# Fuzzy Set Theory

**UNIT-2**

# Introduction

- The word “fuzzy” means “vaguness (ambiguity)”.
- Fuzziness occurs when the boundary of a piece of information is not clear-cut.
- Fuzzy sets - 1965 Lotfi Zadeh as an extension of classical notation set.
- Classical set theory allows the membership of the elements in the set in **binary terms**.
- Fuzzy set theory permits membership function valued in the interval [0,1].

# Introduction

**Example:**

Words like young, tall, good or high are fuzzy.

- There is no single quantitative value which defines the term young.
- For some people, age 25 is young, and for others, age 35 is young.
- The concept young has no clean boundary.
- Age 35 has some possibility of being young and usually depends on the context in which it is being considered.

Fuzzy set theory is an extension of classical set theory where elements have degree of membership.

# Introduction

- In real world, there exist much fuzzy knowledge (i.e. vague, uncertain inexact etc).
- Human **thinking** and **reasoning** (analysis, logic, interpretation) frequently involved **fuzzy** information.
- Human can give satisfactory answers, which are probably true.
- Our systems are unable to answer many question because the systems are designed based upon classical set theory (Unreliable and incomplete).
- We want, our system should be able to cope with unreliable and incomplete information.
- Fuzzy system have been provide solution.

# Introduction

## Classical set theory

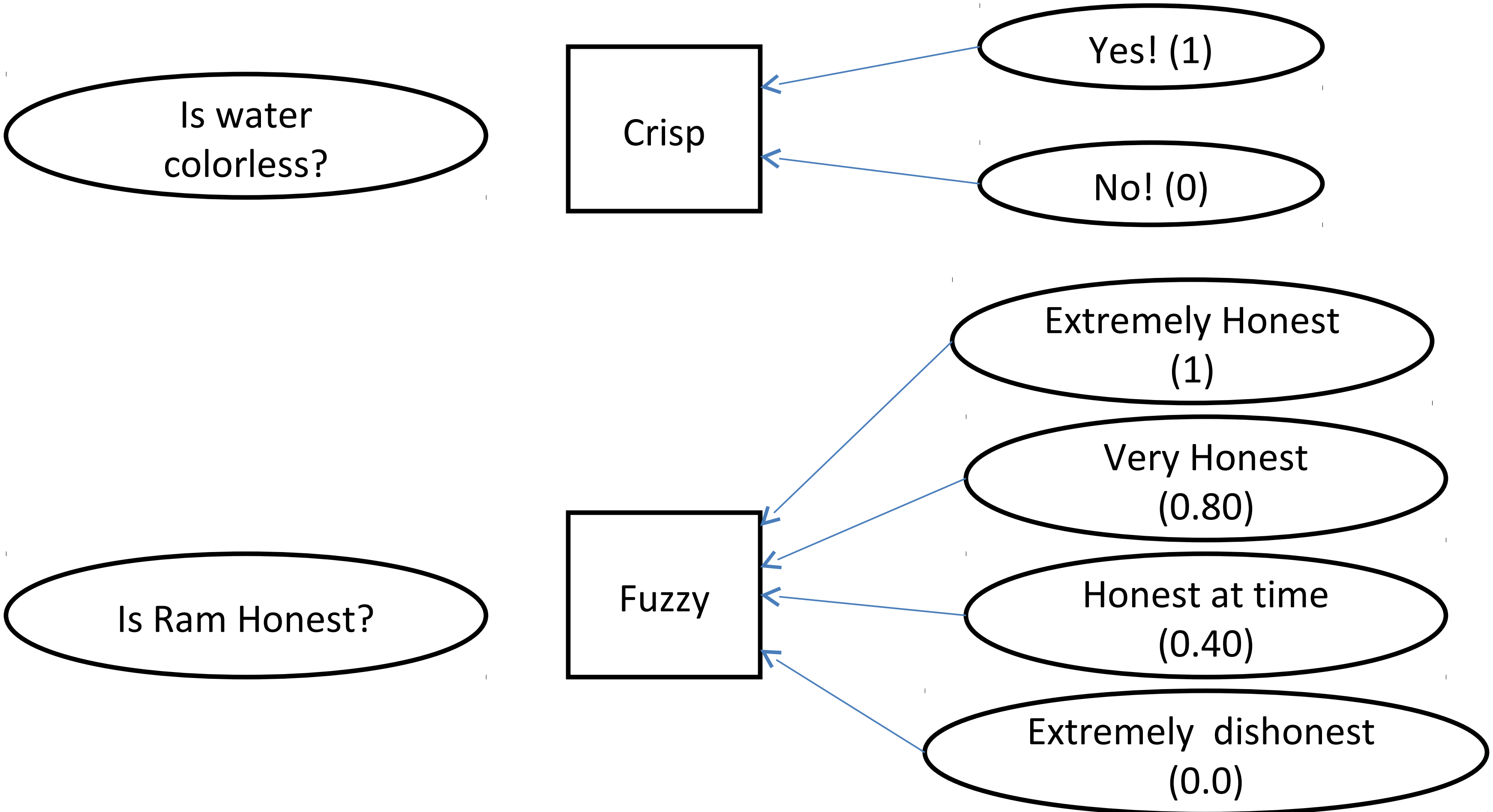
- Classes of objects with sharp boundaries.
- A classical set is defined by crisp(exact) boundaries, i.e., there is no uncertainty about the location of the set boundaries.
- Widely used in digital system design

## Fuzzy set theory

- Classes of objects with un-sharp boundaries.
- A fuzzy set is defined by its ambiguous boundaries, i.e., there exists uncertainty about the location of the set boundaries.
- Used in fuzzy controllers.

# Introduction (Continue)

Example




Fuzzy vs crips

# Classical set theory

- A Set is any well defined collection of objects.
- An object in a set is called an element or member of that set.
- Sets are defined by a simple statement,
- Describing whether a particular element having a certain property belongs to that particular set.

$$A = \{a_1, a_2, a_3, \ldots\ldots, a_n\}$$

- If the elements $a_i$ ($i = 1,2,3,\ldots.,n$) of a set A are subset of universal set X, then set A can be represented for all elements $x \in X$ by its characteristics function

$$\mu_A(x) = 1 \text{ if } x \in X \text{ otherwise } 0$$

# Operations on classical set theory

***Union:*** the union of two sets A and B is given as

$$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$$

***Intersection:*** the intersection of two sets A and B is given as

$$A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$$

***Complement:*** It is denoted by Ã and is defined as

$$\tilde{A} = \{ x \mid x \text{ does not belongs } A \text{ and } x \in X \}$$

# Fuzzy Sets

- Fuzzy sets theory is an extension of classical set theory.
- Elements have varying degree of membership. A logic based on two truth values,
- ***True*** and ***False*** is sometimes insufficient when describing human reasoning.
- Fuzzy Logic uses the whole interval between 0 (false) and 1 (true) to describe human reasoning.
- A Fuzzy Set is any set that allows its members to have different degree of membership, called **membership function**, having interval [0,1].

# Fuzzy Sets

- **Fuzzy Logic** is derived from fuzzy set theory
- Many degree of membership (between 0 to 1) are allowed.
- Thus a membership function $\boldsymbol{\mu}_A^{(x)}$ is associated with a fuzzy sets Ã such that the function maps every element of universe of discourse X to the interval [0,1].
- The mapping is written as: $\mu_{\tilde{A}}(x)$: X → [0,1].

- Fuzzy Logic is capable of handing inherently imprecise (vague or inexact or rough or inaccurate) concepts

# Fuzzy Sets

- **Fuzzy set** is defined as follows:

- If X is an universe of discourse and x is a particular element of X, then a fuzzy set A defined on X and can be written as a collection of ordered pairs

$$A = \{(x, \mu_{\tilde{A}}(x)), x \in X \}$$

# Fuzzy Sets (Continue)

Example

- Let X = {$g_1$, $g_2$, $g_3$, $g_4$, $g_5$} be the reference set of students.
- Let Ã be the fuzzy set of "smart" students, where "smart" is fuzzy term.

$$\tilde{A} = \{(g_1,0.4)(g_2,0.5)(g_3,1)(g_4,0.9)(g_5,0.8)\}$$

Here Ã indicates that the smartness of $g_1$ is 0.4 and so on

# Fuzzy Sets (Continue)

## Membership Function

- The membership function fully defines the fuzzy set
- A membership function provides a measure of *the degree of similarity* of an element to a fuzzy set

## Membership functions can

- either be chosen by the user arbitrarily, based on the user's experience (MF chosen by two users could be different depending upon their experiences, perspectives, etc.)
- Or be designed using machine learning methods (e.g., artificial neural networks, genetic algorithms, etc.)

# Fuzzy Sets (Continue)

There are different shapes of membership functions;

- **Triangular,**
- **Trapezoidal,**
- **Gaussian, etc**

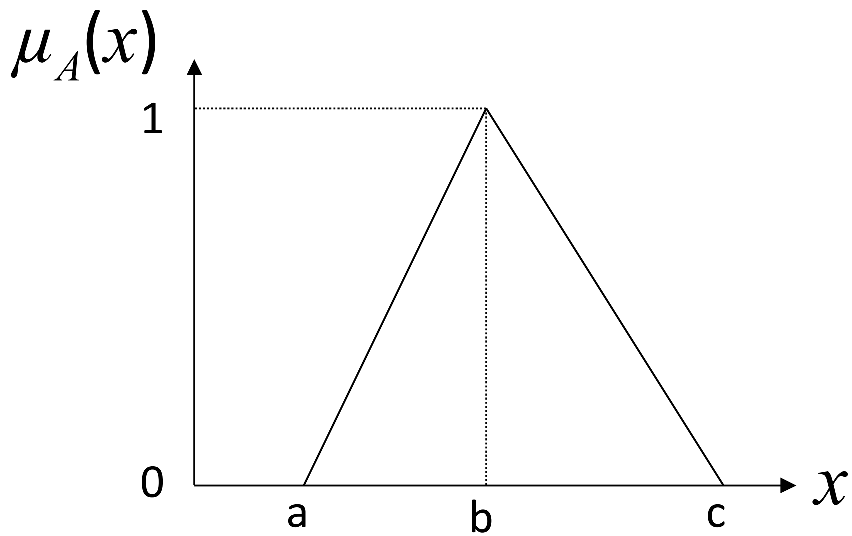
# Fuzzy Sets (Continue)

- **Triangular membership function**

  A *triangular* membership function is specified by three parameters {a, b, c}

  a, b and c represent the *x* coordinates of the three vertices of $\mu_A(x)$ in a fuzzy set A (a: lower boundary and c: upper boundary where membership degree is zero, b: the centre where membership degree is 1)

$$\mu_A(x) = \begin{cases} 0 & if \ x \le a \\ \dfrac{x-a}{b-a} & if \ a \le x \le b \\ \dfrac{c-x}{c-b} & if \ b \le x \le c \\ 0 & if \ x \ge c \end{cases}$$



# Fuzzy Sets (Continue)

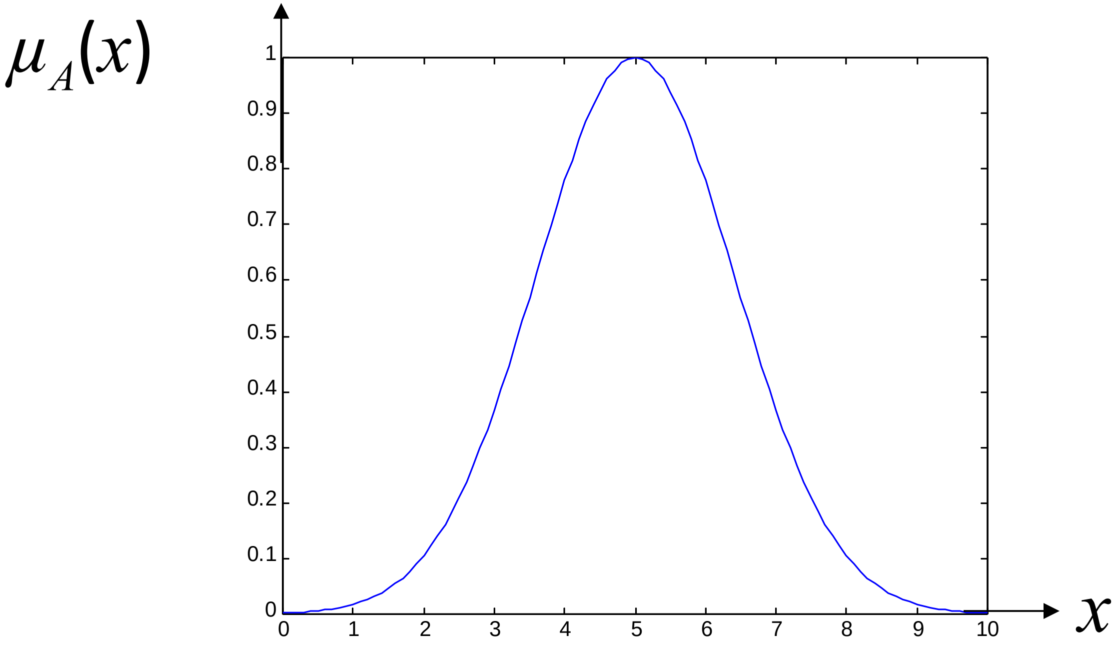
- **Trapezoid membership function**
- A *trapezoidal* membership function is specified by four parameters {a, b, c, d} as follows:

$$\mu_A(x) = \begin{cases} 0 & if\ x \le a \\ \dfrac{x-a}{b-a} & if\ a \le x \le b \\ 1 & if\ b \le x \le c \\ \dfrac{d-x}{d-c} & if\ c \le x \le d \\ 0 & if\ d \le x \end{cases}$$

- **Gaussian membership function**

$$\mu_A(x,c,s,m) = \exp\left[-\frac{1}{2}\left|\frac{x-c}{s}\right|^m\right]$$

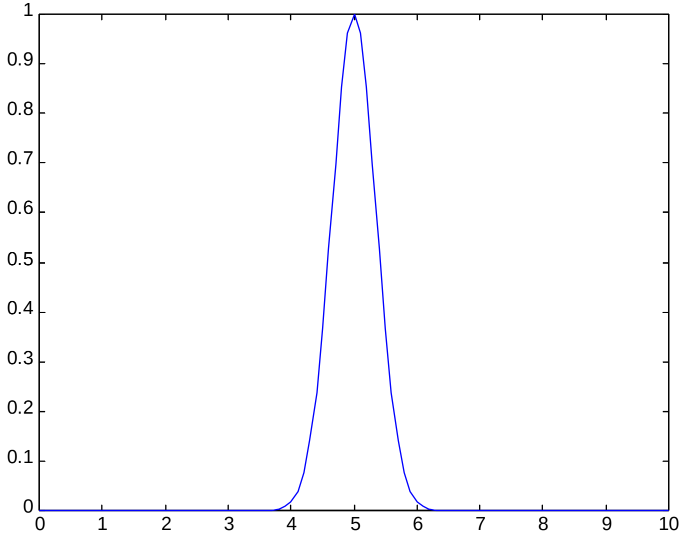
- *c:* centre
- *s:* width
- *m:* fuzzification factor (e.g., *m*=2)




$c=5$

$s=2$

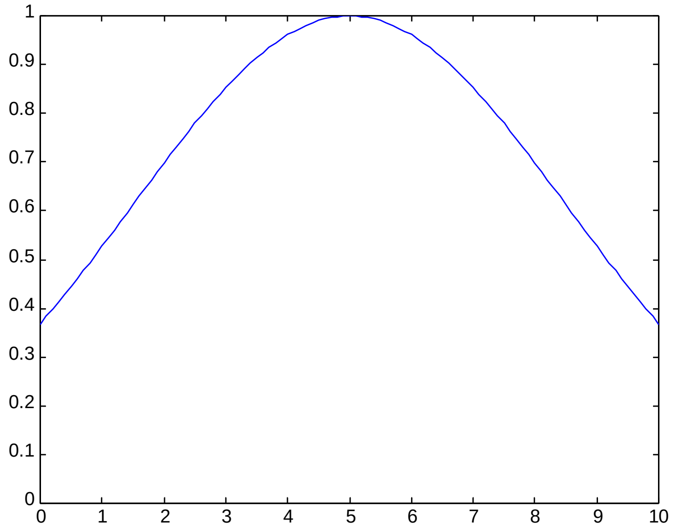
$m=2$




$c=5$

$s=0.5$

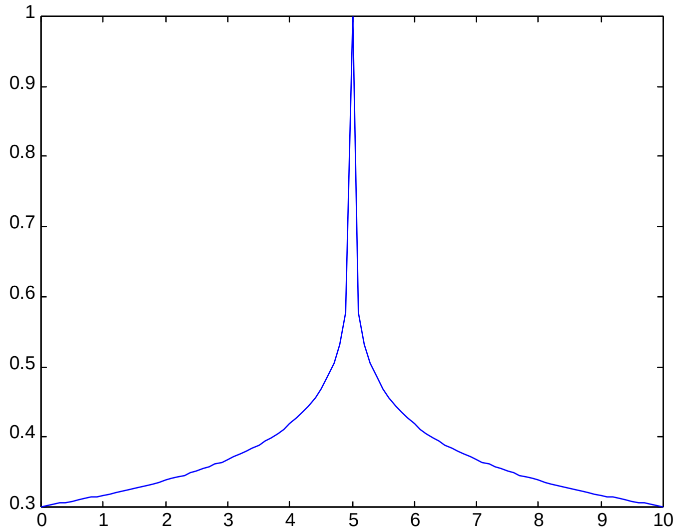
$m=2$




$c=5$

$s=5$

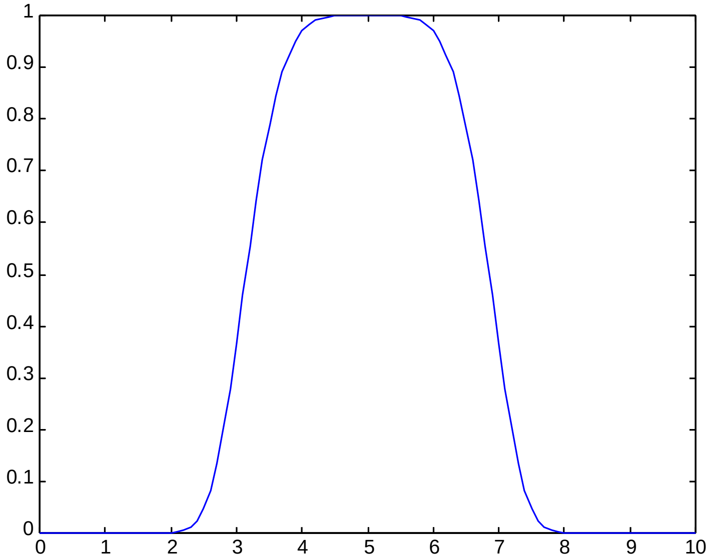
$m=2$




$c=5$

$s=2$

$m=0.2$




$c=5$

$s=5$

$m=5$

# Fuzzy Set Operation

Given X to be the universe of discourse and Ã and Ḃ to be fuzzy sets with $\mu_A(x)$ and $\mu_B(x)$ are their respective membership function, the fuzzy set operations are as follows:

***Union:***

$$\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$$

***Intersection:***

$$\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$$

***Complement:***

$$\mu_A(x) = 1 - \mu_A(x)$$

# Fuzzy Set Operation (Continue)

Example:

$A = \{(x_1,0.5),(x_2,0.7),(x_3,0)\}$  $B = \{(x_1,0.8),(x_2,0.2),(x_3,1)\}$

***Union:***

$$A \cup B = \{(x_1,0.8),(x_2,0.7),(x_3,1)\}$$

Because

$$\mu_{A \cup B}(x_1) = \max(\mu_A(x_1), \mu_B(x_1))$$

$$= \max(0.5,0.8)$$

$$= 0.8$$

$$\mu_{A \cup B}(x_2) = 0.7 \text{ and } \mu_{A \cup B}(x_3) = 1$$

# Fuzzy Set Operation (Continue)

Example:

$A = \{(x_1,0.5),(x_2,0.7),(x_3,0)\}$  $B = \{(x_1,0.8),(x_2,0.2),(x_3,1)\}$

***<u>Intersection:</u>***

$$A \cap B = \{(x_1,0.5),(x_2,0.2),(x_3,0)\}$$

Because

$$\mu_{A \cap B}(x_1) = \min(\mu_A(x_1), \mu_B(x_1))$$

$$= \max(0.5,0.8)$$

$$= 0.5$$

$$\mu_{A \cap B}(x_2) = 0.2 \text{ and } \mu_{A \cap B}(x_3) = 0$$

# Fuzzy Set Operation (Continue)

Example:

$A = \{(x_1,0.5),(x_2,0.7),(x_3,0)\}$

***<u>Complement:</u>***

$$A^c = \{(x_1,0.5),(x_2,0.3),(x_3,1)\}$$

Because

$$\mu_A(x_1) = 1 - \mu_A(x_1)$$

$$= 1 - 0.5$$

$$= 0.5$$

$$\mu_A(x_2) = 0.3 \text{ and } \mu_A(x_3) = 1$$

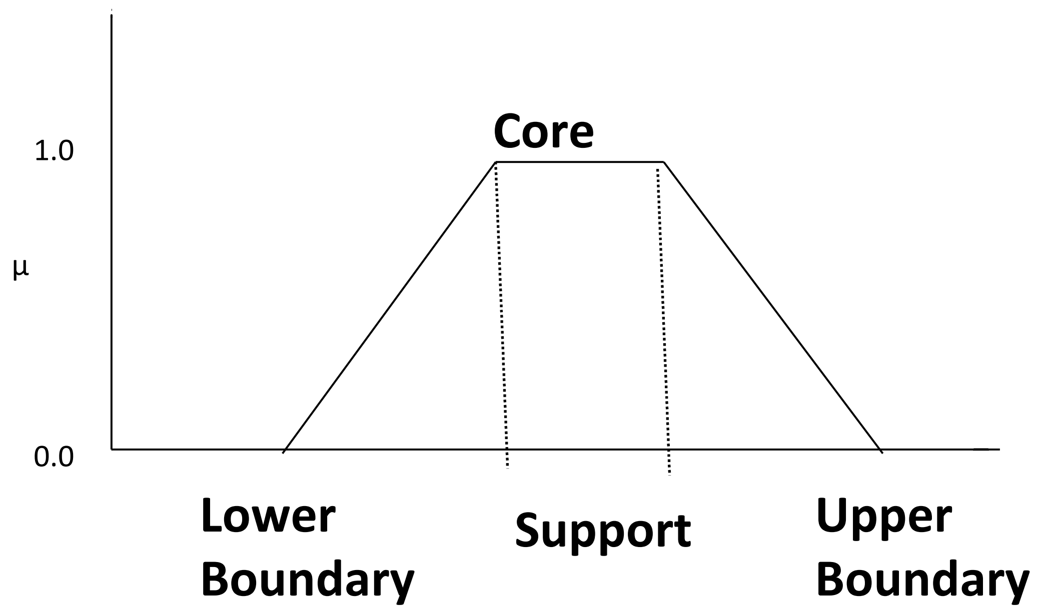
- **Support(A)** is set of all points x in X such that

$$\{(x|\ \mu_A(x) > 0\ \}$$

- **core(A)** is set of all points x in X such that

$$\{(x|\ \mu_A(x) = 1\ \}$$

- Fuzzy set whose support is a single point in X with $\mu_A(x) = 1$ is called fuzzy singleton


Core
1.0
μ
0.0
Lower Boundary
Support
Upper Boundary

# Linguistic variable, linguistic term

- **Linguistic variable: A** *linguistic variable* is a variable whose values are sentences in a natural or artificial language.
- **For example**, the values of the fuzzy variable *height* could be *tall, very tall, very very tall, somewhat tall, not very tall, tall but not very tall, quite tall, more or less tall.*
- *Tall is a linguistic* value or primary term

- If ***age*** is a linguistic variable then its term set is
- *T(age) = { young, not young, very young, not very young,…… middle aged, not middle aged, … old, not old, very old, more or less old, not very old,…not very young and not very old,…}.*

# Fuzzy Rules

- Fuzzy rules are useful for modeling human thinking, perception (Opinion,view) and judgment.
- A fuzzy if-then rule is of the form "If x is A then y is B" where A and B are linguistic values defined by fuzzy sets on universes of discourse *X* and *Y*, respectively.
- "x is A" is called *antecedent* and "y is B" is called *consequent*.

# Examples, for such a rule are

- If pressure is high, then volume is small.
- If the road is slippery, then driving is dangerous.
- If the fruit is ripe, then it is soft.

# Binary fuzzy relation

- A binary fuzzy relation is a fuzzy set in $X \times Y$ which maps each element in $X \times Y$ to a membership value between *0* and *1*.
- If $X$ and $Y$ are two universes of discourse, then
- $R = \{((x,y), \mu_R(x, y)) \mid (x,y) \in X \times Y\}$ is a binary fuzzy relation in $X \times Y$.
- $X \times Y$ indicates cartesian product of X and Y

- The fuzzy rule *"If x is A then y is B"* may be abbreviated as $A \rightarrow B$ and is interpreted as $A \times B$.

- A fuzzy if then rule may be defined (Mamdani) as a binary fuzzy relation $R$ on the product space $X \times Y$.

- $R = A \rightarrow B = A \times B = \int_{X \times Y} \mu_A(x)\ T\text{-}norm\ \mu_B(y) / (x,y)$.

# expert systems: Fuzzy inference

**Mamdani fuzzy inference**

**Sugeno fuzzy inference**

# Fuzzy inference

- The most commonly used fuzzy inference technique is the so-called Mamdani method. In 1975,
- Professor **Ebrahim Mamdani** of London University built one of the first fuzzy systems
- To control a steam engine and boiler combination.
- He applied a set of fuzzy rules supplied by experienced human operators..

# Fuzzy inference

**Mamdani fuzzy inference**

- The Mamdani-style fuzzy inference process is performed in four steps:
- Fuzzification of the input variables,
- Rule evaluation;
- Aggregation of the rule outputs, and finally
- Defuzzification.

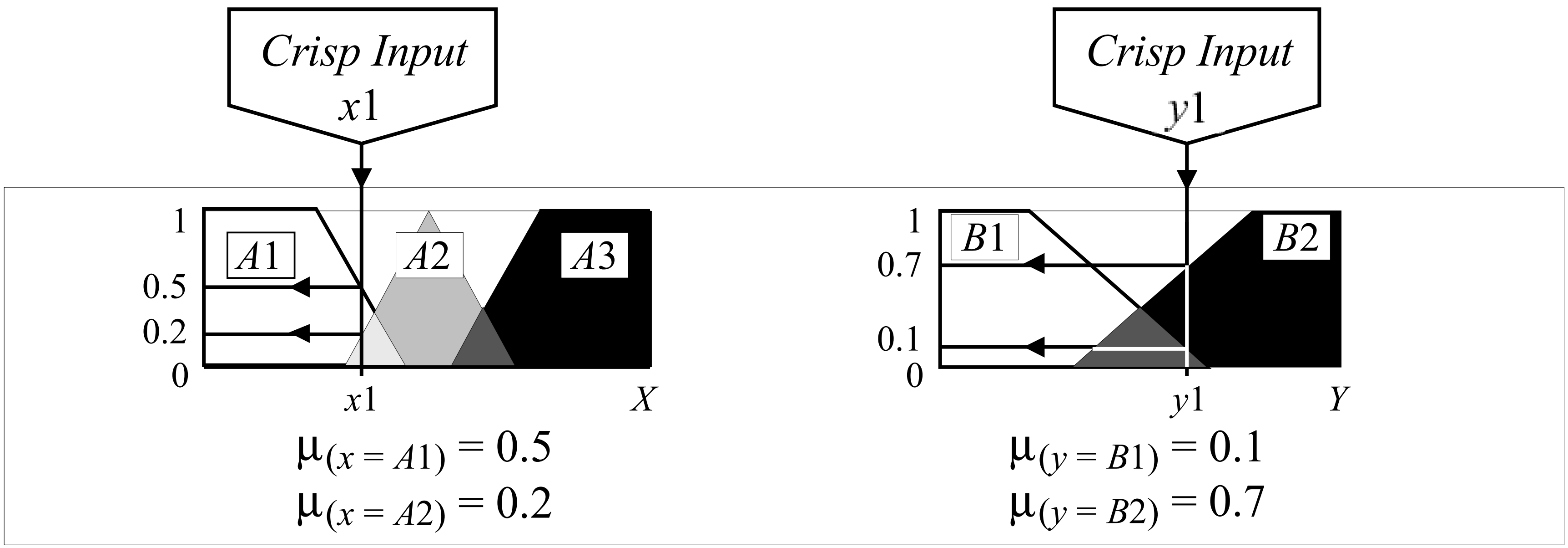
# Fuzzy inference

We examine a simple two-input one-output problem that includes three rules:

| | |
|---|---|
| Rule 1: | Rule 1: |
| IF x is A3 | IF *project_funding is enough* |
| OR y is B1 | OR *project_staffing is small* |
| THEN z is C1 | THEN *risk is low* |
| Rule 2: | Rule 2: |
| IF x is A2 | IF *project_funding is medium* |
| OR y is B2 | OR *project_staffing is large* |
| THEN z is C2 | THEN *risk is normal* |
| Rule 3: | Rule 3: |
| IF x is A1 | IF *project_funding is notenough* |
| THEN z is C3 | THEN *risk is high* |

# Step 1: Fuzzification

The first step is to take the crisp inputs, x1 and y1 (project funding and project staffing), and determine the degree to which these inputs belong to each of the appropriate fuzzy sets.



# Step 2: Rule Evaluation

The second step is to take the fuzzified inputs, $\mu_{(x=A1)} = 0.5$,

$\mu_{(x=A2)} = 0.2$,

$\mu_{(y=B1)} = 0.1$ and $\mu_{(y=B\,2)} = 0.7$,

and apply them to the antecedents of the fuzzy rules.

If a given fuzzy rule has multiple antecedents, the fuzzy operator (AND or OR) is used to obtain a single number that represents the result of the antecedent evaluation. This number (the truth value) is then applied to the consequent membership function.

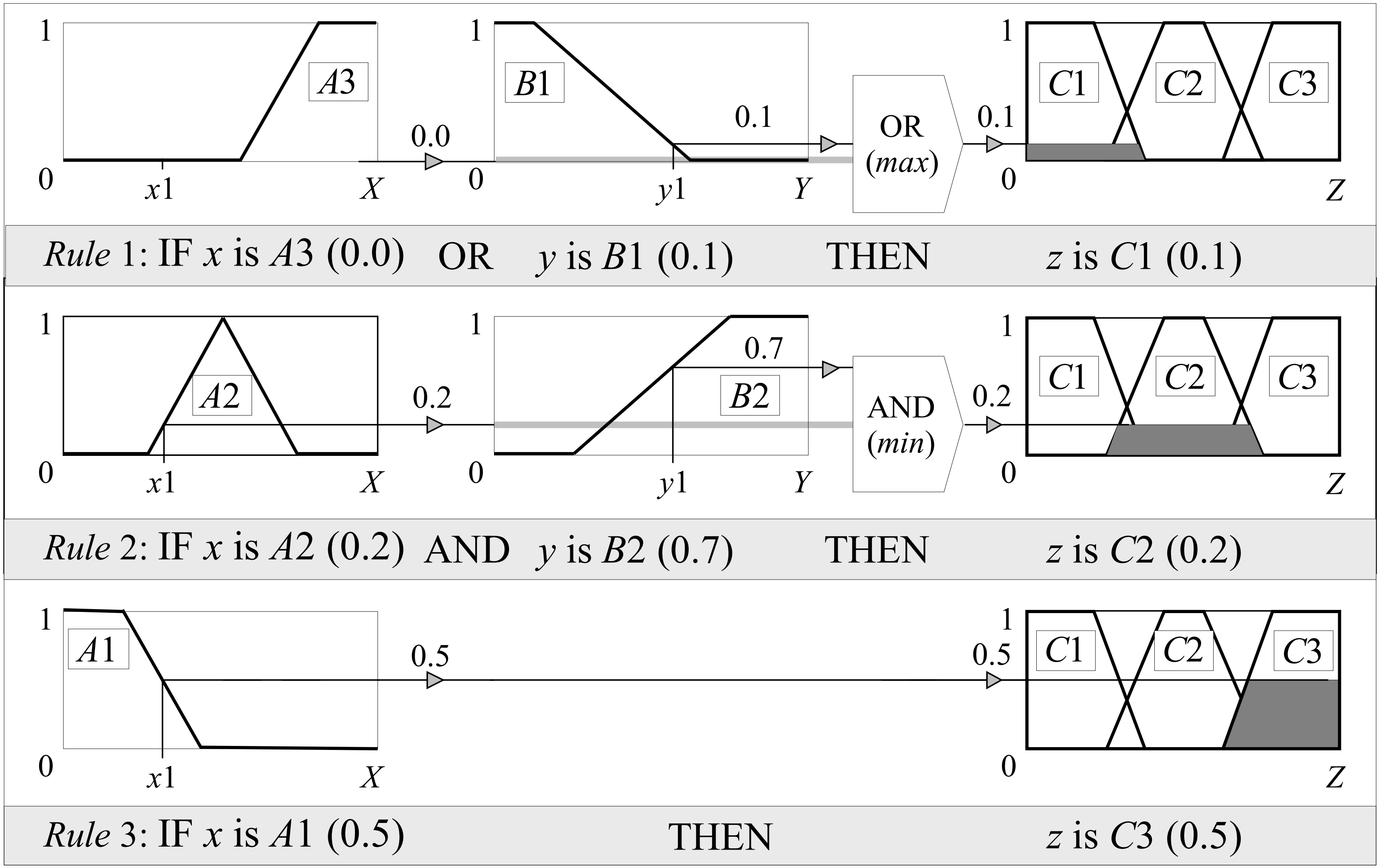
To evaluate the disjunction of the rule antecedents, we use the **OR fuzzy operation**. Typically, fuzzy expert systems make use of the classical fuzzy operation **union**:

$$\mu_{A \cup B}(x) = \max\left(\mu_A(x), \mu_B(x)\right)$$

Similarly, in order to evaluate the conjunction of the rule antecedents, we apply the **AND fuzzy operation intersection**:

$$\mu_{A \cap B}(x) = \min\left(\mu_A(x), \mu_B(x)\right)$$

# Mamdani-style rule evaluation



# Sugeno fuzzy inference

- **Michio Sugeno** suggested to use a single spike, a singleton, as the membership function of the rule

- A **singleton**,, or more precisely a **fuzzy singleton**, is a fuzzy set with a membership function that is unity at a single particular point on the universe of discourse and zero everywhere else.

- Fuzzy set whose support is a single point in X with:

$\boldsymbol{\mu_A(x) = 1}$ is called **fuzzy singleton**

- Sugeno-style fuzzy inference is very similar to the Mamdani method.
- Sugeno changed only a rule consequent (resultant).
- Instead of a fuzzy set, he used a mathematical function of the input variable. The format of the **Sugeno-style fuzzy rule** is

IF $x$ is $A$
AND $y$ is $B$
THEN $z$ is $f(x, y)$

where *x, y* and *z* are linguistic variables; *A* and *B* are fuzzy sets on universe of discourses *X* and *Y*, respectively; and *f (x, y)* is a mathematical function.

The most commonly used **zero-order Sugeno fuzzy model** applies fuzzy rules in the following form:

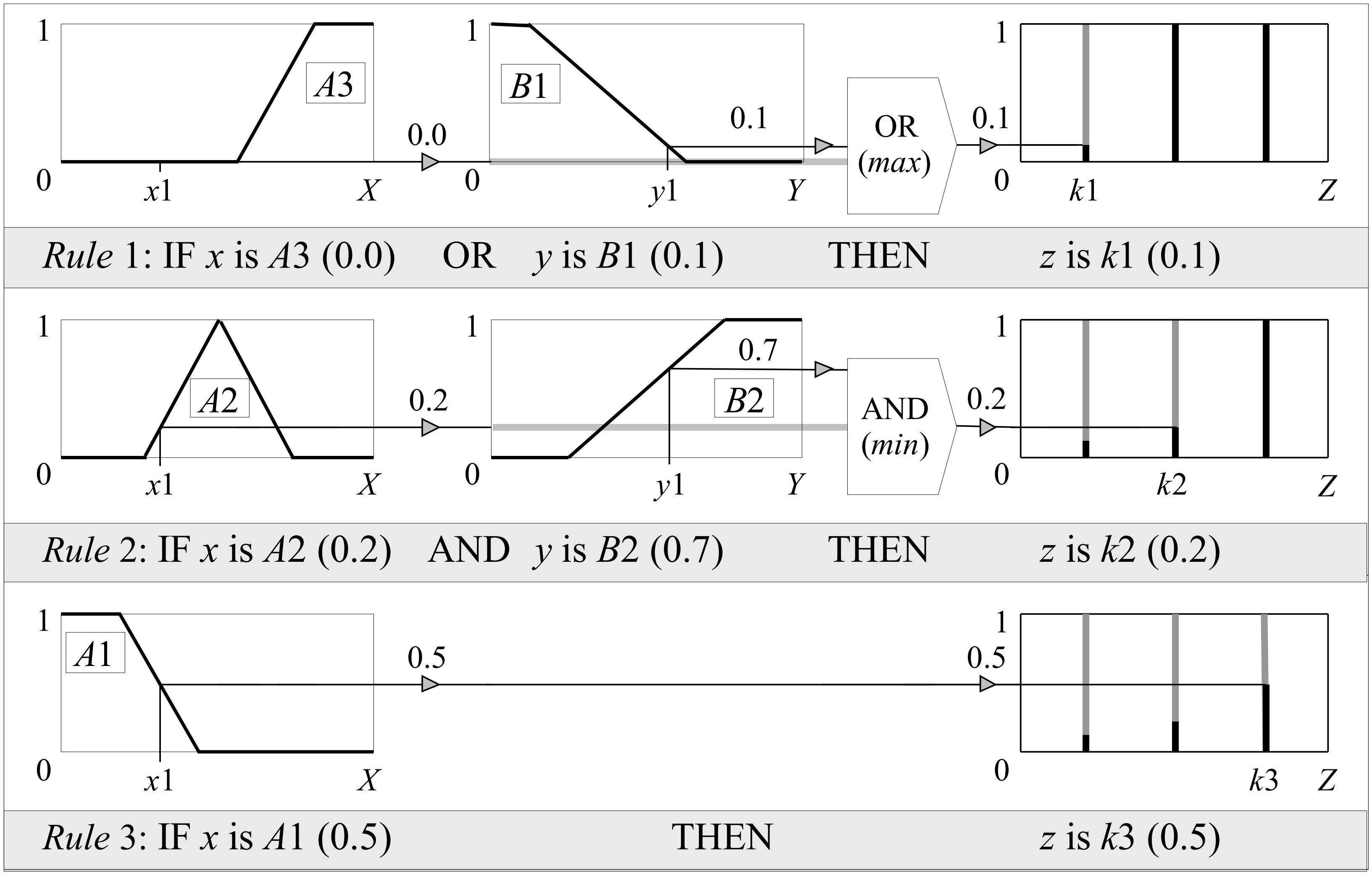
**IF x is A**
**AND y is B**
**THEN z is k**

where k is a constant.

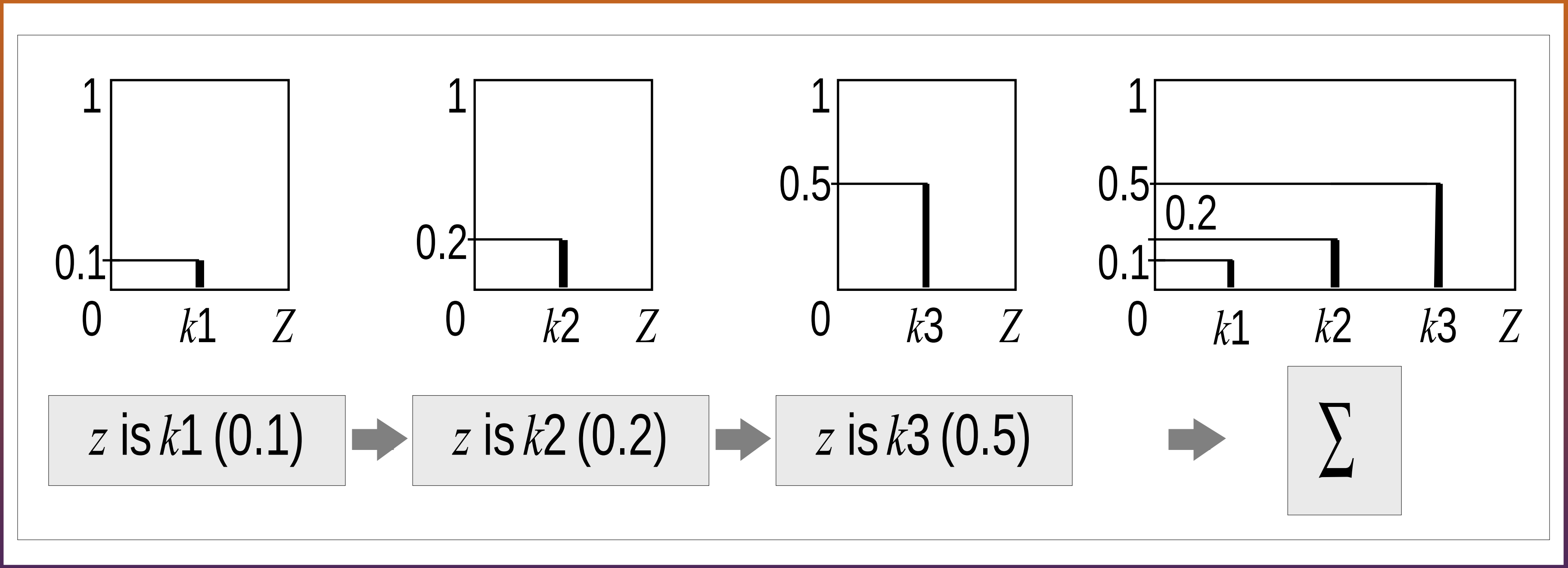
In this case, the output of each fuzzy rule is constant.

All resultant membership functions are represented by singleton spikes.

# Sugeno-style rule evaluation



# Sugeno-style aggregation of the rule outputs



# Weighted average (WA):

$$WA = \frac{\mu(k1)\times k1 + \mu(k2)\times k2 + \mu(k3)\times k3}{\mu(k1)+\mu(k2)+\mu(k3)} = \frac{0.1\times 20 + 0.2\times 50 + 0.5\times 80}{0.1+0.2+0.5} = 65$$

# Sugeno-style defuzzification