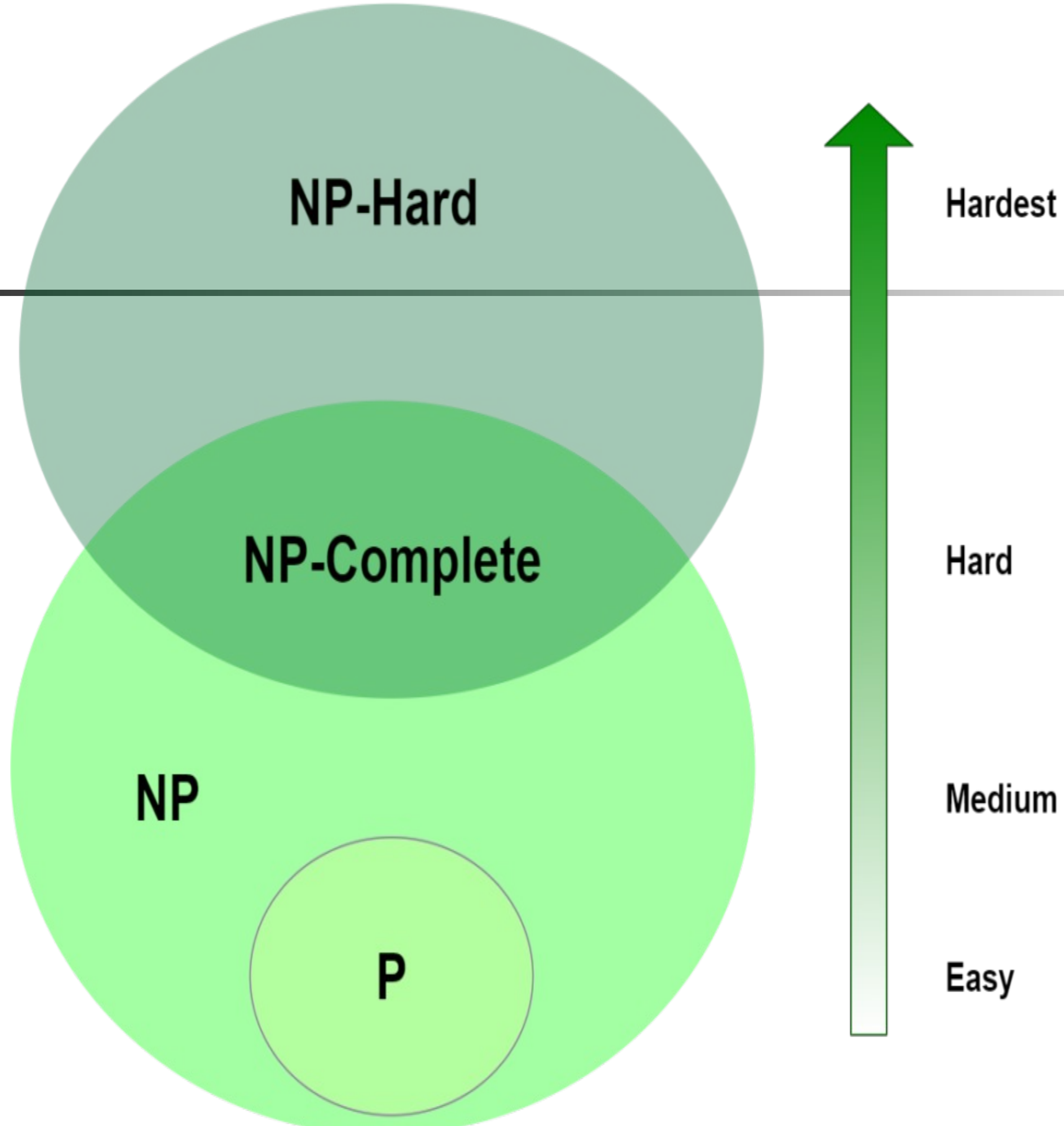
Approximation Algorithms


NP-Hard
NP-Complete
NP
P
Hardest
Hard
Medium
Easy

NP-completeness

"I can't find an efficient algorithm, but neither can all these famous people."

Coping With NP-Hardness

Brute-force Algorithms.

- Develop clever enumeration strategies.
- Guaranteed to find optimal solution.
- No guarantees on running time.

Heuristics.

Develop intuitive algorithms.
Guaranteed to run in polynomial time.
No guarantees on quality of solution.

Approximation Algorithms.

- Guaranteed to run in polynomial time.
- Guaranteed to find "high quality" solution, say within 1% of optimum.

Obstacle: need to prove a solution's value is close to optimum, without even knowing what optimum value is!

Coping with NP-completeness

Q. Suppose I need to solve an **NP**-hard optimization problem. What should I do?

A. Sacrifice one of three desired features.

i. Runs in polynomial time.
ii. Solves arbitrary instances of the problem.
iii. Finds optimal solution to problem.

ρ-approximation algorithm.

- Runs in polynomial time.
- Solves arbitrary instances of the problem
- Finds solution that is within ratio ρ of optimum.

Challenge. Need to prove a solution's value is close to optimum, without even knowing what is optimum value.

Approximation Algorithms

- Up to now, the best algorithm for solving an NP-complete problem requires exponential time in the worst case. It is too time-consuming.
- To reduce the time required for solving a problem, we can relax the problem, and obtain a feasible solution “close” to an optimal solution

Approximation Algorithms

- One compromise is to use heuristic solutions.

- The word “heuristic” may be interpreted as “educated guess.”

Approximation Algorithms

An algorithm that returns near-optimal solutions is called an ***Approximation Algorithm***.

We need to find an *Approximation Ratio Bound* for an approximation algorithm.

Approximation Ratio Bound

We say an approximation algorithm for the problem has a ratio bound of $\rho(n)$ if for any input size n, the cost C of the solution produced by the approximation algorithm is within a factor of $\rho(n)$ of the C^* of the optimal solution:

$$\max\{\frac{C}{C^*}, \frac{C^*}{C}\} = \rho(n)$$

This applies for both minimization and maximization problems.

Performance Guarantees

- An approximation algorithm is bounded by *$\rho(n)$* if, for all input of size *n*, the cost *c* of the solution obtained by the algorithm is within a factor *$\rho(n)$* of the *c^** of an optimal solution

ρ-approximation algorithm

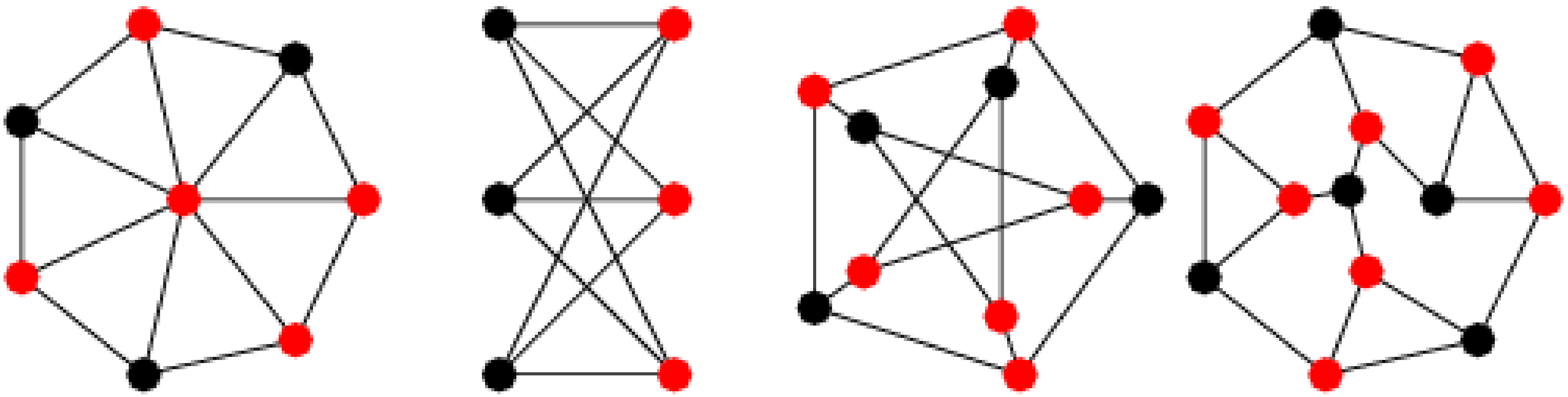
- An approximation algorithm with an approximation ratio bound of ρ is called a ρ-approximation algorithm or a $(1+\varepsilon)$-approximation algorithm.

- Note that ρ is always larger than 1 and $\varepsilon=\rho-1$.

Vertex Cover Problem

- Let G=(V, E). The subset S of V that meets every edge of E is referred to as the Vertex Cover.
- The Vertex Cover Problem is solved for finding a vertex cover of the Minimum size. It is NP-hard Computational Problem or the Optimization Version of an NP-Complete Decision Problem.

Examples of Vertex Cover



APPROX_VERTEX_COVER(G)

$C \leftarrow \phi$

$E' \leftarrow E(G)$

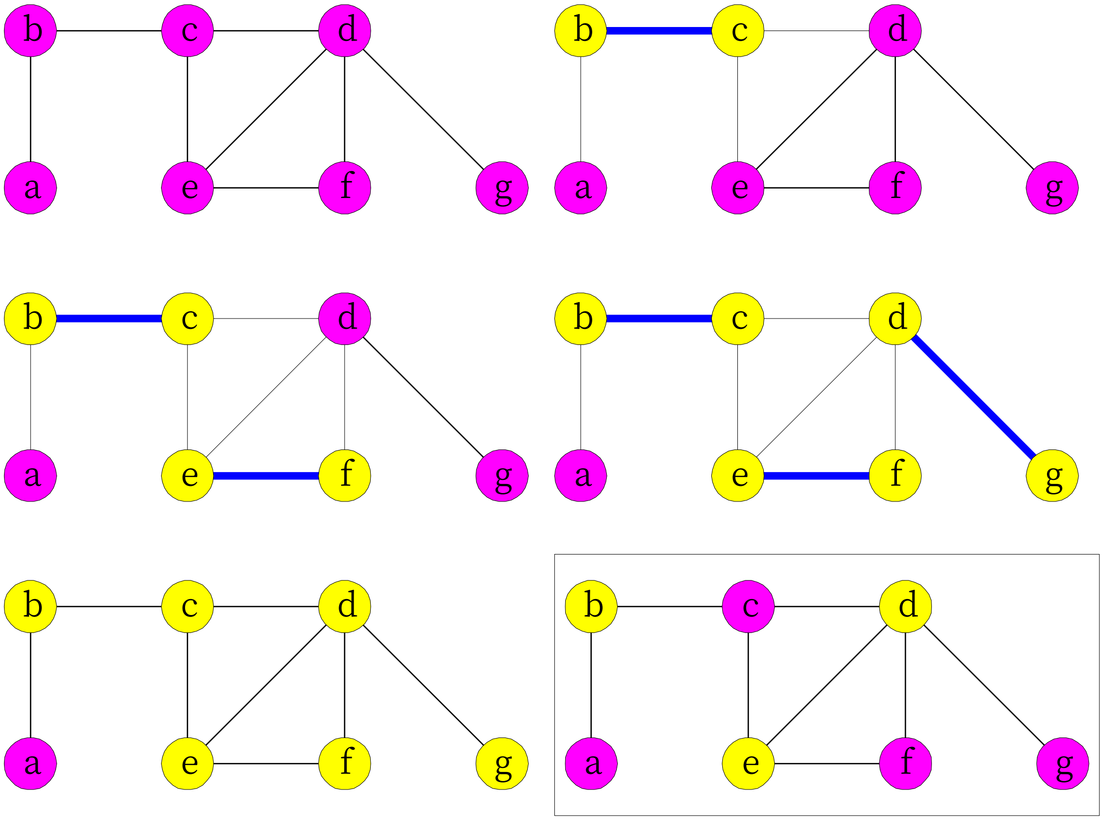
while $E' \neq \phi$

  do let (u,v) be an arbitrary edge of E'

    $C \leftarrow C \cup \{u,v\}$

    remove from E' every edge incident on either u or v

return C



Complexity: O(E)

Theorem: APPROX_VERTEX_COVER has ratio bound of 2.

Proof.

C*: optimal solution
C: approximate solution
A: the set of edges selected in step 4

Let A be the set of selected edges.

$|C| = 2|A|$

When one edge is selected, 2 vertices are added into C.

$|A| \leq |C^*|$

No two edges in A share a common endpoint.

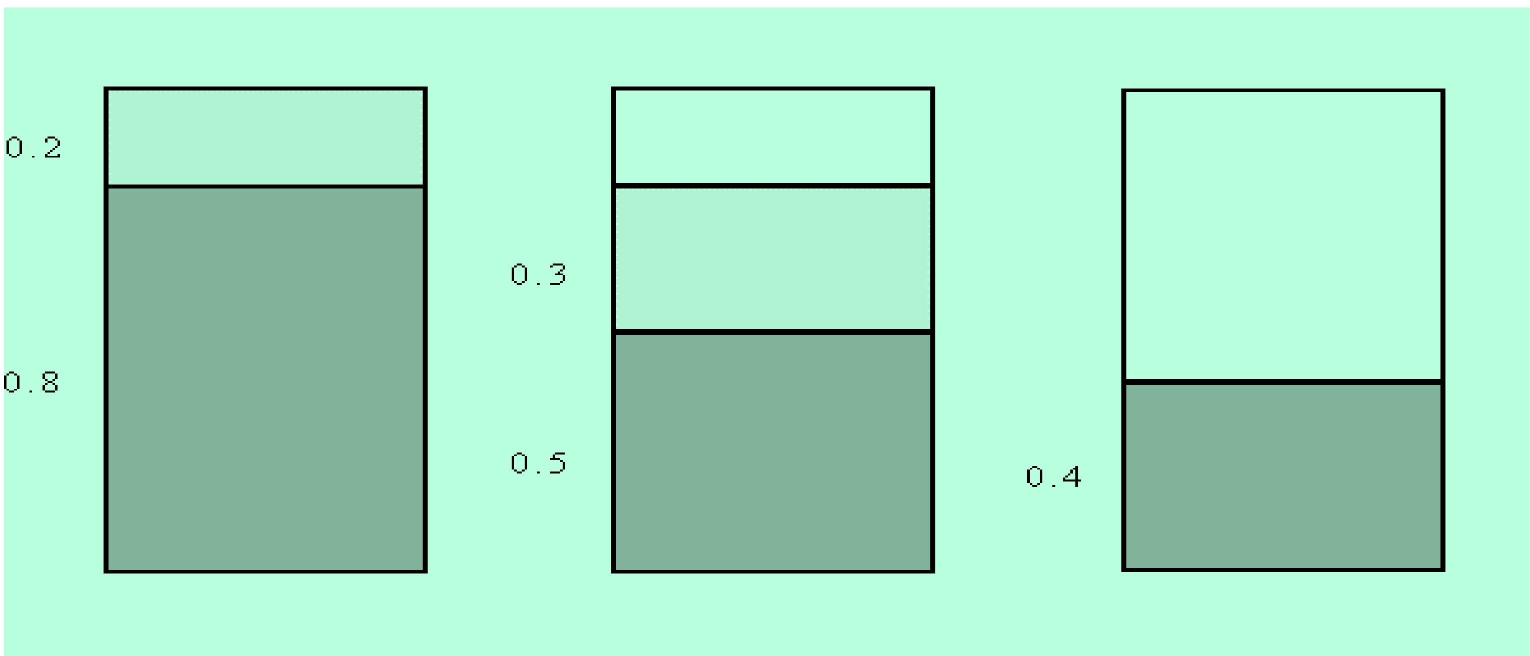
$\Rightarrow |C| \leq 2|C^*|$

Bin Packing Problem

- Given n items of sizes $a_1, a_2, \ldots, a_n$, $0 < a_i \leq 1$ for $1 \leq i \leq n$, which have to be placed in bins of unit capability, the bin packing problem is solved for determining the minimum number of bins to accommodate all items.
- If we consider the items of different sizes to be the lengths of time of executing different jobs on a standard processor, then the problem becomes to use minimum number of processors which can finish all of the jobs within a fixed time. // We can assume the longest job takes one unit time, which equals

Example of Bin Packing Problem

- Ex. Given n = 5 items with sizes 0.3, 0.5, 0.8, 0.2, 0.4, the optimal solution is 3 bins.



The bin packing problem is NP-hard optimization problem.

An Approximation Algorithm for the Bin Packing Problem

- An Approximation Algorithm: (First-Fit (FF)) place the item i into the lowest-indexed bin which can accommodate the item i.
- OPT: The number of bins of the Optimal Solution
- FF: The number of bins in the First-Fit Algorithm
- $C(B_i)$: The sum of the sizes of items packed in bin B_i in the First-Fit Algorithm
- Let FF=m.

An Approximation Algorithm for the Bin Packing Problem

- OPT $\geq \left\lceil \sum_{i=1}^{n} a_i \right\rceil$, ceiling of sum of sizes of all items
- $C(B_i) + C(B_{i+1}) >$ C(Bi): the sum of sizes of items packed in bin B_{i+1} will be put in B_i).
- $C(B_1) + C(B_m) > 1$ (b)(Otherwise, the items in B_m will be put in B_1.)
- For m nonempty bins, $C(B_1)+C(B_2)+\ldots+C(B_m) > m/2$, (a)+(b) for i=1,..,m

  $\Rightarrow FF = m < 2 \sum_{i=1}^{m} C(B_i) = 2 \sum_{i=1}^{n} a_i \leq 2\ OPT$

Load balancing

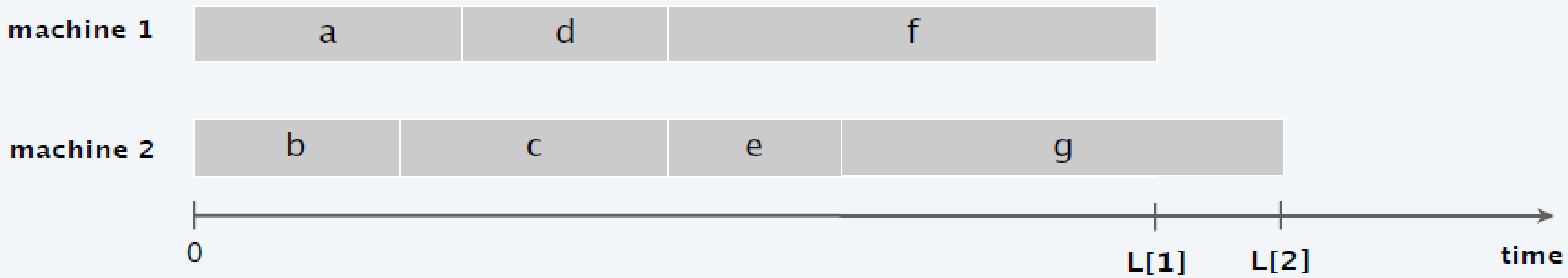
Input. m identical machines; $n \geq m$ jobs, job j has processing time t_j.

- Job j must run contiguously on one machine.
- A machine can process at most one job at a time.

Def. Let $S[i]$ be the subset of jobs assigned to machine i.
The load of machine i is $L[i] = \Sigma_{j \in S[i]} t_j$.

Def. The makespan is the maximum load on any machine $L = \max_i L[i]$.

Load balancing. Assign each job to a machine to minimize makespan.

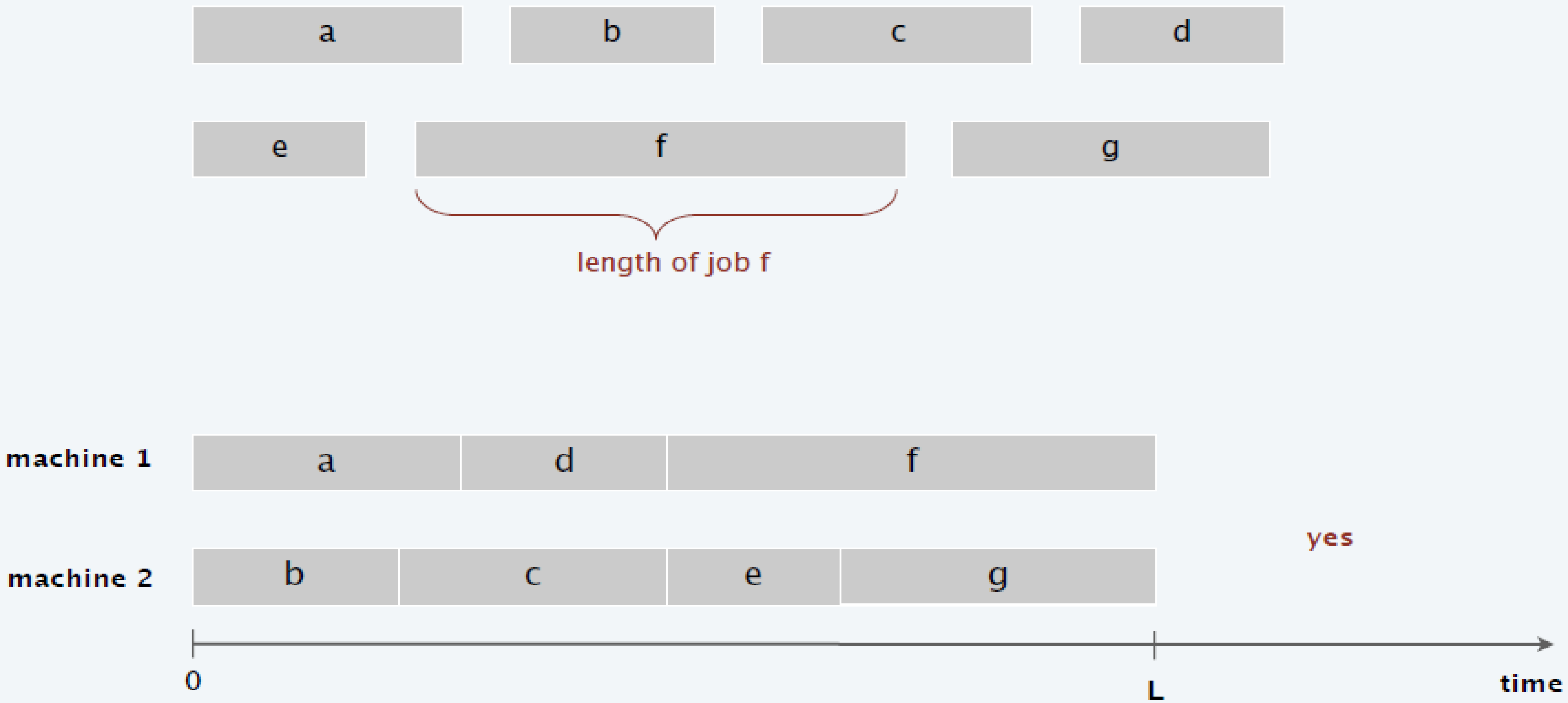


Load balancing on 2 machines is NP-hard

Claim. Load balancing is hard even if $m = 2$ machines.

Pf. PARTITION $\leq_P$ LOAD-BALANCE.

NP-complete by Exercise 8.26



Load balancing: list scheduling

List-scheduling algorithm.

- Consider n jobs in some fixed order.
- Assign job j to machine i whose load is smallest so far.

LIST-SCHEDULING $(m, n, t_1, t_2, \ldots, t_n)$

FOR $i = 1$ TO m

  $L[i] \leftarrow 0.$ ← load on machine i

  $S[i] \leftarrow \varnothing.$ ← jobs assigned to machine i

FOR $j = 1$ TO n

  $i \leftarrow \text{argmin}_k \, L[k].$ ← machine i has smallest load

  $S[i] \leftarrow S[i] \cup \{ j \}.$ ← assign job j to machine i

  $L[i] \leftarrow L[i] + t_j.$ ← update load of machine i

RETURN $S[1], S[2], \ldots, S[m].$

Implementation. $O(n \log m)$ using a priority queue for loads $L[k]$.

Load balancing: list scheduling analysis

Theorem. [Graham 1966] Greedy algorithm is a 2-approximation.

- First worst-case analysis of an approximation algorithm.
- Need to compare resulting solution with optimal makespan L^*.

Lemma 1. For all k: the optimal makespan $L^* \geq t_k$.

Pf. Some machine must process the most time-consuming job. ▪

Lemma 2. The optimal makespan $L^* \geq \frac{1}{m} \sum_k t_k$.

Pf.

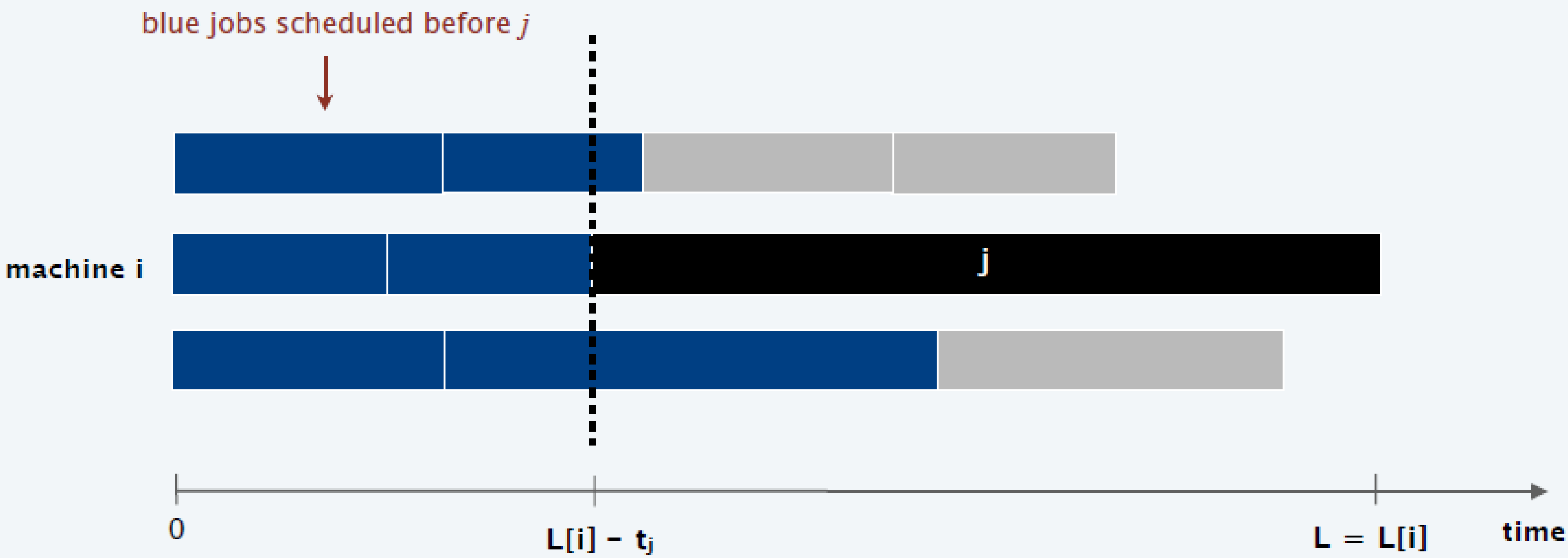
- The total processing time is $\Sigma_k \; t_k$.
- One of m machines must do at least a $1 / m$ fraction of total work. ▪

Load balancing: list scheduling analysis

Theorem. Greedy algorithm is a 2-approximation.

Pf. Consider load $L[i]$ of bottleneck machine i. ← machine that ends up with highest load

- Let j be last job scheduled on machine i.
- When job j assigned to machine i, i had smallest load.
  Its load before assignment is $L[i] - t_j$; hence $L[i] - t_j \leq L[k]$ for all $1 \leq k \leq m$.



Load balancing: list scheduling analysis

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Pf. Consider load $L[i]$ of bottleneck machine i. ← machine that ends up with highest load

- Let j be last job scheduled on machine i.
- When job j assigned to machine i, i had smallest load. Its load before assignment is $L[i] - t_j$; hence $L[i] - t_j \le L[k]$ for all $1 \le k \le m$.
- Sum inequalities over all k and divide by m:

$$\begin{aligned} L[i] - t_j &\le \frac{1}{m} \sum_k L[k] \\ &= \frac{1}{m} \sum_k t_k \\ \text{Lemma 2} \longrightarrow \quad &\le L^*. \end{aligned}$$

- Now, $L = L[i] = \underbrace{(L[i] - t_j)}_{\le L^*} + \underbrace{t_j}_{\le L^*} \le 2L^*$ ▪

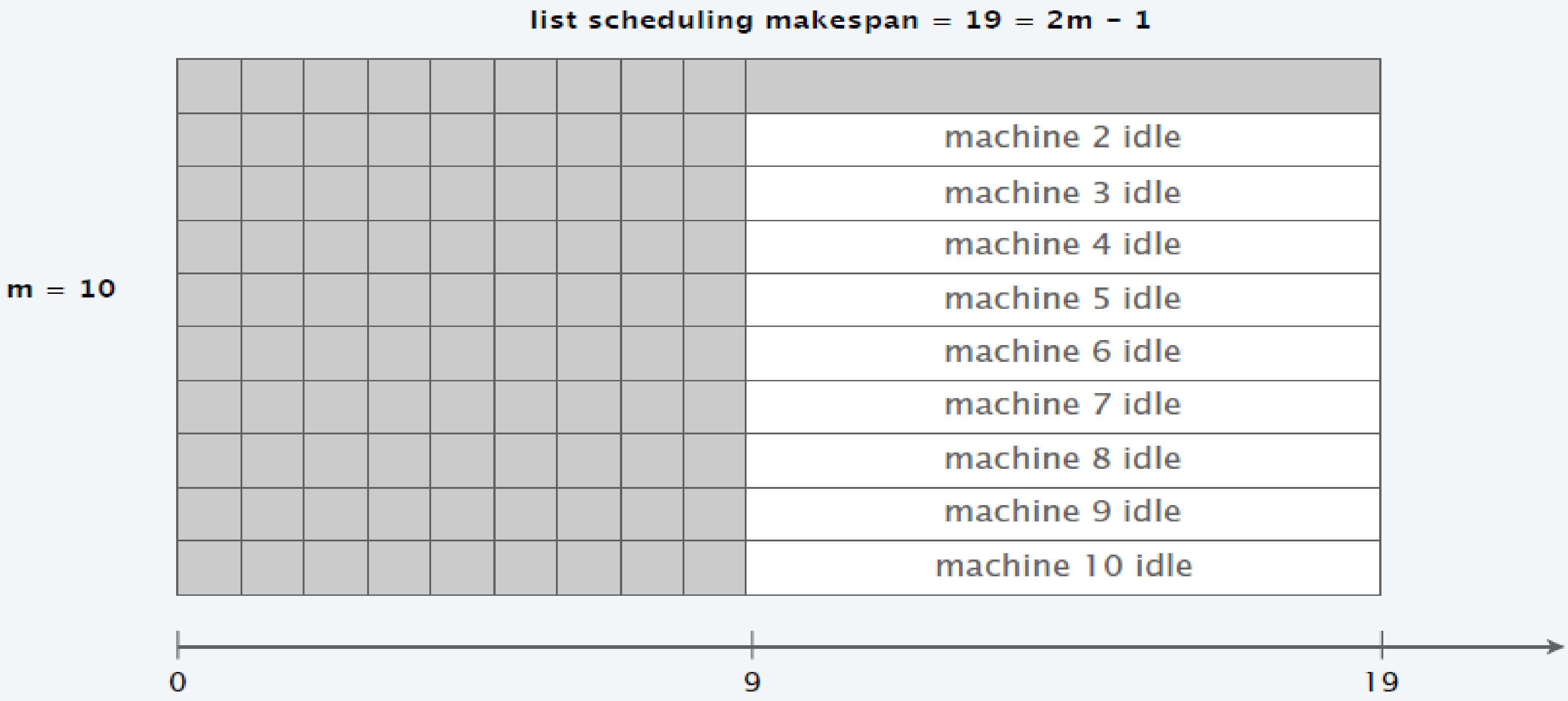
above inequality Lemma 1

Load balancing: list scheduling analysis

Q. Is our analysis tight?

A. Essentially yes.

Ex: m machines, first $m(m-1)$ jobs have length 1, last job has length m.



Load balancing: list scheduling analysis

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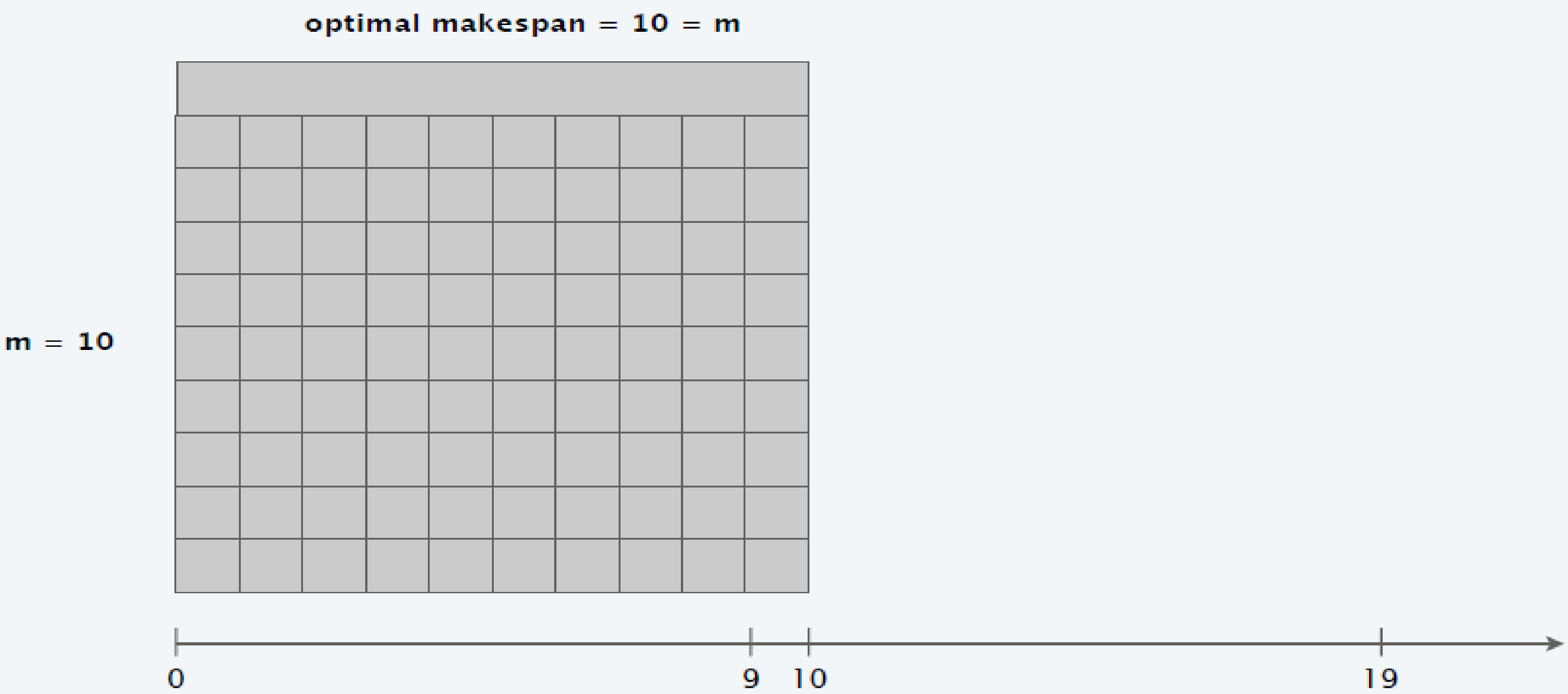


Load balancing: LPT rule

Longest processing time (LPT). Sort n jobs in decreasing order of processing times; then run list scheduling algorithm.

LPT-LIST-SCHEDULING $(m, n, t_1, t_2, \ldots, t_n)$

SORT jobs and renumber so that $t_1 \geq t_2 \geq \ldots \geq t_n$.

FOR $i = 1$ TO m

  $L[i] \leftarrow 0.$ ⟵ load on machine i

  $S[i] \leftarrow \varnothing.$ ⟵ jobs assigned to machine i

FOR $j = 1$ TO n

  $i \leftarrow \text{argmin}_k L[k].$ ⟵ machine i has smallest load

  $S[i] \leftarrow S[i] \cup \{ j \}.$ ⟵ assign job j to machine i

  $L[i] \leftarrow L[i] + t_j.$ ⟵ update load of machine i

RETURN $S[1], S[2], \ldots, S[m].$

Load balancing: LPT rule

Observation. If bottleneck machine i has only 1 job, then optimal.

Pf. Any solution must schedule that job. ▪

Lemma 3. If there are more than m jobs, $L^* \geq 2\, t_{m+1}$.

Pf.

- Consider processing times of first m+1 jobs $t_1 \geq t_2 \geq \ldots \geq t_{m+1}$.
- Each takes at least t_{m+1} time.
- There are $m+1$ jobs and m machines, so by pigeonhole principle, at least one machine gets two jobs. ▪

Theorem. LPT rule is a 3/2-approximation algorithm.

Pf. [similar to proof for list scheduling]

- Consider load $L[i]$ of bottleneck machine i.
- Let j be last job scheduled on machine i. ← assuming machine i has at least 2 jobs, we have $j \geq m + 1$

$$L = L[i] = \underbrace{(L[i] - t_j)}_{\leq L^*} + \underbrace{t_j}_{\leq \frac{1}{2} L^*} \leq \frac{3}{2} L^* \;\blacksquare$$

as before → $\leq L^*$ $\leq ½\, L^*$ ← Lemma 3 (since $t_{m+1} \geq t_j$)

Load balancing: LPT rule

Q. Is our 3/2 analysis tight?

A. No.

Theorem. [Graham 1969] LPT rule is a 4/3-approximation.

Pf. More sophisticated analysis of same algorithm.

Q. Is Graham's 4/3 analysis tight?

A. Essentially yes.

Ex.

- m machines
- $n = 2m + 1$ jobs
- 2 jobs of length $m, m+1, \ldots, 2m-1$ and one more job of length m.
- Then, $L / L^* = (4m - 1) / (3m)$

Generalized load balancing

Input. Set of m machines M; set of n jobs J.

- Job $j \in J$ must run contiguously on an authorized machine in $M_j \subseteq M$.
- Job $j \in J$ has processing time t_j.
- Each machine can process at most one job at a time.

Def. Let J_i be the subset of jobs assigned to machine i.

The load of machine i is $L_i = \Sigma_{j \in J_i} t_j$.

Def. The makespan is the maximum load on any machine $= \max_i L_i$.

Generalized load balancing. Assign each job to an authorized machine to minimize makespan.

Generalized load balancing: integer linear program and relaxation

ILP formulation. x_{ij} = time machine i spends processing job j.

$$
\begin{array}{llllll}
(IP) & \min & L & & & \\
 & \text{s.t.} & \sum_i x_{ij} & = & t_j & \text{for all } j \in J \\
 & & \sum_j x_{ij} & \leq & L & \text{for all } i \in M \\
 & & x_{ij} & \in & \{0, t_j\} & \text{for all } j \in J \text{ and } i \in M_j \\
 & & x_{ij} & = & 0 & \text{for all } j \in J \text{ and } i \notin M_j
\end{array}
$$

LP relaxation.

$$
\begin{array}{llllll}
(LP) & \min & L & & & \\
 & \text{s.t.} & \sum_i x_{ij} & = & t_j & \text{for all } j \in J \\
 & & \sum_j x_{ij} & \leq & L & \text{for all } i \in M \\
 & & x_{ij} & \geq & 0 & \text{for all } j \in J \text{ and } i \in M_j \\
 & & x_{ij} & = & 0 & \text{for all } j \in J \text{ and } i \notin M_j
\end{array}
$$

Generalized load balancing: lower bounds

Lemma 1. The optimal makespan $L^* \geq \max_j t_j$.

Pf. Some machine must process the most time-consuming job. ▪

Lemma 2. Let L be optimal value to the LP. Then, optimal makespan $L^* \geq L$.

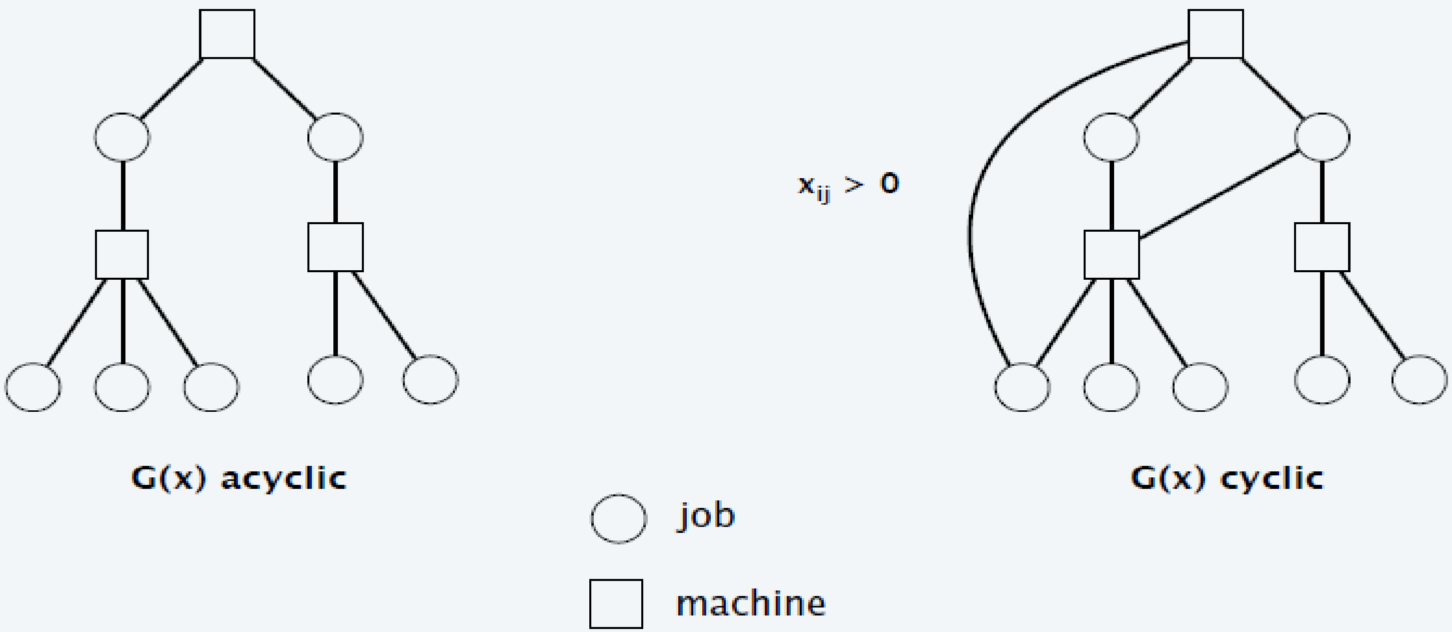
Pf. LP has fewer constraints than ILP formulation. ▪

Generalized load balancing: structure of LP solution

Lemma 3. Let x be solution to LP. Let $G(x)$ be the graph with an edge between machine i and job j if $x_{ij} > 0$. Then $G(x)$ is acyclic.

can transform x into another LP solution where G(x) is acyclic if LP solver doesn't return such an x

Pf. (deferred)



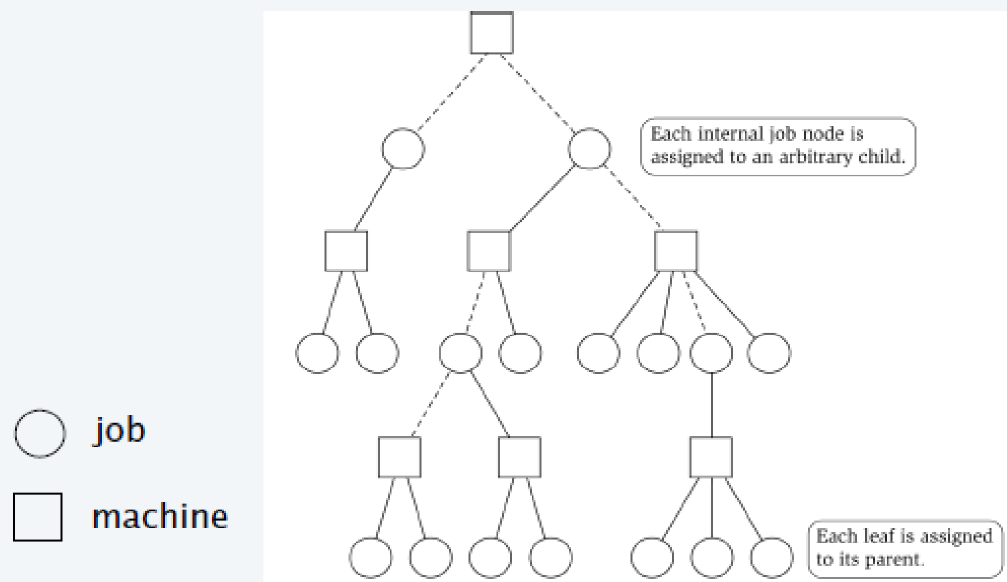
Generalized load balancing: rounding

Rounded solution. Find LP solution x where $G(x)$ is a forest. Root forest $G(x)$ at some arbitrary machine node r.

- If job j is a leaf node, assign j to its parent machine i.
- If job j is not a leaf node, assign j to any one of its children.

Lemma 4. Rounded solution only assigns jobs to authorized machines.

Pf. If job j is assigned to machine i, then $x_{ij} > 0$. LP solution can only assign positive value to authorized machines. ▪



Generalized load balancing: analysis

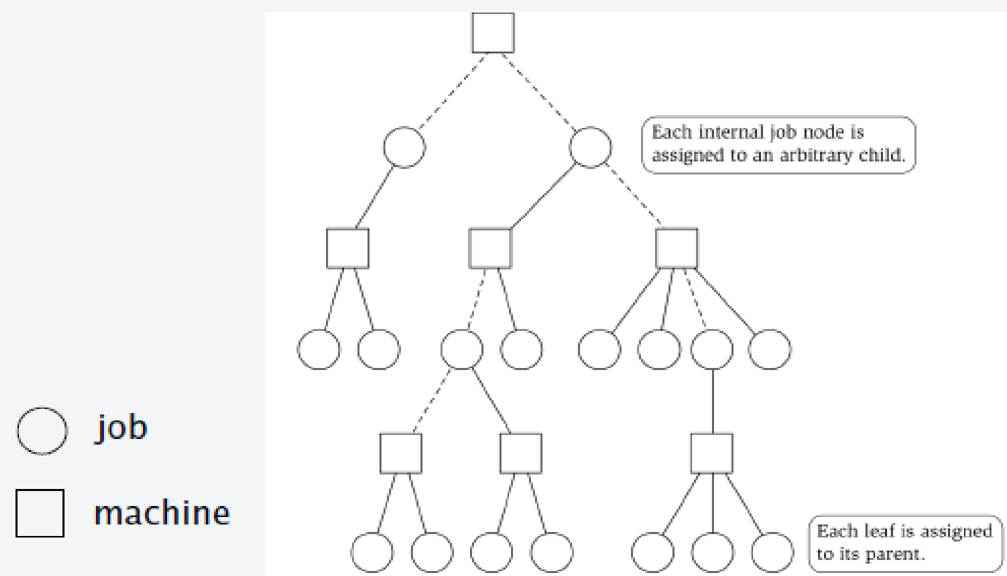
Lemma 5. If job j is a leaf node and machine $i = parent(j)$, then $x_{ij} = t_j$.

Pf.

- Since i is a leaf, $x_{ij} = 0$ for all $j \neq parent(i)$.
- LP constraint guarantees $\Sigma_i \, x_{ij} = t_j$. ▪

Lemma 6. At most one non-leaf job is assigned to a machine.

Pf. The only possible non-leaf job assigned to machine i is $parent(i)$. ▪



Generalized load balancing: analysis

Theorem. Rounded solution is a 2-approximation.

Pf.

- Let $J(i)$ be the jobs assigned to machine i.
- By LEMMA 6, the load L_i on machine i has two components:
  - leaf nodes:

$$\sum_{\substack{j \in J(i) \\ j \text{ is a leaf}}} t_j \overset{\text{Lemma 5}}{=} \sum_{\substack{j \in J(i) \\ j \text{ is a leaf}}} x_{ij} \le \sum_{j \in J} x_{ij} \overset{\text{LP}}{\le} \underset{\text{optimal value of LP}}{L} \overset{\text{Lemma 2 (LP is a relaxation)}}{\le} L^*$$

  - parent:

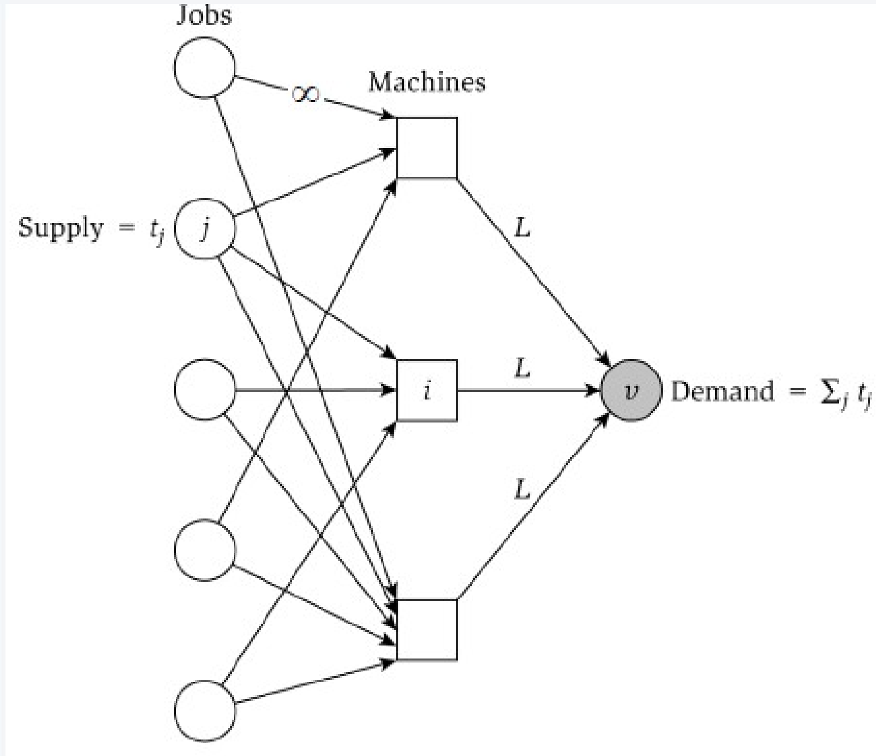
$$t_{\text{parent}(i)} \overset{\text{Lemma 1}}{\le} L^*$$

- Thus, the overall load $L_i \le 2L^*$. ▪

Generalized load balancing: flow formulation

Flow formulation of LP.

$$\begin{array}{llll}
\sum_i x_{ij} & = & t_j & \text{for all } j \in J \\
\sum_j x_{ij} & \leq & L & \text{for all } i \in M \\
x_{ij} & \geq & 0 & \text{for all } j \in J \text{ and } i \in M_j \\
x_{ij} & = & 0 & \text{for all } j \in J \text{ and } i \notin M_j
\end{array}$$



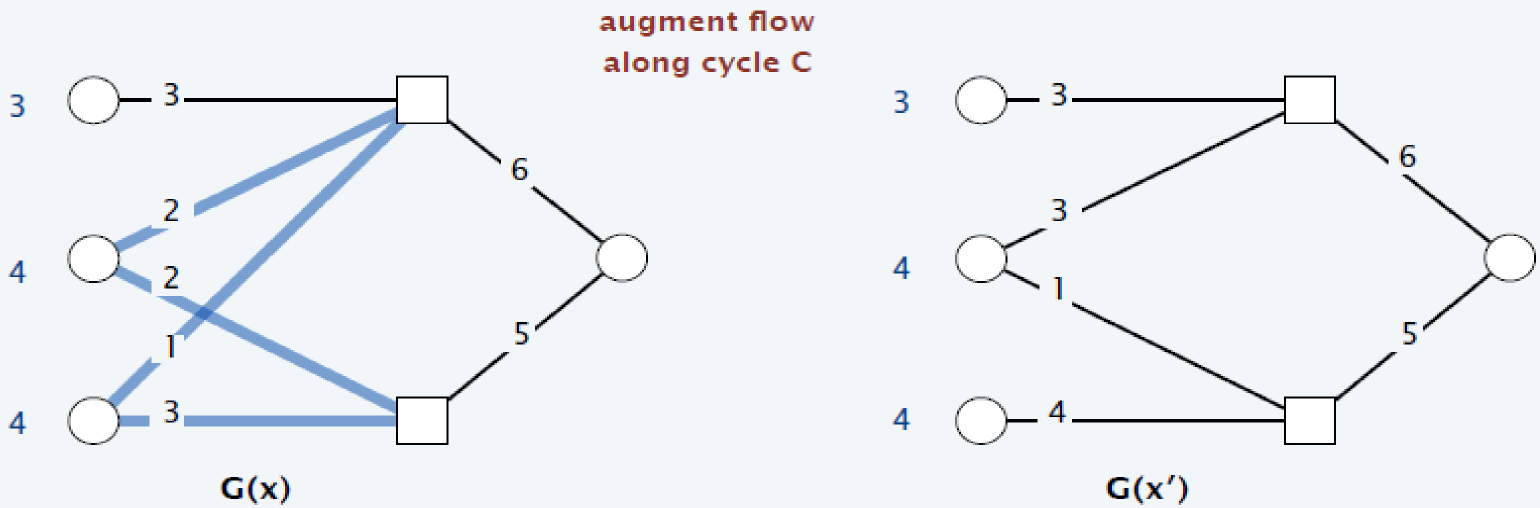
Observation. Solution to feasible flow problem with value L are in 1-to-1 correspondence with LP solutions of value L.

Generalized load balancing: structure of solution

Lemma 3. Let (x, L) be solution to LP. Let $G(x)$ be the graph with an edge from machine i to job j if $x_{ij} > 0$. We can find another solution (x', L) such that $G(x')$ is acyclic.

Pf. Let C be a cycle in $G(x)$.

- Augment flow along the cycle C. ← flow conservation maintained
- At least one edge from C is removed (and none are added).
- Repeat until $G(x')$ is acyclic. ▪



Conclusions

Running time. The bottleneck operation in our 2-approximation is solving one LP with $mn + 1$ variables.

Remark. Can solve LP using flow techniques on a graph with $m + n + 1$ nodes: given L, find feasible flow if it exists. Binary search to find L^*.

Extensions: unrelated parallel machines. [Lenstra–Shmoys–Tardos 1990]

- Job j takes t_{ij} time if processed on machine i.
- 2-approximation algorithm via LP rounding.
- If **P** $\neq$ **NP**, then no no ρ-approximation exists for any $\rho < 3/2$.