Discrete Mathematics

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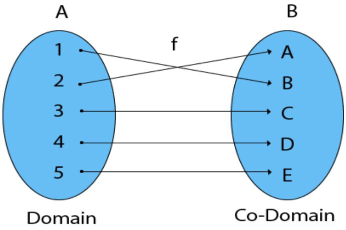
November 17, 2020

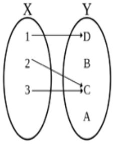
Discrete Mathematics - Functions

- Function - Definition
- It is a mapping in which every element of set A is uniquely associated at the element with set B. The set of A is called Domain of a function and set of B is called Co domain.

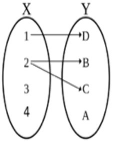


Discrete Mathematics - Functions

- Formally, A function or mapping (Defined as f:X→Y) is a relationship from elements of one set X to elements of another set Y (X and Y are non-empty sets). X is called Domain and Y is called Codomain of function 'f'.
- Function 'f' is a relation on X and Y such that for each $x \in X$, there exists a unique $y \in Y$ such that $(x, y) \in f$. 'x' is called pre-image and 'y' is called image of function f.



Function



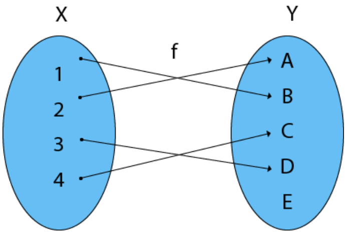
Not a Funtion

Domain, Co-Domain, and Range of a Function:

- Domain of a Function: Let f be a function from P to Q. The set P is called the domain of the function f.
- Co-Domain of a Function: Let f be a function from P to Q. The set Q is called Co-domain of the function f.
- Range of a Function: The range of a function is the set of picture of its domain. In other words, we can say it is a subset of its co-domain. It is denoted as f (domain).
- If f: $P \rightarrow Q$, then $f(P) = \{f(x): x \in P\} = \{y: y \in Q \mid \exists\, x \in P$, such that $f(x) = y\}$.

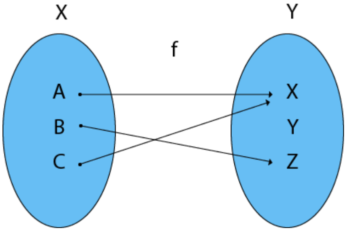
Find the Domain, Co-Domain, and Range of function.

- Let $x = \{1, 2, 3, 4\}$
  $y = \{a, b, c, d, e\}$
  $f = \{(1, b), (2, a), (3, d), (4, c)\}$
- Domain of function: $\{1, 2, 3, 4\}$
  Range of function: $\{a, b, c, d\}$
  Co-Domain of function: $\{a, b, c, d, e\}$



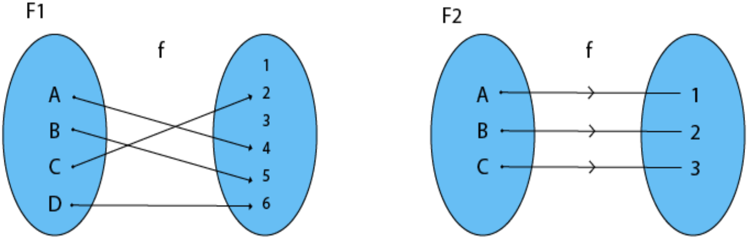
Representation of a Function

- The two sets P and Q are represented by two circles. The function f: $P \rightarrow Q$ is represented by a collection of arrows joining the points which represent the elements of P and corresponds elements of Q
- Let $X = \{a, b, c\}$ and
  $Y = \{x, y, z\}$ and f: $X \rightarrow Y$ such that $f = \{(a, x), (b, z), (c, x)\}$
- Then f can be represented diagrammatically as follows



Types of Fuctions

- **1. Injective (or one-to-one):**
  A function $f : X \rightarrow Y$ is called injective if for every $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$;
  $\forall x_1 \in X\ \forall x_2 \in X(f(x_1) = f(x_2) \rightarrow x_1 = x_2)$
- **Examples:**
- Injective (One-to-One) Functions: A function in which one element of Domain Set is connected to one element of Co-Domain Set.

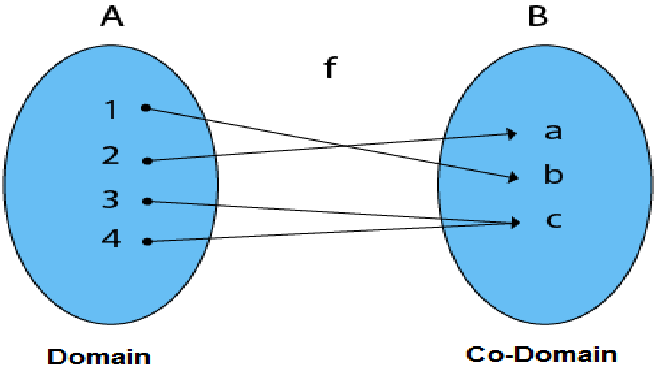


F1 and F2 show one to one Function

Types of Function

- **2. Surjective / Onto function:**
  A function $f : X \rightarrow Y$ is called Surjective if for every $y \in Y$, there exists an $x \in X$ such that f(x) = y: $\forall y \in Y \exists \in X (f(x) = y)$
- Surjective (Onto) Functions: A function in which every element of Co-Domain Set has one pre-image.
- Consider, A = {1, 2, 3, 4},
  B = {a, b, c}
  and f = {(1, b), (2, a), (3, c), (4, c)}.

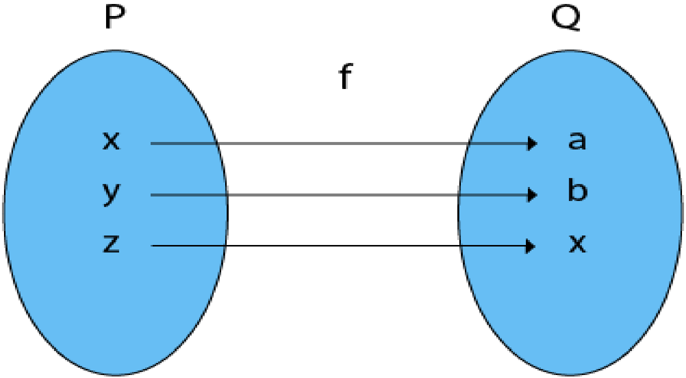
Types of Function



- It is a Surjective Function, as every element of B is the image of some A.
- Note: In an Onto Function, Range is equal to Co-Domain

Types of Function

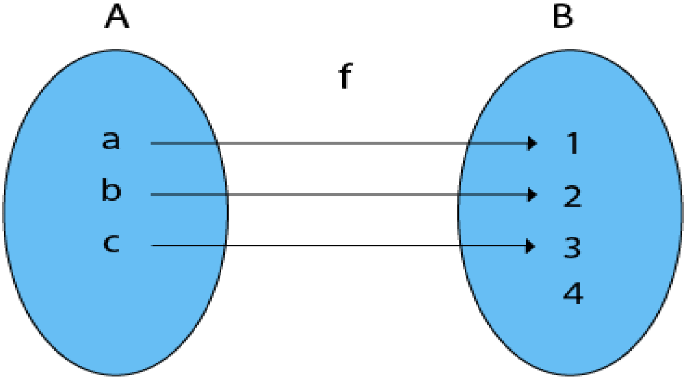
- Bijective (One-to-One Onto) Functions: A function which is both injective (one to - one) and surjective (onto) is called bijective (One-to-One Onto) Function.



- Consider P = {x, y, z} Q = {a, b, c} and f: P $\rightarrow$ Q such that f = {(x, a), (y, b), (z, c)}

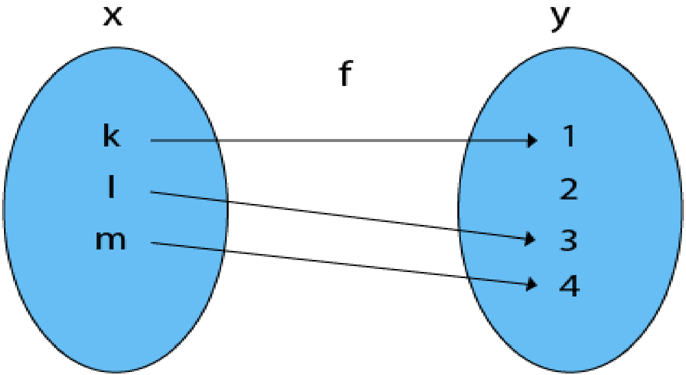
Types of Function

- Into Functions: A function in which there must be an element of co-domain Y does not have a pre-image in domain X.
  Example: Consider, A = {a, b, c} B = {1, 2, 3, 4} and f: $A \to B$ such that f = {(a, 1), (b, 2), (c, 3)} In the function f, the range i.e., {1, 2, 3}$\neq$co-domain of Y i.e., {1, 2, 3, 4}



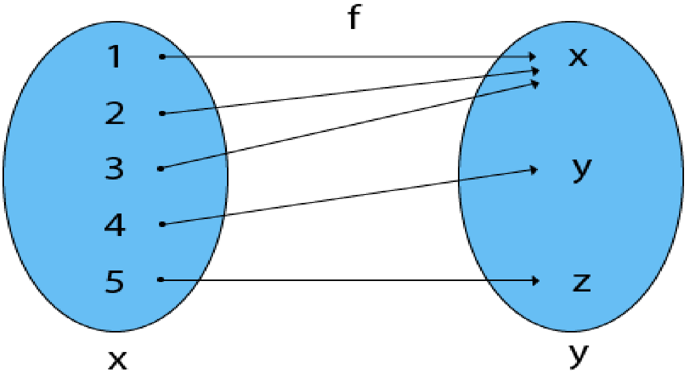
Types of Function

- One-One Into Functions: Let f: $X \rightarrow Y$. The function f is called one-one into function if different elements of X have different unique images of Y.
  Example: Consider, $X = \{k, l, m\}$ $Y = \{1, 2, 3, 4\}$ and f: $X \rightarrow Y$ such that $f = \{(k, 1), (l, 3), (m, 4)\}$



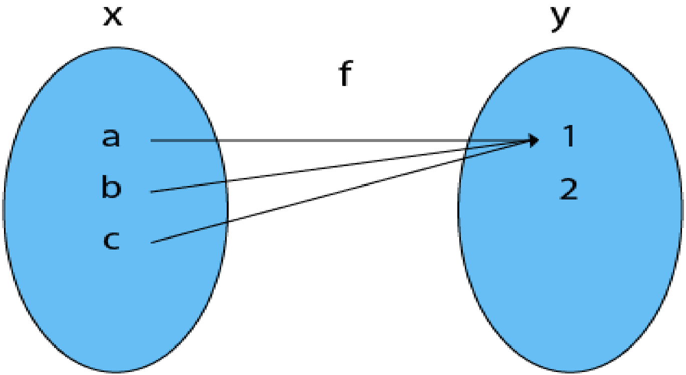
Types of Function

- Many-One Functions: Let f: X→Y. The function f is said to be many-one functions if there exist two or more than two different elements in X having the same image in Y.
  Example: Consider X ={1, 2, 3, 4, 5} Y = {x, y, z} and f: X → Y such that f = {(1, x), (2, x), (3, x), (4, y), (5, z)}



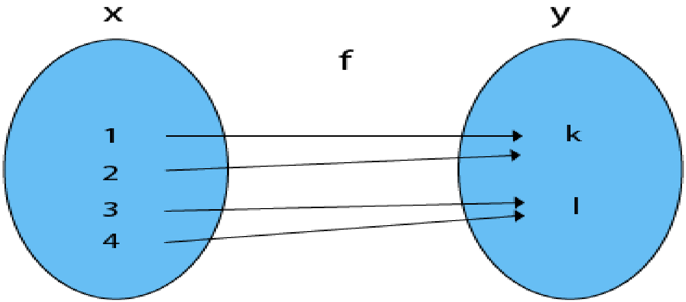
Types of Function

- Many-One Into Functions: Let f: $X \rightarrow Y$. The function f is called the many-one function if and only if is both many one and into function.
  Example:
  Consider $X = \{a, b, c\}$ $Y = \{1, 2\}$ and f: $X \rightarrow Y$ such that $f = \{(a, 1), (b, 1), (c, 1)\}$



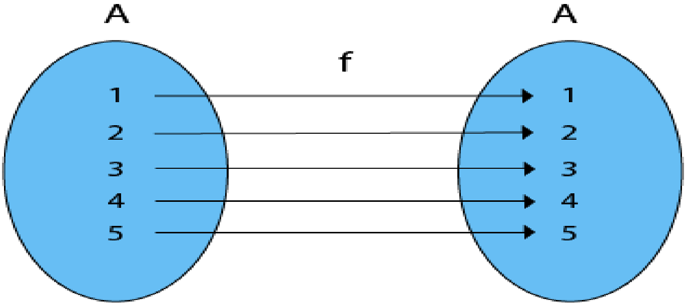
Types of Function

- Many-One Onto Functions: Let f: X $\rightarrow$ Y. The function f is called many-one onto function if and only if is both many one and onto. Example: Consider X = {1, 2, 3, 4} Y = {k, l} and f: X$\rightarrow$ Y such that f = {(1, k), (2, k), (3, l), (4, l)}
- The function f is a many-one (as the two elements have the same image in Y) and it is onto (as every element of Y is the image of some element X). So, it is many-one onto function



Identity Functions

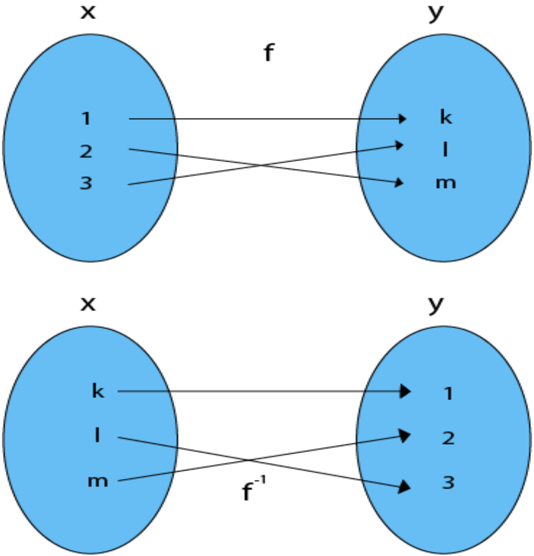
- The function f is called the identity function if each element of set A has an image on itself i.e. f (a) = a $\forall$ a $\in$ A. It is denoted by I.
  Example:
  Consider, A = {1, 2, 3, 4, 5} and f: A $\rightarrow$A such that f = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)}.
- The function f is an identity function as each element of A is mapped onto itself. The function f is a one-one and onto



Invertible (Inverse) Functions

- A function f: X $\rightarrow$ Y is invertible if and only if it is a bijective function.
  Consider the bijective (one to one onto) function f: X $\rightarrow$ Y.
  As f is a one to one, therefore, each element of X corresponds to a distinct element of Y. As f is onto, there is no element of Y which is not the image of any element of X, i.e., range = co-domain Y.
  The inverse function for f exists if f^{-1} is a function from Y to X.
- Consider, X = {1, 2, 3}
  Y = {k, l, m} and f: X$\rightarrow$Y such that
  f = {(1, k), (2, m), (3, l)}

Invertible (Inverse) Functions



Compositions of Functions

Two functions $f : A$ and $g : B$ can be composed to give a composition gof. This is a function from A to C defined by $(gof)(x) = g(f(x))$

Example:

Let $f(x) = x + 2$ and $g(x) = 2x + 1$, find (fog)(x) and (gof)(x).

Solution $(fog)(x) = f(g(x)) = f(2x + 1) = 2x + 1 + 2 = 2x + 3$

$(gof)(x) = g(f(x)) = g(x + 2) = 2(x + 2) + 1 = 2x + 5$

- If f and g are one-to-one then the function (gof) is also one-to-one.
- If f and g are onto then the function (gof) is also onto.
- Composition always holds associative property but does not hold commutative property.

Compositions of Functions

- Consider functions, f: $A \rightarrow B$ and g: $B \rightarrow C$. The composition of f with g is a function from A into C defined by $(g \circ f)(x) = g[f(x)]$ and is defined by $g \circ f$.
  To find the composition of f and g, first find the image of x under f and then find the image of f (x) under g.
- Example1: Let $X = \{1, 2, 3\}$ $Y = \{a, b\}$ $Z = \{5, 6, 7\}$. Consider the function $f = \{(1, a), (2, a), (3, b)\}$ and $g = \{(a, 5), (b, 7)\}$ as in figure. Find the composition of $g \circ f$.
- $(g \circ f)(1) = g[f(1)] = g(a) = 5$, $(g \circ f)(2) = g[f(2)] = g(a) = 5$ $(g \circ f)(3) = g[f(3)] = g(b) = 7$. [refer next figure]

Compositions of Functions