* ## Fuzzy Set Theory :-

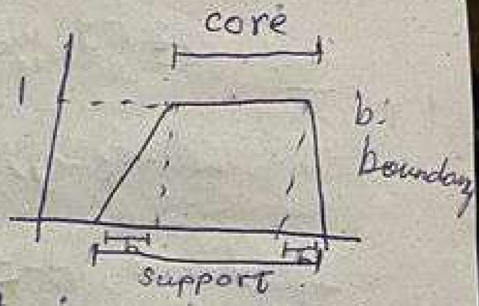
- Permits membership fn. valued in the interval [0,1].
- Membership Function is the degree of membership an element has

$$\mu_A(x) : X \rightarrow [0,1]$$

* Fuzzy Set, $A = \{(x, \mu_A(x)), x \in X\}$
- Diff. shapes of membership fn. :- Triangular, Trapezoidal, Gaussian, etc

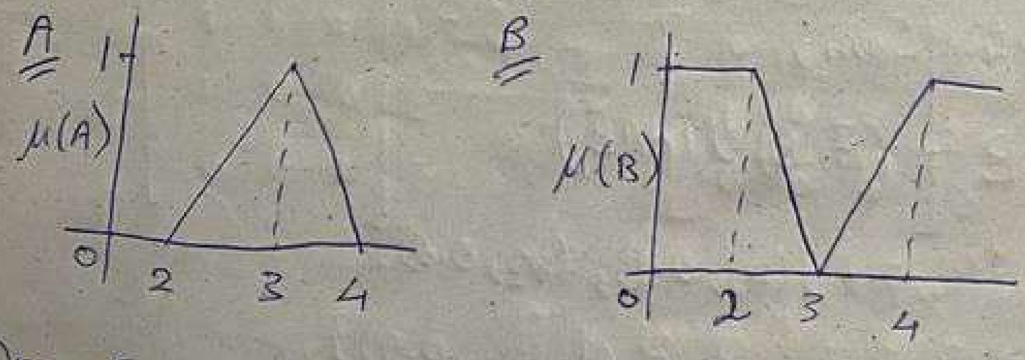
* Operations :- Fuzzy sets A and B
  * Union :- $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$
  * Intersection :- $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$
  * Complement :- $\mu_{\bar{A}}(x) = 1 - \mu_A(x)$

* Support (A) : $\{(x \mid \mu_A(x) > 0)\}$
* Core (A) : $\{(x \mid \mu_A(x) = 1)\}$



* Fuzzy Singleton :- Fuzzy set whose support is a single point in X with $\mu_A(x) = 1$

Eg. :-



Draw Figures for $A \cup B$, $A \cap B$, $\bar{A}$, $\bar{B}$

* More Operations:-

* Product of fuzzy sets,

$$\mu_{A.B}(x) = \mu_A(x) \cdot \mu_B(x)$$

* All operations are applied on the membership value and not on the fuzzy set elements.

* Two fuzzy sets are equal if,

$$\mu_A(x) = \mu_B(x)$$

* Product with crisp no.

$$\mu_{a.A}(x) = a \cdot \mu_A(x)$$ ~~with~~ a:- crisp value

* Power of a fuzzy set

$$\mu_{A^\alpha}(x) = [\mu_A(x)]^\alpha$$

Eg:- $A = \{(x_1, 0.4), (x_2, 0.2), (x_3, 0.7)\}$ $\alpha = 2$

$A^2 = \{(x_1, 0.16), (x_2, 0.04), (x_3, 0.49)\}$

* Difference, $A-B = (A \cap \bar{B})$

~~Do~~ $\bar{B} = 1 - \mu_B \Rightarrow$ ~~max~~ $\min(\mu_A, 1-\mu_B)$

* Disjunctive Sum,

$$A \oplus B = (\bar{A} \cap B) \cup (A \cap \bar{B})$$

Eg:- $A = \{(x_1, 0.4), (x_2, 0.8), (x_3, 0.6)\}$

$B = \{(x_1, 0.2), (x_2, 0.6), (x_3, 0.9)\}$

Soln. $\bar{A} = \{(x_1, 0.6), (x_2, 0.2), (x_3, 0.4)\}$

$\bar{B} = \{(x_1, 0.8), (x_2, 0.4), (x_3, 0.1)\}$

$\bar{A} \cap B = \{(x_1, 0.2), (x_2, 0.2), (x_3, 0.4)\}$

$A \cap \bar{B} = \{(x_1, 0.4), (x_2, 0.4), (x_3, 0.1)\}$

So, $A \oplus B$

$= \{(x_1, 0.4), (x_2, 0.4), (x_3, 0.4)\}$

* Properties of Fuzzy Sets:-

1. Commutative :- $A \cup B = B \cup A$; $A \cap B = B \cap A$
2. Associative :- $(A \cup B) \cup C = A \cup (B \cup C)$

   $(A \cap B) \cap C = A \cap (B \cap C)$

3. Distributive 4. Idempotence 5. Transitivity 6. De Morgan's Law

Crisp Set:-	Fuzzy Set:-
$A \cup \bar{A} = U$	$A \cup \bar{A} \neq U$
$A \cap \bar{A} = \phi$	$A \cap \bar{A} \neq \phi$

* $R(X, Y)$ is a 2-dim matrix where $x \rightarrow$ rows, $y \rightarrow$ columns

  It is a subset of Cartesian product

Eg:- $X = \{1, 2, 3, 4\}$ $X \times X = \{$ all combinations $\}$

Let $R = \{(x, y) \mid y = x + 1, \; x, y \in X\}$

Ans. $R = \{(1,2), (2,3), (3,4)\}$

$$R = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

* Operations on Relations :-

R & S are 2 relations defined on $X \times Y$

* $R \cup S = \max\{R(x,y), S(x,y)\}$
* $R \cap S = \min\{R(x,y), S(x,y)\}$
* $\bar{R}(x,y) = 1 - R(x,y)$

* Composition of Relations :-

Given R to be relation on (X,Y) and
S to be relation on (Y,Z)

Then $R \circ S$ is a composition on X, Z

$$T = R \circ S$$

$$T(x,z) = \max\left[\min\left(R(x,y), S(y,z)\right)\right]$$

Eg:- Let R & S be defined on $\{1,3,5\} \times \{1,3,5\}$

$R = \{(x,y) \mid y = x+2\}$ $\quad$ $S = \{(x,y) \mid x < y\}$

$R = \{(1,3), (3,5)\}$ $\quad$ $S = \{(1,3), (1,5), (3,5)\}$

$$R = \begin{array}{c} \\ 1 \\ 3 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \qquad S = \begin{array}{c} \\ 1 \\ 3 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\Rightarrow R \circ S = \begin{array}{c} \\ 1 \\ 3 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$R \circ S(1,1) = \max[\min(0,0), \min(1,0), \min(0,0)]$

* Fuzzy Relation:-

Let A and B be 2 fuzzy sets defined on X & Y, respectively

$A \times B$ results in a fuzzy relation R,

$$R = A \times B \subseteq X \times Y$$

where R has its membership function given by,

$$\mu_R(x,y) = \mu_{A\times B}(x,y) = \min(\mu_A(x), \mu_B(y))$$

Eg:- $A = \{(x_1, 0.2), (x_2, 0.7), (x_3, 0.4)\}$

$B = \{(y_1, 0.5), (y_2, 0.6)\}$

$$R = A \times B = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.2 & 0.2 \\ 0.5 & 0.6 \\ 0.4 & 0.4 \end{bmatrix} \end{array}$$

consider the minimum membership value for each pair

* Operations on Fuzzy Relation:-

Let R and S be 2 fuzzy relations on $X \times Y$

1. Union, $\mu_{R \cup S} = \max[\mu_R(x,y), \mu_S(x,y)]$
2. Intersection, $\mu_{R \cap S} = \min[\mu_R(x,y), \mu_S(x,y)]$
3. Complement, $\mu_{\bar{R}} = 1 - \mu_R(x,y)$

* Composition of fuzzy relation:-

If R is fuzzy relation on $X \times Y$

S is fuzzy relation on $Y \times Z$

Then $R \circ S$ is a fuzzy relation on $X \times Z$

$$\mu_{R \circ S}(x,z) = \max[\min(\mu_R(x,y), \mu_S(y,z))]$$

Eg:- $X = \{x_1, x_2, x_3\}$ $\quad Y = \{y_1, y_2\}$ $\quad Z = \{z_1, z_2, z_3\}$

$$\text{Let } R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.9 \\ 0.3 & 0.6 \end{bmatrix} \end{array} \quad \text{and} \quad S = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.6 & 0.4 & 0.7 \\ 0.5 & 0.6 & 0.9 \end{bmatrix} \end{array}$$

$$R.S = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.4 & 0.5 \\ 0.5 & 0.6 & 0.9 \\ 0.5 & 0.6 & 0.6 \end{bmatrix} \end{array}$$

$$\mu_{R.S}(x_1, z_1) = \max\left[\min(\overset{x_1,y_1}{0.5}, \overset{y_1,z_1}{0.6}), \min(\overset{x_1,y_2}{0.1}, \overset{y_2,z_1}{0.5})\right]$$
$$= \max(0.5, 0.1) = \underline{\underline{0.5}}$$

* Fuzzification and Defuzzification :-

- Converting crisp to fuzzy with the help of membership function is fuzzification.
- Fuzzy to crisp conversion :- defuzzification

* Methods of Defuzzification :-

1. Centroid Method or Center of Area Method $= \dfrac{\sum A(\bar{x})}{\sum(A)}$ $\quad \bar{x}$:- centroid, $\quad A$:- area of segment

2. Mean of Maxima :-

Considers crisp value with highest degree of membership

$$x = \frac{\sum\limits_{x_i \in M} x_i}{|M|} \qquad M = \{x_i \mid \mu(x_i) \text{ is equal to height of fuzzy set}\}$$

$|M|$:- cardinality ratio

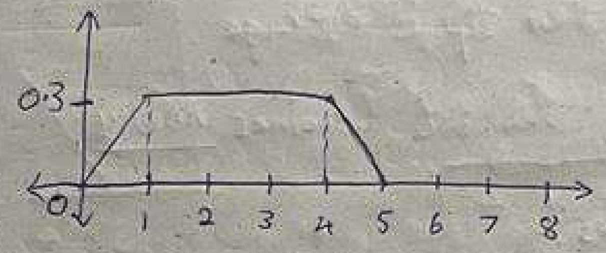
3. Weighted Average Method :-

A single crisp value, $x = \dfrac{\sum(\mu_c(\bar{z}) \cdot \bar{z})}{\sum \mu_c(\bar{z})}$

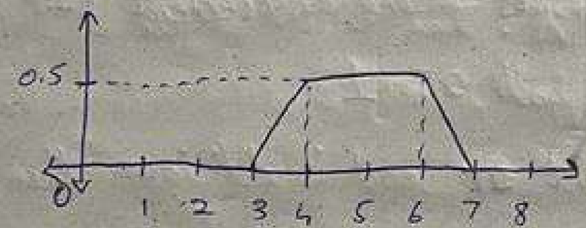
where, $\bar{z}$:- centroid of membership function

$\mu_c(\bar{z})$:- maximum membership value

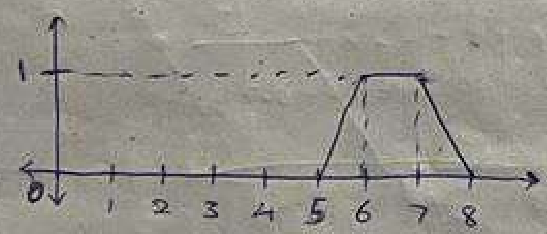
Eg:-



A_1

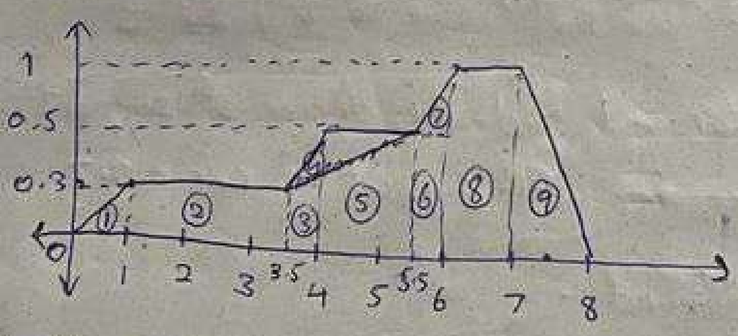


A_2



A_3

Aggregate all 3 fuzzy sets,



Weighted Avg Method,

$$x = \frac{0.3 \times 2.5 + 0.5 \times 5 + 1 \times 6.5}{0.3 + 0.5 + 1}$$

$$= \frac{0.75 + 2.5 + 6.5}{1.8}$$

$$= \frac{9.75}{1.8} = \underline{\underline{5.41}}$$

Ans. Centroid Method, $X = \dfrac{\sum A(\bar{x})}{\sum A} = \dfrac{18.353}{3.715} = \underline{\underline{4.9}}$

Mean of Maxima, $x = \dfrac{6+7}{2} = \underline{\underline{6.5}}$
(from A_3)

* Fuzzy If-Then Rules:-

- Convenient way to represent knowledge
- Highly interpretable

Eg:- If x is A then Y is B, X & Y fuzzy sets

A & B are linguistic (fuzzy) values

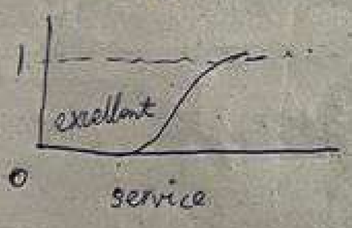
① If service is good then tip is average.

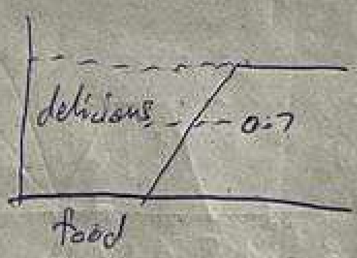
② If wt becomes tall, then wt becomes heavy

* Interpreting fuzzy values:-

① Resolve all fuzzy statements in the antecedent (If part) to degree of membership both 0 to 1

Eg:- If service is Excellent **or** food is delicious then tip is high.





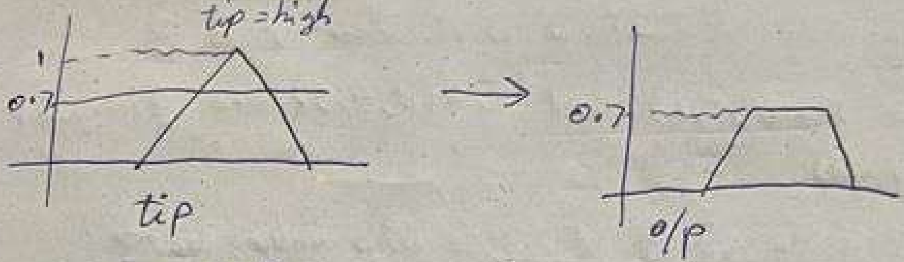
μ(excellent = 0) μ = 0.7 (or) μ(delicious = 0.7)

② Apply fuzzy operator and find the degree of support of rule ~~OR degree of~~ OR support of rule = 0.7

③ Apply Implication Method:-

Truncate the o/p fuzzy set at the level of degree of support of rule.

Eg:-



* Implication Methods :-

① Mamdani Method :- $\mu_A \wedge \mu_B$ ← min operator

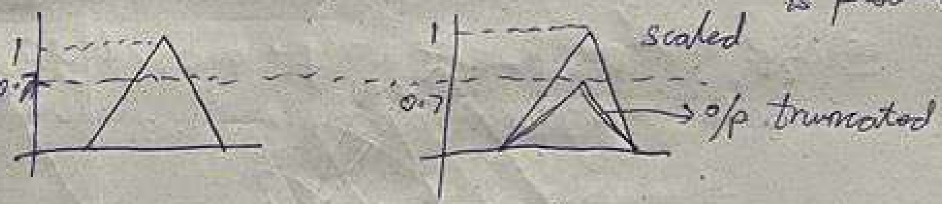
② Larsen Method :- $\mu_A \cdot \mu_B$ (algebraic product)

Eg:- Method 1 :- Mamdani

o/p shape is not preserved

Method 2 :- Larsen

o/p shape is preserved



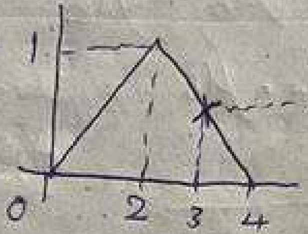
④ If required, apply defuzzification in the o/p fuzzy set

Eg:- Fuzzy Rule R: If u is A then v is B where A = (0,2,4) and B = (3,4,5) are triangular fuzzy sets

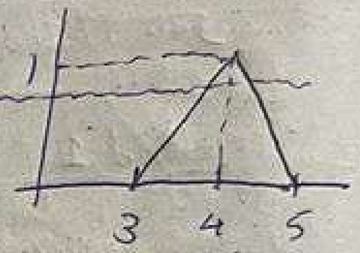
1) What is the o/p B if i/p is crisp value $u_0 = 3$

2) What is the o/p B if i/p is fuzzy set $A' = (0,1,2)$

Ans. Given:- A :-

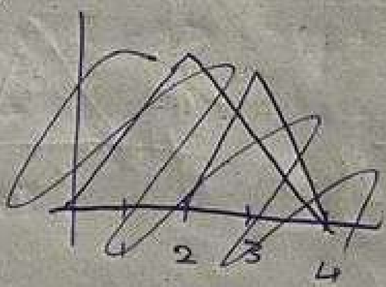


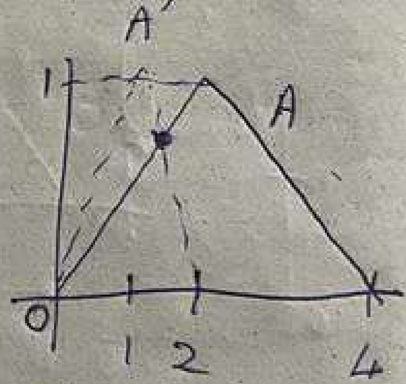
B :-



1) At $x = 3$, cut the Δ in A. That is the degree of support. Using that degree of support, cut B (Mamdani method) and that's the answer.

2) i/p is A'





The point where A' and A intersects, take that as the degree of support. Use that value to truncate B

OA: $x = 2y$ $\qquad$ A'2: $y = -x + 2$

$y = -2y + 2$ $\qquad$ $y = \frac{2}{3}$ $\qquad$ $x = \frac{4}{3}$

A3: $y = \frac{-1}{2}x + 2$

$x = 3$:- $y = 2 - 1.5 = 0.5$