LEC_2
Probability Theory

Dr. Anand Kumar M,
Assistant Professor
Department of Information Technology
National Institute of Technology- Karnataka (NITK)
m_anandkumar@nitk.edu

Outline

- Mathematical models-Deterministic and non Deterministic
- Sets
- Experiment-Sample Space –events
- Finite Sample space
- Equally Likely events
- Conditional Probability
- Bayes Theorem

Mathematical models

Deterministic Phenomena

- There exists a mathematical model that allows "***perfect***" prediction the phenomena's outcome. (Experiment is performed to predict th outcome)
- Many examples exist in Physics, Chemistry (the exact sciences).

Non-deterministic Phenomena

- **No** mathematical model exists that allows "***perfect***" prediction the phenomena's outcome.

Non-deterministic -probabilistic models

Random phenomena

– Unable to predict the outcomes, but in the long-run, the outcomes exhibit statistical regularity.

– Eg: We cannot predict the weather conditions accurately using deterministic – But the probabilistic model describes accurately

– Actual outcome is predicted from the conditions under which the experiments are carried out.

– The conditions of the experimentation determine the ***probabilistic***

Random phenomena

- Unable to predict the outcomes, but in the long-run, the outcomes exhibit statistical regularity.

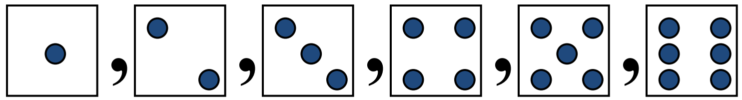
Examples

1. Tossing a coin – outcomes $S = \{\textbf{Head, Tail}\}$

Unable to predict on each toss whether is Head or Tail.

In the long run can predict that 50% of the time heads will occur and 50% of the time tails will occur

2. Rolling a die – outcomes

$S = \{$  $\}$

Unable to predict outcome but in the long run can one can determine that each outcome will occur 1/6 of the time.

Use symmetry. Each side is the same. One side should not occur more frequently than another side in the long run. If the die is not balanced this may not be true.

Sets -review

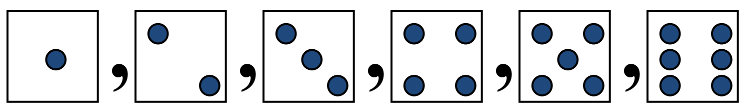
The sample Space, *S*

The **sample space**, *S*, for a random phenomena is the set of all possible outcomes.

Examples

1. Tossing a coin – outcomes $S = \{\textbf{Head, Tail}\}$

2. Rolling a die – outcomes

$S = \{$  $\}$

$= \{1, 2, 3, 4, 5, 6\}$

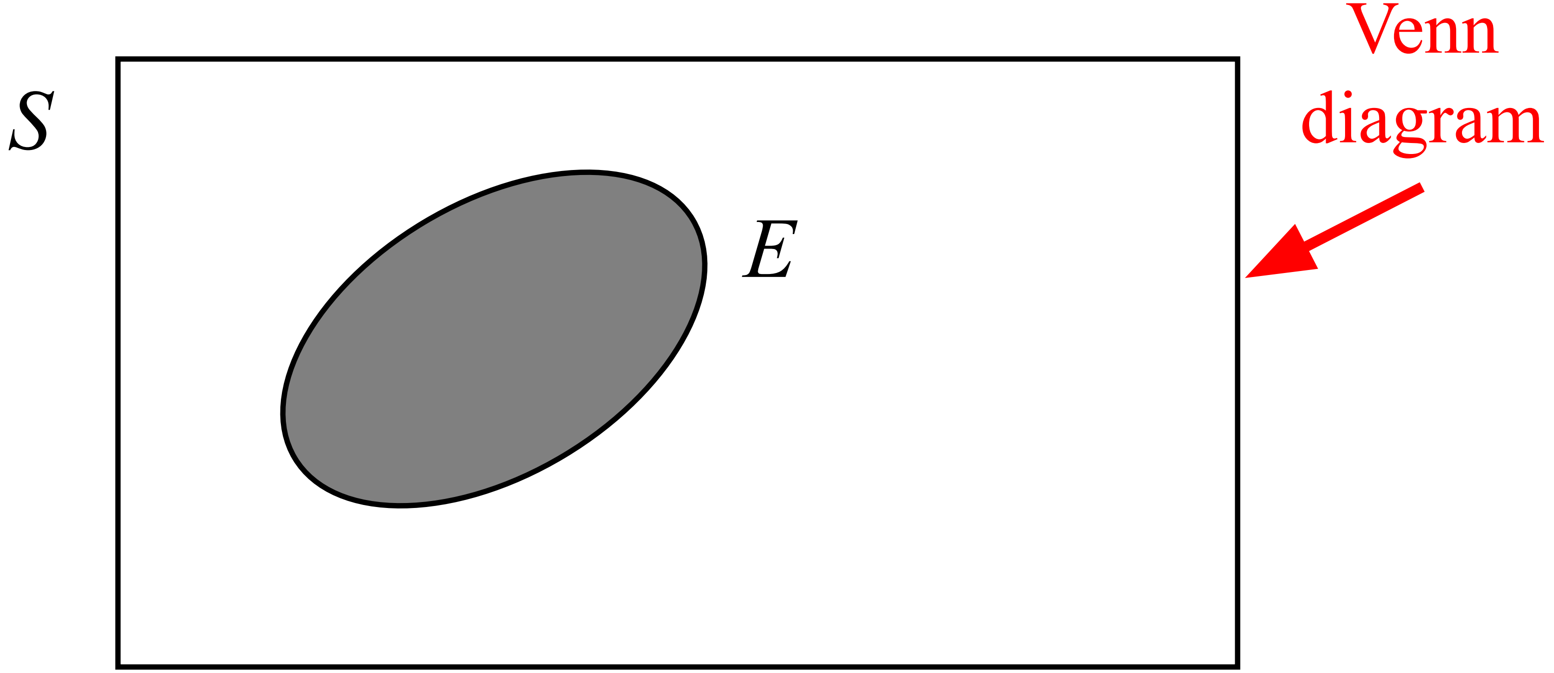
Events -E

- Events are sets:
- Can describe in words
- Can describe in notation
- Can describe with Venn diagrams

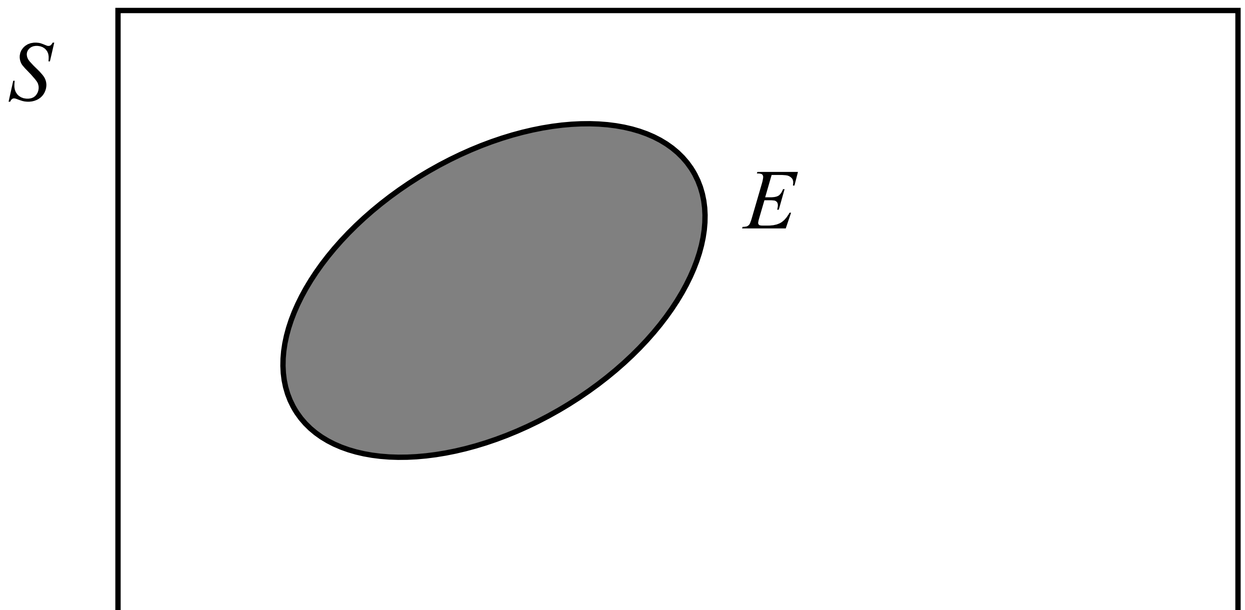
- Experiment: toss a coin 3 times.
- Event: You get 2 or more heads = { HHH, HHT, HTH, THH}

An Event , *E*

The **event**, *E*, is any subset of the **sample space**, *S*. i.e. any set of outcomes (not necessarily all outcomes) of the random phenomena

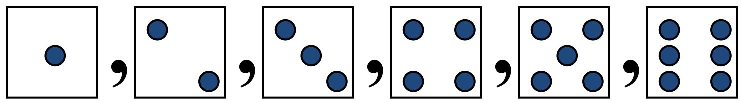


The **event**, *E*, is said to **have occurred** if after the outcome has been observed the outcome lies in *E*.



Examples

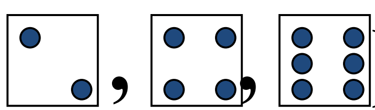
1. Rolling a die – outcomes

$S = \{$  $\}$

$= \{1, 2, 3, 4, 5, 6\}$

E = the event that an even number is rolled

$= \{2, 4, 6\}$

$= \{$  $\}$

Special Events

The Null Event, The empty event - φ

$\varphi = \{\ \} =$ the event that contains no outcomes

The Entire Event, The Sample Space - S

$S =$ the event that contains all outcomes

The empty event, φ , never occurs.

The entire event, S, always occurs.

Experiment-SS-Event

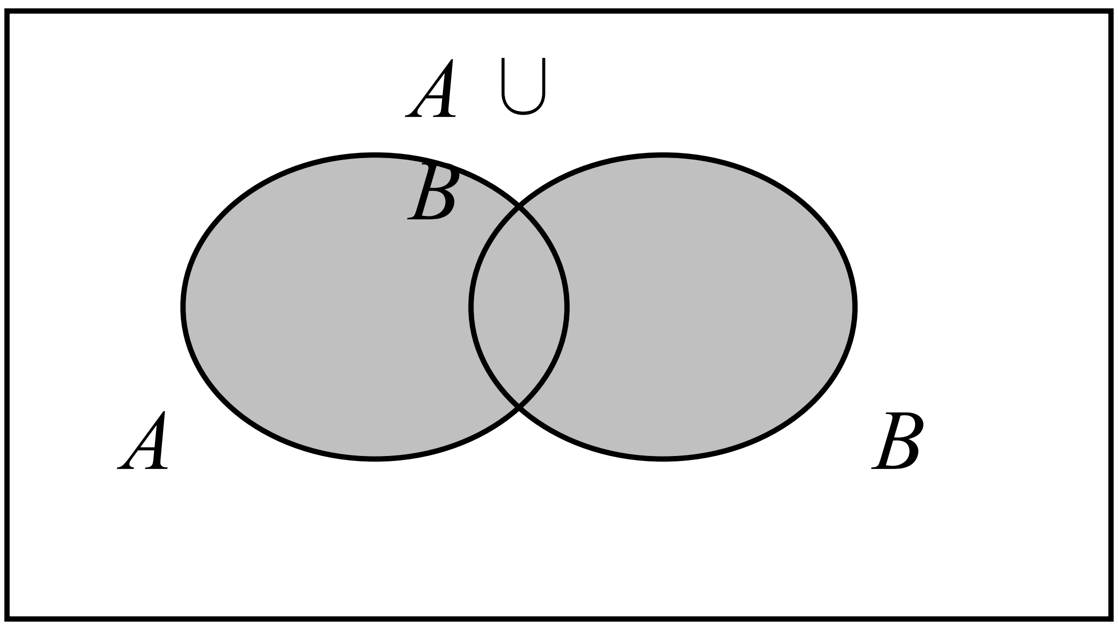
- Experiment: a repeatable procedure
- Sample space: set of all possible outcomes S (or Ω).
- Event: a subset of the sample space.
- Probability function, $P(\omega)$: gives the probability for each outcome $\omega \in S$
- Probability is between 0 and 1
- Total probability of all possible outcomes is 1.

Set operations on Events

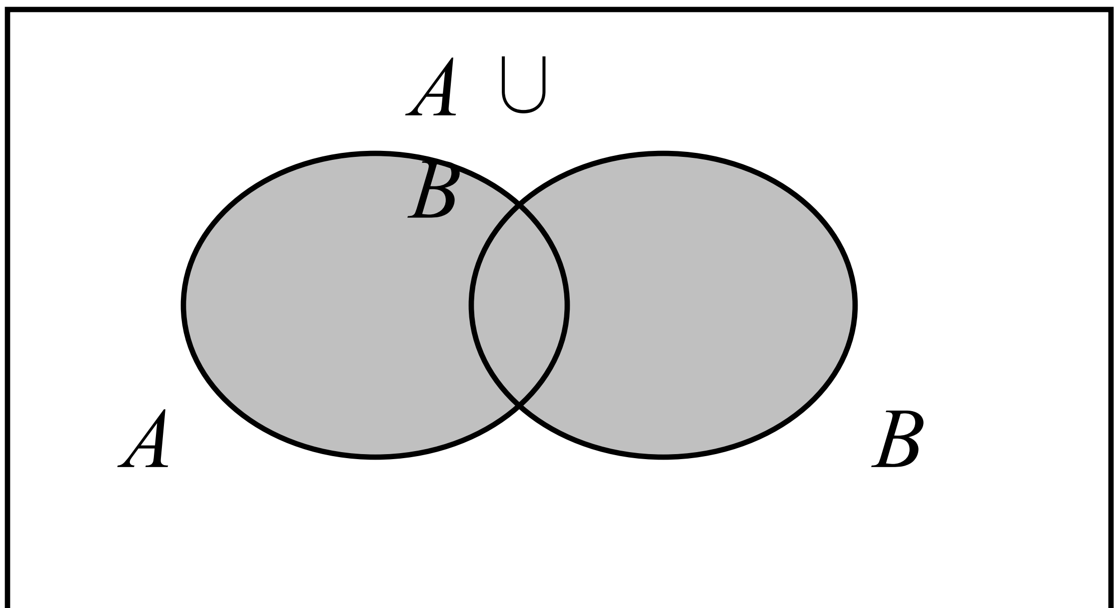
Union

Let A and B be two events, then the **union** of A and B is the event (denoted by $A \cup B$) defined by:

$$A \cup B = \{e \mid e \text{ belongs to } A \textbf{ or } e \text{ belongs to } B\}$$



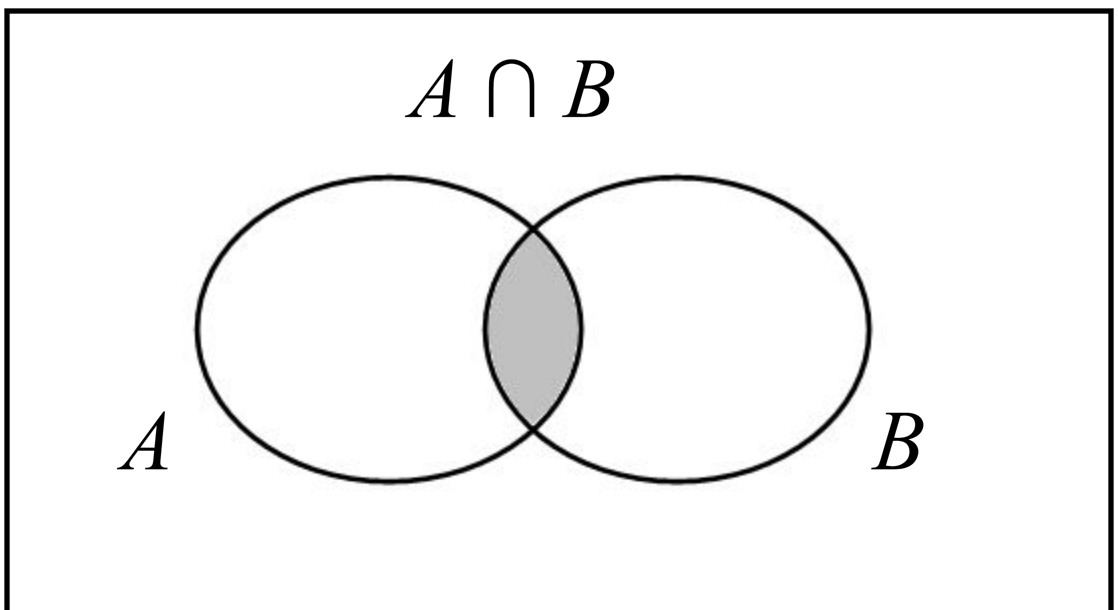
The event $A \cup B$ **occurs** if the event A **occurs or** the event and B **occurs** .



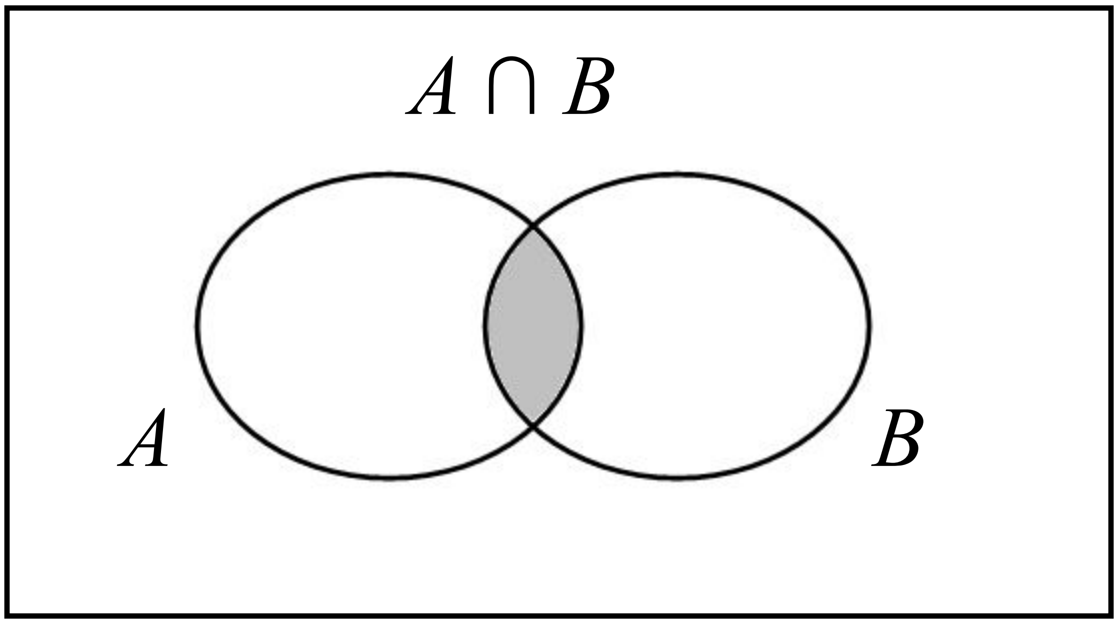
Intersection

Let A and B be two events, then the **intersection** of A and B is the event (denoted by $A \cap B$) defined by:

$$A \cap B = \{e|\ e \text{ belongs to } A \textbf{ and } e \text{ belongs to } B\}$$



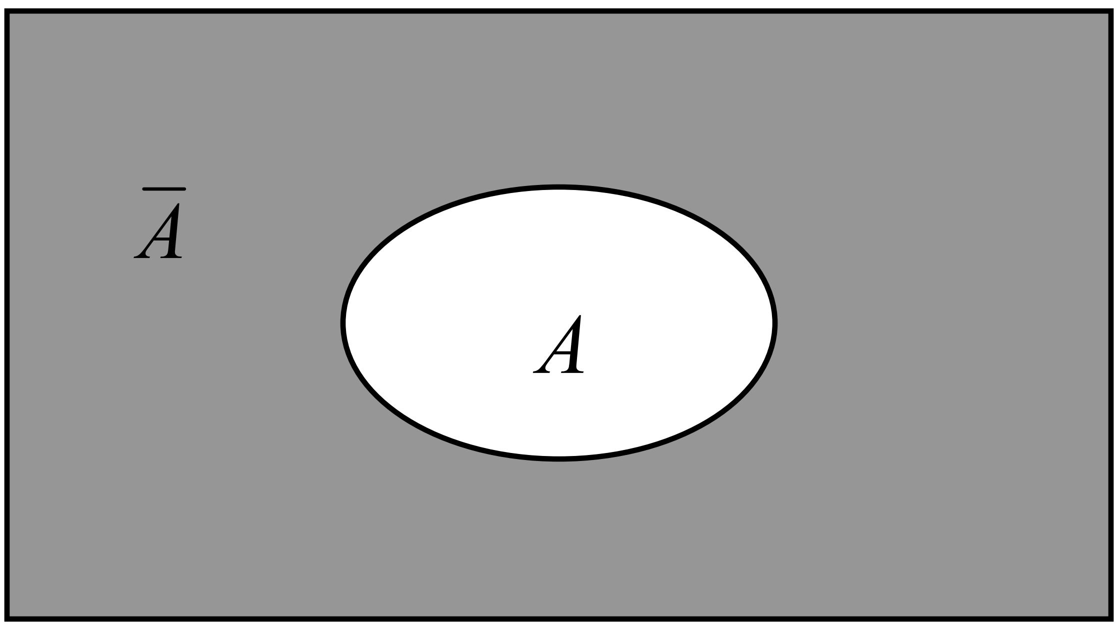
The event $A \cap B$ **occurs** if the event A **occurs and** the event and B **occurs** .



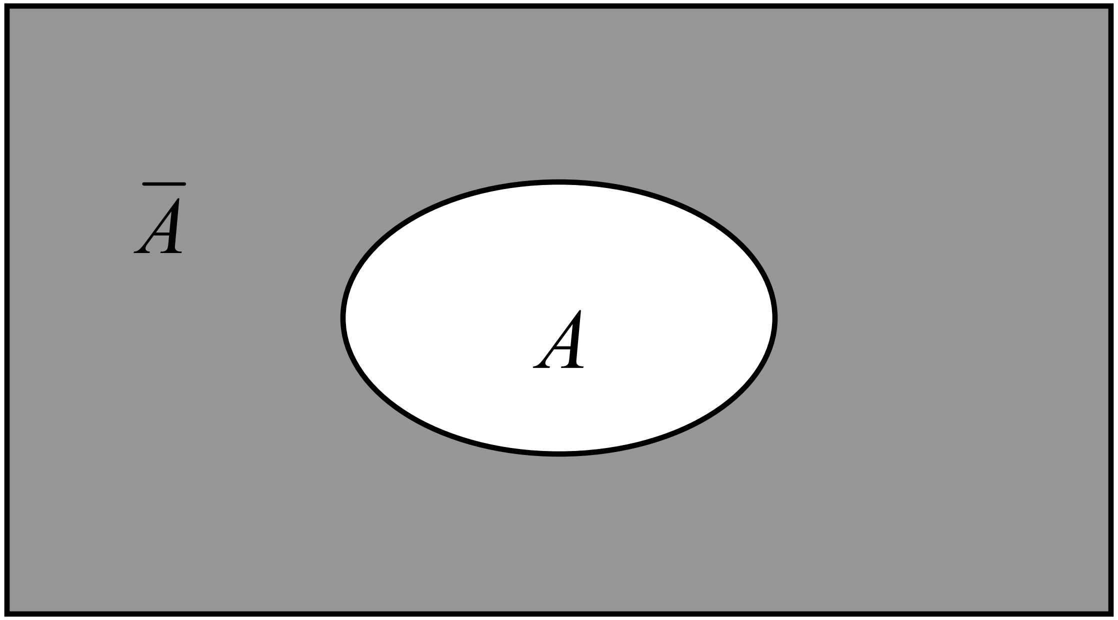
Complement

Let A be any event, then the **complement** of A (denoted by $\overline{A}$) defined by:

$\overline{A} = \{e|\ e$ **does not** belongs to $A\}$



The event $\bar{A}$ **occurs** if the event A **does not occur**



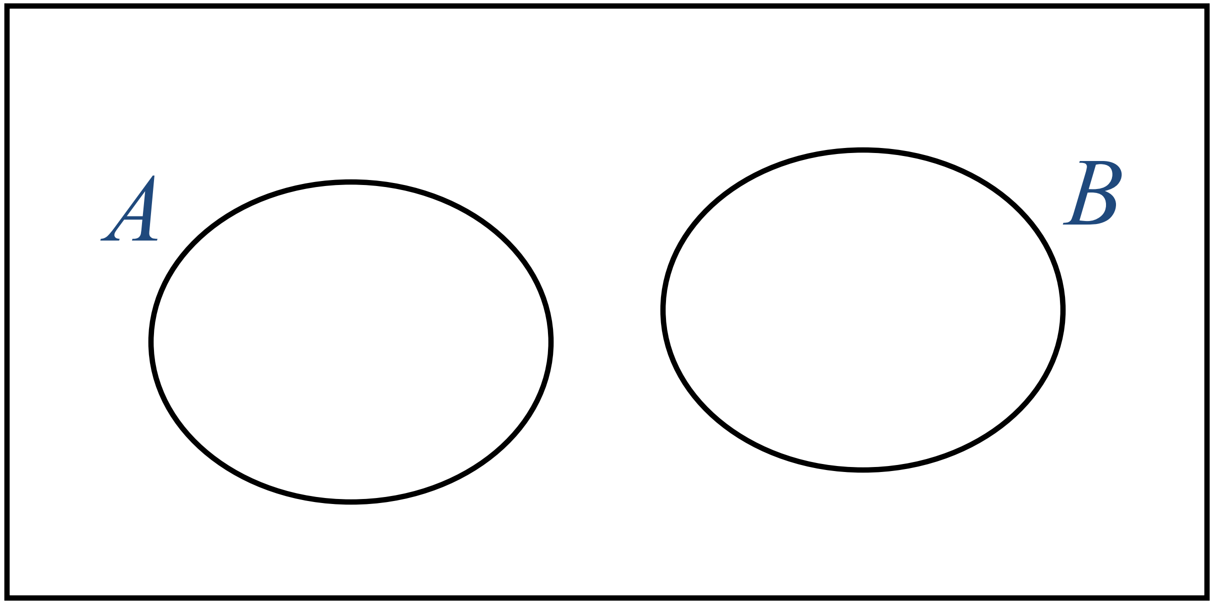
In problems you will recognize that you are working with:

1. **Union** if you see the word **or**,
2. **Intersection** if you see the word **and**,
3. **Complement** if you see the word **not**.

Definition: mutually exclusive

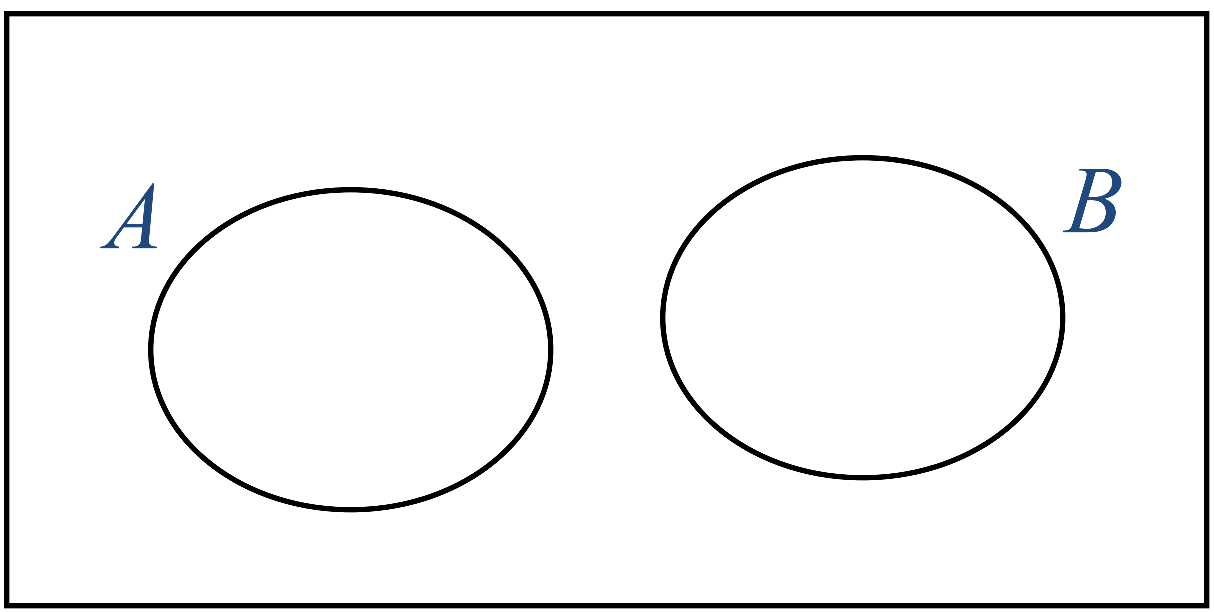
Two events A and B are called **mutually exclusive** if:

$$A \cap B = \phi$$



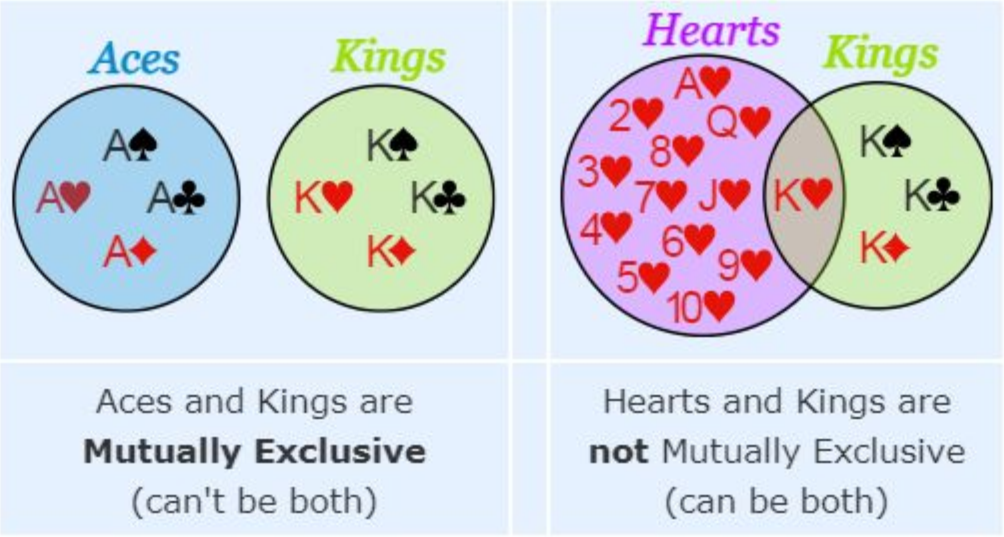
If two events *A* and *B* are are **mutually exclusive** then:

1. They have no outcomes in common.
   They can't occur at the same time. The outcome of the random experiment can not belong to both *A* and *B*.



Examples:

- Turning left and turning right are Mutually Exclusive (you can't do both at the same time)
- Tossing a coin: Heads and Tails are Mutually Exclusive
- Cards: Kings and Aces are Mutually Exclusive
- What is **not** Mutually Exclusive:
- Kings and Hearts, because we can have a King of Hearts!



Probability

Definition: probability of an Event *E*.

Suppose that the sample space $S = \{o_1, o_2, o_3, \ldots o_N\}$ has a finite number, *N*, of oucomes.

Also each of the outcomes is equally likely (because of symmetry).

Then for any event *E*

$$P[E] = \frac{n(E)}{n(S)} = \frac{n(E)}{N} = \frac{\text{no. of outcomes in } E}{\text{total no. of outcomes}}$$

$$\textbf{Note}: \text{ the symbol } n(A) = \text{ no. of elements of } A$$

Thus this definition of $P[E]$, i.e.

$$P[E] = \frac{n(E)}{n(S)} = \frac{n(E)}{N} = \frac{\text{no. of outcomes in } E}{\text{total no. of outcomes}}$$

Applies only to the special case when

1. The sample space has a finite no.of outcomes, and
2. Each outcome is equi-probable

If this is not true a more general definition of probability is required.

Rules of Probability

Rule The additive rule (Mutually exclusive events)

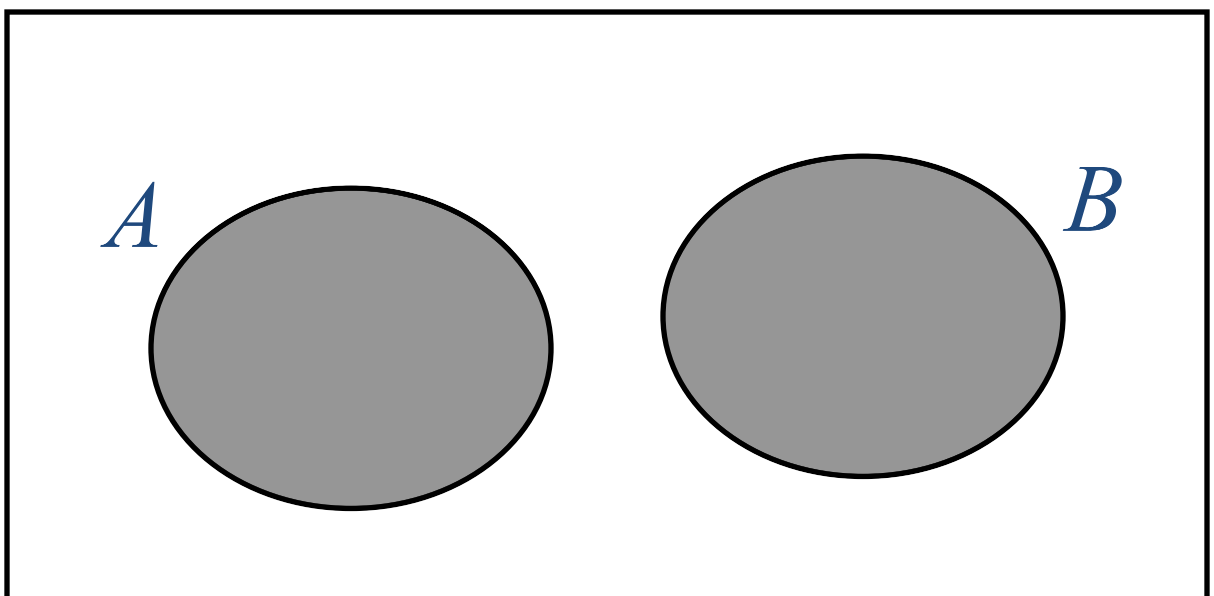
$$\boxed{\begin{gathered} P[A \cup B] = P[A] + P[B] \\ \text{i.e.} \\ P[A \text{ or } B] = P[A] + P[B] \end{gathered}}$$

if $A \cap B = \varphi$

(A and B mutually exclusive)

If two events A and B are are **mutually exclusive** then:

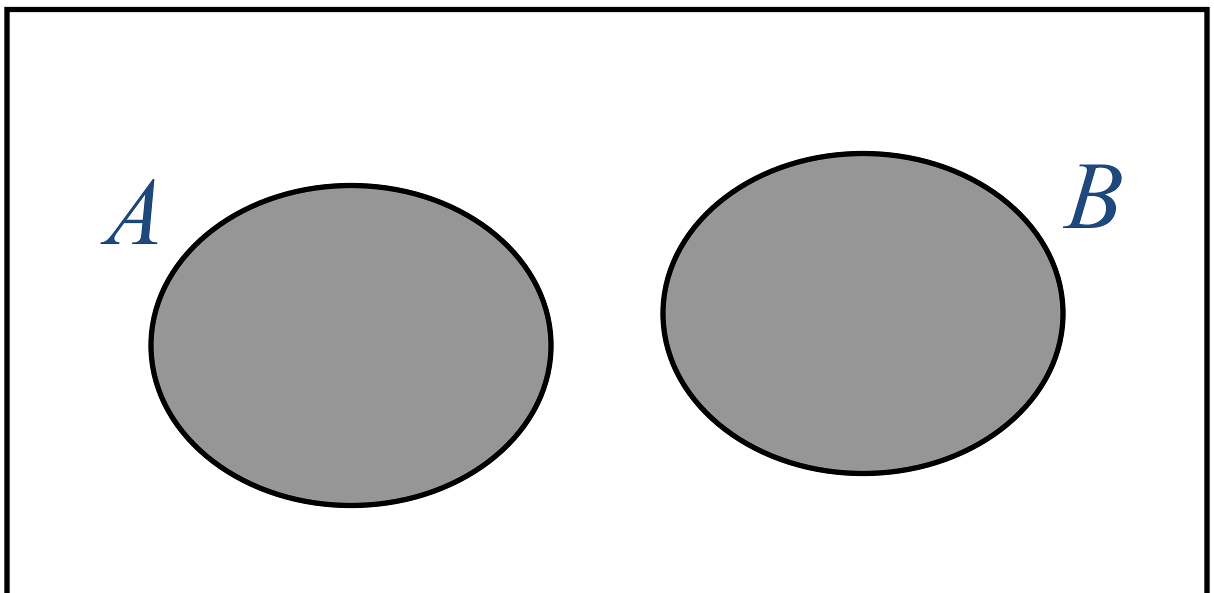
1. They have no outcomes in common.
   They can’t occur at the same time. The outcome of the random experiment can not belong to both A and B.



$$P[A \cup B] = P[A] + P[B]$$

i.e.

$$P[A \text{ or } B] = P[A] + P[B]$$



Rule The additive rule

(In general)

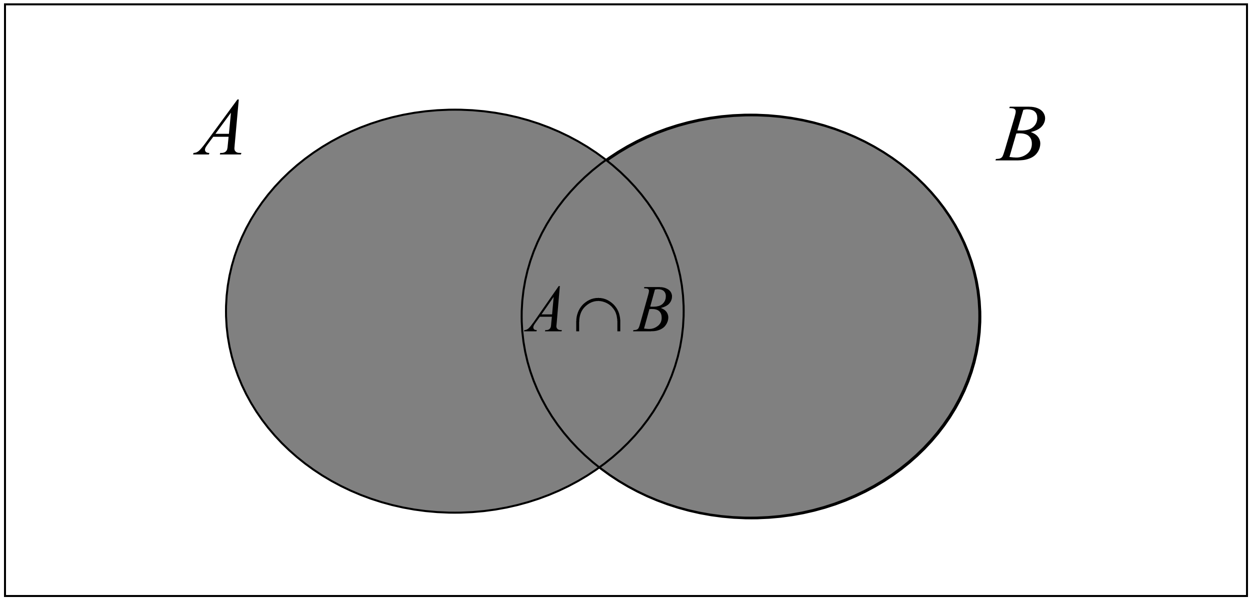
$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

or

$$P[A \text{ or } B] = P[A] + P[B] - P[A \text{ and } B]$$

Logic

$$A \cup B$$



When $P[A]$ is added to $P[B]$ the outcome in $A \cap B$ are counted twice

hence

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

Example:

Saskatoon and Moncton are two of the cities competing for the World university games. (There are also many others). The organizers are narrowing the competition to the **final 5 cities.**

There is a 20% chance that Saskatoon will be amongst the **final 5**. There is a 35% chance that Moncton will be amongst the **final 5** and an 8% chance that both Saskatoon and Moncton will be amongst the **final 5**.

What is the probability that Saskatoon or Moncton will be amongst the **final 5.**

Solution:

Let A = the event that Saskatoon is amongst the **final 5**.

Let B = the event that Moncton is amongst the **final 5**.

Given $P[A] = 0.20$, $P[B] = 0.35$, and $P[A \cap B] = 0.08$

What is $P[A \cup B]$?

Note: "and" $\equiv \cap$**,** "or" $\equiv \cup$.

$$P\left[A \cup B\right] = P\left[A\right] + P\left[B\right] - P\left[A \cap B\right]$$

$$= 0.20 + 0.35 - 0.08 = 0.47$$

Rule for complements

2. $$P\left[\bar{A}\right] = 1 - P\left[A\right]$$

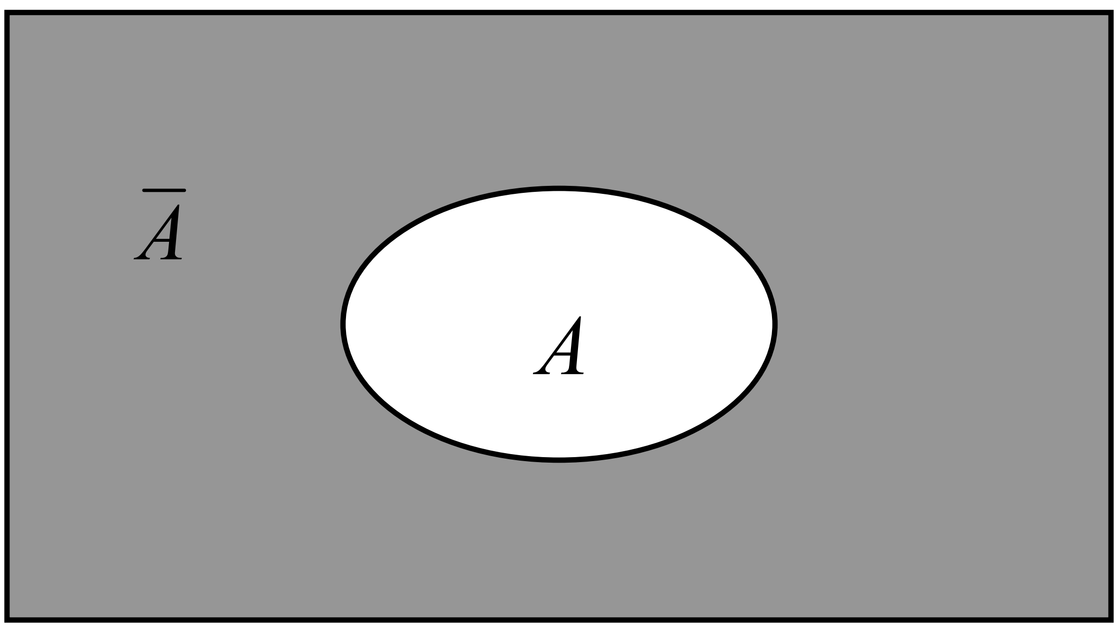
or

$$P\left[\text{not } A\right] = 1 - P\left[A\right]$$

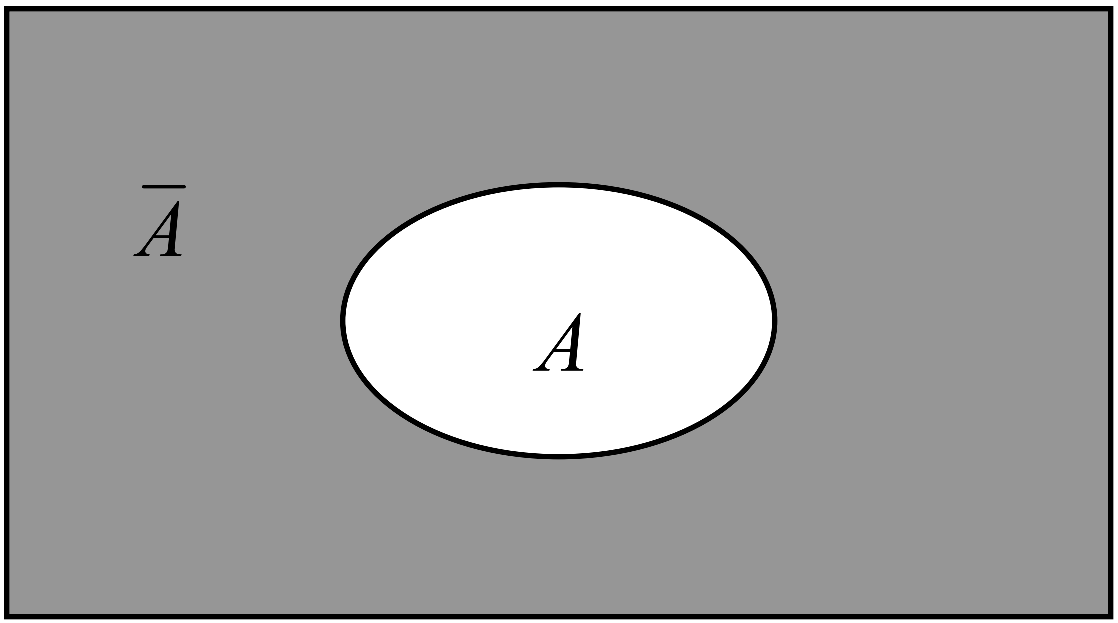
Complement

Let A be any event, then the **complement** of A (denoted by $\overline{A}$) defined by:

$$\overline{A} = \{e |\ e \textbf{ does not } \text{belongs to } A\}$$



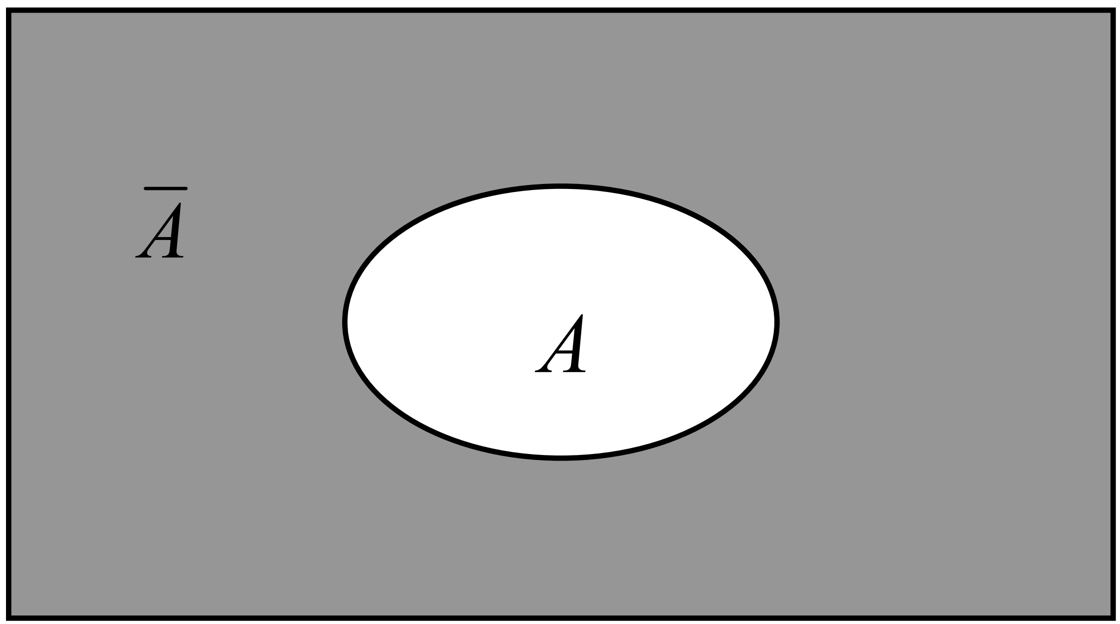
The event $\bar{A}$ **occurs** if the event A **does not occur**



Logic:

$\bar{A}$ and A are **mutually exclusive**.

and $S = A \cup \bar{A}$



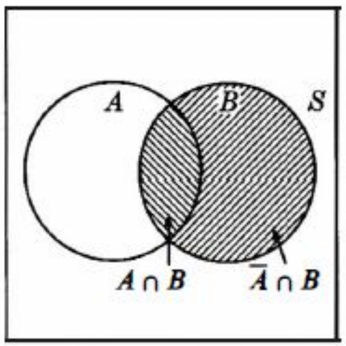
thus $1 = P[S] = P[A] + P[\bar{A}]$

and $P[\bar{A}] = 1 - P[A]$

Proof

If A and B are *any* two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$



Thus we write

$$A \cup B = A \cup (B \cap \bar{A}),$$
$$B = (A \cap B) \cup (B \cap \bar{A}).$$

Hence

$$P(A \cup B) = P(A) + P(B \cap \bar{A}),$$
$$P(B) = P(A \cap B) + P(B \cap \bar{A}).$$

Subtracting the second equation from the first yields

$$P(A \cup B) - P(B) = P(A) - P(A \cap B)$$

EXAMPLE

- An electronic device is tested and its total time of service, say t,
- is recorded. We shall assume the sample space to be {t I t >= O}. Let the three
- events A, B, and C be defined as follows:
- A = {t I t < 100}; B = {t I 50 <= t <= 200}; c = {tIt > 150}.
- AUB, A n B, B U C, B nC, A n C, A U C and A, C

EXAMPLE

- A u B = {t \ t <= 200}; A n B = {t I 50 <= t < 100};
- B u c = {t I t => 50}; B n c = {t I 150 < t :S 200}; A n c = null;
- A U C = {t I t < 100 or t > 150}; $\underline{A}$ = {t I t => 100}; $\underline{C}$ = {t \ t <=150}.

Conditional Probability

Conditional Probability

- Frequently before observing the outcome of a random experiment you are given information regarding the outcome
- How should this information be used in prediction of the outcome.
- Namely, how should probabilities be adjusted to take into account this information
- Usually the information is given in the following form: You are told that the outcome belongs to a given event. (i.e. you are told that a certain event has occurred)

Definition

Suppose that we are interested in computing the probability of event A and we have been told event B has occurred.

Then the conditional probability of A given B is defined to be:

$$P\left[A \middle| B\right] = \frac{P\left[A \cap B\right]}{P\left[B\right]} \quad \text{if} \quad P\left[B\right] \neq 0$$

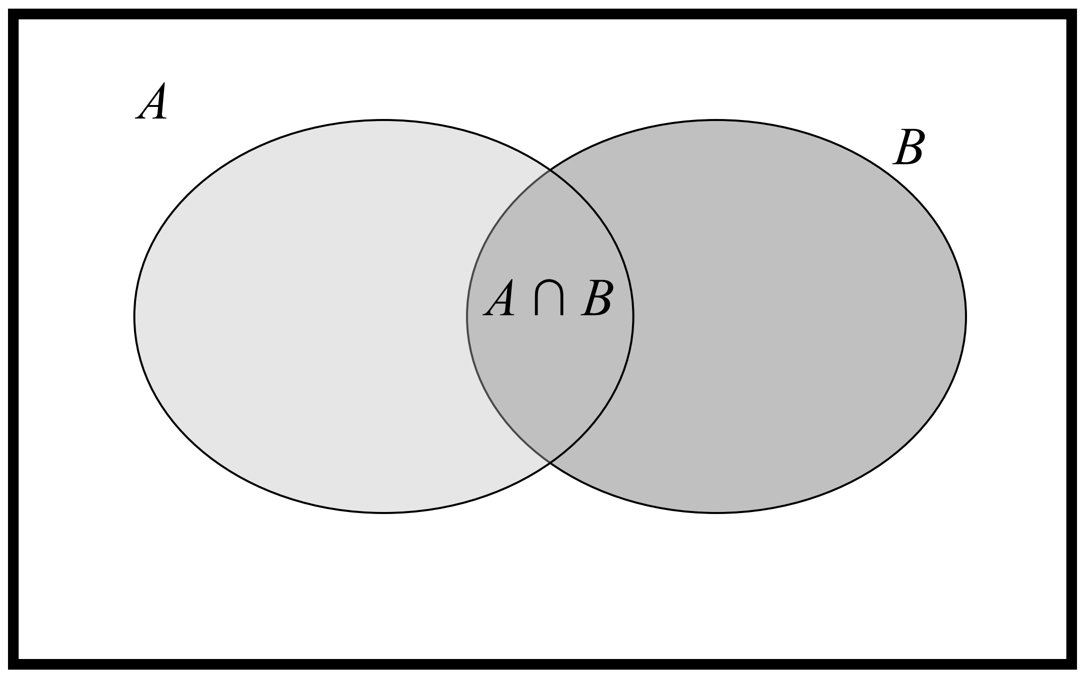
Rationale:

If we're told that event B has occurred then the sample space is restricted to B.

The probability within B has to be normalized, This is achieved by dividing by $P[B]$

The event A can now only occur if the outcome is in of $A \cap B$. Hence the new probability of A is:

$$P\left[A \middle| B\right] = \frac{P\left[A \cap B\right]}{P\left[B\right]}$$



An Example

The academy awards is soon to be shown.

For a specific married couple the probability that the husband watches the show is 80%, the probability that his wife watches the show is 65%, while the probability that they both watch the show is 60%.

If the husband is watching the show, what is the probability that his wife is also watching the show

Solution:

The academy awards is soon to be shown.

Let B = the event that the husband watches the show

$P[B]= 0.80$

Let A = the event that his wife watches the show

$P[A]= 0.65$ and $P[A \cap B]= 0.60$

$$P\left[A \middle| B\right] = \frac{P\left[A \cap B\right]}{P\left[B\right]} = \frac{0.60}{0.80} = 0.75$$

Independence

Definition

Two events *A* and *B* are called **independent** if

$$P\left[A \cap B\right] = P\left[A\right]P\left[B\right]$$

Note if $P[B] \neq 0$ and $P[A] \neq 0$ then

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[A]P[B]}{P[B]} = P[A]$$

and $$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{P[A]P[B]}{P[A]} = P[B]$$

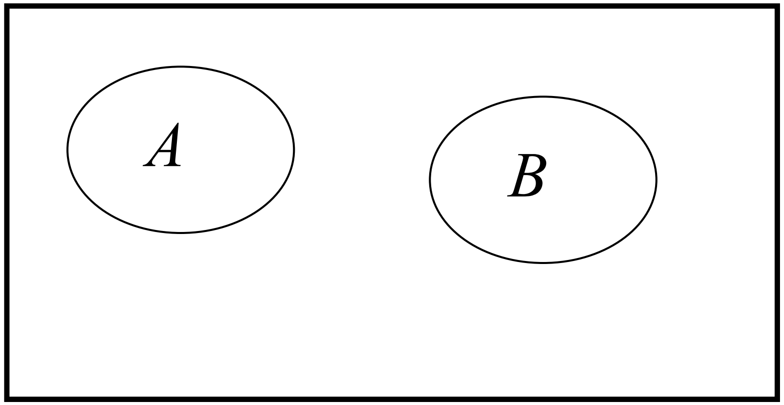
Thus in the case of independence the conditional probability of an event is not affected by the knowledge of the other event

Difference between **independence** and **mutually exclusive**

mutually exclusive

Two mutually exclusive events are independent only in the special case where

$$P[A]=0 \text{ and } P[B]=0. \text{ (also } P[A\cap B]=0$$

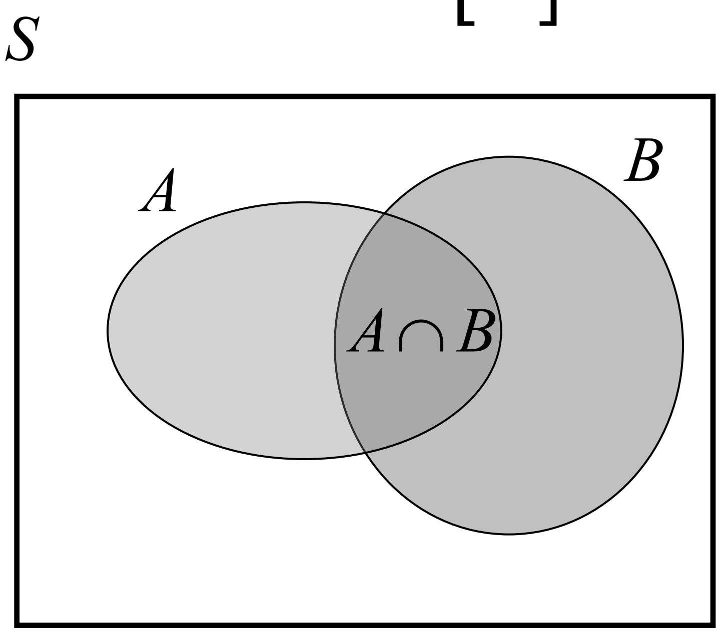


Mutually exclusive events are highly dependent otherwise. A and B **cannot** occur simultaneously. If one event occurs the other event does not occur.

Independent events

$$P[A \cap B] = P[A]P[B]$$

or
$$\frac{P[A \cap B]}{P[B]} = P[A] = \frac{P[A]}{P[S]}$$



The ratio of the probability of the set A within B is the same as the ratio of the probability of the set A within the entire sample S.

The multiplicative rule of probability

$$P\left[A \cap B\right] = \begin{cases} P\left[A\right]P\left[B \middle| A\right] & \text{if } P\left[A\right] \neq 0 \\ P\left[B\right]P\left[A \middle| B\right] & \text{if } P\left[B\right] \neq 0 \end{cases}$$

and

$$P\left[A \cap B\right] = P\left[A\right]P\left[B\right]$$

if A and B are **independent**.

Probability

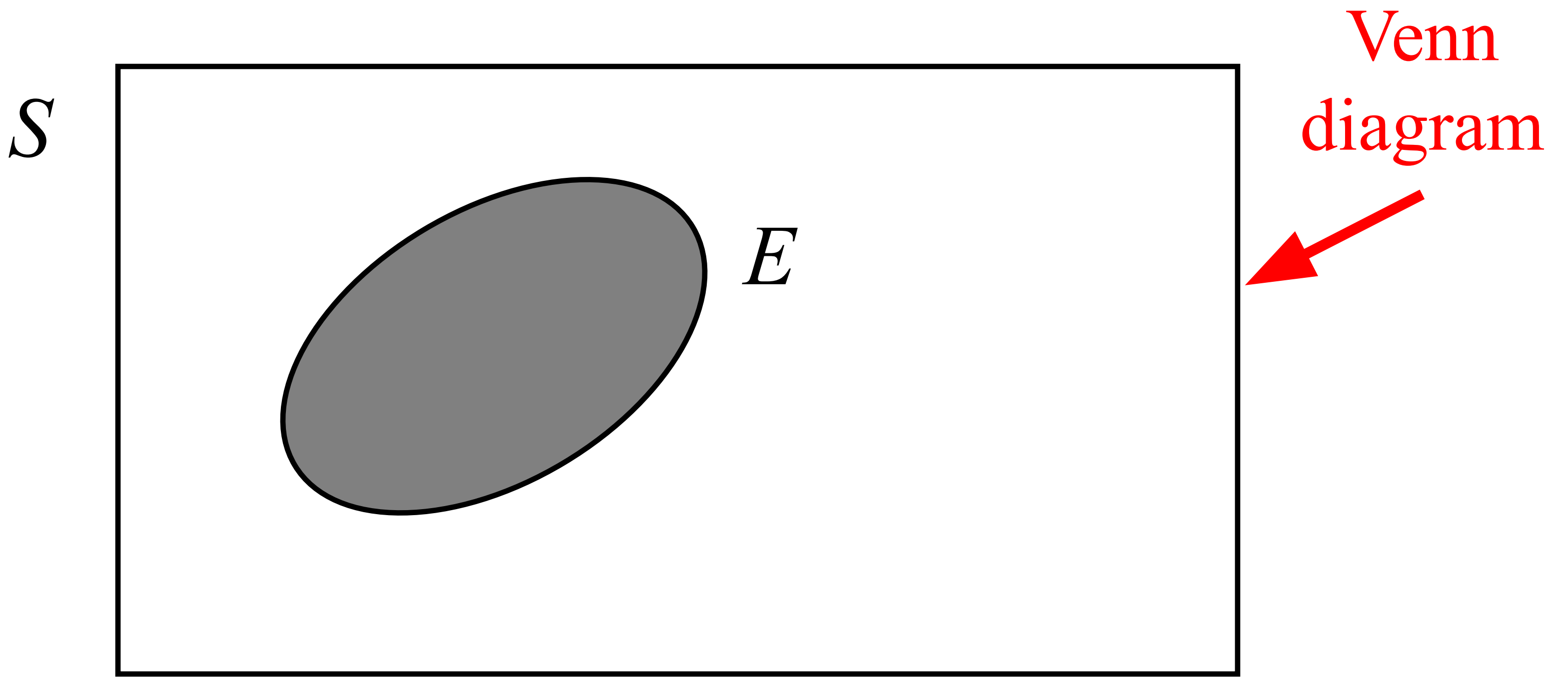
Models for **random** phenomena

The sample Space, *S*

The **sample space**, *S*, for a random phenomena is the set of all possible outcomes.

An Event , *E*

The **event**, *E*, is any subset of the **sample space**, *S*. i.e. any set of outcomes (not necessarily all outcomes) of the random phenomena



Definition: probability of an Event *E*.

Suppose that the sample space $S = \{o_1, o_2, o_3, \ldots o_N\}$ has a finite number, N, of oucomes.

Also each of the outcomes is equally likely (because of symmetry).

Then for any event E

$$P[E] = \frac{n(E)}{n(S)} = \frac{n(E)}{N} = \frac{\text{no. of outcomes in } E}{\text{total no. of outcomes}}$$

$\textbf{Note}$: the symbol $n(A) =$ no. of elements of A

Thus this definition of $P[E]$, i.e.

$$P[E]=\frac{n(E)}{n(S)}=\frac{n(E)}{N}=\frac{\text{no. of outcomes in } E}{\text{total no. of outcomes}}$$

Applies only to the special case when

1. The sample space has a finite no.of outcomes, and
2. Each outcome is equi-probable

If this is not true a more general definition of probability is required.

Summary of the Rules of Probability

The additive rule

$$\boxed{P[A \cup B] = P[A] + P[B] - P[A \cap B]}$$

and

$$P[A \cup B] = P[A] + P[B] \text{ if } A \cap B = \varphi$$

The Rule for complements

for any event E

$$\boxed{P\left[\bar{E}\right]=1-P\left[E\right]}$$

Conditional probability

$$\boxed{P\left[A \middle| B\right] = \frac{P\left[A \cap B\right]}{P\left[B\right]}}$$

The multiplicative rule of probability

$$P[A \cap B] = \begin{cases} P[A]P[B|A] & \text{if } P[A] \neq 0 \\ P[B]P[A|B] & \text{if } P[B] \neq 0 \end{cases}$$

and

$$P[A \cap B] = P[A]P[B]$$

if A and B are **independent**.

This is the definition of independent

Warm-Up

- If P(A) = 0.3 and P(B) = 0.4 and if A and B are mutually exclusive events, find:

  a. $P(\overline{A})$

  b. $P(\overline{B})$

  c. $P(A\, or\, B)$

  d. $P(A\, and\, B)$

- a. 0.7
- b. 0.6
- c. 0.7
- d. 0

Multiplication Rule – Independent Events……

- When 2 events are **independent**, the probability of both occurring is

$$P(A\, and\, B) = P(A) \cdot P(B)$$

General Rule……

- “or” means to add

- “and” means to multiply (unless it is in a contingency table and you can actually see the intersection)

Example……

- If a coin is tossed twice, find the probability of getting 2 heads.

- Answer:

$$P(H\ and\ H) = P(H) \cdot P(H)$$

$$P(H\ and\ H) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Example……

- A coin is flipped and a die is rolled. Find the probability of getting a head on the coin ***and*** a 4 on the die.

- Answer:

$$P(H\ and\ 4) = P(H) \cdot P(4)$$

$$P(H\ and\ 4) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$$

Example……

- A card is drawn from a deck and replaced; then a 2nd card is drawn. Find the probability of getting a queen ***and*** then an ace.

- Answer:

$$P(Q\,and\,A) = P(Q) \cdot P(A)$$

$$P(Q\,and\,A) = \frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$$

Example……

- A box contains 3 red balls, 2 blue balls, and 5 white balls. A ball is selected and its color noted. Then it is replaced. A 2^{nd} ball is selected and its color noted. Find the probability of

a. Selecting 2 blue balls

b. Selecting a blue ball and then a white ball

c. Selecting a red ball and then a blue ball

Answers……

a. Selecting 2 blue balls

$$P(B\,and\,B) = \frac{2}{10} \cdot \frac{2}{10} = \frac{1}{25}$$

b. Selecting a blue ball and then a white ball

$$P(B\,and\,W) = \frac{2}{10} \cdot \frac{5}{10} = \frac{1}{10}$$

c. Selecting a red ball and then a blue ball

$$P(R\,and\,B) = \frac{3}{10} \cdot \frac{2}{10} = \frac{3}{50}$$

Example……

- A poll found that 46% of Americans say they suffer from stress. If 3 people are selected at random, find the probability that **all three** will say they suffer from stress.

- Answer:

$$P(S\ and\ S\ and\ S) = P(S) \cdot P(S) \cdot P(S)$$

$$P(Stress) = (0.46)^3 = 0.097$$

Dependent Events……

- When the outcome or occurrence of the first event affects the outcome or occurrence of the second event in such a way that the probability is changed.

Examples of Dependent Events……

1. Draw a card from a deck. Do not replace it and draw another card.
2. Having high grades and getting a scholarship
3. Parking in a no parking zone and getting a ticket

Multiplication Rule – Dependent Events……

- When 2 events are dependent, the probability of both occurring is

$$P(A\, and\, B) = P(A) \cdot P(BlA)$$

- The slash reads:

  "The probability that B occurs given that A has already occurred."

Example……

- 53% of residents had homeowner’s insurance. Of these, 27% also had car insurance. If a resident is selected at random, find the prob. That the resident has both homeowner’s ***and*** car insurance.

- Answer:

$$P(H\ and\ C) = P(H) \cdot P(C l H)$$

$$P(H\ and\ C) = (.53)(.27) = .1431$$

Example……

- 3 cards are drawn from a deck and **<u>NOT</u>** replaced. Find the following probabilities.
  a. Getting 3 jacks
  b. Getting an ace, king, and queen
  c. Getting a club, spade, and heart
  d. Getting 3 clubs.

a. Getting 3 jacks……

b. Getting an ace, king, queen……

c. Getting a club, spade, and heart……

d. Getting 3 clubs……

Back to Conditional Probability - Remember……

$$P(A\,and\,B)=P(A)\cdot P\left(B/A\right)$$

- **Algebraically change this so that it is now in the form……**

“Given”

$$P\left(B/A\right)=\frac{P(A\,and\;B)}{P(A)}$$

$$P\left(A/B\right)=\frac{P(A\,and\;B)}{P(B)}$$

Example……

- In Rolling Acres Housing Plan, 42% of the houses have a deck **and** a garage; 60% have a deck. Find the probability that a home has a garage, ***given*** that it has a deck.

Answer…….

- Answer:

Independent ≠ mutually exclusive

- Events A and ~A are mutually exclusive, but they are NOT independent.
- $P(A \& \sim A) = 0$
- $P(A) * P(\sim A) \neq 0$

Conceptually, once A has happened, ~A is impossible; thus, they are completely dependent.

Example……

- At an exclusive country club, 83% of the members play bridge; 75% of the members drink champagne ***given*** that he or she plays bridge. Find the probability that members drink champagne **and** play bridge.

Answer……

$$P(bridge) = .83$$

$$P(champ/bridge) = .75$$

$$Find\ P(champ\ and\ bridge)$$

- Answer:

$$P(C/B) = \frac{P(C\ and\ B)}{P(B)}$$

$$.75 = \frac{P(B\ and\ C)}{.83}$$

$$P(C\ and\ B) = (.75)(.83) = .62$$

Example

- Roll two dice and consider the following events A = ‘first die is 3’ B = ‘sum is 6’ C = ‘sum is 7’

A is independent of

(a) B and C (b) B alone (c) C alone (d) Neither B or C.

Examples

Toss a coin 4 times.

Let A = ‘at least three heads’ B = ‘first toss is tails’.

1. What is P(A|B)? (a) 1/16 (b) 1/8 (c) 1/4 (d) 1/5

2. What is P(B|A)? (a) 1/16 (b) 1/8 (c) 1/4 (d) 1/5

Counting techniques

Finite uniform probability space

Many examples fall into this category

1. Finite number of outcomes
2. All outcomes are equally likely
3. $$P[E] = \frac{n(E)}{n(S)} = \frac{n(E)}{N} = \frac{\text{no. of outcomes in } E}{\text{total no. of outcomes}}$$

$$\textbf{Note}: \quad n(A) = \text{ no. of elements of } A$$

To handle problems in case we have to be able to count. Count $n(E)$ and $n(S)$.

Techniques for counting

Rule 1

Suppose we carry out have a sets $A_1, A_2, A_3, \ldots$ and that any pair are mutually exclusive

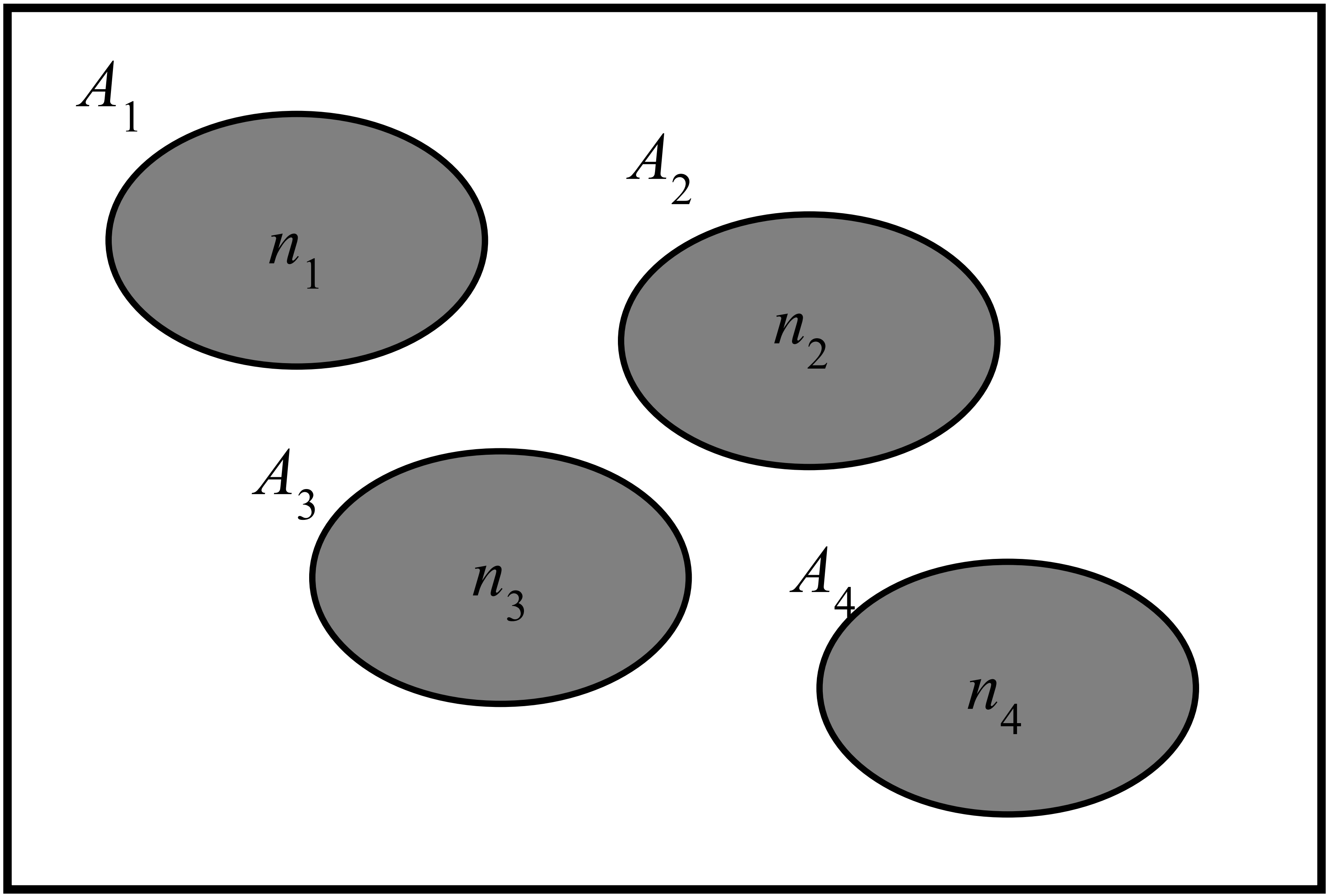
(i.e. $A_1 \cap A_2 = \varphi$) Let

$$n_i = n\,(A_i) = \text{the number of elements in } A_i.$$

Let $A = A_1 \cup A_2 \cup A_3 \cup \ldots.$

Then $N = n(\,A\,) = \text{the number of elements in } A$

$$= n_1 + n_2 + n_3 + \ldots$$


A_1
n_1
A_2
n_2
A_3
n_3
A_4
n_4

Rule 2

Suppose we carry out two operations in sequence

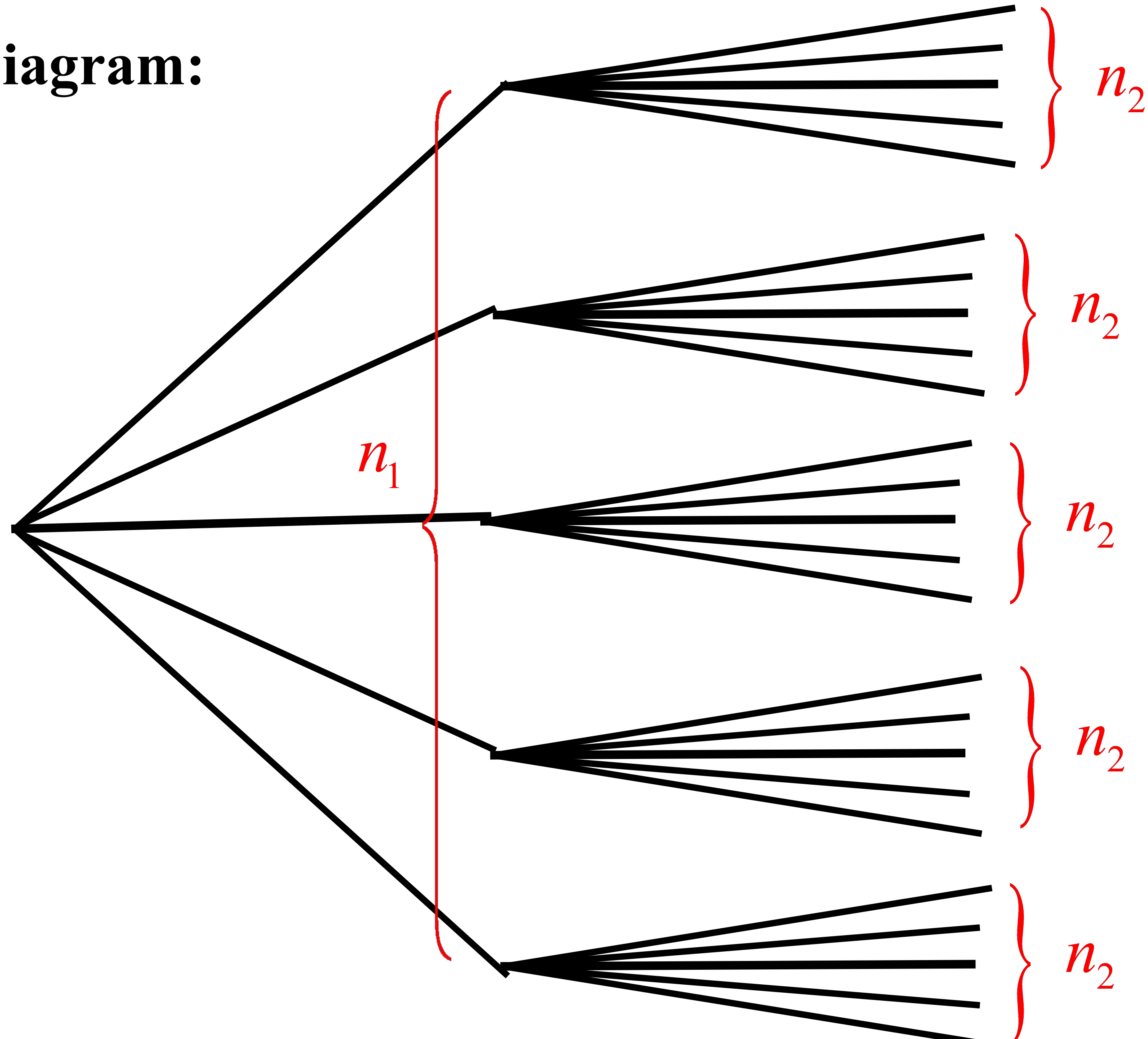
Let

n_1 = the number of ways the first operation can be performed

n_2 = the number of ways the second operation can be performed once the first operation has been completed.

Then $\boxed{N = n_1\, n_2}$ = the number of ways the two operations can be performed in sequence.

Diagram:



Examples

1. We have a committee of 10 people. We choose from this committee, a chairman and a vice chairman. How may ways can this be done?

Solution:

Let n_1 = the number of ways the chairman can be chosen = 10.

Let n_2 = the number of ways the vice-chairman can be chosen once the chair has been chosen = 9.

Then $N = n_1 n_2 = (10)(9) = 90$

2. In **Black Jack** you are dealt 2 cards. What is the probability that you will be dealt a 21?

Solution:

The number of ways that two cards can be selected from a deck of 52 is $N = (52)(51) = 2652$.

A "21" can occur if the first card is an ace and the second card is a face card or a ten {10, J, Q, K} **or** the first card is a face card or a ten and the second card is an ace.

The number of such hands is (4)(16) +(16)(4) =128

Thus the probability of a "21" = 128/2652 = 32/663

The Multiplicative Rule of Counting

Suppose we carry out *k* operations in sequence

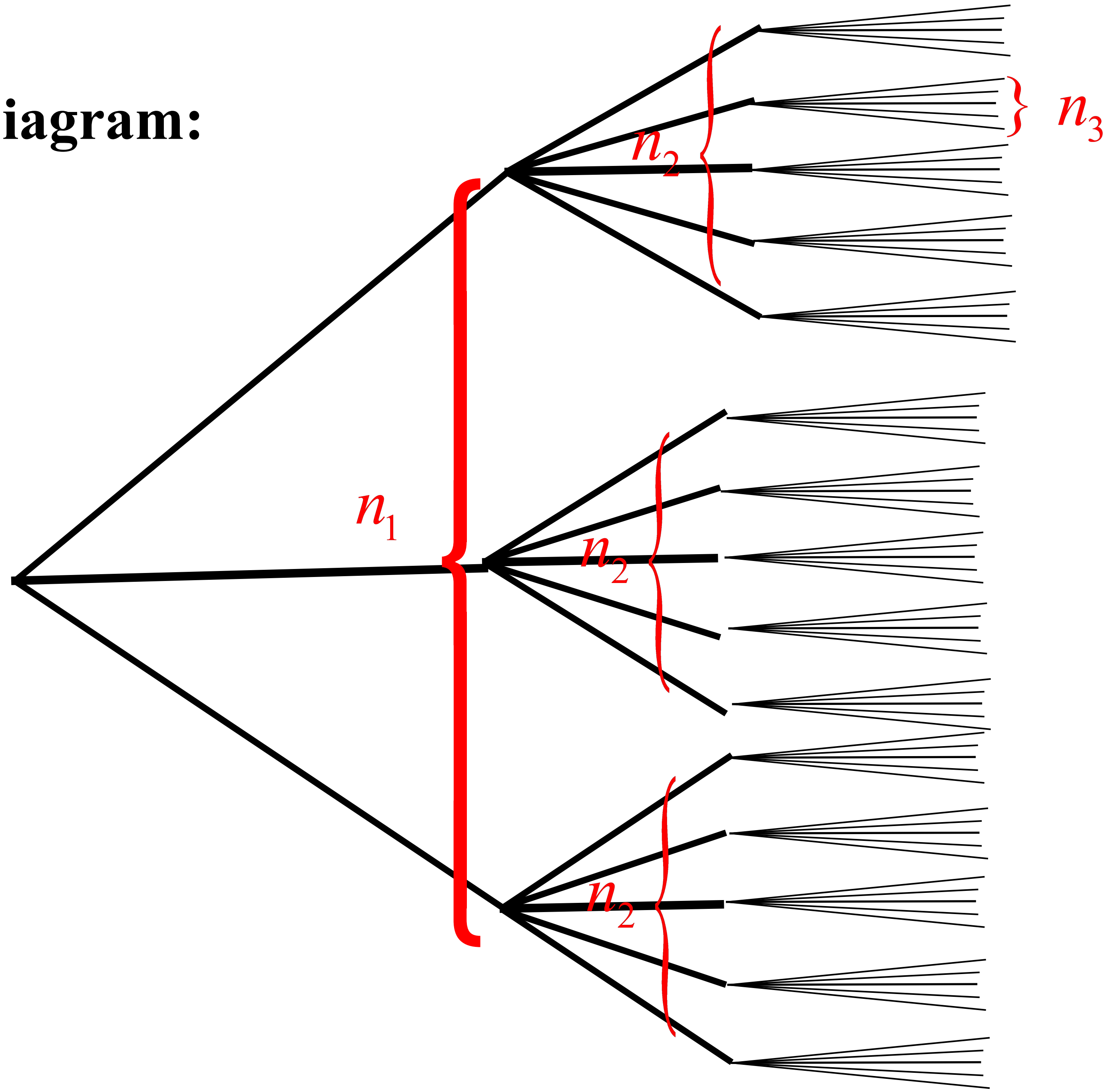
Let

n_1 = the number of ways the first operation can be performed

n_i = the number of ways the i^{th} operation can be performed once the first (i - 1) operations have been completed. $i = 2, 3, \ldots, k$

Then $\boxed{N = n_1 n_2 \ldots n_k}$ = the number of ways the k operations can be performed in sequence.

Diagram:



Examples

1. **Permutations:** How many ways can you order *n* objects

Solution:

Ordering n objects is equivalent to performing n operations in sequence.

1. Choosing the first object in the sequence ($n_1 = n$)
2. Choosing the 2^{nd} object in the sequence ($n_2 = n$ -1).

   …

k. Choosing the k^{th} object in the sequence ($n_k = n - k + 1$)

   …

n. Choosing the n^{th} object in the sequence ($n_n = 1$)

The total number of ways this can be done is:

$$\boxed{N = n(n-1)\ldots(n-k+1)\ldots(3)(2)(1) = n!}$$

Example How many ways can you order the 4 objects
{*A*, *B*, *C*, *D*}

Solution:

$N = 4! = 4(3)(2)(1) = 24$

Here are the orderings.

ABCD	*ABDC*	*ACBD*	*ACDB*	*ADBC*	*ADCB*
BACD	*BADC*	*BCAD*	*BCDA*	*BDAC*	*BDCA*
CABD	*CADB*	*CBAD*	*CBDA*	*CDAB*	*CDBA*
DABC	*DACB*	*DBAC*	*DBCA*	*DCAB*	*DCBA*

Examples - continued

2. **Permutations of size *k* (< *n*):** How many ways can you choose *k* objects from *n* objects in a specific order

Solution: This operation is equivalent to performing k operations in sequence.

1. Choosing the first object in the sequence ($n_1 = n$)
2. Choosing the 2^{nd} object in the sequence ($n_2 = n - 1$).

   …

k. Choosing the k^{th} object in the sequence ($n_k = n - k + 1$)

The total number of ways this can be done is:

$$N = n(n-1)\ldots(n-k+1) = n!/(n-k)!$$

This number is denoted by the symbol

$$_nP_k = n(n-1)\ldots(n-k+1) = \frac{n!}{(n-k)!}$$

Definition: $0! = 1$

This definition is consistent with

$$_{n}P_{k} = n(n-1)\ldots(n-k+1) = \frac{n!}{(n-k)!}$$

for $k = n$

$$_{n}P_{n} = \frac{n!}{0!} = \frac{n!}{1} = n!$$

Example How many permutations of size 3 can be found in the group of 5 objects {*A, B, C, D, E*}

Solution: $${}_5P_3 = \frac{5!}{(5-3)!} = 5(4)(3) = 60$$

ABC	*ABD*	*ABE*	*ACD*	*ACE*	*ADE*	*BCD*	*BCE*	*BDE*	*CDE*
ACB	*ADB*	*AEB*	*ADC*	*AEC*	*AED*	*BDC*	*BEC*	*BED*	*CED*
BAC	*BAD*	*BAE*	*CAD*	*CAE*	*DAE*	*CBD*	*CBE*	*DBE*	*DCE*
BCA	*BDA*	*BEA*	*CDA*	*CEA*	*DEA*	*CDB*	*CEB*	*DEB*	*DEC*
CAB	*DAB*	*EAB*	*DAC*	*EAC*	*EAD*	*DBC*	*EBC*	*EBD*	*ECD*
CAB	*DBA*	*EBA*	*DCA*	*ECA*	*EDA*	*DCB*	*ECB*	*EDB*	*EDC*

Example We have a committee of n = 10 people and we want to choose a **chairperson**, **a vice-chairperson** and a **treasurer**

Solution: Essentually we want to select 3 persons from the committee of 10 in a specific order. (Permutations of size 3 from a group of 10).

$$ {}_{10}P_3 = \frac{10!}{(10-3)!} = \frac{10!}{7!} = 10(9)(8) = 720 $$

Example We have a committee of $n = 10$ people and we want to choose a **chairperson**, **a vice-chairperson** and a **treasurer**. Suppose that 6 of the members of the committee are male and 4 of the members are female. What is the probability that the three executives selected are all male?

Solution: Again we want to select 3 persons from the committee of 10 in a specific order. (Permutations of size 3 from a group of 10).The total number of ways that this can be done is:

$$_{10}P_3 = \frac{10!}{(10-3)!} = \frac{10!}{7!} = 10(9)(8) = 720$$

This is the size, $N = n(S)$, of the sample space S. Assume all outcomes in the sample space are equally likely.

Let E be the event that all three executives are male

$$n(E) = {}_6P_3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = 6(5)(4) = 120$$

Hence

$$P[E] = \frac{n(E)}{n(S)} = \frac{120}{720} = \frac{1}{6}$$

Thus if all candidates are equally likely to be selected to any position on the executive then the probability of selecting an all male executive is:

$$\frac{1}{6}$$

Examples - continued

3. **Combinations of size *k* (≤ *n*):** A combination of size *k* chosen from *n* objects is a subset of size *k* where the order of selection is irrelevant. How many ways can you choose a combination of size *k* objects from *n* objects (order of selection is irrelevant)

Here are the combinations of size 3 selected from the 5 objects $\{A, B, C, D, E\}$

$\{A,B,C\}$	$\{A,B,D\}$	$\{A,B,E\}$	$\{A,C,D\}$	$\{A,C,E\}$
$\{A,D,E\}$	$\{B,C,D\}$	$\{B,C,E\}$	$\{B,D,E\}$	$\{C,D,E\}$

Important Notes

1. In **combinations** ordering is **irrelevant**. Different orderings result in the same combination.
2. In **permutations** order is **relevant**. Different orderings result in the different permutations.

How many ways can you choose a combination of size *k* objects from *n* objects (order of selection is irrelevant)

Solution: Let n_1 denote the number of combinations of size *k*.
One can construct a permutation of size *k* by:

1. Choosing a combination of size *k* (n_1 = unknown)
2. Ordering the elements of the combination to form a permutation ($n_2 = k!$)

Thus $${}_nP_k = \frac{n!}{(n-k)!} = n_1 k!$$

and $$n_1 = \frac{{}_nP_k}{k!} = \frac{n!}{(n-k)!\,k!} = \text{ the \# of combinations of size } k.$$

The number:

$$n_1 = \frac{{}_nP_k}{k!} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots(1)}$$

is denoted by the symbol

$${}_nC_k \quad \text{or} \quad \binom{n}{k} \quad \text{read "}n\text{ choose }k\text{"}$$

It is the number of ways of choosing *k* objects from *n* objects (order of selection irrelevant).

${}_nC_k$ is also called a **binomial coefficient**.

It arises when we expand $(x + y)^n$ **(the binomial theorem)**

Summary of counting rules

Rule 1

$$n(A_1 \cup A_2 \cup A_3 \cup \ldots) = n(A_1) + n(A_2) + n(A_3) + \ldots$$

if the sets $A_1, A_2, A_3, \ldots$ are pairwise mutually exclusive

(i.e. $A_i \cap A_j = \varphi$)

Rule 2

$N = n_1 n_2 =$ the number of ways that two operations can be performed in sequence if

$n_1 =$ the number of ways the first operation can be performed

$n_2 =$ the number of ways the second operation can be performed once the first operation has been completed.

Rule 3 $N = n_1 n_2 \ldots n_k$

= the number of ways the k operations can be performed in sequence if

n_1 = the number of ways the first operation can be performed

n_i = the number of ways the i^{th} operation can be performed once the first $(i - 1)$ operations have been completed. $i = 2, 3, \ldots, k$

Basic counting formulae

1. Orderings

$n! =$ the number of ways you can order n objects

2. Permutations

$${}_nP_k = \frac{n!}{(n-k)!} =$$ The number of ways that you can choose k objects from n in a specific order

3. Combinations

$$\binom{n}{k} = {}_nC_k = \frac{n!}{k!(n-k)!} =$$ The number of ways that you can choose k objects from n (order of selection irrelevant)

Applications to some counting problems

- The trick is to use the basic counting formulae together with the **Rules**
- We will illustrate this with examples
- Counting problems are not easy. The more practice better the techniques

Quick summary of probability