A Mathematical Perspective on The Financial Crisis

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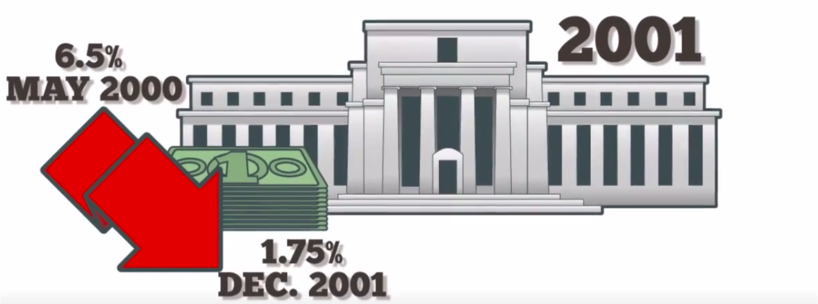
Outline

- The Crisis!
- Banach-Tarski Theorem
- Getting started with Financial Mathematics
- Delbaen Theorem
- Sibuya Theorem
- Fréchet-Höffding Theorem
- Conclusion

The Crisis!

- Seeds sown in 1970s with the Housing and Community Development Act in the United States, fostering market for subprime mortgages
- The Federal Reserve Board lowered the interest rates to 1.75% in Dec. 2001, further to 1% in 2003



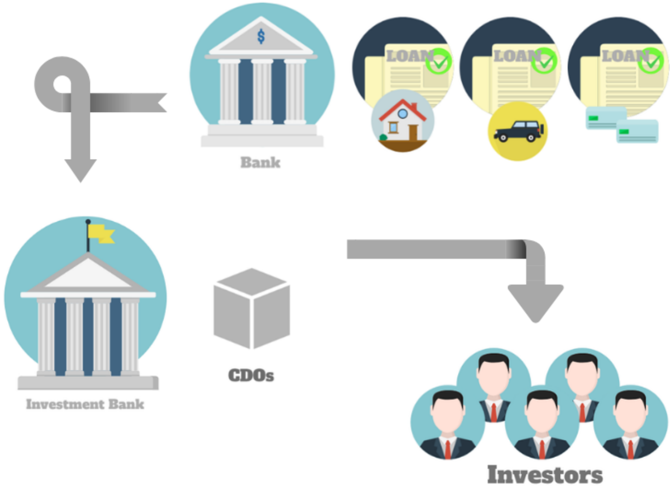
The Crisis!

- Expansion in amount of subprime mortgage debt guaranteed by Freddie Mac[1] and Fannie Mae[2] in the early 2000s
- Loose credit requirements + low interest rates spurred a housing boom, which drove speculation pushing up housing prices and creating a real estate bubble

[1] Federal Home Loan Mortgage Corp
[2] Federal National Mortgage Association

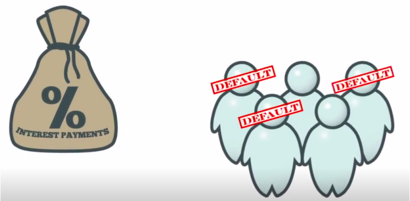
The Crisis!

- Meanwhile, the investment banks in search of easy profits created collateralized debt obligations (CDOs)



The Crisis!

- Housing bubble finally bursted and as housing prices fell, subprime borrowers began to default on loans



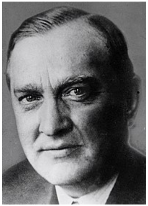
- The subsequent cascade of subprime lender failures created liquidity contagion
- Two major investment banks, Lehman Brothers and Bear Stearns, and more than 450 banks filed for Banktrupcy in 5 years
- Several of the major banks were on the brink of failure and were rescued by a taxpayer-funded bailout

The Crisis!

The consequences of The Crisis deeply impacted the global economy in many ways.

- Stock markets still dropped worldwide
- Housing market suffered, resulting in evictions, foreclosures, and prolonged unemployment
- Led to Credit crisis and Government debt crisis in the United States as well as Europe
- Governments and central banks responded with unprecedented fiscal stimulus

Banach-Tarski Theorem



Stefan Banach

Alfred Tarski

Theorem

Given any two bounded sets A and B in the three-dimensional space $\mathbb{R}^3$, each having non-empty interior, one can partition A into finitely many (at least five) disjoint parts and rearrange them by rigid motions (translation, rotation) to form B.

Interpretation in Finance

Version 1

Given a three-dimensional solid ball (of gold, say), it is possible to cut this ball in finitely many pieces and reassemble these to form two solid balls, each identical in size to the first one.

Version 2

Any solid, a pea, say, can be partitioned into a finite number of pieces, then reassembled to form another solid of any specified shape, say the sun. For this reason, the Banach-Tarski Theorem is often referred to as "The Pea and the Sun Paradox".

Interpretation in Finance

- Proof based on Zermelo-Fraenkel axioms, specifically Axiom of Choice and Non-measurability
- Standard notion of volume does not apply
- Braithwaite wrote: "So by financial alchemy, assets can be transmuted from garbage to gold - and therefore, requires less capital."
- Relevance in Fractional Reserve Banking
- What went wrong in the case of CDOs in 2008?

Getting started with Financial Mathematics

Notations

- A one-period risk will be denoted as a random variable X on the probability space $(\Omega, \mathscr{F}, P)$
- $X \in L^p \;\Leftrightarrow\; \mathbb{E}(|X|^p) < \infty$
- $\bar{F} \in RV_{-\delta}(\infty) \;\Rightarrow\; X \in L^p$ for $p < \delta$ and for $p > \delta$, $\mathbb{E}(|X|^p) = \infty$
- Let E be a vector space of random variables defined on $(\Omega, \mathscr{F}, P)$,
  - *Rearrangement invariant*: If X and Y have the same distribution and $X \in E$, then $Y \in E$ too.
  - *Solid*: If $|Y| < |X|$ and $X \in E$, then too $Y \in E$.

Note: L^p-space is both rearrangement invariant and solid.

Getting started with Financial Mathematics

Risk Measures

A risk measure ρ is only defined as a map from E to $\mathbb{R}$.

- VaR (Value-at-Risk): The VaR at confidence level $\alpha \in (0,1)$ is defined as the α-quantile

$$VaR_\alpha(X) = F^{\leftarrow}(\alpha) := inf\{x \in \mathbb{R} : F(x) \geq \alpha\} \tag{1}$$

- $\mathbb{ES}$: The Expected Shortfall at confidence $\alpha \in (0,1)$ is defined as

$$\mathbb{ES}_\alpha(X) := \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(X)du \quad if\ \mathbb{E}|X| < \infty \tag{2}$$

Getting started with Financial Mathematics

For a risk measure to be reasonable, it should satisfy certain basic properties (axioms).

Convex Risk measure

A convex risk measure $\rho : E \to \mathbb{R}$ satisfies for all $X, Y \in E$:

- $\rho(0) = 0$
- *Monotonicity* : $\;$ *If* $X \leq Y$ *a.s*, *then* $\rho(X) \leq \rho(Y)$
- *TranslationInvariance* : $\;$ *If* $\eta \in \mathbb{R}$, *then* $\rho(X + \eta) = \rho(X) + \eta$.
- *Convexity* : $\;\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$ for $\;0 \leq \lambda \leq 1$.

Coherent risk measures

A subset of the set of convex risk measures, For all $X, Y \in E$:

- *Positive Homogeneity* : $\rho(\lambda X) = \lambda\rho(X)$ *for any* $\lambda \geq 0$.
- *Subadditivity* : $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

Delbaen's Theorem

Freddy Delbaen

Theorem

Let E be a vector space which is rearrangement invariant and solid, and $\rho : E \to \mathbb{R}$ be a convex risk measure, then $E \subset L^1$.

Interpreting Delbaen and relevance to Risk Management

- Delbaen's theorem implies that it is *impossible* to define a coherent risk measure on L^p *for* $p \in (0,1)$.
- $VaR_\alpha(X)$ is non convex and discontinuous function of the confidence level α for discrete distributions
- $\mathbb{ES}$ is convex and continuous with respect to α
- Diversification is beneficial with $\mathbb{ES}_\alpha$ which requires $\mathbb{E}|X| < \infty$. Thereafter, it always admits diversification.
- This theorem raises the question, to what extent can catastrophic events be insured.
- Problems of $VaR_\alpha(X)$ with non-convexity for heavy-tailed or very skewed risks or special dependence
- What role did this play in The Crisis?

Some more Financial-Mathematics tools

Copula

The copula (or probability theory) is a statistical measure that represents a multivariate uniform distribution, which examines the association or dependence between many variables.

Given marginal dfs F_X and F_Y, one can always define a joint df

$$F(x,y) = C(F_X(x), F_Y(y)) \quad \forall (x,y) \in \mathbb{R}^2 \tag{3}$$

The C is exactly a copula, a df on $[0,1]^2$ with standard uniform marginals, C is the df of the random vector $(U_X, U_Y)^T$.

Remark

For F_X and F_Y continuous, C is unique:

$$C(u,v) = F(F_X^{\leftarrow}(u), F_Y^{\leftarrow}(v)) \quad \forall (u,v) \in \mathbb{R}^2$$

More on Copula

- A version of 3 applies also to the joint survival function $\bar{F}(x, y) = \mathbb{P}(X > x, Y > y)$ of the bivariate random vector (X, Y) with distribution F, margins F_X and F_Y , and tails $\bar{F}_X(x) = 1 - F_X(x)$ and $\bar{F}_Y(y) = 1 - F_Y(y)$. Then there exists again a copula $\bar{C}$, the survival copula, such that

$$\bar{F}(x, y) = \bar{C}(\bar{F}_X(x), \bar{F}_Y(y)) \tag{4}$$

- For bivariate case, we have

$$\bar{C}(u, v) = u + v - 1 + C(u, v)$$

- The *coefficients of upper and lower tail* dependence of a bivariate random vector (X, Y) are defined as

$$\lambda_U = \lim_{u \downarrow 0} \frac{\bar{C}(u, u)}{u} \qquad \lambda_L = \lim_{u \downarrow 0} \frac{C(u, u)}{u}$$

Gaussian Copula and Modern Finance

- In finance, no doubt the most (in-)famous copula model is the Gaussian copula:

$$C_{\Phi,\gamma}(u,v) = \Phi_2(\Phi^{\leftarrow}(u), \Phi^{\leftarrow}(v)) \quad \forall (u,v) \in \mathbb{R}^2$$

- For the Gaussian copula, we can check

$$C_{\Phi,\gamma}(u,v) = \bar{C}_{\Phi,\gamma}(u,v)$$

- A Gaussian copula model $C_{\Phi,\gamma}(u,v)$ with correlation coefficient γ applied to the marginal default distribution functions F_A and F_B:

$$\mathbb{P}(T_A < 1, T_B < 1) = C_{\Phi,\gamma}(F_A(1), F_B(1)) \tag{5}$$

- In his web-publication, Salmon (2009), the author puts the title "The formula that killed Wall Street" for equation 5.

Sibuya Theorem

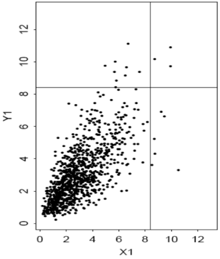
Masaaki Sibuya

Theorem

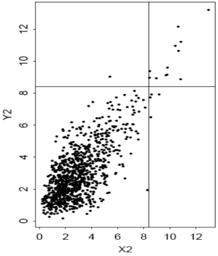
Suppose (X, Y) is a random vector following a bivariate normal distribution with correlation coefficient $\gamma \in [-1, 1)$. Then X and Y are asymptotically independent.

Interpreting Sibuya theorem

- So regardless of how high a correlation γ we choose, if we go far enough in the tails, extreme events occur fairly independently.



Gaussian Copula model



Gumbel Copula model

1000 random variates from two distributions with identical *Gamma*$(3, 1)$ marginals and identical $\gamma = 0.7$, but different *dependence* structures.

- The Gaussian copula model ignores the empirically observed clustering of defaults in extreme circumstances.

Summarising Sibuya theorem and relevance to The Crisis

- Asymptotic tail dependence is strictly a *copula* property, whatever marginals are.
- Under asymptotic independence joint extremes are very rare.
- The very assumption of the Gaussian dependence structure in the standard pricing formulas for tranches of Collateralized Debt Obligations was inconsistent with empirically observed data.
- Model Uncertainty!

Digging deeper to understand Tail Dependencies

- Sibuya's result is insightful, but what actually is happening in the background is the Multivariate Regular Variation!

Multivariate Regular Variation

$(X, Y) \sim F$ is multivariate regularly varying on $E = [0, \infty]^2 \setminus (0, 0)$ if $\exists b(t) \uparrow \infty$ *as* $t \to \infty$ and a Radon measure $\nu \neq 0$ such that

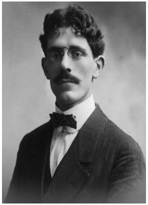
$$t\mathbb{P}\Big(\frac{(X, Y)}{b(t)} \in .\Big) \xrightarrow{v} \nu(.) \quad (t \to \infty).$$

Hidden Regular Variation

Suppose $(X, Y) \in MRV(b, \nu)$.Then $(X, Y) \sim F$ exhibits hidden regularly variation on $E = (0, \infty]^2$ if $\exists b_0(t) \uparrow \infty$ *as* $t \to \infty$ with $\lim_{t\to\infty} \frac{b(t)}{b_0(t)} = \infty$ and a Radon measure $\nu_0 \neq 0$ such that

$$t\mathbb{P}\Big(\frac{(X, Y)}{b_0(t)} \in .\Big) \xrightarrow{v} \nu_0(.) \quad (t \to \infty).$$

Fréchet-Höffding Theorem



Maurice Fréchet

Wassily Höffding

Fréchet-Höffding Theorem

Theorem

Let (X, Y) be a bivariate random vector with finite variances, marginal dfs F_X and FY and an unspecified joint df F ; assume also that X and Y are non-degenerate. The following statements hold.

- *The attainable correlations from any joint model F with the above specifications form a closed interval*

$$[\gamma_{min}, \gamma_{max}] \subset [-1, 1]$$

  with $-1 \leq \gamma_{min} < \gamma_{max} \leq 1$.

- *The minimum correlation min is attained if and only if X and Y are countermonotonic; the maximum correlation max if and only if X and Y are comonotonic.*
- *$\gamma_{min} = -1$ if and only if X and $-Y$ are of the same type; $\gamma_{max} = 1$ if and only if X and Y are of the same type.*

Fréchet-Höffding - Application

F_X, F_Y	$\gamma_{\max}$	$\gamma_{\min}$
$\mathrm{N}(0,\sigma_1^2)$, $\mathrm{N}(0,\sigma_2^2)$, $\sigma_1,\sigma_2>0$	1	-1
$\mathrm{LN}(0,\sigma_1^2)$, $\mathrm{LN}(0,\sigma_2^2)$, $\sigma_1,\sigma_2>0$	$\frac{e^{\sigma_1\sigma_2}-1}{\sqrt{(e^{\sigma_1^2}-1)(e^{\sigma_2^2}-1)}}$	$\frac{e^{-\sigma_1\sigma_2}-1}{\sqrt{(e^{\sigma_1^2}-1)(e^{-\sigma_2^2}-1)}}$
$\mathrm{Pareto}(\alpha)$, $\mathrm{Pareto}(\beta)$, $\alpha,\beta>2$	$\frac{\sqrt{\alpha\beta(\alpha-2)(\beta-2)}}{\alpha\beta-\alpha-\beta}$	$\frac{\sqrt{(\alpha-2)(\beta-2)}\left((\alpha-1)(\beta-1)\mathrm{Beta}(1-\frac{1}{\alpha},1-\frac{1}{\beta})-\alpha\beta\right)}{\sqrt{\alpha\beta}}$
$\mathrm{Beta}(1,1)$, $\mathrm{Beta}(\alpha,1)$, $\alpha>0$	$\frac{\sqrt{3\alpha(\alpha+2)}}{(2\alpha+1)}$	$-\frac{\sqrt{3\alpha(\alpha+2)}}{(2\alpha+1)}$

Figure: Table of $\gamma_{max}(F_X, F_Y)$ and $\gamma_{min}(F_X, F_Y)$ for different pairs of marginal distributions F_X *and* F_Y

Conclusion

- Important is to understand Model Uncertainty.
- An important takeaway is that technical questions, like pricing and hedging, asked in the financial industry often are based on highly incomplete model assumptions.
- More stress on the conditions needed to be fulfilled in order for certain results to be applied

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