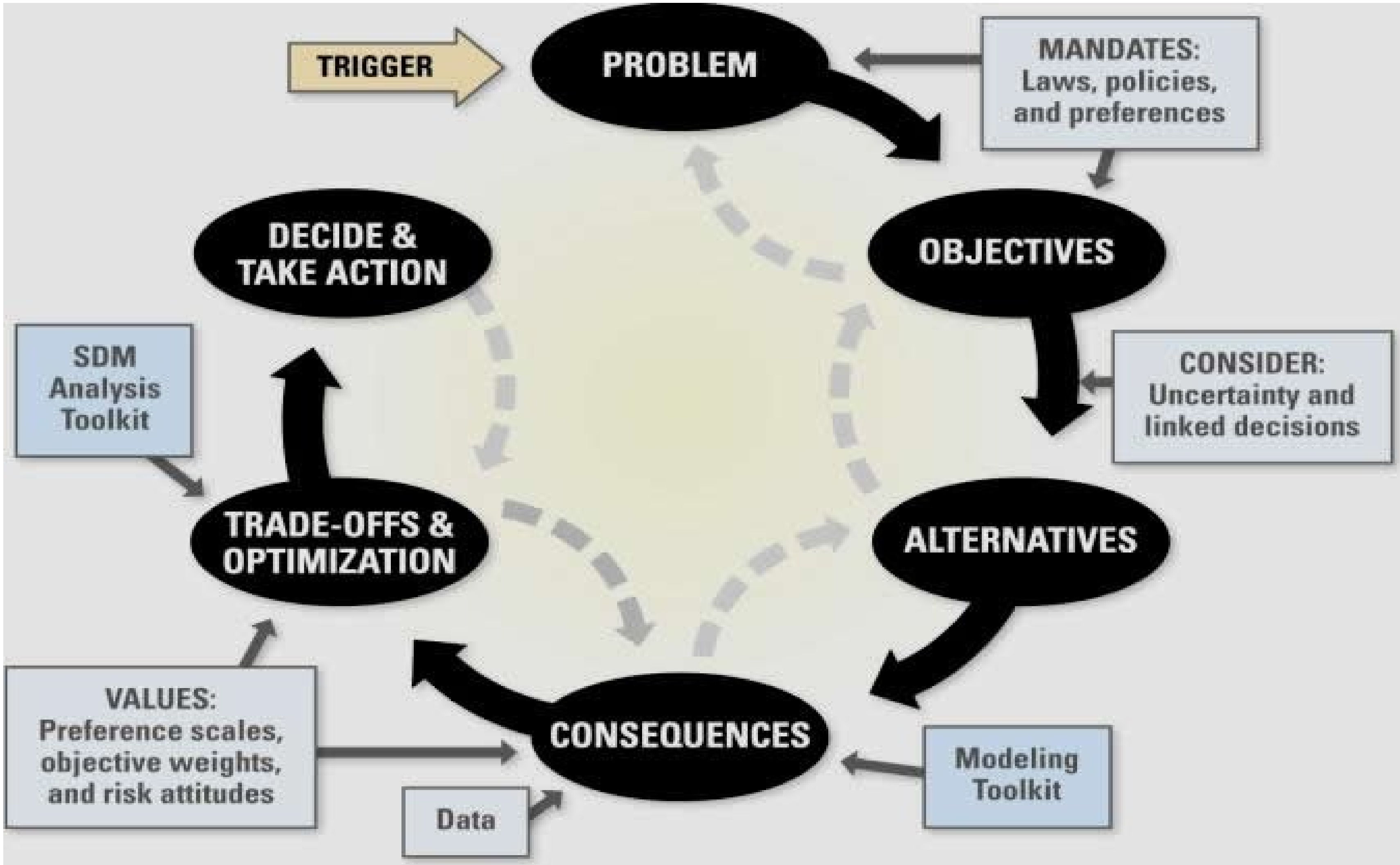
Topics in Computational Sustainability: Introduction to Linear Programming

Core Elements of Strategic Decision Making (SDM)



Inputs

Problem

Outputs

Computational Problems

Decision Problems

(e.g., Yes or No decision: Can it be done and how?)

or

Optimization Problems

(e.g., What's the optimal solution and how?)

Decision Problem
Graph Coloring

Can we color a map with 4 colors?

Decision Problem
Set Covering

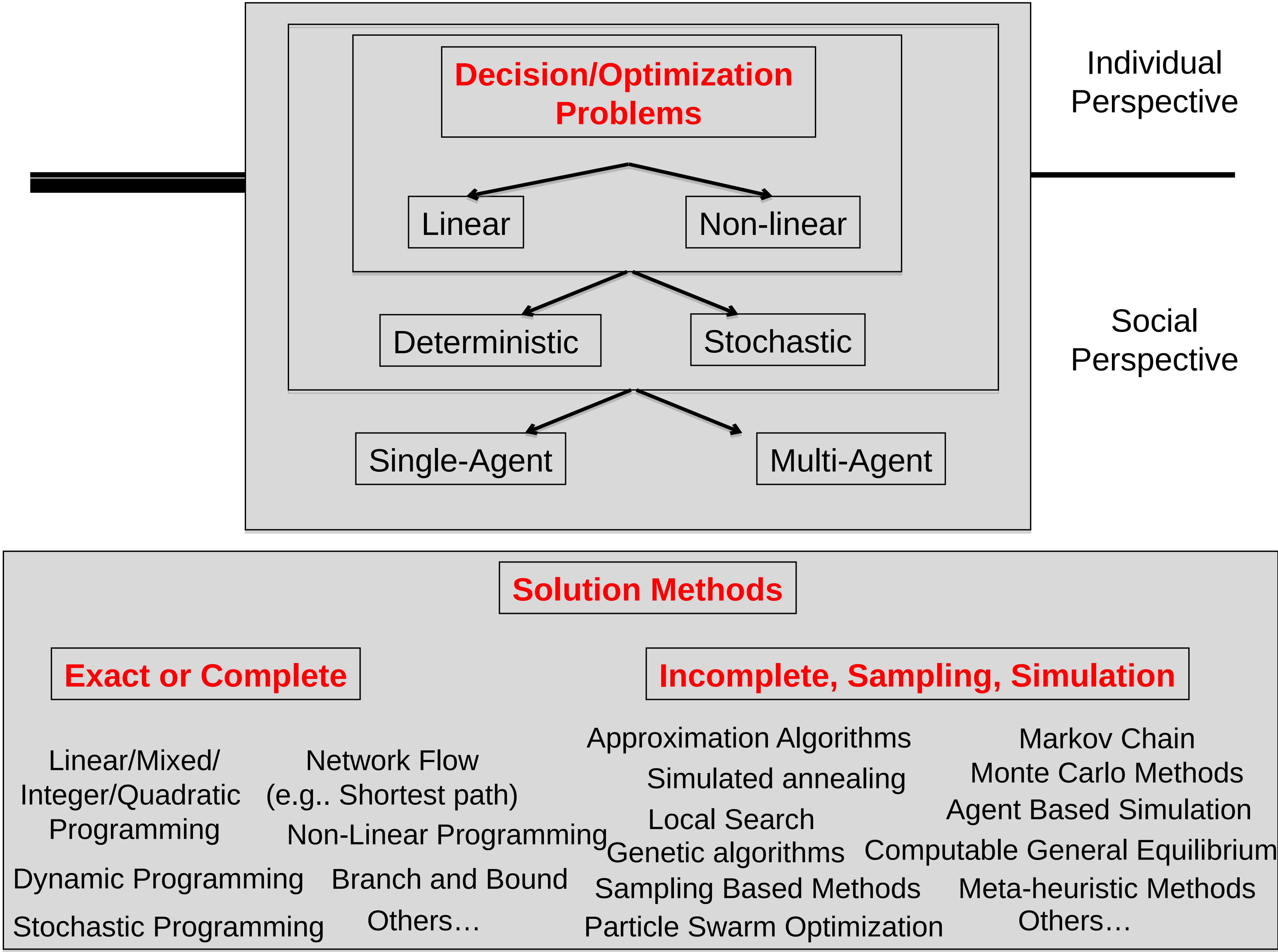
Can we protect the forest with 10 forest rangers?

Optimization Problem
Linear Programming

How to minimize Pulp mill pollution?

Optimization Problem
Knapsack

What Renewable Energy Investments should the Green Power Company pick to maximize profit?


Decision/Optimization Problems
Linear
Non-linear
Deterministic
Stochastic
Single-Agent
Multi-Agent
Individual Perspective
Social Perspective
Solution Methods
Exact or Complete
Linear/Mixed/ Integer/Quadratic Programming
Network Flow (e.g.. Shortest path)
Non-Linear Programming
Dynamic Programming
Branch and Bound
Stochastic Programming
Others…
Incomplete, Sampling, Simulation
Approximation Algorithms
Simulated annealing
Local Search
Genetic algorithms
Sampling Based Methods
Particle Swarm Optimization
Markov Chain
Monte Carlo Methods
Agent Based Simulation
Computable General Equilibrium
Meta-heuristic Methods
Others…

Optimization: Mathematical Program

- Optimization Problem in which the objective and constraints are given as mathematical functions and functional relationships.

Minimize $f(x_1, x_2, \ldots, x_n)$

Subject to:

$$g_1(x_1, x_2, \ldots, x_n) \; = \, , \;\; \geq , \;\; \leq \; b_1$$

$$g_2(x_1, x_2, \ldots, x_n) \; = \, , \;\; \geq , \;\; \leq \; b_2$$

…

$$g_m(x_1, x_2, \ldots, x_n) \; = \, , \;\; \geq \, , \leq \;\; b_m$$

Linear Programming

Linear Programming (LP)

- Linear – All the functions are linear

Ex: $f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2 x_2 + \ldots c_n x_n$

- Programming – does not refer to computer programming but rather “planning” - planning of activities to obtain an optimal result i.e., it reaches the specified goal (best solution) (according to the mathematical model) among all feasible alternatives.

Components of a Linear Programming Model

- A Linear Programming (LP) Model Consists of:
  - A Set of Decision Variables
  - A (Linear) Objective Function
  - A Set of (Linear) Constraints

- **Linear Programming (LP) Problems**
  Both objective function and constraints are linear. Solutions are highly structured and can be rapidly obtained.

Linear Programming (LP)

Linear Programming

- LP has gained widespread industrial acceptance since the 1950s for on-line optimization, blending etc.
- Linear Constraints can arise due to:

1. Production Limitation e.g. Equipment Limitations, Storage Limits, Market Constraints.
2. Raw Material Limitation
3. Safety Restrictions, e.g. Allowable Operating Ranges for Temperature and Pressures.
4. Physical Property Specifications e.g. Product Quality Constraints when a blend property can be calculated as an average of pure component properties:

$$P = \sum_{i=1}^{n} y_i P_i \le \alpha$$

5. Material and Energy Balances
   - Tend to yield equality constraints.
   - Constraints can change frequently (daily or hourly).

- **Effect of Inequality Constraints**
  - Consider the Linear and Quadratic Objective Functions on the next slide.
  - Note that for the LP problem, the optimum must lie on one or more constraints.
- **Generic Statement of the LP Problem:**

$$\max f = \sum_{i=1}^{n} c_i x_i$$

subject to (s. t.):

$$x_i \geq 0 \qquad i = 1, 2, \ldots, n$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1, 2, \ldots, n$$

- **Solution of LP Problems**
  - Simplex Method (Dantzig, 1947)
  - Examine only constraint boundaries
  - Very efficient, even for large problems

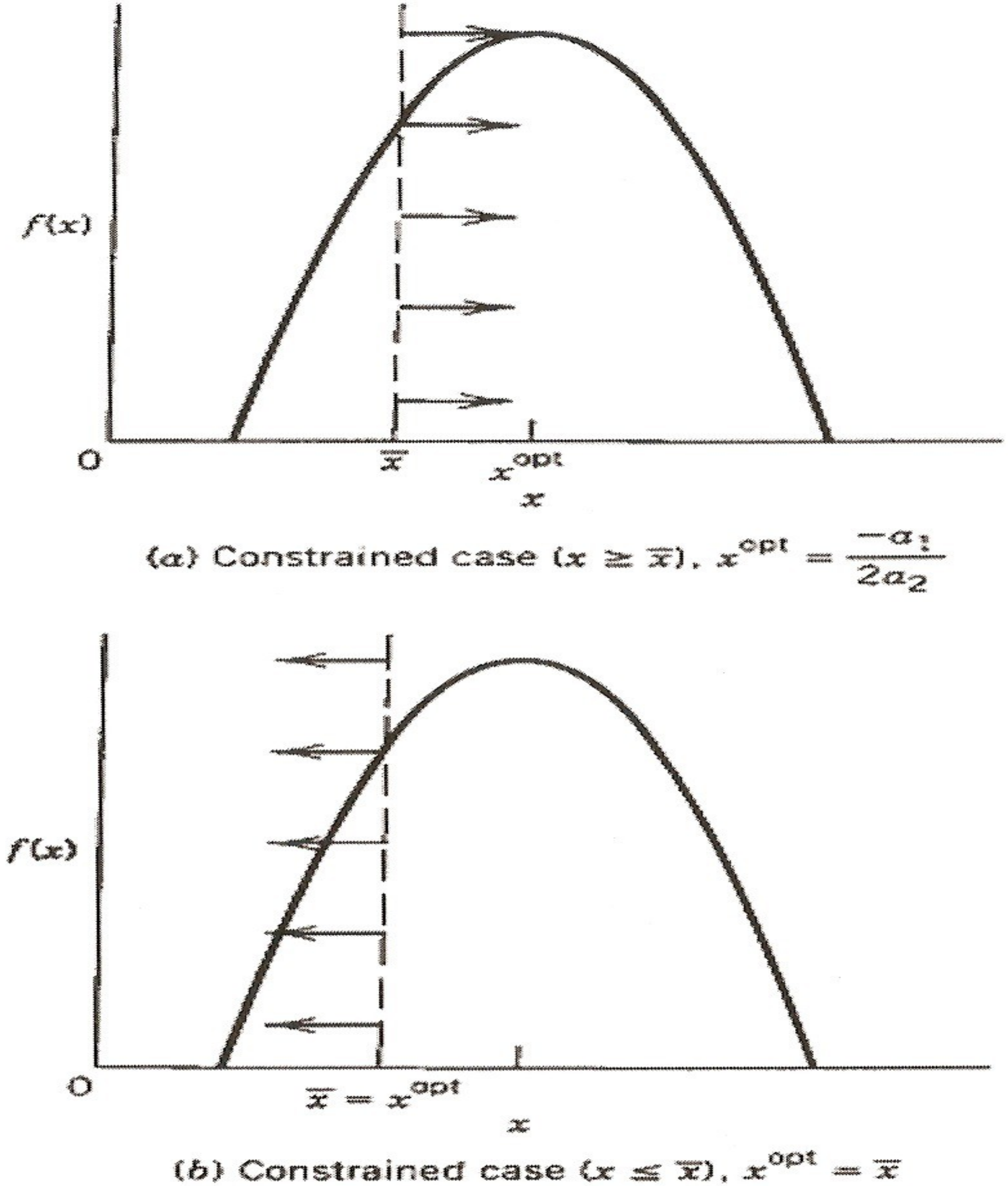
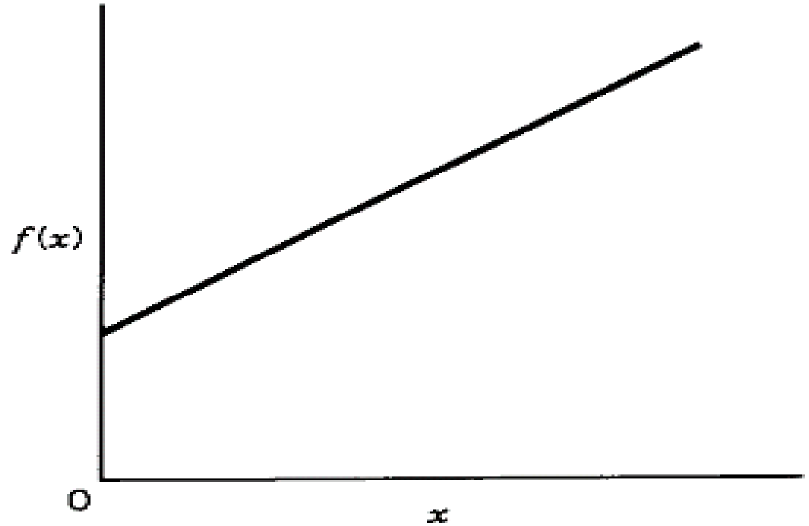
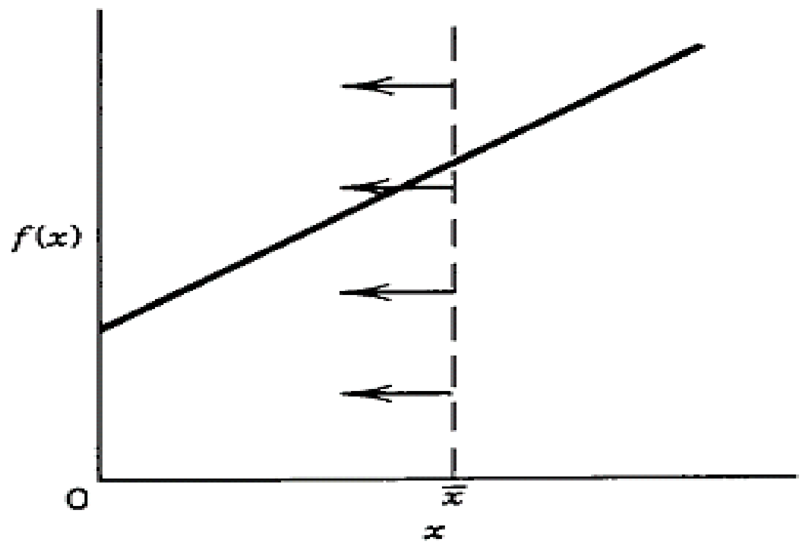


Figure: The effect of an inequality constraint on the maximum of quadratic function, $f(x) = a_0 + a_1 x + a_2 x^2$. The arrows indicate the allowable values of x.



(a) Unconstrained case, $x^{\text{opt}} \to \infty$



(b) Constrained case ($x \leq \bar{x}$), $x^{\text{opt}} = \bar{x}$

The effect of a linear constraint on the maximum of linear objective function, $f(x) = a_0 + a_1 x$.

Steps in Setting up a LP

1. Determine and label the *Decision Variables*.
2. Determine the Objective and use the Decision Variables to write an expression for the *Objective Function*.
3. Determine the Constraints - *Feasible Region*.
    1. Determine the *Explicit Constraints* and write a functional expression for each of them.
    2. Determine the *Implicit Constraints (Nonnegativity Constraints)*.

Nature Connection: Recreational Sites

Nature Connection is planning two new public recreational sites: **a forested wilderness area** and a **sightseeing and hiking park**. They own 80 hectares of forested wilderness area and 20 hectares suitable for the sightseeing and hiking park but they don't have enough resources to make the entire areas available to the public. They have a **budget of $120K** per year. They estimate a yearly management and maintenance cost of $1K per hectare for the forested wilderness area, and $4K per hectare for the sightseeing and hiking park. The expected average number of visiting hours a day per hectare are: 10 for the forest and 20 for the sightseeing and hiking park.

Question: How many hectares should Nature Connection allocate to the public sightseeing and hiking park and to the public forested wilderness area, in order to maximize the amount of recreation, (in average number of visiting hours a day for the total area to be open to the public, for both sites**) given their budget constraint?**

Formulation of the Problem as a Linear Program

1 Decision Variables

$x1$ – # hectares to allocate to the public forested wilderness area
$X2$ – # hectares to allocate the public sightseeing and hiking park

2 Objective Function

$$\text{Max } 10\, x1 + 20\, x2$$

3 Constraints

$$x1 \leq 80$$
$$x2 \leq 20$$
$$x1 + 4\, x2 \leq 120$$

$x1 \geq 0$; $x2 \geq 0$ Non-negativity constraints

Formulation of the Problem as a Linear Program

Nature Connection is planning two new public recreational sites: a forested wilderness area and a sightseeing and hiking park. They own [redacted] 20 hectares suitable for the sightseeing and hiking park but they don't have enough resources to make the entire areas available to the public. They have a budget of $120K per year. They estimate a yearly management and maintenance cost of $1K per hectare, for the forested wilderness area, and $4K per hectare for the sightseeing and hiking park. The expected average number of visiting hours a day per hectare are: 10 for the forest and 20 for the sightseeing and hiking park.

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1 Decision Variables

~~x1 – # hectares to allocate to the public~~ forested wilderness area

X2 – # hectares to allocate the public sightseeing and hiking park

2 Objective Function

Max 10 x1 + 20 x2

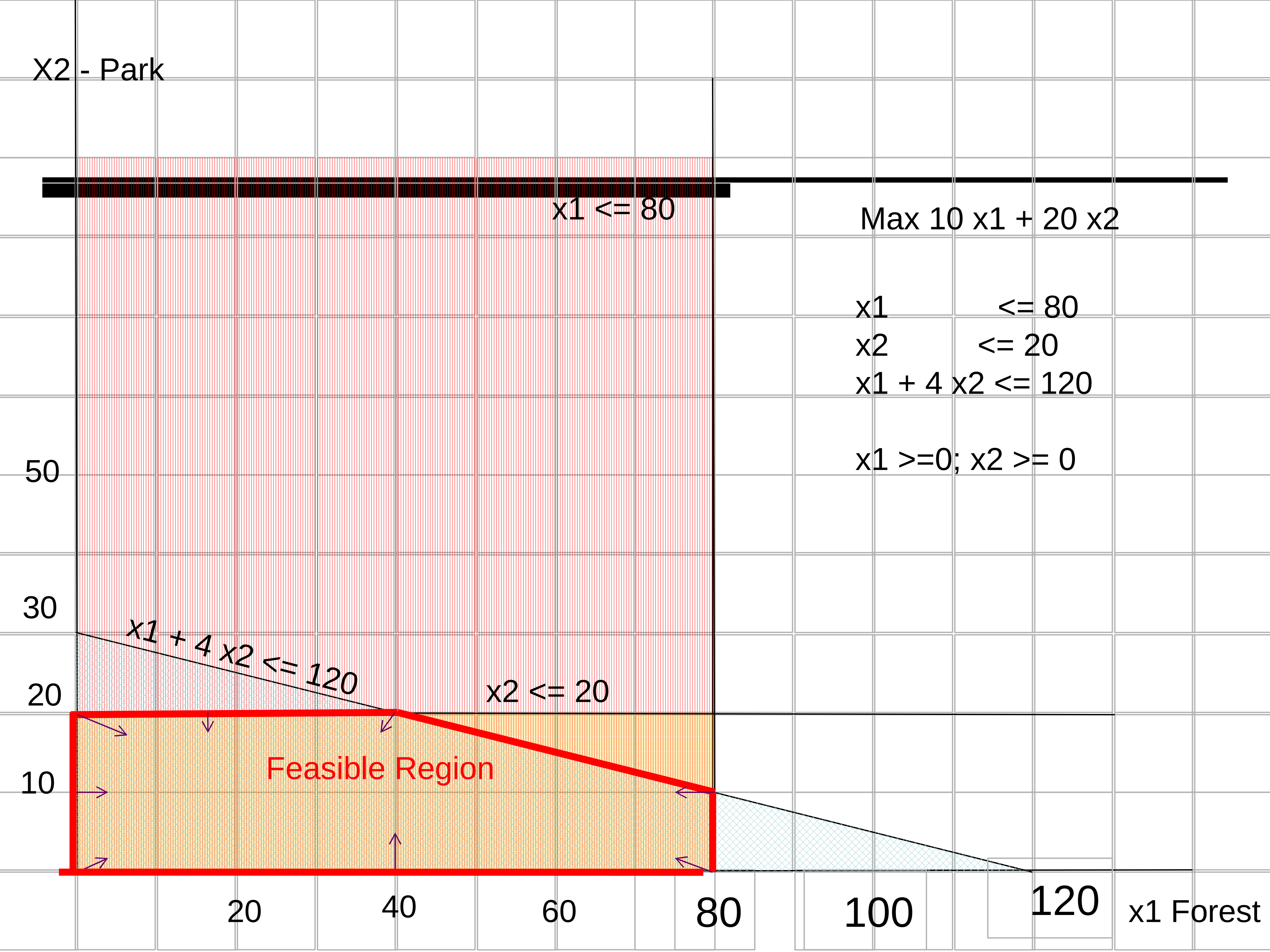
3 Constraints

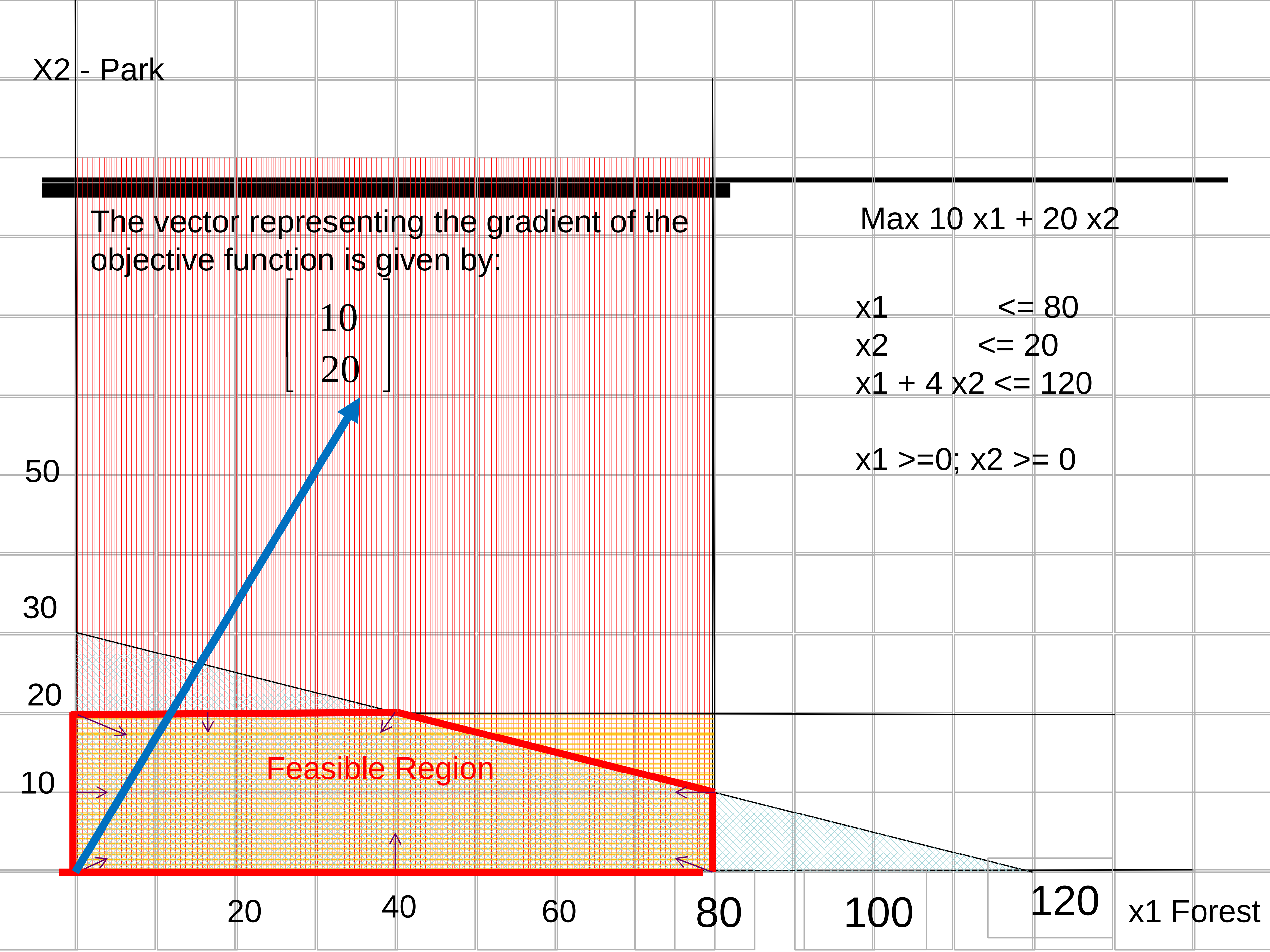
x1 <= 80 Land for forest

x2 <= 20 Land for Park

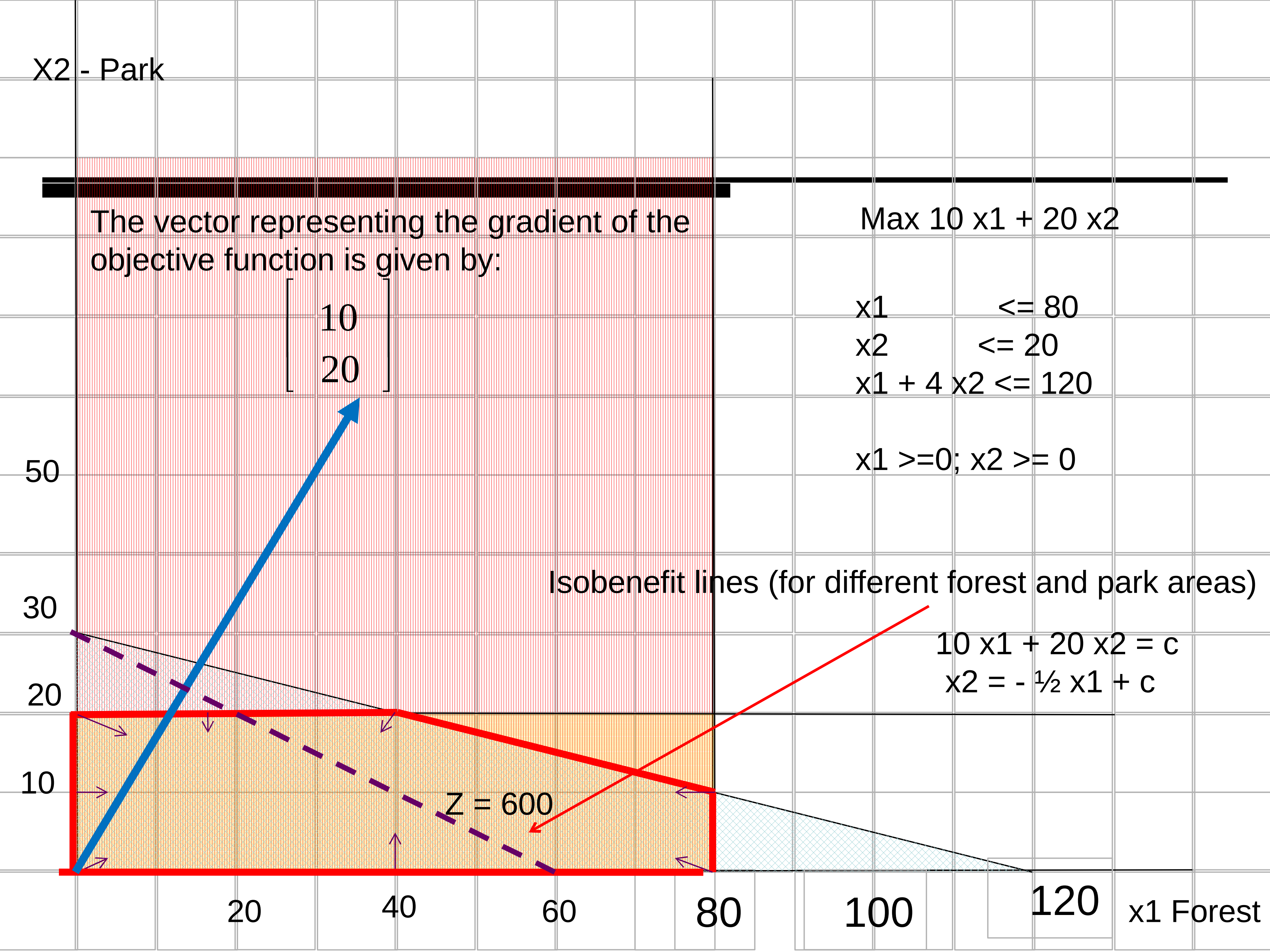
x1 + 4 x2 <= 120 Budget

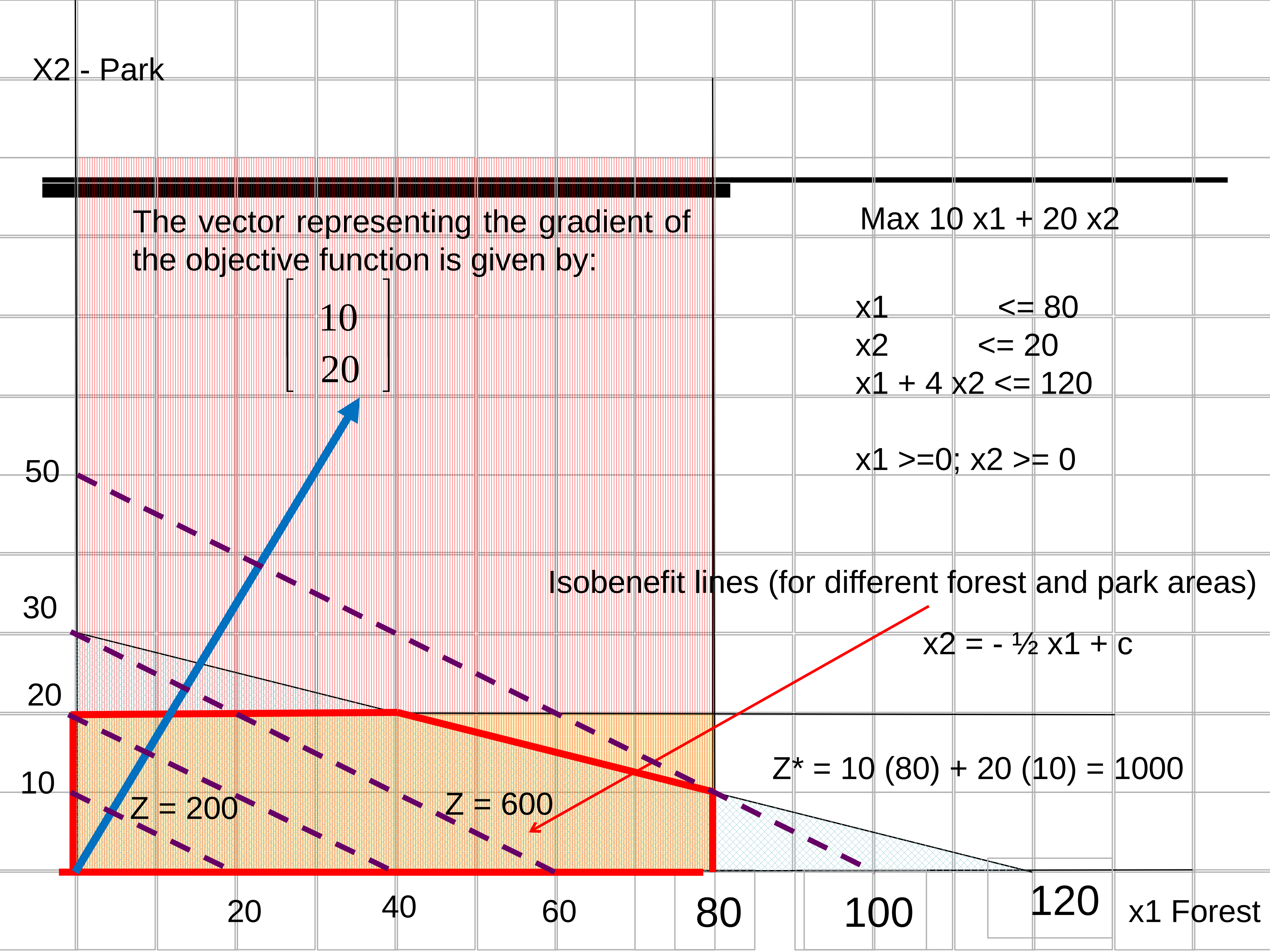
x1 >= 0; x2 >= 0 Non-negativity constraints


X2 - Park
x1 <= 80
Max 10 x1 + 20 x2
x1 <= 80
x2 <= 20
x1 + 4 x2 <= 120
x1 >=0; x2 >= 0
50
30
20
10
x1 + 4 x2 <= 120
x2 <= 20
Feasible Region
20
40
60
80
100
120
x1 Forest



X2 - Park
The vector representing the gradient of the objective function is given by:
$\begin{bmatrix} 10 \\ 20 \end{bmatrix}$
Max 10 x1 + 20 x2
x1 <= 80
x2 <= 20
x1 + 4 x2 <= 120
x1 >=0; x2 >= 0
50
30
20
10
Feasible Region
20
40
60
80
100
120
x1 Forest

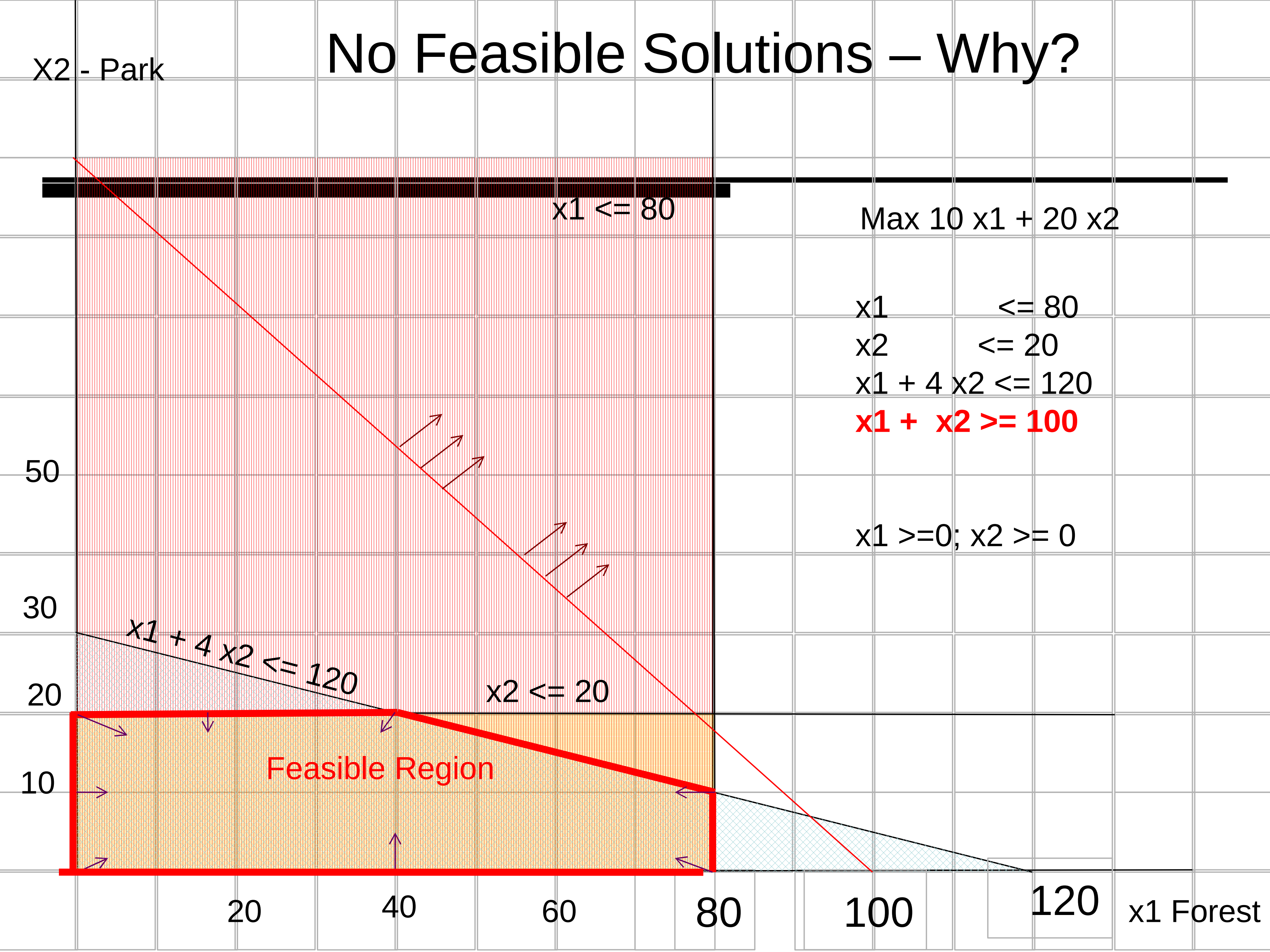

X2 - Park
The vector representing the gradient of the objective function is given by:
$\begin{bmatrix} 10 \\ 20 \end{bmatrix}$
Max 10 x1 + 20 x2
x1 <= 80
x2 <= 20
x1 + 4 x2 <= 120
x1 >=0; x2 >= 0
Isobenefit lines (for different forest and park areas)
10 x1 + 20 x2 = c
x2 = - ½ x1 + c
Z = 600
50
30
20
10
20
40
60
80
100
120
x1 Forest


X2 - Park
The vector representing the gradient of the objective function is given by:
[10 20]
Max 10 x1 + 20 x2
x1 <= 80
x2 <= 20
x1 + 4 x2 <= 120
x1 >=0; x2 >= 0
50
30
20
10
Isobenefit lines (for different forest and park areas)
x2 = - ½ x1 + c
Z* = 10 (80) + 20 (10) = 1000
Z = 200
Z = 600
20
40
60
80
100
120
x1 Forest

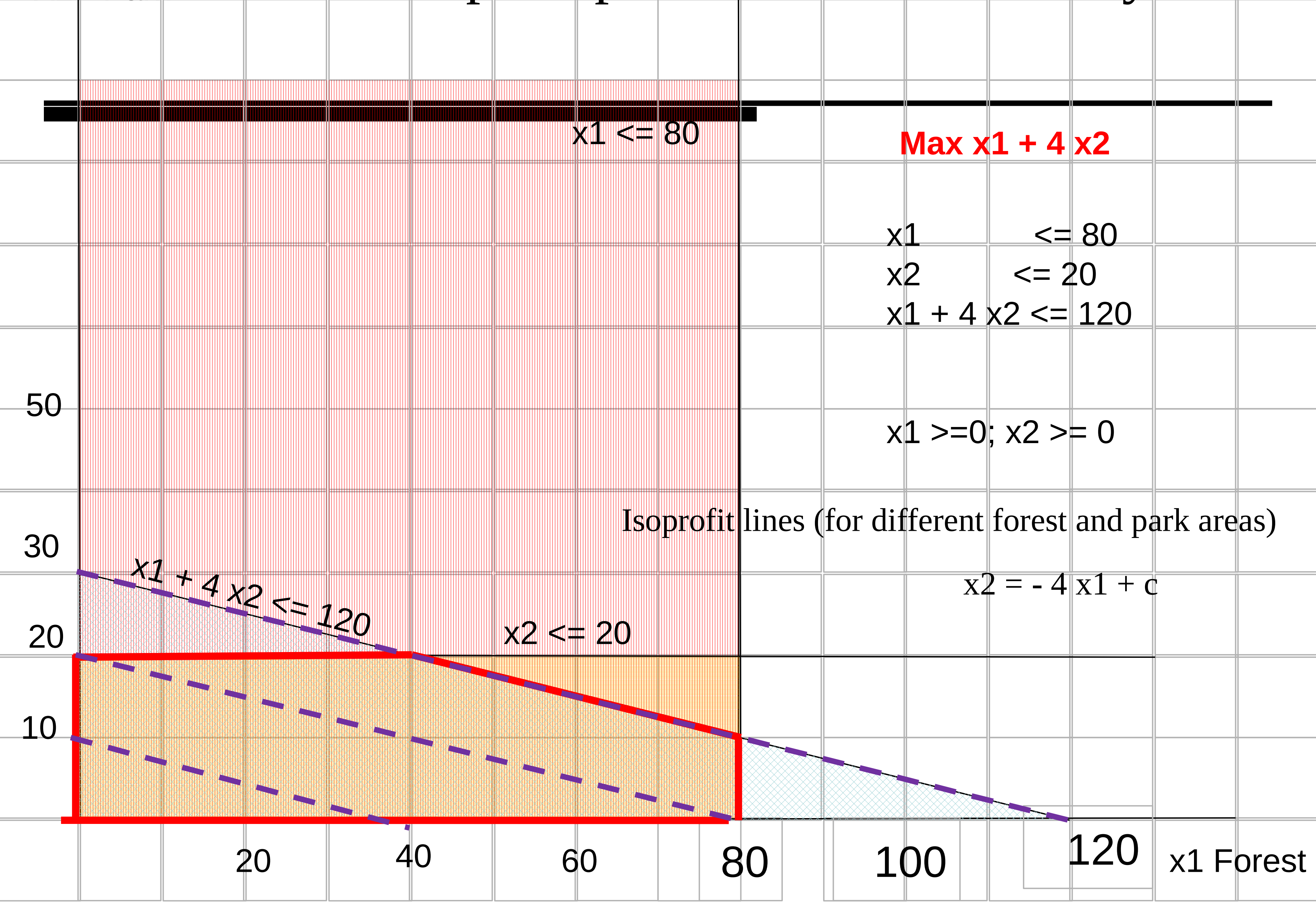
Summary of the Graphical Method

- Draw the constraint boundary line for each constraint. Use the origin (or any point not on the line) to determine which side of the line is permitted by the constraint.
- Find the feasible region by determining where all constraints are satisfied simultaneously.
- Determine the slope of one objective function line (perpendicular to its gradient vector). All other objective function lines will have the same slope.
- Move a straight edge with this slope through the feasible region in the direction of improving values of the objective function (direction of the gradient). Stop at the last instant that the straight edge still passes through a point in the feasible region. This is the optimal objective function line.
- A feasible point on the optimal objective function line is an optimal solution.

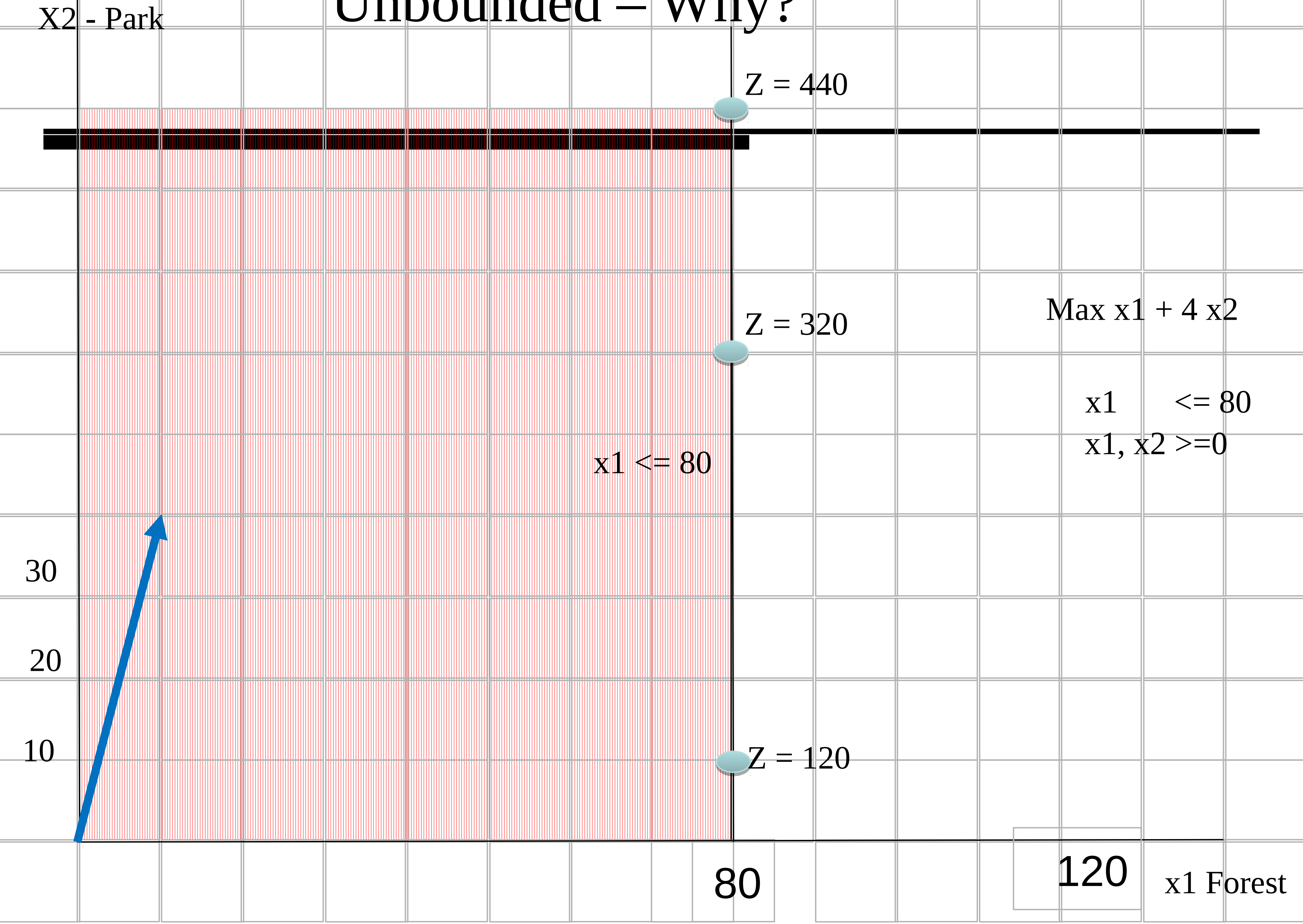
No Feasible Solutions – Why?



Multiple Optimal Solutions – Why



Unbounded – Why?



Key Categories of LP Problems:

- Resource-Allocation Problems
- Cost-benefit-trade-off Problems
- Distribution-Network Problems
- Mixed Problems

Second Example: Keeping the River Clean

Cost-benefit-trade-off problems

Choose the mix of levels of various activities to achieve minimum acceptable levels for various benefits at a minimum cost.

Second Example: Keeping the River Clean

A pulp mill in Maine makes mechanical and chemical pulp, polluting the river in which it spills its spent waters. This has created several problems, leading to a change in management.

The previous owners felt that it would be too expensive to reduce pollution, so they decided to sell the pulp mill. The mill has been bought back by the employees and local businesses, who now own the mill as a cooperative. The new owners have several objectives:

1 – to keep **at least 300 people employed at the mill** (300 workers a day);

2 – to generate at least $40,000 of revenue a day

They estimate that this will be enough to pay operating expenses and yield a return that will keep the mill competitive in the long run. Within these limits, everything possible should be done to **minimize pollution**.

Both chemical and mechanical pulp require the labor of one worker for 1 day (1 workday, wd) per ton produced;

Mechanical pulp sells at $100 per ton; Chemical pulp sells at $200 per ton;

Pollution is measured by the biological oxygen demand (BOD). One ton of mechanical pulp produces 1 unit of BOD; One ton of chemical pulp produces 1.5 units of BOD.

The maximum capacity of the mill to make mechanical pulp is 300 tons per day; for chemical pulp is 200 tons per day. The two manufacturing processes are independent (I.e., the mechanical pulp line cannot be used to make chemical pulp and vice versa).

- Pollution, employment, and revenues result from the production of both types of pulp. So a natural choice for the variables is:

[redacted]

- X1 amount of mechanical pulp produced (in tons per day, or t/d) and
- X2 amount of chemical pulp produce (in tons per day, or t/d
- Min Z = 1 X1 + 1.5 X2

(BOD/day) (BOD/t) (t/d) (BOD/t) (t/d)

Subject to (s.t.):

1 X1 + 1 X2 >= 300 Workers/day

(wd/t) (t/d) (wd/t)(t/d)

100 X1 + 200 X2 >= 40,000 revenue/day

($/t) (t/d) + ($/t)(t/d) $/d

X1 <= 300 (mechanical pulp)

(t/d) (t/d)

X2 <= 200 (mechanical pulp)

(t/d) (t/d)

X1 >=0; X2 >= 0

Distribution Network Problems

Distribution-Network Problem

- The International Hospital Share Organization is a non-profit organization that refurbishes a variety of used equipment for hospitals of developing countries at two international factories (F1 and F2). One of its products is a large X-Ray machine.
- Orders have been received from three large communities for the X-Ray machines.

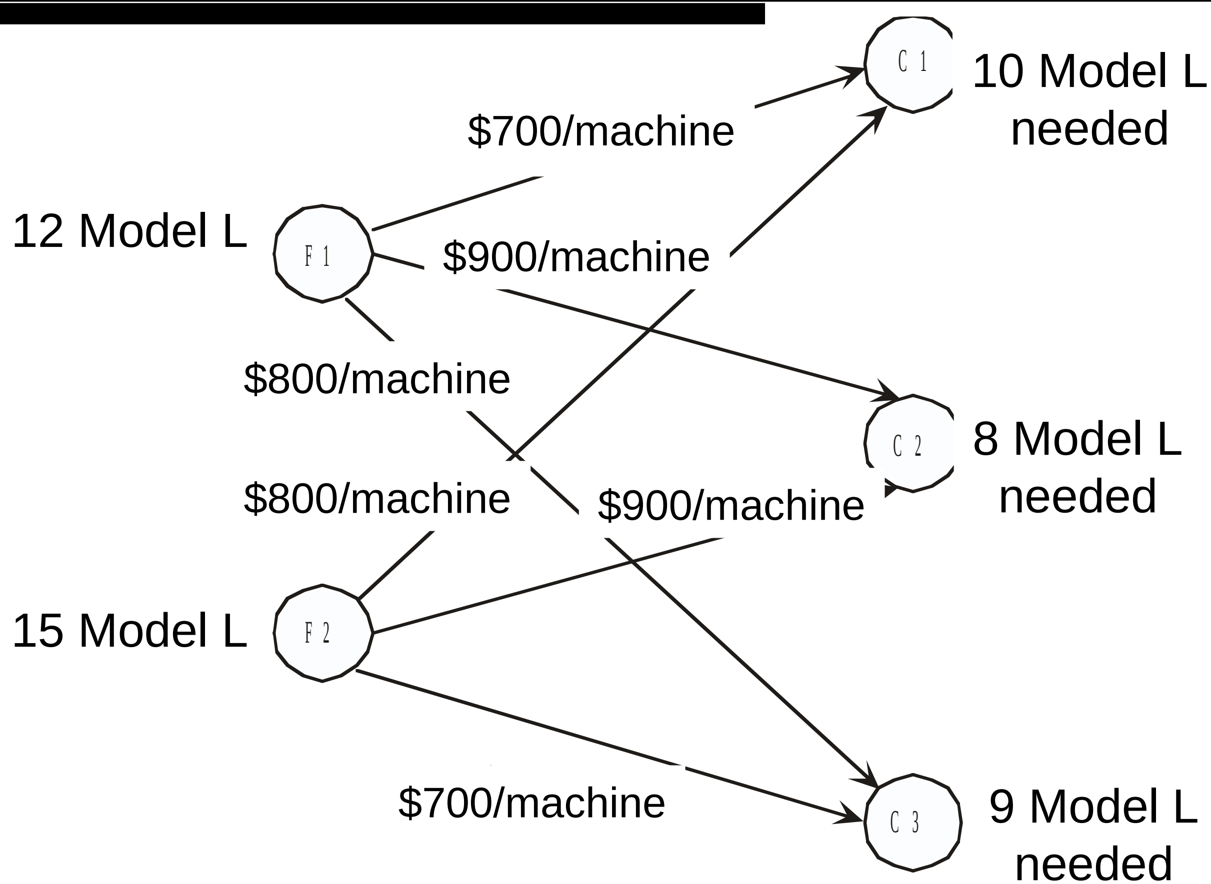
Some Data

Shipping Cost for Each Machine (Model L)

To	**Community 1**	**Community 2**	Community **3**	
From				**Output**
Factory 1	$700	$900	$800	12 X-ray machines
Factory 2	800	900	700	15 X-Ray machines
Order Size	10 X-ray machines	8 X-Ray machines	9 X-Ray machines	

Question: How many X-Ray machines (model L) should be shipped from each factory to each hospital so that shipping costs are minimized?

The Distribution Network



Question: How many machines (model L) should be shipped from each factory to each customer so that sipping costs are minimized?

- Activities – shipping lanes (not the level of production which has already been defined)
  - Level of each activity – number of machines of model L shipped through the corresponding shipping lane.

  → Best mix of shipping amounts

Example:

Requirement 1: Factory 1 must ship 12 machines
Requirement 2: Factory 2 must ship 15 machines
Requirement 3: Customer 1 must receive 10 machines
Requirement 4: Customer 2 must receive 8 machines
Requirement 5: Customer 3 must receive 9 machines

Algebraic Formulation

Let S_{ij} = Number of machines to ship from i to j (i = F1, F2; j = C1, C2, C3).

$$\text{Minimize Cost} = \$700S_{F1\text{-}C1} + \$900S_{F1\text{-}C2} + \$800S_{F1\text{-}C3} + \$800S_{F2\text{-}C1} + \$900S_{F2\text{-}C2} + \$700S_{F2\text{-}C3}$$

subject to

$$S_{F1\text{-}C1} + S_{F1\text{-}C2} + S_{F1\text{-}C3} = 12$$
$$S_{F2\text{-}C1} + S_{F2\text{-}C2} + S_{F2\text{-}C3} = 15$$
$$S_{F1\text{-}C1} + S_{F2\text{-}C1} = 10$$
$$S_{F1\text{-}C2} + S_{F2\text{-}C2} = 8$$
$$S_{F1\text{-}C3} + S_{F2\text{-}C3} = 9$$

and

$S_{ij} \geq 0$ (i = F1, F2; j = C1, C2, C3).

Algebraic Formulation

Let S_{ij} = Number of machines to ship from i to j (i = F1, F2; j = C1, C2, C3).

Minimize Cost = $\$700S_{F1\text{-}C1} + \$900S_{F1\text{-}C2} + \$800S_{F1\text{-}C3} + \$800S_{F2\text{-}C1} + \$900S_{F2\text{-}C2} + \$700S_{F2\text{-}C3}$

subject to

Factory 1: $S_{F1\text{-}C1} + S_{F1\text{-}C2} + S_{F1\text{-}C3} = 12$

Factory 2: $S_{F2\text{-}C1} + S_{F2\text{-}C2} + S_{F2\text{-}C3} = 15$

Customer 1: $S_{F1\text{-}C1} + S_{F2\text{-}C1} = 10$

Customer 2: $S_{F1\text{-}C2} + S_{F2\text{-}C2} = 8$

Customer 3: $S_{F1\text{-}C3} + S_{F2\text{-}C3} = 9$

and

$S_{ij} \geq 0$ (i = F1, F2; j = C1, C2, C3).

Terminology and Notations

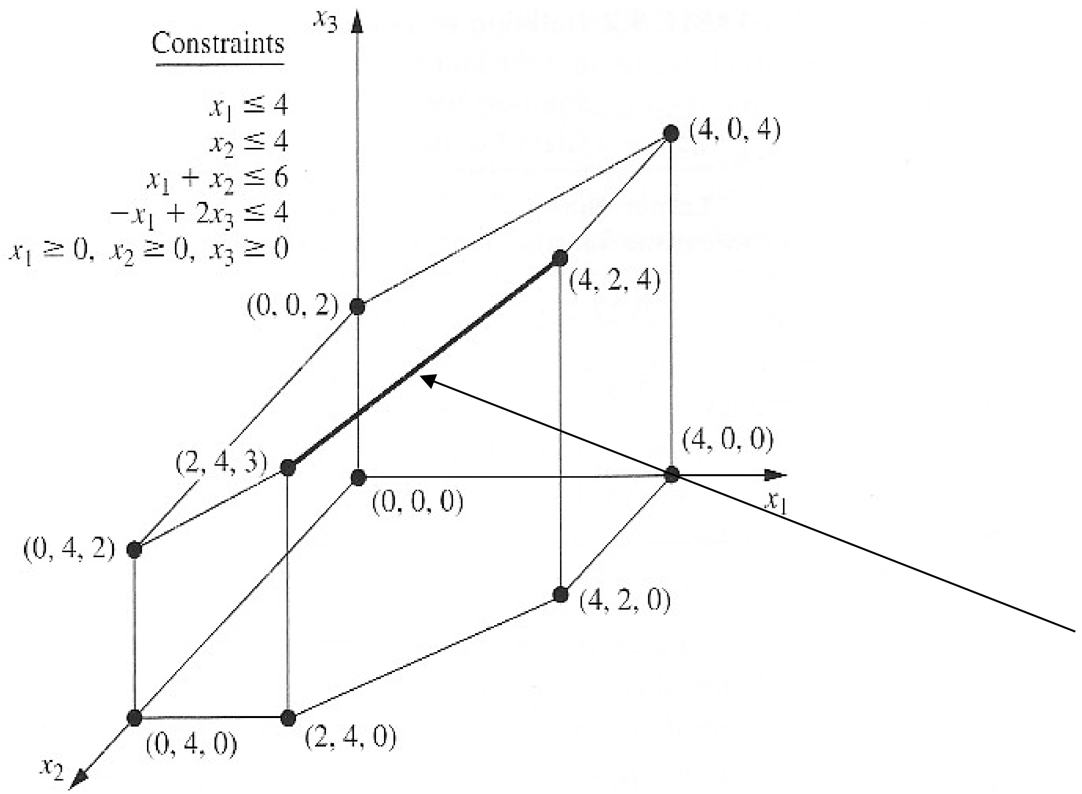
Terminology of Solutions in LP Model

- Solution – not necessarily the final answer to the problem!!!
- [redacted]isfies all the constraints
- Infeasible Solution – Solution for which at least one of the constraints is violated
- Feasible Region – Set of all points that satisfies all constraints (possible to have a problem without any feasible solutions)
- Binding Constraint – The left-hand side (LHS) and the right-hand side (RHS) of the constraint are equal, I.e., constraint is satisfied in equality. Otherwise the constraint is nonbinding.
- Optimal Solution – Feasible Solution that has the Best Value of the Objective Function.

    Largest Value   Maximization Problems

    Smallest Value Minimization Problems
- Multiple Optimal Solutions, no optimal solutions, unbounded Z

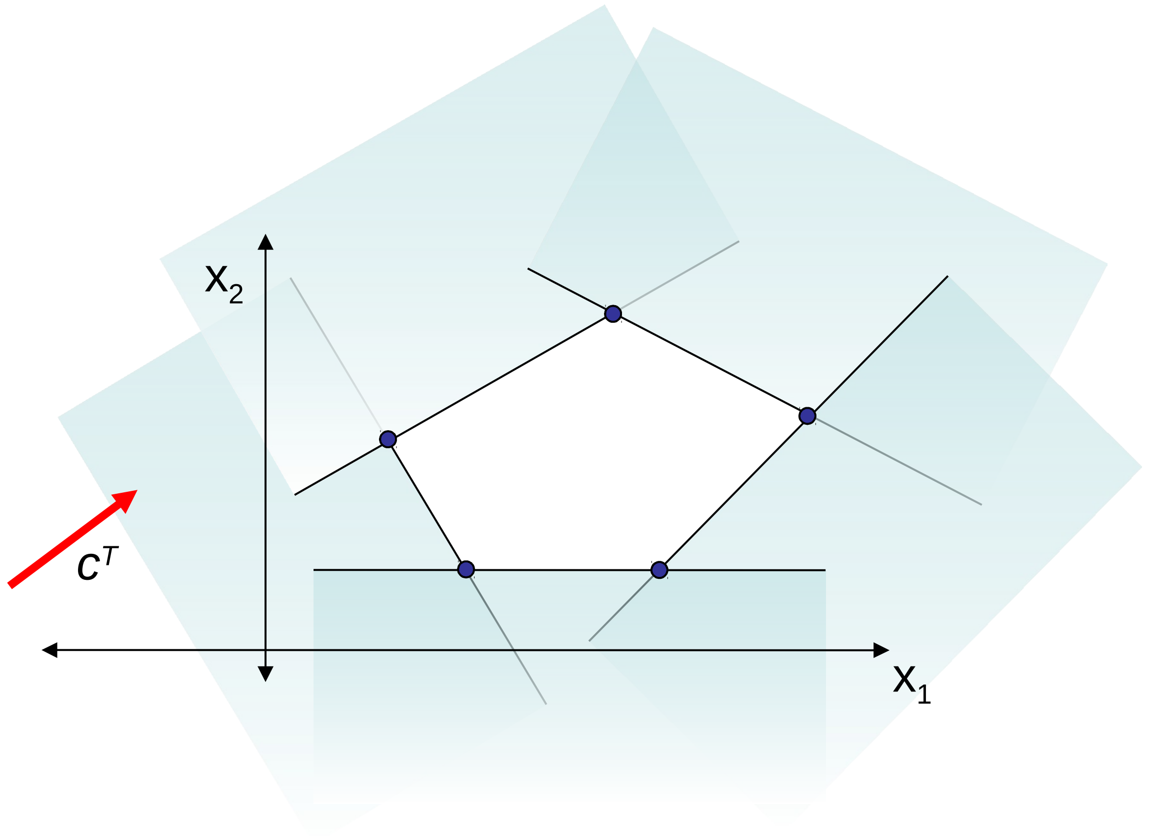
3D Feasible Region



Solving Linear Programs

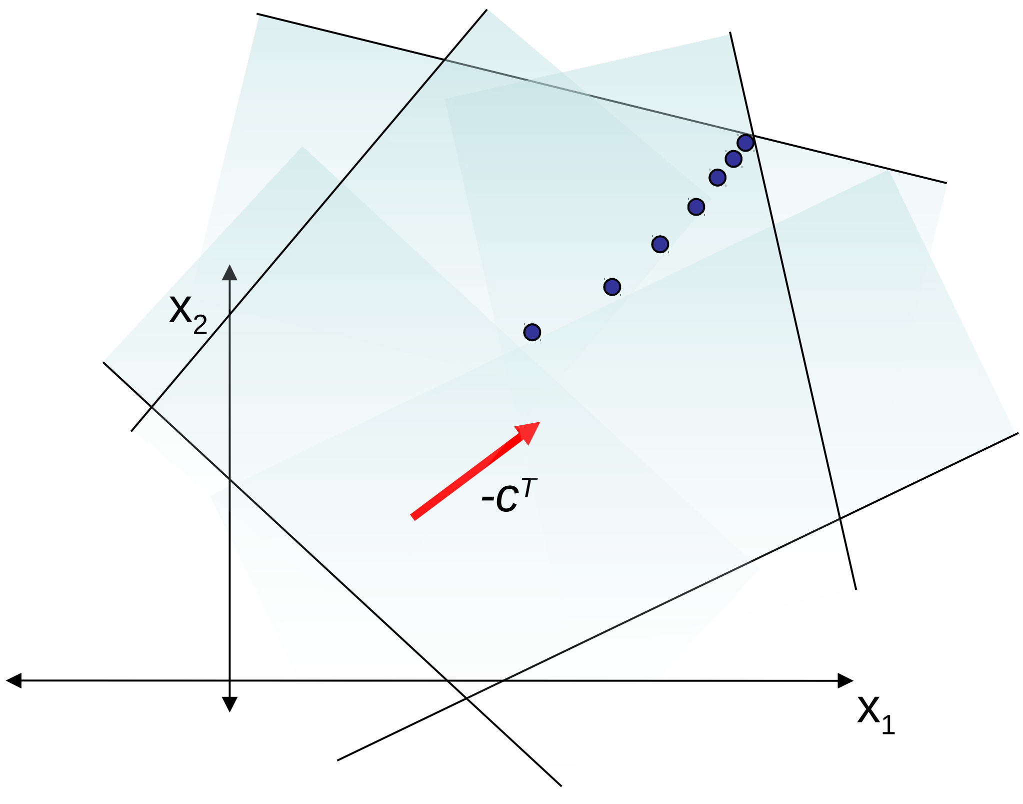
Solution Methods for Linear Programs

- Simplex Method
  - Optimum must be at the intersection of constraints
  - Intersections are easy to find, change inequalities to equalities



Solution Methods for Linear Programs

- Interior Point Methods
- Benefits
  - Scales Better than Simplex



Standard Form of the LP Model

$$\text{minimize} \quad Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n,$$

subject to the restrictions

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m,
\end{aligned}$$

$x_i >= 0$, (i =1,2,…,n)

Other Variants:

Maximize Z (instead of minimizing Z; but Min Z = - Max -Z)

Some constraints have other signs (=; and >=)

Some variables have unrestricted sign, i.e., they are not subject to the non-negativity constraints

Solving Linear Programs

- To solve LPs, typically need to put them in standard form:

$$\begin{aligned} \underset{z}{\text{minimize}} \;\; & c^T z \\ \text{subject to} \;\; & Az \le b \end{aligned}$$

  - $z \in \mathbb{R}^n$, $A \in \mathbb{R}^{N_i \times n}$, $b \in \mathbb{R}^{N_i}$

- For absolute loss LP

$$z = \begin{bmatrix} \theta \\ \nu \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} \Phi & -I \\ -\Phi & -I \end{bmatrix}, \quad b = \begin{bmatrix} y \\ -y \end{bmatrix}$$

$$\underset{\theta}{\text{minimize}} \;\; \sum_{i=1}^m |\theta^T \phi(x_i) - y_i| \quad \Longleftrightarrow \quad \begin{aligned} \underset{\theta,\nu}{\text{minimize}} \;\; & \sum_{i=1}^m \nu_i \\ \text{subject to} \;\; & -\nu_i \le \theta^T \phi(x_i) - y_i \le \nu_i \end{aligned}$$

Violates Divisibility Assumption of LP

- **Divisibility Assumption of Linear Programming (LP)**: Decision variables in a Linear Programming model are allowed to have *any* values, including *fractional (noninteger)* values, that satisfy the functional and nonnegativity constraints i.e., activities can be run at *fractional levels*..

When divisibility assumption is violated then apply Integer Linear Programming!!!