Fuzzy Relations

Crisp relations

To understand the fuzzy relations, it is better to discuss first crisp relation.

Suppose, A and B are two (crisp) sets. Then Cartesian product denoted as $A \times B$ is a collection of order pairs, such that

$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$

Note :

(1) $A \times B \neq B \times A$

(2) $|A \times B|$ = $|A| \times |B|$

(3)$A \times B$ provides a mapping from $a \in A$ to $b \in B$.

The mapping so mentioned is called a relation.

Crisp relations

Example 1:

Consider the two crisp sets A and B as given below. A =$\{$ 1, 2, 3, 4$\}$ B = $\{$3, 5, 7 $\}$.

Then, $A \times B = \{(1,3),(1,5),(1,7),(2,3),(2,5),(2,7),(3,3),(3,5),(3,7),(4,3),(4,5),(4,7)\}$

Let us define a relation R as $R = \{(a,b)|b = a+1,(a,b) \in A \times B\}$

Then, $R = \{(2,3),(4,5)\}$ in this case.

We can represent the relation R in a matrix form as follows.

$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} 3 & 5 & 7 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Operations on crisp relations

Suppose, $R(x, y)$ and $S(x, y)$ are the two relations define over two crisp sets $x \in A$ and $y \in B$

Union:

$$R(x, y) \cup S(x, y) = max(R(x, y), S(x, y));$$

Intersection:

$$R(x, y) \cap S(x, y) = min(R(x, y), S(x, y));$$

Complement:

$$\overline{R(x, y)} = 1 - R(x, y)$$

Example: Operations on crisp relations

Example:
Suppose, $R(x, y)$ and $S(x, y)$ are the two relations define over two crisp sets $x \in A$ and $y \in B$

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

Find the following:

1. $R \cup S$
2. $R \cap S$
3. $\overline{R}$

Composition of two crisp relations

Given R is a relation on X, Y and S is another relation on Y,Z. Then $R \circ S$ is called a composition of relation on X and Z which is defined as follows.

$$R \circ S = \{(x,z)|(x,y) \in R \text{ and } (y,z) \in S \text{ and } \forall y \in Y\}$$

Max-Min Composition

Given the two relation matrices R and S, the max-min composition is defined as $T = R \circ S$;

$$T(x,z) = max\{min\{R(x,y), S(y,z) \text{ and } \forall y \in Y\}\}$$

Example:

Given

$X = \{1,3,5\}; Y = \{1,3,5\}; R = \{(x,y)|y = x+2\}; S = \{(x,y)|x < y\}$

Here, R and S is on $X \times Y$.

Thus, we have

$R = \{(1,3),(3,5)\}$

$S = \{(1,3),(1,5),(3,5)\}$

$$R= \begin{array}{c c} & \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \text{ and } S= \begin{array}{c c} & \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Using max-min composition $R \circ S$=

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Fuzzy relations

- Fuzzy relation is a fuzzy set defined on the Cartesian product of crisp set $X_1, X_2, ..., X_n$
- Here, n-tuples $(x_1, x_2, ..., x_n)$ may have varying degree of memberships within the relationship.
- The membership values indicate the strength of the relation between the tuples.

Example:

X = { typhoid, viral, cold } and Y = { running nose, high temp, shivering }

The fuzzy relation R is defined as

$$\begin{array}{c} \\ typhoid \\ viral \\ cold \end{array} \begin{array}{c} \begin{array}{ccc} runningnose & hightemperature & shivering \end{array} \\ \left[\begin{array}{ccc} 0.1 & 0.9 & 0.8 \\ 0.2 & 0.9 & 0.7 \\ 0.9 & 0.4 & 0.6 \end{array}\right] \end{array}$$

Fuzzy Cartesian product

Suppose
A is a fuzzy set on the universe of discourse X with $\mu_A(x)|x \in X$

B is a fuzzy set on the universe of discourse Y with $\mu_B(y)|y \in Y$

Then $R = A \times B \subset X \times Y$; where R has its membership function given by $\mu_R(x, y) = \mu_{A\times B}(x, y) = min\{\mu_A(x), \mu_B(y)\}$

Example :

$A = \{(a_1, 0.2), (a_2, 0.7), (a_3, 0.4)\}$and $B = \{(b_1, 0.5), (b_2, 0.6)\}$

$$R = A \times B = \begin{array}{c c} & \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} & \begin{bmatrix} 0.2 & 0.2 \\ 0.5 & 0.6 \\ 0.4 & 0.4 \end{bmatrix} \end{array}$$

Operations on Fuzzy relations

Let R and S be two fuzzy relations on $A \times B$.

Union:

$$\mu_{R \cup S}(a, b) = max\{\mu_R(a, b), \mu_S(a, b)\}$$

Intersection:

$$\mu_{R \cap S}(a, b) = min\{\mu_R(a, b), \mu_S(a, b)\}$$

Complement:

$$\mu_{\overline{R}}(a, b) = 1 - \mu_R(a, b)$$

Composition

$$T = R \circ S$$
$$\mu_{R \circ S} = max_{y \in Y}\{min(\mu_R(x, y), \mu_S(y, z))\}$$

Operations on Fuzzy relations: Examples

Example:

$X = (x_1, x_2, x_3)$; $Y = (y_1, y_2)$; $Z = (z_1, z_2, z_3)$;

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.9 \\ 0.8 & 0.6 \end{bmatrix} \end{array}$$

$$S = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.6 & 0.4 & 0.7 \\ 0.5 & 0.8 & 0.9 \end{bmatrix} \end{array}$$

$$R \circ S = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.4 & 0.5 \\ 0.5 & 0.8 & 0.9 \\ 0.6 & 0.6 & 0.7 \end{bmatrix} \end{array}$$

$\mu_{R \circ S}(x_1, y_1) = max\{min(x_1, y_1), min(y_1, z_1), min(x_1, y_2), min(y_2, z_1)\}$
$= max\{min(0.5, 0.6), min(0.1, 0.5)\} = max\{0.5, 0.1\} = 0.5$ and so on.