Discrete Mathematics

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Discrete Mathematics - Sets

- Set - Definition
- A Set is an unordered collection of different elements. A set can be written explicitly by listing its elements using set bracket. If the order of the elements is changed or any element of a set is repeated, it does not make any changes in the set.
- Some Example of Sets
- A set of all positive integers
- A set of all the planets in the solar system
- A set of all the states in India
- A set of all the lowercase letters of the alphabet

Discrete Mathematics - Relations

- Whenever sets are being discussed, the relationship between the elements of the sets is the next thing that comes up. Relations may exist between objects of the same set or between objects of two or more sets.
- Binary Relation.
- Let P and Q be two non- empty sets. A binary relation R is defined to be a subset of P x Q from a set P to Q. If (a, b)$\in$ R and R$\subseteq$ P x Q then a is related to b by R i.e., aRb. If sets P and Q are equal, then we say R $\subseteq$ P x P is a relation on P.
- (i) Let A = {a, b, c} B = {r, s, t} Then R = {(a, r), (b, r), (b, t), (c, s)} is a relation from A to B.
  (ii) Let A = {1, 2, 3} and B = A R = {(1, 1), (2, 2), (3, 3)} is a relation (equal) on A.

Discrete Mathematics - Relations

- If a set has n elements, how many relations are there from A to A.
- If a set A has n elements, A x A has n$^2 elements.$

So, there are 2$^{n^2} relations from A to A.$

- If A has m elements and B has n elements. How many relations are there from A to B and vice versa?
- There are m x n elements
  hence there are 2$^{mxn} relations from A to A.$
- If a set A $=$ $\{$1, 2$\}$. Determine all relations from A to A.
- There are 2^2 $= 4 elements$
  i.e., $\{$(1, 2), (2, 1), (1, 1), (2, 2)$\}$ in A x A.
  So, there are 2^4 $= 16 relations from A to A$

Domain and Range of Relation

- Domain of Relation: The Domain of relation R is the set of elements in P which are related to some elements in Q, or it is the set of all first entries of the ordered pairs in R. It is denoted by DOM (R).
- Range of Relation: The range of relation R is the set of elements in Q which are related to some element in P, or it is the set of all second entries of the ordered pairs in R. It is denoted by RAN (R).
- Let A = {1, 2, 3, 4}
  B = {a, b, c, d}
  R = {(1, a), (1, b), (1, c), (2, b), (2, c), (2, d)}.
- DOM (R) = {1, 2}
  RAN (R) = {a, b, c, d}

Examples:Domain and Range of Relation

- Let, A=$\{1,2,9\}$ and B=$\{1,3,7\}$

  Case 1 If relation R is 'equal to' then R=$\{(1,1),(3,3)\}$

  DOM(R) = $\{1,3\}$,
  RAN(R)=$\{1,3\}$

  Case 2 If relation R is 'less than' then R=$\{(1,3),(1,7),(2,3),(2,7)\}$
  DOM(R) = $\{1,2\}$,

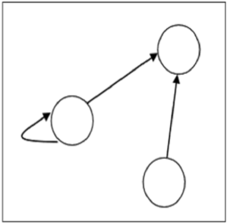
  RAN(R)=$\{3,7\}$

  Case 3 If relation R is 'greater than' then R=$\{(2,1),(9,1),(9,3),(9,7)\}$
  DOM(R) = $\{2,9\}$,

  RAN(R)=$\{1,3,7\}$'

Representation of Relations using Graph

- A relation can be represented using a directed graph.
  The number of vertices in the graph is equal to the number of elements in the set from which the relation has been defined.
  For each ordered pair (x, y) in the relation R, there will be a directed edge from the vertex ‘x’ to vertex ‘y’. If there is an ordered pair (x, x), there will be self- loop on vertex ‘x’.
  Suppose, there is a relation R={(1,1),(1,2),(3,2)} on set S={1,2,3}



Representation of Relations

- Relations can be represented in many ways. Some of which are as follows:

  1. Relation as a Matrix: Let P = [a$_1, a_2, a_3, ...a_m]and$

Q = [b$_1, b_2, b_3......b_n]arefinitesets, containingmandnnumberofelementsre$

R is a relation from P to Q.

The relation R can be represented by m x n matrix M = [M$_{ij}], definedas$

M$_{ij} = 0if(a_i, b_j) \notin$ R

1 if (a$_i, b_j) \in$ R

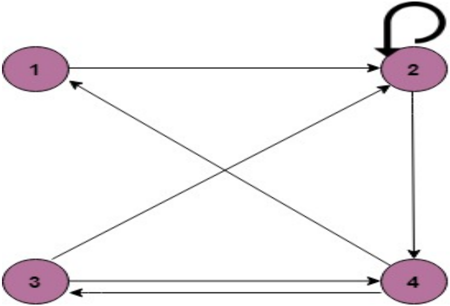
Relation as a Matrix

- Let P = {1, 2, 3, 4}, Q = {a, b, c, d} and R = {(1, a), (1, b), (1, c), (2, b), (2, c), (2, d)}.
- The matrix of relation R is shown as fig:

$$M_R = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{Bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Bmatrix} \end{array}$$

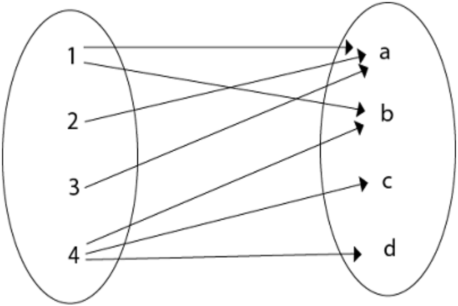
Relation as a Directed Graph

- There is another way of picturing a relation R when R is a relation from a finite set to itself.
- A = {1, 2, 3, 4}
  R = {(1, 2) (2, 2) (2, 4) (3, 2) (3, 4) (4, 1) (4, 3)}



Relation as an Arrow Diagram

- If P and Q are finite sets and R is a relation from P to Q. Relation R can be represented as an arrow diagram as follows.
  Draw two ellipses for the sets P and Q. Write down the elements of P and elements of Q column-wise in three ellipses. Then draw an arrow from the first ellipse to the second ellipse if a is related to b and a$\in$ P and b$\in$ Q.
  Let P = {1, 2, 3, 4} Q = {a, b, c, d} R = {(1, a), (2, a), (3, a), (1, b), (4, b), (4, c), (4, d)}



Relation as a Table

- If P and Q are finite sets and R is a relation from P to Q. Relation R can be represented in tabular form.
  Make the table which contains rows equivalent to an element of P and columns equivalent to the element of Q. Then place a cross (X) in the boxes which represent relations of elements on set P to set Q. Let P = {1, 2, 3, 4} Q = {x, y, z, k} R = {(1, x), (1, y), (2, z), (3, z), (4, k)}.

	x	y	z	k
1	x	x		
2			x	
3			x	
4				x

Types of Relations

- Reflexive Relation:A relation R on set A is said to be a reflexive if (a, a)$\in$ R for every a$\in$ A.
  Example: If A $= \{1, 2, 3, 4\}$ then R $= \{(1, 1)$ (2, 2), (1, 3), (2, 4), (3, 3), (3, 4), (4, 4)$\}$. Is a relation reflexive?
  Solution: The relation is reflexive as for every a $\in$ A. (a, a) $\in$ R, i.e. (1, 1), (2, 2), (3, 3), (4, 4) $\in$ R.
- Irreflexive Relation: A relation R on set A is said to be irreflexive if (a, a) $\notin$ R for every a $\in$ A.
  Example: Let A $= \{1, 2, 3\}$ and R $=\{(1, 2), (2, 2), (3, 1), (1, 3)\}$. Is the relation R reflexive or irreflexive?
  Solution: The relation R is not reflexive as for every a $\in$ A, (a, a) $\notin$ R, i.e., (1, 1) and (3, 3)$\notin$ R. The relation R is not irreflexive as (a, a)$\notin$ R, for some a $\in$ A, i.e., (2, 2) $\in$ R.

Types of Relations

- Symmetric Relation: A relation R on set A is said to be symmetric iff (a, b) $\in$ R (b, a) $\in$ R.
  Example: Let A = {1, 2, 3} and R = {(1, 1), (2, 2), (1, 2), (2, 1), (2, 3), (3, 2)}. Is a relation R symmetric or not?
  Solution: The relation is symmetric as for every (a, b)$\in$ R, we have (b, a) $\in$ R, i.e., (1, 2), (2, 1), (2, 3), (3, 2) $\in$ R but not reflexive because (3, 3) $\notin$ R.

- Antisymmetric Relation: A relation R on a set A is antisymmetric iff (a, b) $\in$ R and (b, a) $\in$ R then a = b.
  Example1: Let A = {1, 2, 3} and R = {(1, 1), (2, 2)}. Is the relation R antisymmetric?
  Solution: The relation R is antisymmetric as a = b when (a, b) and (b, a) both belong to R.
  Example2: Let A = {4, 5, 6} and R = {(4, 4), (4, 5), (5, 4), (5, 6), (4, 6)}. Is the relation R antisymmetric?
  Solution: The relation R is not antisymmetric as 4 5 but (4, 5) and

Types of Relations

- Asymmetric Relation: A relation R on a set A is called an Asymmetric Relation if for every (a, b)$\in$ R implies that (b, a) does not belong to R.
  Transitive Relations: A Relation R on set A is said to be transitive iff (a, b)$\in$ R and (b, c) $\in$ R (a, c) $\in$ R.
  Example1: Let A $=\{1, 2, 3\}$ and R $= \{(1, 2), (2, 1), (1, 1), (2, 2)\}$. Is the relation transitive?
  Solution: The relation R is transitive as for every (a, b) (b, c) belong to R, we have (a, c)$\in$ R i.e, (1, 2) (2, 1)$\in$ R (1, 1)$\in$ R.
- Identity Relation: Identity relation I on set A is reflexive, transitive and symmetric. So identity relation I is an Equivalence Relation.
  Example: A$= \{1, 2, 3\} = \{(1, 1), (2, 2), (3, 3)\}$

Types of Relations

- Void Relation: It is given by $R: \, : A \to B$ such that $R = \emptyset \ A \subseteq AxB$ is a null relation. Void Relation $R = \emptyset$ is symmetric and transitive but not reflexive.
- Universal Relation: A relation R: A $\to$ B such that R $=$ A x B ($\subseteq AxB$) is a universal relation. Universal Relation from A $\to$ B is reflexive, symmetric and transitive. So this is an equivalence relation.
- A relation is an Equivalence Relation if it is reflexive, symmetric, and transitive.
  The relation R$=${(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2),(1,3),(3,1)} on set A$=${1,2,3} is an equivalence relation since it is reflexive, symmetric, and transitive.

Equivalence Relations

- A relation R on a set A is called an equivalence relation if it satisfies following three properties:

  Relation R is Reflexive, i.e. aRa $a \in A$.

  Relation R is Symmetric, i.e., aRb $\Longrightarrow$ bRa

  Relation R is transitive, i.e., aRb and bRc $\Longrightarrow$ aRc.

Equivalence Relations Example

- Example: Let A = {1, 2, 3, 4} and R = {(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3), (4, 2), (4, 4)}.
  Show that R is an Equivalence Relation.
- Solution:
  Reflexive: Relation R is reflexive as (1, 1), (2, 2), (3, 3) and (4, 4)$\in$ R.
  Symmetric: Relation R is symmetric because whenever (a, b) $\in$ R, (b, a) also belongs to R.
  Example: (2, 4)$\in$ R$\Longrightarrow$ (4, 2) $\in$ R.
  Transitive: Relation R is transitive because whenever (a, b) and (b, c) belongs to R, (a, c) also belongs to R.
  Example: (3, 1)$\in$ R and (1, 3)$\in$ R$\Longrightarrow$ (3, 3) So, as R is reflexive, symmetric and transitive, hence, R is an Equivalence Relation.

Equivalence Relations

- Note1: If R_1 andR_2 are equivalence relation then$R_1\cap$ $R_2$is also an equivalence relation.
- Example: A $= \{1, 2, 3\}$
  R1 $= \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$
  R2 $=\{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$
  R1 $\cap$ R2 $= \{(1, 1), (2, 2), (3, 3)\}$
- Note2: If R1 and R2 are equivalence relation then $R_1\cup$ R_2 may or may not be an equivalence relation.
- Hence, Reflexive or Symmetric are Equivalence Relation but transitive may or may not be an equivalence relation

Composition of Relations

- Let A, B, and C be sets, and let R be a relation from A to B and let S be a relation from B to C. That is, R is a subset of A $\times$ B and S is a subset of B $\times$ C. Then R and S give rise to a relation from A to C indicated by R∘S and defined by
- a (R∘S)c if for some b $\in$ B we have aRb and bSc. is,
  R ∘S = {(a, c)— there exists b$\in$ B for which (a, b)$\in$ R and (b, c) $\in$ S}
- The relation R∘S is known the composition of R and S; it is sometimes denoted simply by RS.
  Let R is a relation on a set A, that is, R is a relation from a set A to itself. Then R∘R, the composition of R with itself, is always represented. Also, R∘R is sometimes denoted by R^2$.Similarly, R^3 = R^2$∘R = R∘R∘R, and so on. Thus R^n is defined for all positive n.

Composition of Relations Example1

- Let X = {4, 5, 6}, Y = {a, b, c} and Z = {l, m, n}. Consider the relation R1 from X to Y and R2 from Y to Z.
  R1 = {(4, a), (4, b), (5, c), (6, a), (6, c)}
  R2 = {(a, l), (a, n), (b, l), (b, m), (c, l), (c, m), (c, n)}
  Find the composition of relation (i) R1 o R2
  R1∘R2 = {(4, l), (4, n), (4, m), (5, l), (5, m), (5, n), (6, l), (6, m), (6, n)}

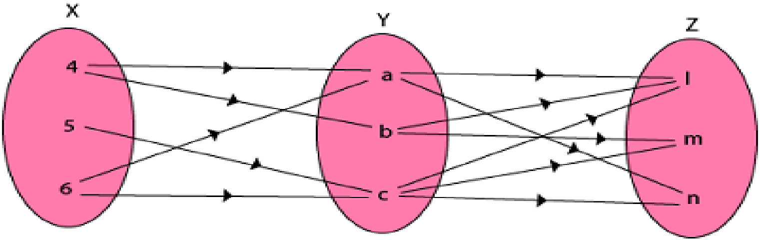


Fig : R_1 o R_2

Inverse Relation

- Let R be any relation from set A to set B. The inverse of R denoted by R^{-1} is the relations from B to A which consist of those ordered pairs which when reversed belong to R that is:

  R^{-1}= {(b, a): (a, b)$\in$ R}

  Example1: A ={1, 2, 3}
  B = {x, y, z}

  Solution: R = {(1, y), (1, z), (3, y)}
  R^{-1} = {(y, 1), (z, 1), (y, 3)}
  Clearly $(R^{-1})^{-1} = R$

Inverse Relation

- Note1: Domain and Range of R^{-1} is equal to range and domain of R.
- Note2: If R is an Equivalence Relation then R^{-1} is always an Equivalence Relation.
- Note3: If R is a Symmetric Relation then R^{-1}=R and vice-versa.

Partial Order Relations

- A relation R on a set A is called a partial order relation if it satisfies the following three properties:

  Relation R is Reflexive, i.e. aRa $\forall$ a$\in$ A.

  Relation R is Antisymmetric, i.e., aRb and bRa$\implies$ a $=$ b.

  Relation R is transitive, i.e., aRb and bRc$\implies$ aRc.

Partial Order Relations Examples

- Example1: Show whether the relation (x, y)$\in$ R, if, x $\geqslant$ y defined on the set of +ve integers is a partial order relation.
  Solution: Consider the set A = $\{1, 2, 3, 4\}$ containing four +ve integers.
  Find the relation for this set such as
  R = $\{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3), (1, 1), (2, 2), (3, 3), (4, 4)\}$.
- Reflexive: The relation is reflexive as for every a $\in$ A. (a, a)$\in$ R, i.e. (1, 1), (2, 2), (3, 3), (4, 4) $\in$ R.
  Antisymmetric: The relation is antisymmetric as whenever (a, b) and (b, a) $\in$ R, we have a = b.
  Transitive: The relation is transitive as whenever (a, b) and (b, c)$\in$ R, we have (a, c) $\in$ R.
  Example: (4, 2) $\in$ R and (2, 1)$\in$ R, implies (4, 1$\in$) R.
  As the relation is reflexive, antisymmetric and transitive. Hence, it is a partial order relation.