Discrete Mathematics

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November 17, 2020

Discrete Mathematics - Recurrence Relation

In this chapter, we will discuss how recursive techniques can derive sequences and be used for solving counting problems. The procedure for finding the terms of a sequence in a recursive manner is called **recurrence relation**. We study the theory of linear recurrence relations and their solutions. Finally, we introduce generating functions for solving recurrence relations.

Definition

A recurrence relation is an equation that recursively defines a sequence where the next term is a function of the previous terms (Expressing F_n as some combination of F_i with $i < n$).

Example – Fibonacci series – $F_n = F_{n-1} + F_{n-2}$, Tower of Hanoi – $F_n = 2F_{n-1} + 1$

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Linear Recurrence Relations

A linear recurrence equation of degree k or order k is a recurrence equation which is in the format $x_n = A_1 x_{n-1} + A_2 x_{n-1} + A_3 x_{n-1} + \ldots A_k x_{n-k}$ (A_n is a constant and $A_k \neq 0$) on a sequence of numbers as a first-degree polynomial.

These are some examples of linear recurrence equations –

Recurrence relations	Initial values	Solutions
$F_n = F_{n-1} + F_{n-2}$	$a_1 = a_2 = 1$	Fibonacci number
$F_n = F_{n-1} + F_{n-2}$	$a_1 = 1, a_2 = 3$	Lucas Number
$F_n = F_{n-2} + F_{n-3}$	$a_1 = a_2 = a_3 = 1$	Padovan sequence
$F_n = 2F_{n-1} + F_{n-2}$	$a_1 = 0, a_2 = 1$	Pell number

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How to solve linear recurrence relation

Suppose, a two ordered linear recurrence relation is – $F_n = AF_{n-1} + BF_{n-2}$ where A and B are real numbers.

The characteristic equation for the above recurrence relation is –

$$x^2 - Ax - B = 0$$

Three cases may occur while finding the roots –

Case 1 – If this equation factors as $(x - x_1)(x - x_1) = 0$ and it produces two distinct real roots x_1 and x_2 , then $F_n = ax_1^n + bx_2^n$ is the solution. [Here, a and b are constants]

Case 2 – If this equation factors as $(x - x_1)^2 = 0$ and it produces single real root x_1 , then $F_n = ax_1^n + bnx_1^n$ is the solution.

Case 3 – If the equation produces two distinct complex roots, x_1 and x_2 in polar form $x_1 = r\angle\theta$ and $x_2 = r\angle(-\theta)$, then $F_n = r^n(acos(n\theta) + bsin(n\theta))$ is the solution.

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Problem 1

Solve the recurrence relation $F_n = 5F_{n-1} - 6F_{n-2}$ where $F_0 = 1$ and $F_1 = 4$

Solution

The characteristic equation of the recurrence relation is –

$$x^2 - 5x + 6 = 0,$$

So, $(x - 3)(x - 2) = 0$

Hence, the roots are –

$x_1 = 3$ and $x_2 = 2$

The roots are real and distinct. So, this is in the form of case 1

Hence, the solution is –

$$F_n = ax_1^n + bx_2^n$$

Here, $F_n = a3^n + b2^n \ (As\ x_1 = 3\ and\ x_2 = 2)$

Therefore,

$1 = F_0 = a3^0 + b2^0 = a + b$

$4 = F_1 = a3^1 + b2^1 = 3a + 2b$

Solving these two equations, we get $a = 2$ and $b = -1$

Hence, the final solution is –

$$F_n = 2.3^n + (-1).2^n = 2.3^n - 2^n$$

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Problem 2

Solve the recurrence relation – $F_n = 10F_{n-1} - 25F_{n-2}$ where $F_0 = 3$ and $F_1 = 17$

Solution

The characteristic equation of the recurrence relation is –

$$x^2 - 10x - 25 = 0$$

So $(x-5)^2 = 0$

Hence, there is single real root $x_1 = 5$

As there is single real valued root, this is in the form of case 2

Hence, the solution is –

$$F_n = ax_1^n + bnx_1^n$$

$$3 = F_0 = a.5^0 + (b)(0.5)^0 = a$$

$$17 = F_1 = a.5^1 + b.1.5^1 = 5a + 5b$$

Solving these two equations, we get $a = 3$ and $b = 2/5$

Hence, the final solution is – $F_n = 3.5^n + (2/5).n.2^n$

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Generating Functions

Generating Functions represents sequences where each term of a sequence is expressed as a coefficient of a variable x in a formal power series.

Mathematically, for an infinite sequence, say $a_0, a_1, a_2, \dots, a_k, \dots,$ the generating function will be –

$$G_x = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

Some Areas of Application

Generating functions can be used for the following purposes –

- For solving a variety of counting problems. For example, the number of ways to make change for a Rs. 100 note with the notes of denominations Rs.1, Rs.2, Rs.5, Rs.10, Rs.20 and Rs.50
- For solving recurrence relations
- For proving some of the combinatorial identities
- For finding asymptotic formulae for terms of sequences

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Problem 1

What are the generating functions for the sequences $\{a_k\}$ with $a_k = 2$ and $a_k = 3k$?

Solution

When $a_k = 2$, generating function, $G(x) = \sum_{k=0}^{\infty} 2x^k = 2 + 2x + 2x^2 + 2x^3 + \ldots$

When $a_k = 3k, G(x) = \sum_{k=0}^{\infty} 3kx^k = 0 + 3x + 6x^2 + 9x^3 + \ldots\ldots$

Problem 2

What is the generating function of the infinite series; $1, 1, 1, 1, \ldots$?

Solution

Here, $a_k = 1$, for $0 \leq k \leq \infty$

Hence, $G(x) = 1 + x + x^2 + x^3 + \ldots\cdots = \frac{1}{(1-x)}$

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Some Useful Generating Functions

- For $a_k = a^k, G(x) = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \ldots\ldots\cdots = 1/(1-ax)$
- For $a_k = (k+1), G(x) = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 \ldots\ldots\cdots = \frac{1}{(1-x)^2}$
- For

  $a_k = c_k^n, G(x) = \sum_{k=0}^{\infty} c_k^n x^k = 1 + c_1^n x + c_2^n x^2 + \ldots\ldots\cdots + x^2 = (1+x)^n$
- For $a_k = \frac{1}{k!}, G(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots\ldots\cdots = e^x$