1.1 Q.1

→ $P(A)=0.6$, $P(B)=0.5$, $P((A\cup B)^c)=0.3$

a. → $P(A\cup B)+P((A\cup B)^c)=1$

$\therefore P(A\cup B)=1-P((A\cup B)^c)$

$\downarrow$ either today or tomorrow

$=1-0.3=$

$=\underline{\underline{0.7}}$

b. → $P(A\cap B)=P(A)+P(B)-P(A\cup B)$

$\downarrow$ and

$=0.6+0.5-0.7$

$=\underline{\underline{0.4}}$

c. → $P(\text{only } A)=P(A)-P(A\cap B)$

$\downarrow$ only today

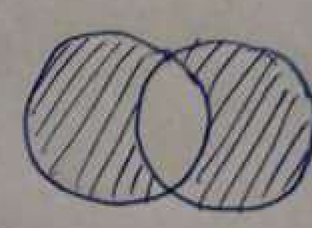
$=0.6-0.4$

$=0.2$

d. → $P(\text{m.e. } A\ B)=P(A\cup B)-P(A\cap B)$

$=0.7-0.4$

$=\underline{\underline{0.3}}$



1.2 Q.2

$\longrightarrow$ $S = \{(x_1, x_2) \mid x_1 = 1, 2, \ldots 6;\ x_2 = 1, 2, \ldots 6\}$ .4

$P(A) = P(x_1 + x_2 = 8) = \boxed{\frac{5}{36}}$ $\qquad n(S) = 36$ 1.

$(2,6), (3,5), (4,4), (5,3), (6,2)$

1.3 Q3

$\longrightarrow$ $P(A), P(B)$

A = all n children are girls

B = random child picked and it was a girl

Find $P(A \mid B)$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B \mid A)\, P(A)}{P(B)}$$

$$P(A) = \frac{1}{2} \times \frac{1}{2} \times \ldots n \text{ times} = \left(\frac{1}{2}\right)^n$$

$P(B \mid A) = 1$, obvious; if all children are girls then then any random child is a girl.

$$P(B) = \frac{1}{2}$$

$$\therefore P(A \mid B) = \frac{P(B \mid A)\, P(A)}{P(B)} = \frac{1 \times \left(\frac{1}{2}\right)^n}{\frac{1}{2}} = \frac{1}{2^n} \times 2 = \frac{1}{2^{n-1}}$$

$$\therefore \boxed{P(A \mid B) = \frac{1}{2^{n-1}}}$$

1. $\rightarrow$ $F_X(x) = P(X \leq x)$

$$\cancel{= P(X \leq x)}$$

$$= \underbrace{P(X \leq x \mid H)}_{F_d(x)} P(H) + \underbrace{P(X \leq x \mid T)}_{F_c(x)} P(T)$$

$$F_X(x) = pF_d(x) + (1-p) F_c(x)$$

2. $\rightarrow$ $f_X(x) = \frac{d}{dx} F_X(x)$

$$= \frac{d}{dx} [pF_d(x) + (1-p) F_c(x)]$$

$$f_X(x) = pf_d(x) + (1-p) f_c(x)$$

3. $\rightarrow$ $E[X] = \sum x P(x) = \int_{-\infty}^{\infty} x f_X(x) dx$

$$= \int_{-\infty}^{\infty} x [pf_d(x) + (1-p) f_c(x)] dx$$

$$= p \int_{-\infty}^{\infty} x f_d(x) dx + (1-p) \int_{-\infty}^{\infty} x f_c(x) dx$$

$$= pE[X_d] + (1-p) E[X_c]$$

4. $\rightarrow$ $var(X) = E[X^2] - E^2[X]$

$$E[X^2] = pE[X_d^2] + (1-p) E[X_c^2]$$

$$= p(var(X_d) + E^2[X_d]) + (1-p)(var(X_c) + E^2[X_c])$$

$$= pvar(X_d) + (1-p) var(X_c) + p(1-p)(E[X_d] - E[X_c])^2$$

1.5 Q5

$\longrightarrow Z = 1 + X + XY^2, \quad W = 1 + X$

$X, Y \sim N(0,1)$

For random variable with normal dist.

$X \sim N(\mu, \sigma^2)$

$\therefore \mu = 0, \ \sigma^2 = 1$

$\Rightarrow E[X] = E[Y] = 0$

$\Rightarrow \text{var}(X) = 1 = \text{var}(Y)$

$\text{cov}(Z, W) = \text{cov}(1 + X + XY^2, 1 + X)$

$= \text{cov}(X + XY^2, X) \quad \ldots \text{cov}(a + X) = \text{cov}(X)$

$= \underbrace{\text{cov}(X, X)}_{\text{var}(X)} + \underbrace{\text{cov}(XY^2, X)}$

use $\text{cov}(A,B) = E[AB] - E[A]E[B]$

$\hookrightarrow \text{cov}(XY^2, X) = E[X^2Y^2] - E[XY^2]E[X]$

$\text{var}(X) = E[X^2] - E^2[X]$

$\therefore 1 = E[X^2] - 0$

$\therefore E[X^2] = E[Y^2] = 1$

$E[X^2Y^2] = E[X^2]E[Y^2]$ as X, Y are independent

$\therefore \text{cov}(XY^2, X) = E[X^2]E[Y^2] - E[XY^2]E[X]$

$= 1 \times 1 - 0$

$= 1$

$\therefore$ ~~COZ~~ $\text{cov}(Z, W) = \text{cov}(X, X) + \text{cov}(XY^2, X)$

$= 1 + 1$

$\boxed{\text{cov}(Z, W) = 2}$

.6 Q6

→ 2

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1.6 Q6

→ 20% chance from each company. → event A

90% chance from at least one company.

$P(A) = 0.2$, $P(A^c) = 1 - P(A) = 0.8$

The offer from each company is independent i.e they can't be directly added

$$P(\text{no offer received}) = 0.8 \times 0.8 \times 0.8 \times 0.8$$
$$= (0.8)^4$$
$$= 0.4096$$

∴ 40.96% chance that he gets zero offer from all companies.

$$P(\text{Atleast one offer}) = 1 - P(\text{No offer})$$
$$= 1 - 0.4096$$
$$= 0.5904$$

59% → probability of atleast one offer.

He claims 90%, which is far from the expected value.

∴ His claim is wrong.

1.7 Q7

$\longrightarrow$ Each bit can be with $\underbrace{\text{an error or no error}}_{\text{2 possibilities.}}$

There are thousand bits no no. of trials is fixed.

$P(\text{error}) = 0.1 \longrightarrow$ failure

$P(\text{no error}) = 0.9 \longrightarrow$ success

Let x R.V. denote number of errors. We are asked $P(x > 120)$

$$P(x>120) = \underbrace{P(x=121) + P(x=122) + \dots + P(x=1000)}_{\text{880 terms}}$$

not possible to add all

$$P(x=k) = {}^{1000}C_k \, (0.9)^k (0.1)^{1000-k}$$

mean, variance $= np, npq$

$np = 1000 \times 0.1 = 100$

$npq = 1000 \times 0.1 \times 0.9 = 100 \times 0.9 = 90$

$\sigma^2 = 90 \quad \therefore \text{S.D.} = \sigma = \sqrt{90} = 3\sqrt{10} \approx \cancel{3}\ 9.49$

Since here n is large, we have to add 880 terms, we will normal distribution as the distribution is continuous.

$$Z = \frac{x-\mu}{\sigma} = \frac{121-100}{9.49} = 2.21$$

$\therefore P(Z<2.21) = 0.9864$

$\therefore P(Z \geqslant 2.21) = 1 - 0.9864$

$= 0.0136$

$\therefore P(x>120) = 0.0136$

$\therefore$ Probability $= \boxed{1.36\%}$

1.8

$\longrightarrow$

→ Let x denote no. of sandwich

$P(x=0)=\frac{1}{4}$, $P(x=1)=\frac{1}{2}$, $P(x=2)=\frac{1}{4}$

$$E[x] = x P(X=x) = 0\times\frac{1}{4} + 1\times\frac{1}{2} + 2\times\frac{1}{4}$$

$$= \frac{1}{2}+\frac{1}{2} = 1$$

$$\therefore E[x]=1$$

$$E[x^2] = x^2 P(x) = 0^2\times\frac{1}{4} + 1^2\times\frac{1}{2} + 2^2\times\frac{1}{4}$$

$$= \frac{1}{2} + 1 = 1.5 = \frac{3}{2}$$

$$\text{var}(x) = E[x^2] - E^2[x]$$

$$= \frac{3}{2} - 1 = \frac{1}{2} = 0.5$$

This mean and variance are for a single guest.

$\therefore \mu_T = 64\times 1 = 64$, $\sigma_T^2 = 64\times 0.5 = 32$

$$\therefore \sigma_T = \sqrt{32} = \sqrt{8\times 4} = 2\times 2\sqrt{2} = 4\sqrt{2} = 4\times 1.41 = \underline{\underline{5.64}}$$

$$Z = \frac{X-\mu}{\sigma}$$

$$\therefore Z = \frac{S-\mu_T}{\sigma_T}$$

$1.64 \to 0.9495$ $\quad$ $1.65 \to 0.9505$ $\quad \to \frac{0.9495+0.9505}{2} = 0.95 = \underline{\underline{95\%}}$

Take average $= \frac{1.64+1.65}{2} = 1.645$

$$\therefore Z = 1.645 = \frac{S-64}{\sqrt{32}} \quad \therefore S = 1.645\times\sqrt{32} + 64$$

$$= 1.645\times 5.64 + 64 = 73.2778$$

min. sandwich required

~~= 74.7583 ≈ 75~~

1.9 Q9

$\longrightarrow \begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$

$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N(\mu, \Sigma)$

$\therefore \mu = \begin{pmatrix} E[X] \\ E[Y] \end{pmatrix}, \Sigma = \begin{pmatrix} var(X) & cov(X,Y) \\ cov(Y,X) & var(Y) \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$

On comparing.

1. $E[X] = E[Y] = 0$

2. $var(X) = 1$, $var(Y) = 1$

$cov(X,Y) = \rho$

1.10 Q10.

$\longrightarrow \begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}\right)$

$E[X] = 1$, $E[Y] = 2$, $var(X) = 4$, $var(Y) = 3$

$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} E[X] \\ E[Y] \end{pmatrix}, \begin{pmatrix} var(X) & \overset{\sigma_{XY}}{\cancel{\sqrt{var(XY)}}} \\ \underset{\sigma_{XY}}{\cancel{\sqrt{var(XY)}}} & var(Y) \end{pmatrix}\right)$

Let $E[X] = \mu_X$, $E[Y] = \mu_Y$, $var(X) = \sigma_X^2$, $var(Y) = \sigma_Y^2$, covariance $= \sigma_{XY}$

$E[X|Y] = \mu_X + \frac{\sigma_{XY}}{\sigma_Y^2}(y - \mu_Y) = 1 + \frac{1}{3}(y-2)$

$= \boxed{\frac{y}{3} + \frac{1}{3}}$

$\sigma^2_{X|Y} = \sigma_X^2 - \frac{\sigma_{XY}^2}{\sigma_Y^2} = 4 - \frac{1^2}{3} = \boxed{\frac{11}{3}}$

1.11 Q11

$\longrightarrow \quad Z = 3X - 2Y$

$E[Z] = E[3X - 2Y]$

$= 3E(X) - 2E(Y)$

$= 3 \times 0 - 2 \times 0$

$E[Z] = 0$

$var(Z) = var(3X - 2Y)$

$= 3^2 var(X) - 2^2 var(Y) + 2 \times 3(-2) cov(X,Y)$

$= 9 \times 1 - 4 \times 4 - 12 \times 1$

$\therefore \boxed{var(Z) = 13}$

$Z \sim N(\mu, \sigma^2)$

$\boxed{Z \sim N(0, 13)}$

~~cov~~

$corr(Z,X) = \dfrac{cov(Z,X)}{\sqrt{var(Z) var(X)}}$

$cov(Z,X) = cov(3X - 2Y, X)$

$= 3cov(X,X) - 2cov(Y,X)$

$= 3var(X) - 2cov(X,Y)$

$= 3 \times 1 - 2 \times 1$

$= 1$

$\therefore corr(Z,X) = \dfrac{1}{\sqrt{13 \times 1}} = \boxed{\dfrac{1}{\sqrt{13}}}$

1.12 Q12

$\rightarrow$ mean $= 0$

$$\Sigma = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 3 & -1 \\ 1 & -1 & 5 \end{pmatrix}$$

$var(X) = 4$, $cov(X, (Y, Z)) = \begin{pmatrix} 2 & 1 \end{pmatrix}$

$$cov(Y, Z) = \begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix}$$

$E[\;] = 0$

$$\therefore \mu_{X|Y=y, Z=z} = \Sigma_{X(Y,Z)} (\Sigma_{Y,Z})^{-1} \begin{pmatrix} y \\ z \end{pmatrix}$$

$$\Sigma_{Y,Z} = \begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix}$$

$$\det(\Sigma_{Y,Z}) = 15 - 1 = 14$$

$$(\Sigma_{Y,Z})^{-1} = \frac{1}{14} \begin{pmatrix} 5 & 1 \\ 1 & 3 \end{pmatrix}$$

$$\therefore \mu_{X|Y=y, Z=z} = \begin{pmatrix} 2 & 1 \end{pmatrix} \frac{1}{14} \begin{pmatrix} 5 & 1 \\ 1 & 3 \end{pmatrix}$$

$$= \frac{1}{14} \begin{pmatrix} (2 \times 5 + 1 \times 1) & (2 \times 1 + 1 \times 3) \end{pmatrix}$$

$$\boxed{= \frac{1}{14} \begin{pmatrix} 11 & 5 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}} = \frac{11y + 5z}{14}$$

$$var(X | Y = y, Z = z) = \Sigma_{XX} - \underbrace{\Sigma_{X(Y,Z)} (\Sigma_{(Y,Z),(Y,Z)})^{-1}}_{\frac{1}{14}(11 \;\; 5)} \Sigma_{(Y,Z)X}$$

$$\therefore \Sigma_{X(Y,Z)} (\Sigma_{(Y,Z)(Y,Z)})^{-1} = \frac{1}{14} \begin{pmatrix} 11 & 5 \end{pmatrix}$$

$$\therefore var(X | Y, Z) = 4 - \frac{1}{14} \begin{pmatrix} 11 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= 4 - \frac{27}{14} = \boxed{\frac{29}{14}}$$