML

r: You are not hired by another company

$$p \to (q \vee r)$$

~~If~~ r: You are hired by another company

$$p \to (q \vee (\sim r))$$

You cannot ride the bike if you are under 4 feet tall unless you are older than 16 years

p: You can ride the bike

q: You are under 4 feet tall

r: You are older than 16 years

$$(q \wedge (\sim r)) \to (\sim p)$$