

Data Structures and Algorithms-I

Algorithm Complexity

About the course

- **My name: Dr. Prakash Raghavendra**
- **Areas of Interest:**
 - **Compilers, HPC, Optimization for Machine Learning**
- **In this course:**
 - **Complexity, Stacks, Lists**
 - **Hash Tables**
 - **Binary Search Trees**
 - **Graphs, Balanced Search Trees**
 - **Searching, Sorting**
 - **Hard Problems**

Evaluation

- **Quiz: 20%**
- **Mid Sem: 30%**
- **End Sem: 50%**

What is an algorithm?

- **An algorithm is “a finite set of precise/well defined steps/instructions for performing a computation or for solving a problem”**
 - **A program is an implementation of algorithm**
 - **Directions to somebody’s house is an algorithm**
 - **A recipe for cooking a cake is an algorithm**
 - **The steps to compute the cosine of 90° is an algorithm**
 - **Next, you can take left/right turn – not an algorithm step**

Some algorithms are harder than others

- **Some algorithms are easy**
 - Finding the largest (or smallest) value in a list
 - Finding a specific value in a list
- **Some algorithms are a bit harder**
 - Sorting a list
- **Some algorithms are very hard**
 - Finding the shortest path between Miami and Seattle
- **Some algorithms are essentially impossible**
 - Factoring large composite numbers
- **In this course, we'll see how to rate how “hard” algorithms are**

Algorithm : Maximum element

- **Given a list, how do we find the maximum element in the list?**
- **To express the algorithm, we'll use pseudocode**
 - **Pseudocode is kind of a programming language, but not really**

Algorithm : Maximum element

- **Algorithm for finding the maximum element in a list:**

procedure max (a_1, a_2, \dots, a_n : integers)

max := a_1

for i := 2 to n

if ($a_i > \text{max}$) then max := a_i

{max is the largest element}

Algorithm 1: Maximum element

procedure max (a_1, a_2, \dots, a_n : integers)

$\text{max} := a_1$

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for $i := 2$ to n

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if $\text{max} < a_i$ then $\text{max} := a_i$

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max

9

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
4	1	7	0	5	2	9	3	6	8

i

10

Maximum element running time

- **How long does this take? Rather, how many operations does it take to find the max?**
- **If the list has n elements, worst case scenario is that it takes n “steps”**
 - **Here, a step is considered a single step through the list**

Properties of algorithms

- **Algorithms generally share a set of properties:**
 - **Input:** what the algorithm takes in as input
 - **Output:** what the algorithm produces as output
 - **Definiteness:** the steps are defined precisely
 - **Correctness:** should produce the correct output
 - **Finiteness:** the steps required should be finite
 - **Effectiveness:** each step must be able to be performed in a finite amount of time
 - **Generality:** the algorithm *should* be applicable to all problems of a similar form

A General Portable Performance Metric

- *Informally*, Time to solve a problem of size, n ,

$$T(n) \text{ is } O(\log n)$$

$$\square T(n) = c \log_2 n$$

- **Formally:**

- $O(g(n))$ is **the set of functions, f** , such that

$$f(n) < c g(n)$$

for some constant, $c > 0$, and $n > N$

ie for sufficiently large n

- **Alternatively,**
we may write
and say

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} \leq c$$

g is an upper bound for f

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- $g(n)$ describes the **worst case behaviour** of an algorithm that is $O(g)$

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- $\Theta(g) = O(g) \cap \Omega(g)$

Set of functions growing
at the same rate as g

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 - If f is $O(g)$, then $f + g$ is $O(g)$
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- Polynomial's growth rate is determined by leading term
 - If f is a polynomial of degree d ,
then f is $O(n^d)$

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- *Exponential functions grow faster than powers*
 - *n^k is $O(b^n) \forall b > 1$ and $k \geq 0$*
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- Logarithms grow more slowly than powers
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Important!

Properties of the O notation

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 - $\log_b n$ is $O(\log_d n) \forall b, d > 1$
- Sum of first n r^{th} powers grows as the $(r+1)^{th}$ power

- $\sum_{k=1}^n k^r$ is $\Theta(n^{r+1})$

eg $\sum_{k=1}^n i = \frac{n(n+1)}{2}$ is $\Theta(n^2)$

Polynomial and Intractable Algorithms

- **Polynomial Time complexity**
 - An algorithm is said to be polynomial if it is $O(n^d)$ for some integer d
 - Polynomial algorithms are said to be **efficient**
 - They solve problems in reasonable times!
- **Intractable algorithms**
 - Algorithms for which there is no **known** polynomial time algorithm
 - *We will come back to this important class later in the course*

Analysing an Algorithm

- Simple statement sequence

$s_1; s_2; \dots; s_k$

- $O(1)$ as long as k is constant

- Simple loops

`for (i=0; i<n; i++) { s; }`

where s is $O(1)$

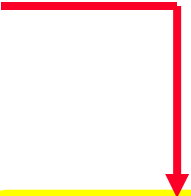
- Time complexity is $n O(1)$ or $O(n)$

- Nested loops

`for (i=0; i<n; i++)`

`for (j=0; j<n; j++) { s; }`

- Complexity is $n O(n)$ or $O(n^2)$



This part is
 $O(n)$

Analysing an Algorithm

- **Loop index doesn't vary linearly**

```
h = 1;
while ( h <= n ) {
    s;
    h = 2 * h;
}
```

- **h takes values 1, 2, 4, ... until it exceeds n**
- **There are $1 + \log_2 n$ iterations**
- **Complexity $O(\log n)$**

Analysing an Algorithm

- Loop index depends on outer loop index

```
for (j=0 ; j<n ; j++)  
    for (k=0 ; k<j ; k++) {  
        s ;  
    }
```

- Inner loop executed
 - 1, 2, 3,, n times

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

∴ Complexity $O(n^2)$

Distinguish this case -
where the iteration count
increases (decreases) by a
constant $\zeta O(n^k)$
from the previous one -
where it changes by a factor
 $\zeta O(\log n)$