From Mahan

Chap 2 Green's fn at Zero Temp

$$S(t,t') = T\exp\left[-i\int_{t'}^{t} dt_1\, \hat{V}(t_1)\right]$$

Gell-Mann Low Theorem

$$\hat{\Psi}(0) = S(0,-\infty)\,\Phi_0$$

$U(t) = e^{iH_0 t} e^{-iHt}$

Green's fns

$$G(\lambda, t-t') = -i\langle | T\, C_\lambda(t)\, C_\lambda^\dagger(t') \rangle$$

quantum numbers

At zero temp. $|\rangle$ is G.S. of H

However, H's eigenstates are unknown $\Rightarrow$ $|\rangle$ is unknown

- Write $H = H_0 + V$
  $\hookrightarrow$ chosen s.t. eigenstates are known

C_λ are defined wrt complete set of state
$\Psi_\lambda \rightarrow$ eigenstate of H_0.

NB:

Defined in Heisenberg repr.
$\therefore$ $|\rangle$ independent of time

$$C_\lambda(t) = e^{iHt}\, C_\lambda\, e^{-iHt}$$

$\Big\downarrow$ Change to interaction repr.

$$C_\lambda(t) = S(0,t)\, \underbrace{\hat{C}_\lambda(t)}_{\text{Interaction}}\, S(t,0)$$

$$|\rangle = S(0,-\infty)\,|\rangle_0$$

$\hookrightarrow$ H_0 G.S.

$\rightarrow G(\lambda, t-t')$

$$= \frac{-i}{{}_0\langle|S(\infty,-\infty)|\rangle_0}\Big[\theta(t-t')\;{}_0\langle|S(\infty,t)\,\hat{C}_\lambda(t)\,S(t,t')\,\hat{C}^\dagger_\lambda(t')\,S(t',-\infty)|\rangle_0$$

$$- \theta(t'-t)\;{}_0\langle|S(\infty,t')\,\hat{C}^\dagger_\lambda(t')\,S(t',t)\,\hat{C}_\lambda(t)\,S(t,-\infty)|\rangle_0\Big]$$

$\downarrow$ Simplify 1st term

$$\theta(t-t')\;{}_0\langle|\,T\big(\hat{C}_\lambda(t)\,\hat{C}^\dagger_\lambda(t')\,S(\infty,-\infty)\,|\rangle_0$$

$$\Rightarrow \boxed{G(\lambda, t-t') = -i\,\frac{{}_0\langle|\,T\,\hat{C}_\lambda(t)\,\hat{C}^\dagger_\lambda(t')\,S(\infty,-\infty)|\rangle_0}{{}_0\langle|\,T\,S(\infty,-\infty)\,|\rangle_0}}$$

Where $V=0$

unperturbed Green's fn. $\leftarrow$

$$\boxed{G^{(0)}(\lambda, t-t') = -i\;{}_0\langle|\,T\,\hat{C}_\lambda(t)\,\hat{C}^\dagger_\lambda(t')|\rangle_0} \quad -(2.51)$$

$\downarrow$ free propagator

1. An empty band : Analyzing single electron in a conduction band of semiconductor / insulator.

G.S. $\rightarrow |0\rangle$

$$C_{\vec{p}}|0\rangle = 0 \quad \text{[No } \bar{e}$$
$$a_{\vec{q}}|0\rangle = 0 \quad \text{No phonon]}$$

both H_0 & V give zero when operating upon vacuum

$$S(t,-\infty)|0\rangle = |0\rangle$$

$$\therefore |\rangle_0 = |\rangle = |0\rangle$$

$$\boxed{G(\lambda, t-t') = -i\theta(t-t')\langle 0|\,\hat{C}_\lambda(t)\,\hat{C}^\dagger_\lambda(t')|0\rangle}$$

$\Rightarrow$ The other one annihilates vacuum

$$G^{(0)} = -i\,\Theta(t-t')\,e^{-i\epsilon_\lambda(t-t')}$$

$$\boxed{G(\lambda,E) = \int_{-\infty}^{\infty} dt\, e^{iEt}\, G(\lambda,t)}$$

$$\boxed{G^{0}(\lambda,E) = \frac{1}{E-\epsilon_\lambda + i\delta}}$$

Added to make integrals converge.

2. <u>Degenerate e^- gas</u>: Define $\xi_{\vec{k}} = \epsilon_{\vec{k}} - \mu$

where in G.S $\epsilon_{\vec{k}} < \mu$ are filled

$\epsilon_{\vec{k}} > \mu$ unfilled

For spherical Fermi surface with p_F,

$$\langle | c_{\vec{k}}^\dagger c_{\vec{k}} | \rangle = \Theta(p_F - k)$$
$$\langle | c_{\vec{k}} c_{\vec{k}}^\dagger | \rangle = \Theta(k - p_F)$$

$$\rightarrow \langle | c_k^\dagger c_k | \rangle = \Theta(-\xi_{\vec{k}}) = \lim_{\beta\to\infty} \frac{1}{e^{\beta\xi_k}+1}$$

$$= n_F(\xi_k)$$

$$G^{(0)}(\lambda, t-t') = -i\left[\theta(t-t')\,\theta(\xi_k) - \theta(t'-t)\,\Theta(-\xi_k)\right] e^{-i\xi_k(t-t')}$$

$\downarrow$ F.T.

$$G^{(0)}(\vec{k},E) = \frac{\Theta(\xi_{\vec{k}})}{E-\xi_{\vec{k}}+i\delta} + \frac{\theta(-\xi_{\vec{k}})}{E-\xi_{\vec{k}}-i\delta}$$

$$G^{(0)}(\vec{k},E) = \frac{1}{E-\xi_{\vec{k}}+i\delta_{\vec{k}}}$$

$$\delta_{\vec{k}} = \delta\, \mathrm{sgn}(\xi_{\vec{k}})$$

3. Phonons

Green fn. defined as

$$D(\vec{q}, \lambda, t-t') = -i \langle | T A_{\vec{q}\lambda}(t) A_{-\vec{q}\lambda}(t') | \rangle$$

Refers to polarization $A_{q\lambda} = a_{\vec{q}\lambda} + a^{\dagger}_{-\vec{q}\lambda}$

~~Not~~ Most applications have only one kind of polarization that do not mix → can be omitted.

In interaction repn.,

$$\boxed{D(\vec{q}, t-t') = -i \frac{{}_0\langle | T \hat{A}_{\vec{q}}(t) \hat{A}_{-q}(t') S(\infty, -\infty) | \rangle_0}{{}_0\langle | S(\infty, -\infty) | \rangle_0}}$$

At zero T, no phonons. $|\rangle_0 = |\rangle = |0\rangle$ → particle vacuum.

$$D^{(0)}(\vec{q}, t-t') = -i \, {}_0\langle | T \hat{A}_q(t) \hat{A}_{-q}(t') | 0 \rangle_0$$

$$\hat{A}_q(t) = a_{\vec{q}} e^{-i\omega_q t} + a^{\dagger}_{-\vec{q}} e^{i\omega_q t}$$

$$\hat{A}_{-q}(t') = a_{-q} e^{-i\omega_q t'} + a_q^{\dagger} e^{i\omega_q t'}$$

Now ${}_0\langle | a_q a_q^{\dagger} | \rangle_0 = 1$ ${}_0\langle | a_q^{\dagger} a_q | \rangle_0 = 0$

$$\Rightarrow D^0(\vec{q}, t-t') = -i \left[\Theta(t-t') e^{-i\omega_q (t-t')} + \Theta(t'-t) e^{i\omega_q (t-t')} \right]$$

↓

$$\boxed{D^{(0)}(\vec{q}, \omega) = \frac{2\omega_{\vec{q}}}{\omega^2 - \omega_{\vec{q}}^2 + i\delta}}$$

3. Phonons

Green fn. defined as

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At zero T, no phonons. $|\rangle_0 = |\rangle = |0\rangle$ → particle vacuum.

$$D^{(0)}(\vec{q}, t-t') = -i \, {}_0\langle | T \hat{A}_q(t) \hat{A}_{-q}(t') | 0\rangle_0$$

$$\hat{A}_q(t) = a_q e^{-i\omega_q t} + a^{\dagger}_{-\vec{q}} e^{i\omega_q t}$$

$$\hat{A}_{-q}(t') = a_{-q} e^{-i\omega_q t'} + a^{\dagger}_q e^{i\omega_q t'}$$

Now ${}_0\langle | a_q a_q^{\dagger} | \rangle_0 = 1$ ${}_0\langle | a_q^{\dagger} a_q | \rangle_0 = 0$

$$\Rightarrow D^{0}(\vec{q}, t-t') = -i \left[\Theta(t-t') e^{-i\omega_q (t-t')} + \Theta(t'-t) e^{i\omega_q (t-t')} \right]$$

↓

$$\boxed{D^{(0)}(\vec{q}, \omega) = \frac{2\omega_{\vec{q}}}{\omega^2 - \omega_{\vec{q}}^2 + i\delta}}$$

Non-zero T, Thermal avg. is needed

$${}_0\langle | a_{\vec q}\, a^\dagger_{\vec q} | \rangle_0 = N_{\vec q} + 1$$

$${}_0\langle | a^\dagger_{\vec q}\, a_{\vec q} | \rangle_0 = N_{\vec q} = \frac{1}{e^{\beta \omega_{\vec q}} - 1}$$

$$\downarrow$$

$$D^{(0)}(\vec q, t-t') = -i\left[(N_{\vec q}+1)\, e^{-i\omega_q |t-t'|} + N_q\, e^{i\omega_q |t-t'|}\right]$$

2.4 Wick's Theorem

expand $S(\infty,-\infty)$ in Green fn. in terms of

$$S(t_1, t_2) = T \exp\left(-i\int_{t_1}^{t_2} dt'\, V(t')\right)$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^{n}}{n!} \int_{t_1}^{t_2} dt_1' \int_{t_1}^{t_2} dt_2' \ldots \int_{t_1}^{t_2} dt_n'\, T[V(t_1')\, V(t_2') \ldots V(t_n')]$$

$$\boxed{\therefore\ G(\vec p, t-t') = \sum_{n=0}^{\infty} \frac{(-i)^{n+1}}{n!} \int_{-\infty}^{\infty} dt_1 \ldots \int_{-\infty}^{\infty} dt_n \times \frac{{}_0\langle | T\, \hat C_p(t)\, \hat V(t_1)\, \hat V(t_2) \ldots \hat V(t_n)\, \hat C_p^\dagger(t') | \rangle_0}{{}_0\langle | S(\infty,-\infty) | \rangle_0}}$$

To learn: How to evaluate ${}_0\langle | T\, \hat C_p(t_1)\, \hat V(t_1)\, \hat V(t_2) \ldots \hat V(t_n)\, \hat C_p^\dagger(t') |$

Will usually have n creation & n destruction operators.
Will be different creation annihilation pairings.

Acc to Wick's theorem, all pairings should be time ordered.

for eg. $_0\langle 1| T\, \hat{c}_\alpha(t)\, \hat{c}_\beta^\dagger(t_1)\, \hat{c}_\gamma(t_2)\, \hat{c}_\delta^\dagger(t')|1\rangle_0$

$= {}_0\langle 1| T\, \hat{c}_\alpha(t)\, \hat{c}_\beta^\dagger(t_1)|1\rangle_0 \; {}_0\langle 1| T\, \hat{c}_\gamma(t_2)\, \hat{c}_\delta^\dagger(t')|1\rangle_0$

Due to commutation $(-)\; {}_0\langle 1| T\, \hat{c}_\alpha(t)\, \hat{c}_\delta^\dagger(t')|1\rangle_0 \; {}_0\langle 1| T\, \hat{c}_\gamma(t_2)\, \hat{c}_\beta^\dagger(t_1)|1\rangle_0$

There are always $n!$ possible pairings.
Within pairing, states must be same

for phonons no - signs due to commutation.
~~mom~~ wave vectors must be equal & opposite

i.e.

$$\boxed{{}_0\langle 1| T\, A_{q_1}(t_1)\, A_{q_2}(t_2)|1\rangle_0 = \delta_{\vec{q}_1+\vec{q}_2}\; {}_0\langle 1| T\, A_{\vec{q}_1}(t_1)\, A_{-\vec{q}_1}(t_2)|1\rangle_0}$$

for ${}_0\langle 1| T\, c^\dagger_{\vec{k}_1}(t_1)\, c_{\vec{k}_2}(t_1)|1\rangle$

same time

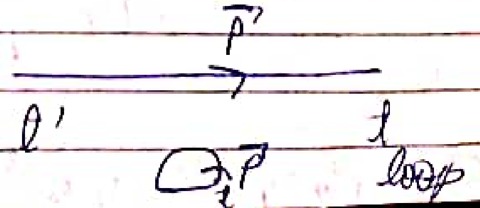
destruction goes to right [i.e like normal ordering

Wick's theorem valid when H_0 is billinear in creation & destruction operators.
fails if H_0 contains pair-wise interactions.

2.5 FEYNMAN DIAGRAMS

Phonons don't have directional arrows (- - - - dashed lines) $D^{(0)}(\vec{q}, t-t') = D^{(0)}(-\vec{q}, t-t')$

$G^0(\vec{p}, t-t')$



$V_{\vec{q}}$ -

$(f(\xi_p))$ $\langle c_p^\dagger(t)\, c_p(t)\rangle$

$\vec{q}=0$ terms are zero as there are no phonons with $\vec{q}=0$.

$\vec{q}=0$ phonon → · translation of crystal
or,
· permanent strain.

Neither of these is supposed to be in Hamiltonian

Term (a) & (b)

Disconnected diagrams (Provide just multiplicative constants f_1, which multiply contribution from connected parts.

2.6 VACUUM POLARIZATION GRAPHS

Consider $_0\langle|S(\infty,-\infty)|\rangle_0 = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}dt_2\cdots\int dt_n$
$\times\, _0\langle|T\hat{V}(t_1)\ldots\hat{V}(t_n)|\rangle_0$

upto order $n=2$

Order $n=1$ vanishes due to $_0\langle|T A_{q_1}(t_1)\rangle_0=0$

$_0\langle|S|\rangle_0 = 1+\frac{(-i)^2}{2!}\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}dt_2\; _0\langle|T\hat{V}_1(t_1)\hat{V}_2(t_2)|0\rangle$

$i\delta(\vec{q}_1+\vec{q}_2)\,D^{(0)}(q_1,t_1-t_2)$ $\quad\sum_{\vec{q}_1\vec{q}_2}M_{q_1}M_{q_2}\langle 0|T\hat{A}_{q_1}(t_1)\hat{A}_{q_2}(t_2)|\rangle_0$

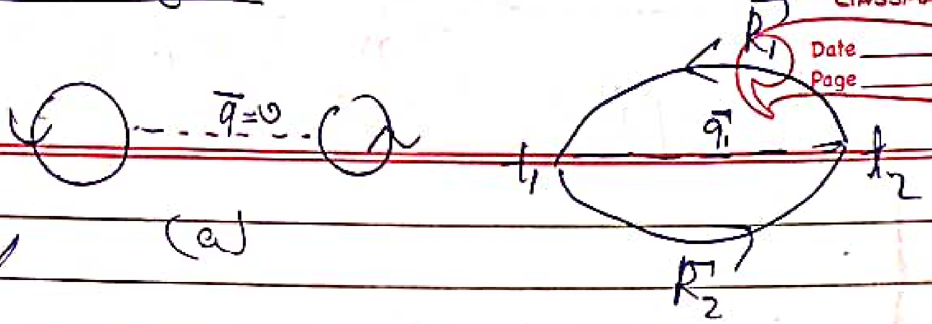
$\times\sum_{k_1k_2ss'}\, _0\langle|T\,\hat{C}^{\dagger}_{k_1+q_1,s}(t_1)\,\hat{C}_{k_1s}(t_1)$

$\delta_{q_1}\, n_F(\xi_{k_1})n_F(\xi_{k_2})$ Wick's theorem

$\hat{C}^{\dagger}_{k_2+q_2,s'}(t_2)\quad\hat{C}_{k_2s'}(t_2)|\rangle$

$\delta_{k_1=k_2-q_1}(i^2)\,G^0(k_2,t_2-t_1)\,G^0(k_1,t_1-t_2)$

Feynman diagrams:



0 contribution since no $\vec{q}=0$ phonons in Hamiltonian

Contribution is

$$= \frac{-i}{2!}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2$$

$$\sum_{\vec{k}\vec{q}} |M_q|^2 \, D^{(0)}(\vec{q}, t_1-t_2)$$

$$G^{(0)}(\vec{k}_1, t_1-t_2)\, G^{(0)}(\vec{k}_1+\vec{q}, t_2=t_1)$$

$$\therefore {}_0\langle |S(\infty,-\infty)|\rangle_0 = 1 + \underbrace{F_1 + \ldots}$$

Vacuum polarization terms
each is a constant F_j
to be evaluated by doing required time & wave vector integrals.

Cancellation Theorem:

Vacuum polarization diagrams exactly cancel the disconnected diagrams in expansion for $G(\vec{p}, t-t')$

$\Rightarrow$ To evaluate $G(\vec{p}, t-t')$, just calculate connected diagrams

In nth term expansion, there are $n!$ terms exactly alike.

$$\therefore G(\vec{p}, t-t') = -i \sum_{n=0}^{\infty} (-i)^n \int_{-\infty}^{\infty} dt_1 \ldots \int_{-\infty}^{\infty} dt_n \, {}_0\langle | T \hat{C}_{\vec{p}\sigma}(t) \hat{C}^{\dagger}_{\vec{p}\sigma}(t')$$

$$\times \hat{V}(t_1) \ldots \hat{V}(t_n) |\rangle_0 \text{ (different connected)}$$

DYSON'S EQUATION

$$\Sigma^{(1)}(\vec{p},E) = i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\sum_{\vec{q}}|M_{\vec{q}}|^2 D^{(0)}(\vec{q},\omega)\, G^{(0)}(\vec{p}-\vec{q},E-\omega)$$

→ Dyson's eqn

$$G(\vec{p},E) = \frac{G^{(0)}(\vec{p},E)}{1 - G^{(0)}(\vec{p},E)\,\Sigma(\vec{p},E)}$$

where $\Sigma(\vec{p},E) = \sum_j \Sigma^{(j)}(\vec{p},E)$

total self - energy

e^- in fermi sea at zero Temp.

$$\boxed{G(\vec{p},E) = \frac{1}{E - \epsilon_{\vec{p}} + i\delta_{\vec{p}} - \Sigma(\vec{p},E)}}$$

$$\boxed{D(\vec{q},\omega) = \frac{D^{(0)}(\vec{q},\omega)}{1 - D^{(0)}(\vec{q},\omega)\,\boxed{\Pi(\vec{q},\omega)}}}$$

Self energy fn for phonons

called polarization operator

$$\boxed{D(\vec{q},\omega) = \frac{2\omega_{\vec{q}}}{\omega^2 - \omega_{\vec{q}}^2 + i\delta - 2\omega_{\vec{q}}\,\Pi(\vec{q},\omega)}}$$

Real (Σ/Π) ⟶ energy shifts of excitation

Im (Σ/Π) ⟶ causes damping of particle motion, related to mean free path of excitation

DYSON'S EQUATION

$$\Sigma^{(1)}(\vec{p},E) = i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\sum_{\vec{q}}|M_{\vec{q}}|^2 D^{(0)}(\vec{q},\omega)\, G^{(0)}(\vec{p}-\vec{q},E-\omega)$$

→ Dyson's eqn

$$G(\vec{p},E) = \frac{G^{(0)}(\vec{p},E)}{1 - G^{(0)}(\vec{p},E)\,\Sigma(\vec{p},E)}$$

where $\Sigma(\vec{p},E) = \sum_j \Sigma^{(j)}(\vec{p},E)$

Σ → total self-energy

e^- in fermi sea at zero Temp.,

$$\boxed{G(\vec{p},E) = \frac{1}{E - \epsilon_{\vec{p}} + i\delta_{\vec{p}} - \Sigma(\vec{p},E)}}$$

$$\boxed{D(\vec{q},\omega) = \frac{D^{(0)}(\vec{q},\omega)}{1 - D^{(0)}(\vec{q},\omega)\,\boxed{\pi(\vec{q},\omega)}}}$$

π → Self energy fn for phonons → called polarization operator

$$\boxed{D(\vec{q},\omega) = \frac{2\omega_{\vec{q}}}{\omega^2 - \omega_{\vec{q}}^2 + i\delta - 2\omega_{\vec{q}}\,\pi(\vec{q},\omega)}}$$

Real (Σ / π) ⟶ energy shifts of excitation

Im (Σ / π) ⟶ causes damping of particle motion, related to mean free path of excitation

2.8 RULES FOR CONSTRUCTING DIAGRAMS

1. Draw Feynman diagram for self energy term, with all phonon / Coulomb / electron lines

2. For e^- line, introduce Green fn:

$$G^{(0)}_{\alpha\beta}(\vec{p}, E) = \frac{\delta_{\alpha\beta}}{E - \epsilon_{\vec{p}} + i\delta_{\vec{p}}}$$

$\delta_{\alpha\beta} \to$ Conserves spin index

δ for one e^- in a band

$\delta(\xi_p)$ for degenerate fermi systems

3. For phonon line,

$$D^{(0)}(\vec{q}, \omega) = \frac{2\omega_{\vec{q}}}{\omega^2 - \omega_{\vec{q}}^2 + i\delta}$$

Also add $|M_{\vec{q}}|^2$ for each phonon Green's fn.

$|M_{\vec{q}}|^2 \to e^-$ - phonon interaction matrix element

4. Add Coulomb potential $v_q = \frac{4\pi e^2}{q^2}$ for each coulomb interaction

5. Conserve energy & momentum at each vertex.

6. Sum over internal degrees of freedom: momentum, energy & spin

7. Multiply by factor $\frac{i^m}{(2\pi)^{4m}}(-1)^F(2S+1)^F$ $F \to$ no. of closed Fermion loops

'm' chosen as follows:

(a) For e^- self energy: m = no. of internal phonon & coulomb lines

(b) For phonon self energy: $m = \frac{1}{2} \times$ no. of vertices

8. For each photon line,

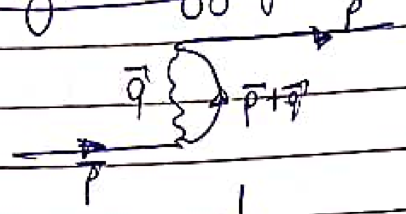
$\vec{j}\cdot\vec{A}$ interaction $\left[\frac{e^2}{m^2}\sum_{\mu\nu}\left(\vec{k} + \frac{\vec{q}}{2}\right)_\mu D_{\mu\nu}(\vec{q}, \omega)\left(\vec{k}' + \frac{1}{2}\vec{q}\right)_\nu\right.$

$D_{\mu\nu}(\vec{q}, \omega) \to$ photon's Green fn.

$\vec{k}, \vec{k}' \to$ wave vecs. of scattered particles at 2 vertices

Identity : $\boxed{i\int \frac{d\omega}{2\pi}\, G^0(\vec{p}+\vec{q}, \epsilon+\omega) = -n_f(\xi_{\vec{p}+\vec{q}})}$

Unscreened exchange energy for e^-



$$\boxed{\Sigma_x(\vec{p}) = -\int \frac{d^3q}{(2\pi)^3}\, v_q\, n_f(\xi_{\vec{p}+\vec{q}})}$$

2.10 PHOTON GREEN'S FUNCTIONS

$$\epsilon(\vec{q},\omega) = 1 - v_q\, P(\vec{q},\omega)$$

self energy parts of coulomb potential.

$$D_{\mu\nu}(\vec{k}, t-t') = -i\sum_{\lambda} \langle T A_\mu(\vec{k},\lambda,t)\, A_\nu(-\vec{k},\lambda,t')\rangle$$

sum over transverse polarization

polarization vec. for each component

$$A_\mu(\vec{k},\lambda,t) = \left(\frac{2\pi}{\omega_{\vec{k}}}\right)^{\frac{1}{2}} \xi_\mu(\vec{k}\lambda)\left[a_{\vec{k}\lambda} e^{-i\omega_{\vec{k}} t} + a^\dagger_{-\vec{k}\lambda} e^{i\omega_{\vec{k}} t}\right]$$

$$\boxed{D^{(0)}_{\mu\nu}(\vec{k}, t-t') = -\frac{2\pi i}{\omega_k}\, e^{-i\omega_k|t-t'|} \sum_\lambda \xi_\mu(\vec{k}\lambda)\,\xi_\nu(\vec{k}\lambda)}$$

$\downarrow$ F.T.

$$\boxed{D^0_{\mu\nu}(\vec{k},\omega) = \frac{4\pi}{\omega^2-\omega_{\vec{k}}^2+i\delta} \sum_\lambda \xi_\mu \xi_\nu}$$

Use $\boxed{\delta_{\mu\nu} = \sum_{\lambda} \xi_\mu \xi_\nu + \hat{k}\hat{k}}$

$$\Rightarrow \boxed{D_{\mu\nu}^{(0)} = \frac{4\pi\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)}{\omega^2 - \omega_k^2 + i\delta}}$$

Photon's Green fn.

Dyson's eqn $\boxed{D_{\mu\nu} = D_{\mu\nu}^{(0)} + \sum_{\lambda\delta} D_{\mu\lambda}^{(0)} \underbrace{\Pi_{\lambda\delta}}_{\text{Self energy fn. (3×3 matrix)}} D_{\delta\nu}}$

$$D_{\mu\nu}^{(0)} = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) D^{(0)} \rightarrow \text{Scalar}$$

$$D_{\mu\nu} = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) D$$

$$\Pi_{\mu\nu} = \delta_{\mu\nu}\, \Pi^{(1)} + \frac{k_\mu k_\nu}{k^2}\, \Pi^{(2)}$$

Scalars

Now $\boxed{\begin{aligned} &\sum_{\lambda\delta} \left(\delta_{\mu\lambda} - \hat{k}_\mu \hat{k}_\lambda\right)\left(\delta_{\lambda\delta}\Pi^{(1)} + \hat{k}_\lambda \hat{k}_\delta \Pi^{(2)}\right)\left(\delta_{\delta\nu} - \hat{k}_\delta \hat{k}_\nu\right) \\ &= \left(\delta_{\mu\nu} - \hat{k}_\mu \hat{k}_\nu\right)\Pi^{(1)} \end{aligned}}$

$$\Downarrow$$

$$\boxed{\begin{aligned} D &= \frac{D^{(0)}}{1 - D^{(0)}\Pi^{(1)}} \\ D_{\mu\nu} &= \frac{4\pi\left(\delta_{\mu\nu} - k_\mu k_\nu / k^2\right)}{\omega^2 - \omega_k^2 - 4\pi \Pi^{(1)}} \end{aligned}}$$

109 - 141
109 - 141
3.1 - 3.5

CH3
Non-zero Temperatures

To treat t & β as real & imaginary parts of one complex variable, → only one S-matrix expansion.

$$n_F(\xi_{\vec{p}}) = \frac{1}{e^{\beta\xi_{\vec{p}}}+1} = \frac{1}{2} + \frac{1}{\beta}\sum_{n=-\infty}^{\infty}\frac{1}{\frac{(2n+1)i\pi}{\beta}-\xi_{\vec{p}}}$$

$$n_B(\omega_q) = \frac{1}{e^{\beta\omega_q}-1} = -\frac{1}{2} + \frac{1}{\beta}\sum_{n=-\infty}^{\infty}\frac{1}{\frac{2\pi i n}{\beta}-\omega_q}$$

Any meromorphic fn. can be expanded as summation over its poles & residue at poles.

pole frequencies:

$$\omega_n = \frac{(2n+1)\pi}{\beta} \quad \text{fermions}$$

$$\omega_n = \frac{2n\pi}{\beta} \quad \text{bosons}$$

$\frac{1}{i\omega_n - \omega_q}$ → non-interacting Green's fn. in Matsubara method.

n odd fermions
n even bosons

$\tau = it$

Green fns are fns of τ with domain $-\beta \leq \tau \leq \beta$

Fourier expansion of fn $f(\tau)$ over $-\beta \leq \tau < \beta$,

$$f(\tau) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi\tau}{\beta}\right) + b_n\sin\left(\frac{n\pi\tau}{\beta}\right)\right]$$

where
$$\boxed{\begin{aligned} a_n &= \frac{1}{\beta}\int_{-\beta}^{\beta} d\tau\, f(\tau)\cos\left(\frac{n\pi\tau}{\beta}\right) \\ b_n &= \frac{1}{\beta}\int_{-\beta}^{\beta} d\tau\, f(\tau)\sin\left(\frac{n\pi\tau}{\beta}\right) \end{aligned}}$$

Define $f(i\omega_n) = \frac{1}{2}\beta(a_n + i b_n)$

$$\boxed{f(\tau) = \frac{1}{\beta}\sum_{n=-\infty}^{\infty} e^{-\frac{i n\pi\tau}{\beta}} f(i\omega_n)}$$

$$\boxed{f(i\omega_n) = \frac{1}{2}\int_{-\beta}^{\beta} d\tau\, f(\tau)\, e^{\frac{i n\pi\tau}{\beta}}}$$

for bosons $f(\tau) = f(\tau+\beta)$ when $-\beta < \tau < 0$

$\Rightarrow$ for bosons, $\boxed{f(i\omega_n) = \int_0^{\beta} d\tau\, e^{i\omega_n\tau} f(\tau)}$

$$\boxed{f(\tau) = \frac{1}{\beta}\sum e^{-i\omega_n\tau} f(i\omega_n)}$$

$\omega_n = 2n\pi k_B T$ ~~[illegible]~~

Diff. b/w

Same

fermions

3.23

$$f(i\omega_n) = \int_0^{\beta} d\tau\, e^{i\omega_n\tau} f(\tau)$$

$$f(\tau) = \frac{1}{\beta}\sum_n e^{-i\omega_n\tau} f(i\omega_n)$$

$\omega_n = (2n+1)\pi k_B T$

Matsubara fn

$f(i\omega_n)$ $\xrightarrow[\text{Analytic continuation}]{i\omega_n \to \omega + i\delta}$ Retarded Green's fn

δ infinitesimal

complex frequencies

Define $f(i\omega_n) = \frac{1}{2}\beta(a_n + i b_n)$

$$\boxed{\begin{aligned} f(\tau) &= \frac{1}{\beta}\sum_{n=-\infty}^{\infty} e^{-\frac{i n\pi\tau}{\beta}} f(i\omega_n) \\ f(i\omega_n) &= \frac{1}{2}\int_{-\beta}^{\beta} d\tau\, f(\tau)\, e^{\frac{i n\pi\tau}{\beta}} \end{aligned}}$$

for bosons $f(\tau) = f(\tau+\beta)$ when $-\beta<\tau<0$

$\Rightarrow$ for bosons, $\boxed{f(i\omega_n) = \int_0^{\beta} d\tau\, e^{i\omega_n\tau} f(\tau)}$

$$\boxed{f(\tau) = \frac{1}{\beta}\sum e^{-i\omega_n\tau} f(i\omega_n)}$$

$\omega_n = 2n\pi k_B T$

Same

fermions

diff.

(3.23)

$$f(i\omega_n) = \int_0^{\beta} d\tau\, e^{i\omega_n\tau} f(\tau)$$

$$f(\tau) = \frac{1}{\beta}\sum_n e^{-i\omega_n\tau} f(i\omega_n)$$

$\omega_n = (2n+1)\pi k_B T$

Matsubara fn

$f(i\omega_n)$

complex frequencies

$i\omega_n \to \omega + i\delta$

infinitesimal

Analytic continuation

Retarded Green's fn

3.2 MATSUBARA's GREEN fn.

$\langle\ \rangle \to$ thermodynamic avg.

Define (3.24) $$\mathcal{G}(\vec{p}, \tau-\tau') = -\langle T_\tau\, C_{\vec{p}\sigma}(\tau)\, C^\dagger_{\vec{p}\sigma}(\tau')\rangle$$

(3.25) $$\mathcal{G}(\vec{p}, \tau-\tau') = \cancel{\ }\mathrm{Tr}\left[e^{-\beta(H-\mu N-\Omega)}\, T_\tau\left(e^{\tau(H-\mu N)}\, C_{\vec{p}\sigma}\, e^{-(\tau-\tau')(H-\mu N)}\, C^\dagger_{\vec{p}\sigma}\, e^{-\tau'(H-\mu N)}\right)\right]$$

Normalization factor for thermodynamic avg $\leftarrow$ $\boxed{e^{-\beta\Omega}} = \mathrm{Tr}\left(e^{-\beta(H-\mu N)}\right)$ $\quad \Omega \to$ thermodynamic grand potential $\downarrow$ Scalar fn.

In one e^- system,
Analytic continuation $i\omega_n \to E+\mu+i\delta$ removes μ from all expressions

Many e^- system, μ retained in formalism
$i\omega_n \to E+i\delta$
energy measured from chem. potential

$T_\tau \to$ Time ordering (earliest τ to right)

is a fn. of only diff $(\tau-\tau')$ despite appearance in (3.24).
$\therefore$ only one of the variables is necessary only

$$\mathcal{G}(\vec{p}, \tau) = -\langle T_\tau\, C_{\vec{p}\sigma}(\tau)\, C^\dagger_{\vec{p}\sigma}(0)\rangle$$

$$= -\mathrm{Tr}\left[e^{-\beta(K-\Omega)}\, T_\tau\left(e^{\tau K}\, C_{\vec{p}\sigma}\, e^{-\tau K}\, C^\dagger_{\vec{p}\sigma}\right)\right]$$

$$K = H - \mu N$$

It can be shown
$$-\beta < \tau < 0 \ : \ \boxed{\mathcal{G}(\vec{p}, \tau) = -\mathcal{G}(\vec{p}, \tau+\beta)}$$

As asserted earlier.

$\Rightarrow$ Green's fn can be expanded as Fourier series of the type in (3.23)

$$\mathcal{G}(\vec{p}, i\omega_n) = \int_0^\beta d\tau \, e^{i\omega_n \tau} \mathcal{G}(\vec{p}, \tau) \quad (3.39)$$

$$\mathcal{G}(\vec{p}, \tau) = \frac{1}{\beta} \sum_n e^{-i\omega_n \tau} \mathcal{G}(\vec{p}, i\omega_n)$$

$i\omega_n \rightarrow$ odd multiple of $\frac{\pi}{\beta}$ for fermions

for free-particle (non-interacting) Green's fn

$$\boxed{K = \sum_{\vec{p}\sigma} \xi_{\vec{p}} \, c^{\dagger}_{\vec{p}\sigma} c_{\vec{p}\sigma}} \quad (3.42)$$

$$\xi_{\vec{p}} = \epsilon_{\vec{p}} - \mu$$

$\therefore$ Using Baker-Hausdorff theorem

$$c_{\vec{p}\sigma}(\tau) = e^{\tau K_0} c_{\vec{p}\sigma} e^{-\tau K_0} = e^{-\xi_{\vec{p}}\tau} c_{\vec{p}\sigma}$$

$$c^{\dagger}_{\vec{p}\sigma}(\tau) = e^{\tau K_0} c^{\dagger}_{\vec{p}\sigma} e^{-\tau K_0} = e^{\xi_p \tau} c^{\dagger}_{\vec{p}\sigma}$$

$$\Rightarrow \boxed{\mathcal{G}^{(0)}(\vec{p}, \tau) = -e^{-\xi_{\vec{p}}\tau}\left[\theta(\tau) - n_f(\xi_{\vec{p}})\right]}$$

$$n_f(\xi_{\vec{p}}) = \langle c^{\dagger}_{\vec{p}\sigma} c_{\vec{p}\sigma} \rangle = \frac{1}{e^{\beta\xi_p} + 1}$$

$\downarrow$

$$\mathcal{G}^{(0)}(\vec{p}, \omega) = \int_0^\beta d\tau \, e^{i\omega_n \tau} \mathcal{G}^{(0)}(\vec{p}, \tau) = -(1 - n_f)\int_0^\beta d\tau \, e^{\tau(i\omega_n - \xi_p)}$$

$$\boxed{\mathcal{G}^{(0)}(\vec{p}, \omega) = -(1 - n_f)\frac{\left[e^{\beta(i\omega_n - \xi_p)} - 1\right]}{i\omega_n - \xi_p}} \quad - 3.51$$

Since $i\beta\omega_n = i(2n+1)\pi$,

$e^{i\beta\omega_n} = -1$

$$\Rightarrow \mathcal{G}^{(0)}(\vec{p}, i\omega_n) = \frac{(1-n_{\vec{p}})(e^{-\beta\xi_p}+1)}{i\omega_n - \xi_{\vec{p}}}$$

$$\downarrow$$

$$\boxed{\frac{1}{i\omega_n - \xi_{\vec{p}}}} \quad - (3.55)$$

Phonons → have no chemical potential (since one can make arbitrary number of them)

$$\mathcal{D}(\vec{q}, \tau-\tau') = -\langle T_\tau A(\vec{q},\tau) A(-\vec{q},\tau')\rangle$$

where $A(\vec{q},\tau) = e^{\tau H}(a_{\vec{q}} + a^\dagger_{-\vec{q}})e^{-\tau H}$

↓ Keeping only one variable

$$\mathcal{D}(\vec{q},\tau) = -\langle T_\tau A(\vec{q},\tau) A(-\vec{q},0)\rangle$$

for $-\beta<\tau<0$: $\mathcal{D}(\vec{q},\tau) = \mathcal{D}(\vec{q},\tau+\beta)$

∴ F.T.:

$\omega_n = 2n\pi k_B T$

$$\boxed{\begin{aligned} \mathcal{D}(q, i\omega_n) &= \int_0^\beta d\tau\, e^{i\omega_n\tau} \mathcal{D}(\vec{q},\tau) \\ \mathcal{D}(\vec{q},\tau) &= \frac{1}{\beta}\sum_n e^{-i\omega_n\tau}\mathcal{D}(\vec{q}, i\omega_n) \end{aligned}} \quad - (3.65)$$

Non interacting Green's fn $\quad H = \sum_{\vec{q}} \omega_{\vec{q}}\, a_q^\dagger a_q$

$a_{\vec{q}}(\tau) = e^{-\tau\omega_q} a_{\vec{q}} \qquad a^\dagger_{\vec{q}}(\tau) = e^{\tau\omega_q} a^\dagger_{\vec{q}}$

$N_q = \langle a_q^\dagger a_q\rangle = \dfrac{1}{e^{\beta\omega_q}-1} \qquad \langle a_q a_q^\dagger\rangle = N_q + 1$

$$\boxed{\langle a_q a_q\rangle = \langle a_q^\dagger a_q^\dagger\rangle = 0}$$

Leads to

$$D^{(0)}(\vec{q}, \tau) = -\Theta(\tau)\left[(N_q+1)e^{-\tau\omega_q} + N_q e^{\tau\omega_q}\right]$$

$$-\Theta(-\tau)\left[N_q e^{\tau\omega_q} + (N_q+1)e^{-\tau\omega_q}\right]$$

$$D^{(0)}(q, i\omega_n) = -\left[(N_q+1)\frac{\left(e^{\beta(i\omega_n-\omega_q)}-1\right)}{i\omega_n-\omega_q} + \frac{N_q\left(e^{\beta(i\omega_n+\omega_q)}\right)}{i\omega_n+\omega_q}\right]$$

$$= -\left[\frac{-1}{i\omega_n-\omega_q} + \frac{1}{i\omega_n+\omega_q}\right]$$

$$\Rightarrow \boxed{D^{(0)}(\vec{q}, i\omega_n) = -\frac{2\omega_{\vec{q}}}{\omega_n^2+\omega_q^2}} \quad (3.76)$$

Photon's Green fn.

$$D_{\mu\nu}(\vec{k}, \tau) = -\sum_\lambda \langle T_\tau A_\mu(\vec{k}, \lambda, \tau) A_\nu(-\vec{k}, \lambda, 0)\rangle$$

$$A_\mu(\vec{k}, \lambda, 0) = \xi_\mu(\vec{k}, \lambda)\left(\frac{2\pi}{\omega_{\vec{k}}}\right)\left(a_{k\lambda} + a^{\dagger}_{k\lambda}\right)$$

$$\boxed{D^{0}_{\mu\nu}(\vec{k}, i\omega_n) = \frac{-4\pi\left(\delta_{\mu\nu} - k_\mu k_\nu/k^2\right)}{\omega_n^2+\omega_k^2}} \quad (3.79)$$

$$\mathcal{G}(\vec{p}, ip_n) = \mathcal{G}(p) \qquad D(\vec{q}, i\omega_n) = D(q)$$

4 vector notation $p = (\vec{p}, ip_n)$

$q = (\vec{q}, i\omega_n)$

3.3 RETARDED & ADVANCED GREEN'S FUNCTIONS

classmate

for e^- in ~~state~~ $\vec{p}$

$$G_{ret}(\vec{p}, t-t') = -i\,\Theta(t-t')\,\langle [C_{p\sigma}(t)\, C^{\dagger}_{p\sigma}(t') + C^{\dagger}_{p\sigma}(t')\, C_{p\sigma}(t)]\rangle$$

on real time, not tau

(3.82) — $$= -i\,\underset{\text{Causal}}{\Theta(t-t')}\, Tr\{e^{-\beta(K-\Omega)}[C_{p\sigma}(t)\, C^{\dagger}_{p\sigma}(t') + C^{\dagger}_{p\sigma}(t')\, C_{p\sigma}(t)]\}$$

(3.83) — $K = H - \mu N$ $\qquad C_{p\sigma}(t) = e^{iKt} C_{p\sigma} e^{-itK}$

(3.84) — $$\lim_{t \to t'} \{C_{p\sigma}(t)\, C^{\dagger}_{p\sigma}(t') + C^{\dagger}_{p\sigma}(t')\, C_{p\sigma}(t)\} = 1$$

→ Anticommutation relation

for phonons,

(3.85) — $$D_{ret}(\vec{q}, t-t') = -i\,\Theta(t-t')\,\langle A(\vec{q},t)\, A(-\vec{q},t') - A(-\vec{q},t')\, A(\vec{q},t)\rangle$$

Define $$\boxed{U = \sum_{ij} M_{ij}\, C_i^{\dagger} C_j}$$

billinear in C_i → is bosonic irrespective of whether C_i's are all bosonic or all fermionic

~~used~~ encountered often ←

(3.88) — $$\boxed{U_{ret}(t-t') = -i\,\Theta(t-t')\,\langle [U(t)\, U^{\dagger}(t') - U^{\dagger}(t')\, U(t)]\rangle}$$

(3.89) $$\boxed{V = \sum_{ijk} M_{ijk}\, C_i^{\dagger} C_j C_k}$$ → fermionic if product of odd no. of fermions

$$V_{ret}(t-t') = -i\,\Theta(t-t')\,\langle [V(t)\, V^{\dagger}(t') + V^{\dagger}(t')\, V(t)]\rangle$$

All have F.T.!

$$G_{ret}(\bar{p}, E) = \int_{-\infty}^{\infty} dt\, e^{iE(t-t')}\, G_{ret}(\bar{p}, t-t')$$

$$D_{ret}(\bar{q}, \omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega(t-t')}\, D_{ret}(\bar{q}, t-t')$$

$$\bar{U}_{ret}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, \bar{U}_{ret}(t)$$

3.90 - 3.92

<u>Advanced Green fn</u>

$$G_{adv}(\bar{p}, t-t') = i\theta(t'-t)\langle [C_p(t) C_p^{\dagger}(t') + C_p^{\dagger}(t') C_p(t)]\rangle$$

$$D_{adv}(\bar{q}, t-t') = i\theta(t'-t)\langle A(\bar{q}, t) A(-\bar{q}, t') - A(-\bar{q}, t') A(\bar{q}, t)\rangle$$

$$\bar{U}_{adv}(t-t') = i\theta(t'-t)\langle [U(t) U^{\dagger}(t') - U^{\dagger}(t') U(t)]\rangle$$

F.T. defined same ways as (3.90 - 3.92).

Advanced fns. of energy $\xleftrightarrow{\text{complex conjugate}}$ Ret. fn. of energy

$$\boxed{\bar{U}_{ret}(\omega) = \bar{U}_{adv}(\omega)^{*}} \quad \text{— (3.98)}$$

Introduce: a particular repn. for these green fn.
Use complete set of states $|m\rangle$ which are <u>exact eigenstates</u> of $K = H - \mu N$.

$$K|m\rangle = E_m |m\rangle \qquad 1 = \sum_m |m\rangle\langle m|$$

$$\therefore \bar{U}_{ret}(t-t') = -i\theta(t-t') e^{\beta\Omega} \sum_n \langle n| e^{-\beta K} [U(t)\, \underline{1}\, U^{\dagger}(t') - U^{\dagger}(t')\, \underline{1}\, U(t)]|n\rangle$$

$$\sum_m |m\rangle\langle m|$$

All have F.T. !

$$G_{ret}(\bar{p},E) = \int_{-\infty}^{\infty} dt\, e^{iE(t-t')}\, G_{ret}(\bar{p},t-t')$$ — 3.90

$$D_{ret}(\bar{q},\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega(t-t')}\, D_{ret}(\bar{q},t-t')$$

$$\bar{U}_{ret}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, \bar{U}_{ret}(t)$$ — 3.92

Advanced Green fn

$$G_{adv}(\bar{p},t-t') = i\theta(t'-t)\langle[C_p(t)\,C_p^{\dagger}(t') + C_p^{\dagger}(t')C_p(t)]\rangle$$

$$D_{adv}(\bar{q},t-t') = i\theta(t'-t)\langle A(\bar{q},t)\,A(-\bar{q},t') - A(-\bar{q},t')\,A(\bar{q},t)\rangle$$

$$\bar{U}_{adv}(t-t') = i\theta(t'-t)\langle[U(t)\,U^{\dagger}(t') - U^{\dagger}(t')U(t)]\rangle$$

F.T. defined same ways as (3.90 - 3.92).

Advanced fns. of energy $\xleftrightarrow{\text{complex conjugate}}$ Ret. fn. of energy

$$\boxed{\bar{U}_{ret}(\omega) = \bar{U}_{adv}(\omega)^{*}} \quad \text{— } 3.98$$

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$$K|m\rangle = E_m|m\rangle \qquad 1 = \sum_m |m\rangle\langle m|$$

$$\therefore \bar{U}_{ret}(t-t') = -i\theta(t-t')e^{\beta\Omega}\sum_n \langle n|e^{-\beta K}[U(t)\,\underline{1}\,U^{\dagger}(t') - U^{\dagger}(t')\,\underline{1}\,U(t)]|n\rangle$$

$\sum_m |m\rangle\langle m|$

(3.102) $$\langle n|U(t)|m\rangle = \langle n|e^{itK} U e^{-itK}|m\rangle = \langle n|U|m\rangle e^{it(E_n-E_m)}$$

$$\downarrow$$

$$U_{ret}(t-t') = -i\theta(t-t') e^{\beta\Omega} \sum_{m,n} e^{-\beta E_n} \Big[e^{i(t-t')(E_n-E_m)} |\langle n|U(0)|m\rangle|^2 - e^{-i(t-t')(E_n-E_m)} |\langle m|U(0)|n\rangle|^2 \Big]$$

$\downarrow$ in 2nd term exchange n & m

$$\boxed{U_{ret}(t-t') = -i\theta(t-t') e^{\beta\Omega} \sum_{m,n} |\langle n|U|m\rangle|^2 e^{i(t-t')(E_n-E_m)} \times \left[e^{-\beta E_n} - e^{-\beta E_m} \right]}$$

$\downarrow$ F.T.

(3.104) $$\boxed{U_{ret}(\omega) = e^{\beta\Omega} \sum_{n,m} |\langle n|U|m\rangle|^2 \frac{e^{-\beta E_n} - e^{-\beta E_m}}{\omega + E_n - E_m + i\delta}}$$

δ: To ensure convergence at large times

Equivalent Matsubara fn $\mathcal{U}(\tau) = -\langle T_\tau U(\tau) U^\dagger(0)\rangle$

$$\mathcal{U}(i\omega_n) = \int_0^\beta d\tau\, e^{i\omega_n\tau} \mathcal{U}(\tau)$$

Apply same repn.

$\tau > 0$: $$\mathcal{U}(\tau) = -e^{\beta\Omega} \sum_{n,m} |\langle n|U|m\rangle|^2 e^{-\beta E_n} e^{\tau(E_n-E_m)}$$

$$\downarrow$$

$$\mathcal{U}(i\omega_n) = e^{\beta\Omega} \sum_{n,m} |\langle n|U|m\rangle|^2 \frac{e^{-\beta E_n} - e^{-\beta E_m}}{i\omega_n + E_n - E_m}$$

Same as ret(ω) except for $i\omega_n \to \omega + i\delta$

$$\therefore \quad \mathcal{U}(i\omega_n) \xrightarrow{\boxed{i\omega_n \to \omega + i\delta}} \bar{U}_{ret}(\omega) \qquad -(3.110)$$

↳ Analytic continuation

$$G(\vec{p}, i\omega_n) \to G_{ret}(\vec{p}, \omega) \qquad -(3.111)$$

$$\mathcal{D}(\vec{q}, i\omega_n) \to D_{ret}(\vec{q}, \omega) \qquad -(3.112)$$

for adv. fns. $\quad i\omega_n \to \omega - i\delta$.

Spectral fn. (Spectral density fn.)

$$\left.\begin{aligned} R(\omega) &= -2\,\mathrm{Im}[\bar{U}_{ret}(\omega)] \\ B(\vec{q}, \omega) &= -2\,\mathrm{Im}[D_{ret}(\vec{q}, \omega)] \\ A(\vec{p}, \omega) &= -2\,\mathrm{Im}[G_{ret}(\vec{p}, \omega)] \end{aligned}\right\} \quad 3.113 - 3.115$$

⇒ from expr in (3.104), only complex part:

$$\boxed{\frac{1}{\omega + E_n - E_m + i\delta} = P\,\frac{1}{\omega + E_n - E_m} - i\pi\,\delta(\omega + E_n - E_m)}$$

$$\Rightarrow R(\omega) = e^{\beta\Omega} \sum_{n,m} |\langle n|U|m\rangle|^2 \underbrace{\{e^{-\beta E_n} - e^{-\beta E_m}\}}_{e^{-\beta E_n}\{1 - e^{-\beta(E_m - E_n)}\}} 2\pi\,\delta(\omega + E_n - E_m)$$

$$\boxed{R(\omega) = 2\pi\,(1 - e^{-\beta\omega})\, e^{\beta\Omega} \sum_{n,n} |\langle n|U|m\rangle|^2\, \delta(\omega + E_n - E_m)}$$

↳ Missing factor $e^{-\beta E_n}$ ⇒

Lehmann Representation

$$\Rightarrow \bar{U}_{ret}(\omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi}\, \frac{R(\omega')}{\omega - \omega' + i\delta} \qquad \mathcal{U}(i\omega_n) = \int \frac{d\omega'}{2\pi}\, \frac{R(\omega')}{(i\omega_n - \omega')}$$

Electrons

(3.1A)

$$G_{ret}(\vec{p},\omega) = e^{\beta\Omega} \sum_{n,m} |\langle n|C_{\vec{p}\sigma}|m\rangle|^2 \frac{e^{-\beta E_n} + e^{-\beta E_m}}{\omega + E_n - E_m + i\delta}$$

$\downarrow$

Spectral fn.

$$\boxed{A(\vec{p},\omega) = 2\pi e^{\beta\Omega} \sum_{n,m} |\langle n|C_{\vec{p}\sigma}|m\rangle|^2 \left(e^{-\beta E_n} + e^{-\beta E_m}\right) \delta(\omega + E_n - E_m)}$$

Doesn't hold for bosons

$A(\vec{p},\omega) > 0$ Since RHS has only +ve values.

IMP : $A(\vec{p},\omega)$ interpreted as probability fn.

Also, $$\boxed{\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A(\vec{p},\omega) = 1} \quad \text{— (3.121)}$$

Spectral fn. for non-interacting electrons

$$C_{\vec{p}\sigma}(t) = e^{-it\xi_{\vec{p}}} C_{\vec{p}\sigma}$$

leads to

$$G^{(0)}_{ret}(\vec{p}, t-t') = -i\Theta(t-t')\, e^{-i(t-t')\xi_p}$$

$\downarrow$

$$G^{(0)}_{ret}(p,E) = \frac{1}{E - \xi_{\vec{p}} + i\delta}$$

Never have $\delta_{\vec{p}}$ like zero-temp Green fn. $\Rightarrow$ easier to use

$\downarrow$

$$\boxed{A^{(0)}(\vec{p},\omega) = 2\pi\,\delta(E - \xi_p)}$$

Interpreted as Probability that an e^- has momentum $\vec{p}$ & energy E.

For interacting e^- system, we assume

$$n_{\vec{p}} = \langle C^{\dagger}_{p\sigma} C_{p\sigma} \rangle$$

$$\boxed{n_{\vec{p}} = \int_{-\infty}^{\infty} \frac{dE}{2\pi} n_F(E)\, A(\vec{p},E)} \quad \text{— (3.135)}$$

Similarly for phonons, $N_q = \langle a_q^\dagger a_q \rangle$

(3.136) $\leftarrow$ $$\boxed{2N_q + 1 = \langle A_q^\dagger A_q \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} n_B(\omega) B(\vec{q}, \omega)}$$

Always +ve for phonons

temp. dependent probability of having phonons with $\vec{q}$ & ω

Non-interacting spectral fn

(3.137) $$B^{(0)}(\vec{q}, \omega) = 2\pi [\delta(\omega - \omega_q) - \delta(\omega + \omega_q)]$$

Matsubara fn. have Dyson eqn. of form

(3.138 - 3.139)
$$\mathcal{G}(\vec{p}, ip_n) = \frac{1}{ip_n - \xi_{\vec{p}} - \Sigma(\vec{p}, ip_n)}$$

$$\mathcal{D}(\vec{q}, i\omega_n) = \frac{-2\omega_{\vec{q}}}{\omega_n^2 + \omega_q^2 + 2\omega_{\vec{q}} P(\vec{q}, i\omega_n)}$$

Self energy Σ & P rules will be described later.

Define retarded self energies

$$\Sigma(\vec{p}, ip_n) \xrightarrow{ip_n \to E + i\delta} \Sigma_{ret}(p, E)$$

$$= Re\, \Sigma_{ret}(\vec{p}, E) + i\, Im\, \Sigma_{ret}(\vec{p}, E)$$

$$P(\vec{q}, i\omega_n) \xrightarrow{i\omega_n \to \omega + i\delta} P_{ret}(\vec{q}, \omega)$$

$\therefore$ $$G_{ret}(p, E) = \frac{1}{E + i\delta - \xi_p - \Sigma_{ret}(\vec{p}, E)}$$

$\downarrow$

$$A(\vec{p}, E) = \frac{-2\, Im\,(\Sigma_{ret}(\vec{p}, E))}{[E - \xi_p - Re\, \Sigma_{ret}(p, E)]^2 + [Im\, \Sigma_{ret}(p, E)]^2}$$

where $Im\, \Sigma_{ret} < 0$ s.t. $A > 0$.

Example complex variable

Take $\Sigma(\vec{p}, \bar{z}) = C \ln[f(\vec{p}) - z]$

Matsubara self energy $\Sigma(\vec{p}, ip_n) = C \ln[f(p) - ip_n]$

(3.145) $$\boxed{\Sigma_{ret}(\vec{p}, E) = C \ln|f(\vec{p}) - E| - i\pi C\Theta(E - f(\vec{p}))}$$

ln of complex fns ?

$$\Sigma_{adv}(\vec{p}, E) = C \ln|f(\vec{p}) - E| + i\pi C\Theta[E - f(\vec{p})]$$

$$\boxed{\Sigma_{ret}^{*}(p, E) = \Sigma_{adv}(p, E)}$$

In region where $Im(\Sigma) = 0$

$$A(p,E) = \lim_{Im(\Sigma)\to 0} A(p,E) = 2\pi\delta(E - \xi_p - Re\Sigma_{ret}(p,E))$$

Take eqn. $\boxed{E(\vec{p}) - \mu = \xi_{\vec{p}} + Re\,\Sigma_{ret}(\vec{p}, E(\vec{p}) - \mu)}$

Recall, that if $g(x) = 0$ at $x = x_0$

$$\boxed{\delta[g(x)] = \frac{\delta(x - x_0)}{|g'(x_0)|}} \rightarrow (3.153)$$

$$\Rightarrow \boxed{A(\vec{p}, E) = 2\pi Z(\vec{p})\,\delta(E - E(\vec{p}) + \mu)} \quad — (3.154)$$

$$Z(p) = \frac{1}{1 - \frac{d}{dE}\Sigma_{ret}(p,E)\Big|_{E = E(\vec{p}) - \mu}}$$

called renormalization factor

Since $\boxed{Z(p) < 1}$

due to ~~to~~ dependence of $\Sigma(p)$

To define effective mass

Assume non-interacting states are free particles

$$- \xi_p = \frac{p^2}{2m} - \mu = \epsilon_p - \mu$$

Assume at low p, $\quad \mathcal{E}_p = \mathcal{E}_0 + \frac{p^2}{2m^*} + O(p^4)$

$$\frac{m}{m^*} = \frac{\partial \mathcal{E}_p}{\partial \epsilon_p} = \lim_{\epsilon_p \to 0} \left(1 + \frac{\partial}{\partial \epsilon_p} \text{Re}\, \Sigma(\vec{p}, \mathcal{E}) + \frac{\partial}{\partial \mathcal{E}} \text{Re}\, \Sigma(\vec{p}, \mathcal{E}) \frac{\partial \mathcal{E}_p}{\partial \epsilon_p} \right)$$

(3.160) $\Rightarrow$ $$\boxed{\frac{m}{m^*} = \lim_{\epsilon_p \to 0} \left[\frac{1 + \left(\frac{\partial}{\partial \epsilon_p}\right) \text{Re}\, \Sigma(\vec{p}, \mathcal{E}_0 - \mu)}{1 - \left(\frac{\partial}{\partial \mathcal{E}_0}\right) \text{Re}\, \Sigma(\vec{p}, \mathcal{E}_0 - \mu)} \right]}$$

3.4 DYSON'S EQUATION

(3.161) $$\mathcal{G}(\vec{p}, \tau) = - e^{\beta \Omega}\, \text{Tr} \left[e^{-\beta K} T_\tau \left(e^{\tau K} C_{\vec{p}\sigma} e^{-\tau K} \right) C^\dagger_{p\sigma} \right]$$

$$K = K_0 + V \qquad K_0 = H_0 - \mu N + V$$

$\cancel{K_0 = H_0 - \mu N}$ $\qquad H = H_0 + V$

problem that can be solved [we know complete set of states]

Usually,

(3.65) $[H_0, N] = 0$ $\qquad \therefore$ simultaneous eigenstates of
$[H, N] = 0$ $\qquad H_0$ & N can be defined.

$H_0 / K_0 \rightarrow$ non-interacting problem.

Consider interaction repn., $\boxed{\text{Define}}$ $U(\tau) = e^{\tau K_0} e^{-\tau K}$

$\cancel{U(\tau) = e^{\tau K_0} e^{-\tau K}}$ $\quad U^{-1}(\tau) = e^{\tau K} e^{-\tau K_0}$

(3.64) $- \hat{C}_{p\sigma}(\tau) = e^{\tau K_0} C_{\vec{p}\sigma} e^{-\tau K_0} = e^{-\tau \xi_{\vec{p}}} C_{\vec{p}\sigma}$

18/6/21

132 - 153
8.1 - 8.2 Bruus
~~8.3 - 8.5~~

3.161 is rewritten as: (for $\tau > 0$)

$$G(\bar{p},\tau) = -\frac{\mathrm{Tr}\left(e^{-\beta K_0} U(\beta) U^{-1}(\tau)\, \hat{C}_{\bar{p}\sigma}(\tau)\, U(\tau)\, \hat{C}^{\dagger}_{\bar{p}\sigma}\right)}{\mathrm{Tr}\left[e^{-\beta K_0} U(\beta)\right]}$$

3.170 ✓

Consider

$$\frac{\partial}{\partial \tau} U(\tau) = e^{\tau K_0} \underbrace{(K_0 - K)}_{V} e^{-\tau K} \qquad \hat{V}(\tau) = e^{\tau K_0} V e^{-\tau K_0}$$

$$= -\hat{V}(\tau)\, U(\tau)$$

138 - 153
8.1 - 8.5 Bruus

solved by repeated integration & take $U(0) = 1$

$$\Rightarrow U(\tau) = 1 - \int_0^{\tau} d\tau_1\, \hat{V}(\tau_1)\, U(\tau_1)$$

3.175 —
$$= \sum_{n=0}^{\infty} (-1)^n \int_0^{\tau} d\tau_1 \cdots \int_0^{\tau_{n-1}} \hat{V}(\tau_1) \cdots \hat{V}(\tau_n)$$

3.177: ←
$$\boxed{U(\tau) = T_\tau \exp\left[-\int_0^{\tau} d\tau_1\, \hat{V}(\tau_1)\right]}$$

3.178 — Define $S(\tau_1, \tau_2) = T_\tau \exp\left[-\int_{\tau_1}^{\tau_2} d\tau_1\, \hat{V}(\tau_1)\right]$

Which has identities:

$$S(\tau_2, \tau_1) = U(\tau_2)\, U^{-1}(\tau_1)$$
$$S(\tau_3, \tau_2)\, S(\tau_2, \tau_1) = S(\tau_3, \tau_1)$$

3.170 rewritten as:

3.182 —
$$\Rightarrow \boxed{G(\bar{p},\tau) = -\frac{\mathrm{Tr}\left[e^{-\beta K_0}\, T_\tau\, S(\beta,\tau)\, \hat{C}_{\bar{p}\sigma}(\tau)\, S(\tau,0)\, C^{\dagger}_{\bar{p}\sigma}(0)\right]}{\mathrm{Tr}\left[e^{-\beta K_0} S(\beta)\right]}}$$

$$\mathcal{G}(\vec{p},\tau) = -\frac{Tr[e^{-\beta K_0} T_\tau S(\beta)\hat{C}_{p\sigma}(\tau)\hat{C}^{\dagger}_{p\sigma}(0)]}{Tr[e^{-\beta K_0} S(\beta)]}$$

Introduce notation $Tr[e^{-\beta K_0}] = {}_0\langle\ \rangle$

$$\boxed{\mathcal{G}(\vec{p},\tau) = \frac{{}_0\langle T_\tau S(\beta)\hat{C}_{\vec{p}\sigma}(\tau)\hat{C}^{\dagger}_{\vec{p}\sigma}(0)\rangle}{{}_0\langle S(\beta)\rangle}} \quad (3.185)$$

Green fn. evaluated, (formally) by expanding S-matrix in numerator:

$${}_0\langle T_\tau S(\beta)\hat{C}_{p\sigma}(\tau)\hat{C}^{\dagger}_{p\sigma}(0)\rangle$$

$$(3.186) - \quad = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^\beta d\tau_1 \cdots \int_0^\beta d\tau_n \; {}_0\langle T_\tau \hat{C}_{p\sigma}(\tau)\hat{V}(\tau_1)\cdots\hat{V}(\tau_n)\hat{C}^{\dagger}_{p\sigma}(0)\rangle$$

evaluated by applying Wick's theorem & expressing brackets as combinations of non-interacting $\mathcal{G}(0)$ & $\mathcal{D}(0)$

Now usually ${}_0\langle AB\rangle \neq {}_0\langle A\rangle_0\langle B\rangle$ for thermodynamic averages

but while using Wick's theorem we do just that:

~~As long~~

$\therefore$ we get right answer only as long as H_0 is billinear in the operators. Errors vanish in limit of infinite volume

Exception: Macroscopic quantum states as in superfluids.

In S-matrix expansion → connected diagrams

↓ disconnected diagrams

↓ cancelled by vacuum polarization diagrams (come from denominator $\langle S(\beta)\rangle_0$)

↓ retained in expansion to obtain Green's fn.

$$e^{-\beta \Omega_1} = \langle S(\beta)\rangle_0 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta d\tau_1 \ldots \int_0^\beta d\tau_n \langle T_\tau \hat{V}(\tau_1) \ldots$$

useful quantity

↳ thermodynamic potential.

↳ calculated from linked cluster expansion

$$\therefore \quad \mathcal{G}(\vec{p},\tau) = -\sum_{n=0}^{\infty} (-1)^n \int_0^\beta d\tau_1 \ldots \int_0^\beta d\tau_n \langle T_\tau \hat{C}_{p\sigma}(\tau) \hat{V}(\tau_1) \ldots \qquad (3.196)$$

where ONLY different, connected diagrams are evaluated

Fourier

$$\boxed{\mathcal{G}(\vec{p}, ip_n) = \int_0^\beta d\tau \, e^{ip_n \tau} \mathcal{G}(\vec{p},\tau)} \qquad (3.197)$$

$p_n = (2n+1)\pi k_B T$

Terms in (3.196) yield $\boxed{\text{self energy diagrams}}$

↓ collected into Dyson's equations

(3.138) - (3.139)

In general, complex frequency is conserved at each vertex in feynman diagram.

& does maintain oddness for fermions
evenness for bosons

To find Matsubara Green fn.
→ Evaluate terms in expansion for $\mathcal{G}(\vec{p}, ip_n)$
- Draw feynman diagrams, conserve momentum & frequency at each vertex & then integrate / sum over internal variables

$\langle V(\tau_n) \rangle$

$\langle \hat{V}(\tau_n) \hat{c}^{\dagger}_{p\sigma}(0) \rangle$

Rules for constructing diagrams:

1) for each $\bar{e}$ internal line, $\mathcal{G}^{(0)}(\vec{p}, ip_n)$
2) " " " " phonon " " " " $M_q^2 D^{(0)}(\vec{q}, i\omega_m)$
3) " " " " Coulomb " " " " $v_q = \frac{4\pi e^2}{q^2}$
4) Conserve momentum & complex frequency at each vertex.

fermion freq. → $(2n+1)\pi k_B T$
Boson freq. → $2n\pi k_B T$

5) Sum over internal degrees of freedom. (momentum & frequency)
6) Multiply expression by

$$\frac{(-1)^{m+f}(2S+1)^f}{(\beta V)^m}$$

$(2S+1)^f$ → summation over spin degrees of freedom

m → order of diagram

for e^- self energy
m = no. of internal phonon + coulomb lines

f → no. of closed fermion loops.

for phonon self energy
$m = \frac{1}{2} \times$ no. of vertices

3.5 FREQUENCY SUMMATIONS

When using Matsubara Green fn.s, one must often evaluate frequency summations over combinations of unperturbed Green's fns.

TABLE OF RESULTS (for some common combinations)

Summation	Results
$-\frac{1}{\beta}\sum_m D^{(0)}(q, i\omega_m)\, G^{(0)}(\vec{p}, ip_n + i\omega_m)$	$\frac{N_{\vec{q}} + n_f(\xi_{\vec{p}})}{ip_n - \xi_{\vec{p}} + \omega_{\vec{q}}} + \frac{N_{\vec{q}} + 1 - n_f(\xi_{\vec{p}})}{ip_n - \xi_{\vec{p}} - \omega_{\vec{q}}}$
$\frac{1}{\beta}\sum_n G^{(0)}(p, ip_n)\, G^{(0)}(k, ip_n + i\omega_m)$	$\frac{n_f(\xi_p) - n_f(\xi_k)}{i\omega_m + \xi_{\vec{p}} - \xi_{\vec{k}}}$
$-\frac{1}{\beta}\sum_n G^{(0)}(\vec{p}, ip_n)\, G^{(0)}(k, i\omega_m - ip_n)$	$\frac{1 - n_f(\xi_p) - n_f(\xi_k)}{i\omega_m - \xi_p - \xi_k}$
$\frac{1}{\beta}\sum_n G^{(0)}(\vec{p}, ip_n)$	$n_f(\xi_p)$

Take boson series (even integer $\omega_m = 2\pi m k_B T$)

Consider $S = \frac{1}{\beta}\sum_m \frac{2\omega_{\vec{q}}}{\omega_m^2 + \omega_{\vec{q}}^2} \cdot \frac{1}{ip_n + i\omega_m - \xi_{\vec{p}}}$

(3.220)

$\downarrow$

$$-\frac{1}{\beta}\sum_m \underbrace{f(i\omega_m)}_{\text{Product of Green's fns}}$$

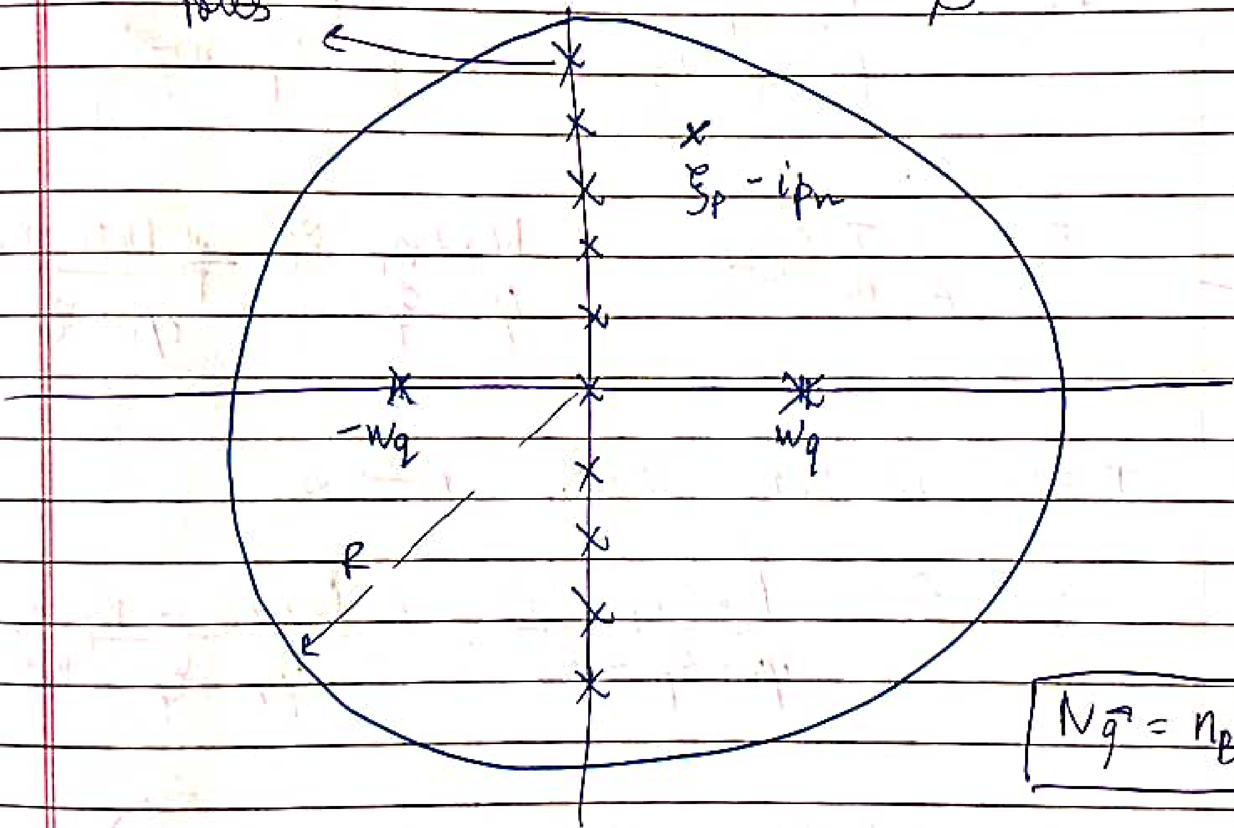
Contour integration is used

$$I = \lim_{R \to \infty} \oint \frac{dz}{2\pi i} f(z)\, n_B(z) \quad \text{— (3.222)}$$

$n_B(z) \to$ Chosen to generate poles at points $i\omega_n$ for all even integer m.
↳ fn. which does this

$$n_B(z) = \frac{1}{e^{\beta z} - 1} \quad \text{— (3.223)}$$

Poles at at the points $i2\pi \cdot m k_B T$ ~~with~~ $\forall$ m (positive - negative integer & $m = 0$) -
Residue at these poles is $\frac{1}{\beta}$.



$$\boxed{N_q = n_B(\omega_q)}$$

$$f(z) = \underbrace{\frac{2\omega_q}{z^2 - \omega_q^2}}_{\text{Poles at } \pm \omega_q} \cdot \underbrace{\frac{1}{ip_n + z - \xi_p}}_{\text{Poles at } \xi_p - ip_n}$$

∴ Poles & their residue for I in (3.222)

$z_m = i2\pi m k_B T$ $\qquad R_i = \frac{1}{\beta} f(i\omega_m)$

$z_1 = \omega_q$ $\qquad R_1 = \frac{N_q}{ip_n - \xi_p + \omega_q}$

$$z_2 = -w_{\vec{q}} \qquad R_2 = \frac{N_{\vec{q}}+1}{ip_n - \xi_{\vec{p}} - w_{\vec{q}}}$$

(3.225) $$z_3 = \xi_{\vec{p}} - ip_n, \qquad R_3 = \frac{-2w_{\vec{q}}\, n_F(\xi_{\vec{p}})}{(ip_n - \xi_{\vec{p}})^2 - w_{\vec{q}}^2}$$

because $\exp(ip_n\beta) = -1$,

$$\boxed{n_B(\xi_p - ip_n) = \frac{1}{e^{\beta(\xi_p - ip_n)} - 1} = \frac{-1}{e^{\beta\xi_p}+1} = -n_F(\xi_{\vec{p}})} \quad (3.226)$$

Partial fractions,

$$R_3 = \frac{n_F(\xi_{\vec{p}})}{ip_n - \xi_p + w_q} - \frac{n_F(\xi_{\vec{p}})}{ip_n - \xi_p - w_{\vec{q}}}$$

$$\therefore\ I = \frac{1}{\beta}\sum_m f(i\omega_m) + \frac{N_q + n_F(\xi_{\vec{p}})}{ip_n - \xi_p + w_q} + \frac{N_q + 1 - n_F(\xi_{\vec{p}})}{ip_n - \xi_{\vec{p}} - w_{\vec{q}}}$$

Now, $I = 0$ in the limit $R \to \infty$,

$$\Rightarrow\ S = \frac{N_q + n_F(\xi_{\vec{p}})}{ip_n - \xi_p + w_q} + \frac{N_q + 1 - n_F(\xi_{\vec{p}})}{ip_n - \xi_{\vec{p}} - w_{\vec{q}}}$$

$\therefore$ To evaluate series such as (3.221) one simply finds simple poles of $f(z)$ which are at points z_j with residue r_j

$$S = \sum_j r_j\, n_B(z_j) \quad (3.229)$$

For fermions, identify

(3.230) — $$S = -\frac{1}{\beta}\sum_n f(ip_n)$$

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Date
Page

fn. $n_F(z)$ is used $\frac{1}{e^{\beta z}+1}$ → poles at ip_n with $-\frac{1}{\beta}$ residues

(3.231) — $$S = -\sum_i q_i\, n_F(z_i)$$

4th result (3.219) $$\frac{1}{\beta}\sum_n G^{(0)}(\vec{p}, ip_n) = n_F(\xi_p)$$

LHS is from F.T. of $\mathcal{G}^{(0)}(\vec{p},\tau)$

$$\mathcal{G}^{(0)}(\vec{p},\tau) = \frac{1}{\beta}\sum_n e^{-ip_n\tau}\, G^{(0)}(\vec{p}, ip_n)$$

$$= -{}_0\langle T_\tau\, c_{p\sigma}(\tau) c^\dagger_{p\sigma}(0)\rangle$$

(3.219) is just this as limit $\tau \to 0$.

Diff. result when approached from +ve or -ve side

(3.238)
$$\mathcal{G}^{(0)}(\vec{p}, \tau = 0^+) = -{}_0\langle c_{p\sigma} c^\dagger_{p\sigma}\rangle = -[1 - n_F(\xi_{\vec{p}})]$$

(3.239)
$$\mathcal{G}^{(0)}(\vec{p}, \tau = 0^-) = -{}_0\langle c^\dagger_{p\sigma} c_{p\sigma}\rangle = n_F(\xi_{\vec{p}})$$

convention of adopting $\tau = 0^-$.

Next: How to handle when $\mathcal{G}$ is present instead of $\mathcal{G}^{(0)}$. (I integral has branch cuts)

$$\mathcal{G}(\vec{p}, ip_n) = \frac{1}{ip_n - \xi_{\vec{p}} - \Sigma(\vec{p}, ip_n)}$$

Leads to branch cuts when $z = ip_n$. Contour integrals become more complicated

Most general possibility → All Green fns fully dressed.

↓ Proceed by expressing in Lehmann representation.

Example

$$S = -\frac{1}{\beta} \sum_{n} D(\vec{q}, i\omega_m)\, G(\vec{p}, ip_n + i\omega_m)$$

Express Green fns. as freq. integral over spectral fns.

(3.24)
$$D(\vec{q}, i\omega_m) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{B(\vec{q},\omega)}{i\omega_m - \omega}$$
$$G_2(\vec{p}+\vec{q}, ip_n + i\omega_m) = \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \frac{A(\vec{q}+\vec{p},\epsilon)}{ip_n + i\omega_m - \epsilon}$$

$$\Longrightarrow \quad S = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} B(\vec{q},\omega) \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} A(\vec{p}+\vec{q},\epsilon)\, S_0(ip_n,\omega,\epsilon)$$

$$\left[S_0(ip_n,\omega,\epsilon) = -\frac{1}{\beta} \sum_{m} \frac{1}{i\omega_m - \omega} \cdot \frac{1}{ip_n + i\omega_m - \epsilon} = \frac{n_B(\omega) + n_F(\epsilon)}{ip_n + \omega - \epsilon} \right.$$

Simple sum over Matsubara frequencies

$$\longrightarrow \boxed{S = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} B(\vec{q},\omega) \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} A(\vec{p}+\vec{q},\epsilon) \left[\frac{n_B(\omega) + n_F(\epsilon)}{ip_n + \omega - \epsilon} \right]}$$

Electron in disordered potential

$$H = \sum_{\vec{k}} \epsilon_{\vec{k}} c^{\dagger}_{\vec{k}} c_{\vec{k}} + V_{\text{disorder}}$$

$$\int d^3x \; U(\vec{x})\, \psi^{\dagger}(\vec{x})\, \psi(\vec{x})$$

$U(\vec{x})$ scattering potential by random array of N_i impurities located at position $\vec{R}_j$, each with atomic potential $\mathcal{U}(\vec{x}-\vec{R}_j)$

$$U(\vec{x}) = \sum_j \mathcal{U}(\vec{x}-\vec{R}_j)$$

<u>Note</u>: • H contains no. interaction b/w e^-
$\Rightarrow$ energy of each individual e^- is conserved
All interactions are elastic.

We're interested in calculating physical quantities <u>averaged</u> over locations of impurities

$$\overline{\langle A \rangle} = \int \prod_j \frac{1}{V} d^3R_j \; \langle \hat{A}[\{R_j\}] \rangle$$

quenched average (impurity average takes place <u>after</u> thermodynamic average)

<u>To calculate</u>: Impurity averaged Green's fn

Fluctuations of impurity scattering potential about its average scatter the electrons.

Avg. impurity potential $\overline{U(\vec{x})}$ plays role of shifted chemical potential.
If shift chem. potential by $\Delta\mu$,

$$U(\vec{x}) \longrightarrow U(\vec{x}) - \Delta\mu$$

we can choose $\Delta\mu = \overline{U(\vec{x})}$

s.t. $\overline{U'(\vec{x})} = \overline{U(\vec{x})} - \overline{U(\vec{x})}$
$= 0$

Residual $\delta U(\vec{x}) = U(\vec{x}) - \overline{U(\vec{x})}$
describes fluctuations in scattering potential

we shall show:

(8.78) — $$\boxed{\overline{\delta U(\vec{x})\,\delta U(\vec{x}')} = \int_{\vec{q}} e^{i\vec{q}\cdot(\vec{x}-\vec{x}')}\, n_i\, |u(\vec{q})|^2}$$

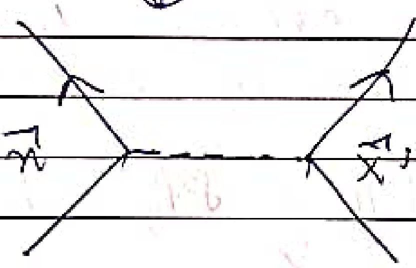
ie fluctuations are spatially correlated.

$$\boxed{u(\vec{q}) = \int d^3x\, U(\vec{x})\, e^{-i\vec{q}\cdot\vec{x}}}$$

$n_i = \frac{N_i}{V}$ → concentration of impurities

F.T. of scattering potential

Spatial correlations induce attractive interaction denoted by diagram



$$\int n_i\, |u(\vec{q})|^2 e^{i\vec{q}\cdot(\vec{x}-\vec{x}')}$$
$$= -V_{eff}(\vec{x}-\vec{x}')$$

(8.79)

Note: · fluctuations in scattering potential are short range — extend over range of scattering potential.

Assume impurity scattering dominated by low-energy s-wave scattering, then $u(\vec{q}) = u_0$

here, fluctuations are purely local

$$\overline{\delta U(\vec{x})\,\delta U(\vec{x}')} = u_0^2 n_i \int_{\vec{q}} e^{i\vec{q}\cdot(\vec{x}-\vec{x}')} = n_i u_0^2\, \delta(\vec{x}-\vec{x}')$$

white noise potential

Here, we neglect higher-order moments → assuming it is purely Gaussian

$$U(\vec{q}) = \sum_j e^{-i\vec{q}\cdot\vec{R}_j} \int d^3x \, \mathcal{U}(x-\vec{R}_j)\, e^{-i\vec{q}\cdot(x-\vec{R}_j)}$$

$$= u(\vec{q}) \underbrace{\sum_j e^{-i\vec{q}\cdot\vec{R}_j}}$$ — (8.80)

locations of impurities encoded in phase shifts that multiply $u(\vec{q})$

Carry out average

$$\boxed{\overline{\delta u(x)\delta u(x')} = \int_{\vec{q},\vec{q}'} e^{i(\vec{q}\cdot x-\vec{q}'\cdot x')}\left[\overline{U(\vec{q})U(-\vec{q}')} - \left(\overline{U(\vec{q})}\right)\left(\overline{U(-\vec{q}')}\right)\right]}$$

(8.81)

$$= \int_{\vec{q},\vec{q}'} e^{i(\vec{q}\cdot\vec{x}-\vec{q}'\cdot\vec{x}')} u(\vec{q})u(-\vec{q}') \sum_{i,j}\left(\overline{e^{-i\vec{q}\cdot\vec{R}_i} e^{i\vec{q}'\cdot\vec{R}_j}} - \overline{e^{-i\vec{q}\cdot\vec{R}_i}} \cdot \overline{e^{i\vec{q}'\cdot\vec{R}_j}}\right)$$

Phase terms independent at diff. sites
⇒ Variance of random phase term vanishes unless $i=j$

$$\Rightarrow \sum_{i,j}\left(\overline{e^{-i\vec{q}\cdot\vec{R}_i} e^{i\vec{q}'\cdot\vec{R}_j}} - \overline{e^{-i\vec{q}\cdot\vec{R}_i}} \cdot \overline{e^{i\vec{q}'\cdot\vec{R}_j}}\right)$$

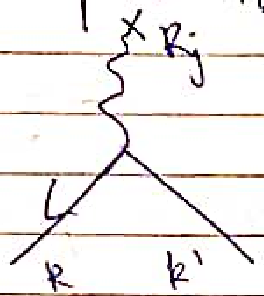
$$= N_i \int \frac{1}{V} d^3R_j \, e^{-i(\vec{q}-\vec{q}')\cdot\vec{R}_j}$$

$$= n_i (2\pi)^3 \delta^3(\vec{q}-\vec{q}')$$

$$\Rightarrow \overline{U(\vec{q})U(-\vec{q}')} - \overline{U(\vec{q})}\,\overline{U(-\vec{q}')} = n_i |u(\vec{q})|^2 (2\pi)^3 \delta^3(\vec{q}-\vec{q}')$$

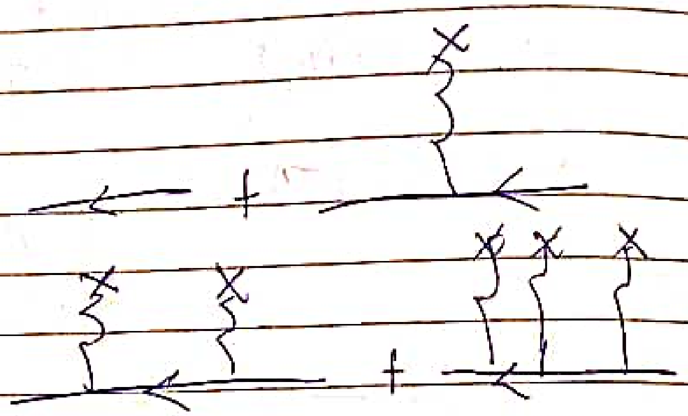
$$\hat{V} = \int_{\vec{k},\vec{k}'} c^{\dagger}_{\vec{k}} c_{\vec{k}'}\, \underbrace{\delta U(\vec{k}-\vec{k}')}_{u(\vec{k}-\vec{k}')\sum_j e^{i(\vec{k}-\vec{k}')\cdot\vec{R}_j}} - \Delta\mu\, \delta_{\vec{k}-\vec{k}'} \quad (8.83)$$

Represented by



freq. conserved along e^- line because elastic scattering

$$\mathcal{G}(\vec{k},\vec{k}',i\omega_n) = \longleftarrow + \ldots + \ldots + \ldots + \ldots$$



$$= \mathcal{G}^{(0)}(\vec{k},i\omega_n)\,\delta_{\vec{k},\vec{k}'} + \mathcal{G}^{(0)}(\vec{k},i\omega_n)\,\delta U(\vec{k}-\vec{k}')\,\mathcal{G}^{(0)}(\vec{k}',i\omega_n)$$

(8.84) —

$$+ \int_{\vec{k}_1} \mathcal{G}^{(0)}(\vec{k},i\omega_n)\,\delta U(\vec{k}-\vec{k}_1)\,\mathcal{G}^{(0)}(\vec{k}_1,i\omega_n)\,\delta U(\vec{k}_1-\vec{k}')\,\mathcal{G}^{(0)}(\vec{k}',i\omega_n)$$

$$+ \ldots$$

$\mathcal{G}$ is a fn. of each impurity position!

Want to calculate $\overline{\mathcal{G}(\vec{k},\vec{k}',i\omega_n)}$ quenched average

→ need to average each Feynman diagram in above series.

when we average,

Single scattering event $\quad \overline{\delta U(\vec{k}-\vec{k}')} = 0$

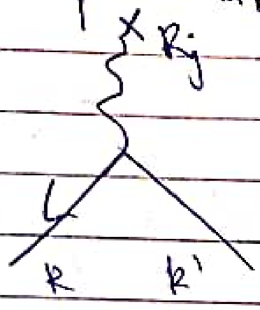
Double scattering event $\quad \overline{\delta U(k-k_1)\,\delta U(k_1-k')} = n_i\,|u_{\vec{k}-\vec{k}'}|^2\,\delta_{k}$

$$\hat{V} = \int_{\vec{k},\vec{k}'} c^\dagger_{\vec{k}}\, c_{\vec{k}'}\, \underline{\delta U(\vec{k}-\vec{k}')}$$

$$\downarrow$$

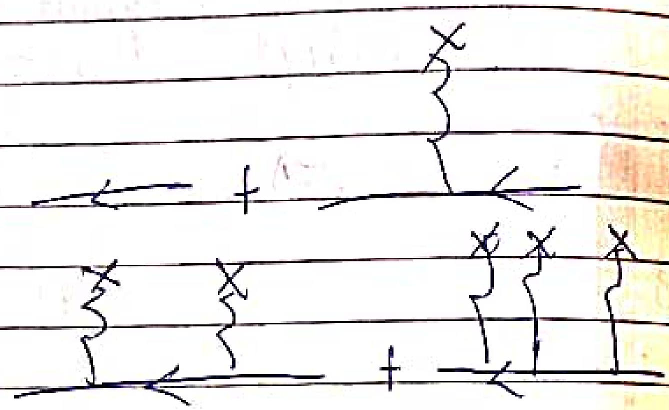
$$u(\vec{k}-\vec{k}') \sum_j e^{i(\vec{k}-\vec{k}')\cdot\vec{R}_j} - \Delta\mu\, \delta_{\vec{k}-\vec{k}'} \quad \text{— (8.83)}$$

Represented by



freq. conserved along e^- line because elastic scattering

$$\mathcal{G}(\vec{k},\vec{k}',i\omega_n) =$$



$$+ \cdots$$

$$= \mathcal{G}^{(0)}(\vec{k},i\omega_n)\, \delta_{\vec{k},\vec{k}'} + \mathcal{G}^{(0)}(\vec{k},i\omega_n)\, \delta U(\vec{k}-\vec{k}')\, \mathcal{G}^{(0)}(\vec{k}',i\omega_n)$$

(8.84) —

$$+ \int_{\vec{k}_1} \mathcal{G}^{(0)}(\vec{k},i\omega_n)\, \delta U(\vec{k}-\vec{k}_1)\, \mathcal{G}^{(0)}(\vec{k}_1,i\omega_n)\, \delta U(\vec{k}_1-\vec{k}')\, \mathcal{G}^{(0)}(\vec{k}',i\omega_n)$$

$$+ \cdots$$

$\mathcal{G}$ is a fn. of each impurity position!

Want to calculate $\overline{\mathcal{G}(\vec{k},\vec{k}',i\omega_n)}$ quenched average
$\Rightarrow$ need to average each Feynman diagram in above series.

whenever we average,

Single scattering event $\quad \overline{\delta U(\vec{k}-\vec{k}')} = 0$

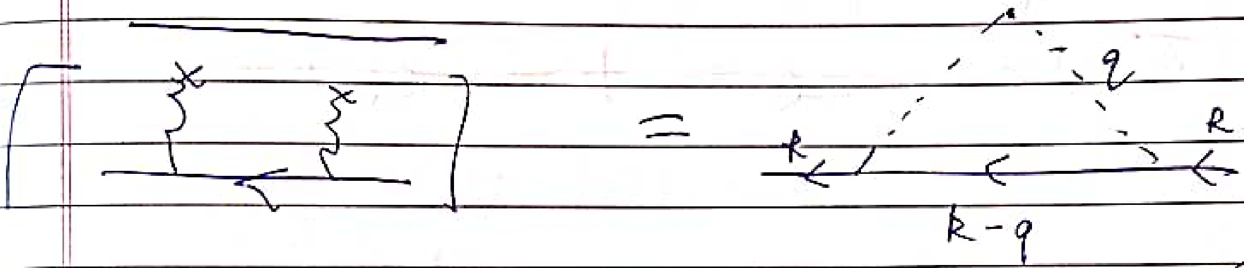
Double scattering event $\quad \overline{\delta U(k-k_1)\,\delta U(k_1-k')} = n_i |u_{\vec{k}-\vec{k}_1}|^2 \delta_{\vec{k}-\vec{k}'}$

∴ 3rd term in expansion

$$= \int_{\vec{k}_1} \overbrace{\delta U(\vec{k}-\vec{k}_1)\,\delta U(\vec{k}_1-\vec{k}')}^{n_i |u(\vec{k}-\vec{k}')|^2 \delta_{\vec{k}-\vec{k}'}}\, \mathcal{G}^{(0)}(\vec{k}, i\omega_n)\,\mathcal{G}^0(\vec{k}_1, i\omega_n)\,\mathcal{G}^0(\vec{k}', i\omega_n)$$

$$= \delta_{\vec{k}-\vec{k}'} \times \mathcal{G}^0(\vec{k}, i\omega_n)^2\, n_i \sum_{\vec{k}_1} u(\vec{k}-\vec{k}_1)^2\, \mathcal{G}^0(\vec{k}_1, i\omega_n)\, \cancel{\mathcal{G}^0(\vec{k}, i\omega_n)}$$

Note: After impurity avg. ⇒ momentum is now conserved!



(8.86)

quenched disorder can be thought of as interaction with an effective potential

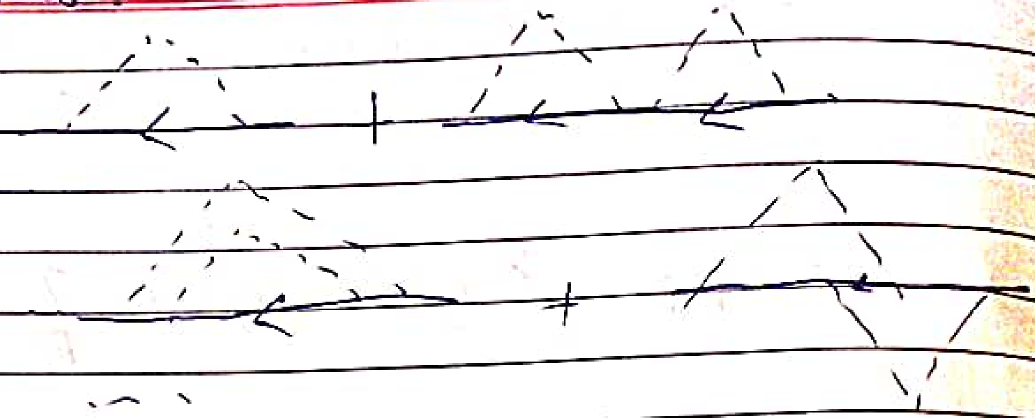
$$V_{eff}(\vec{q}, i\nu_n) = \int_0^\beta d\tau\, e^{i\nu_n \tau}\, \underbrace{V_{eff}(\vec{q},\tau)}_{-n_i |u(\vec{q})|^2}$$

$$= -\beta\,\delta_{n0}\, n_i\, |u(\vec{q})|^2$$

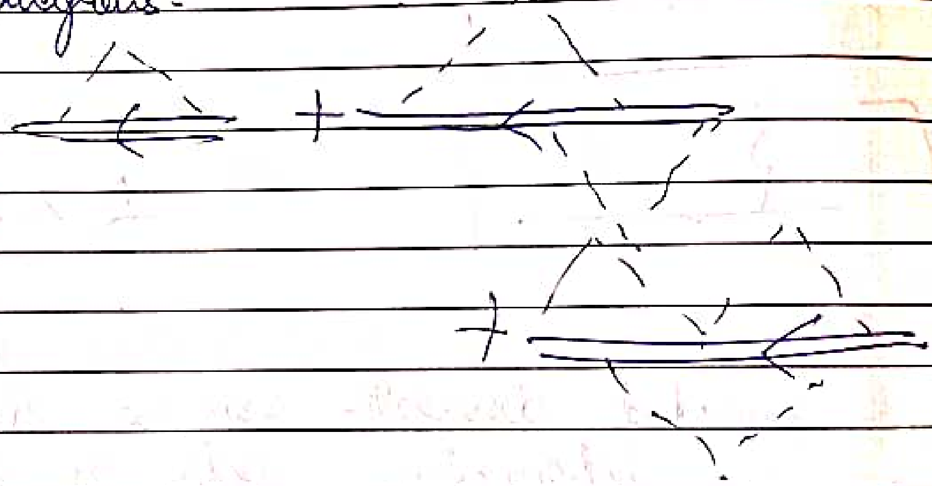
$$\boxed{\beta\,\delta_{n0} = \int d\tau\, e^{i\nu_n \tau}}$$

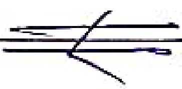
So $\mathcal{G}(\vec{k}, \vec{k}', i\nu_n) = \delta_{\vec{k}-\vec{k}'}\, \mathcal{G}(\vec{k}, i\nu_n)$

If we take account of all scattering events induced by Gaussian fluctuations, we generate a series of diagrams:

$G(k) =$ 

We can group all scattering into connected self energy diagrams:

$\Sigma(k) =$ 

$G(k) = \frac{1}{i\omega_n - \epsilon_k - \Sigma(k)}$ 

For s wave scattering, $\Sigma(k) = \Sigma(i\omega_n)$ & momentum dependence is lost.

Self-consistent solving: $\Sigma(k)$ calculated, then used for $G(k)$, which is used to update $\Sigma(k)$ & so on.

Calculate 1st order diagrams for self-energy (without imposing self consistency)

$$\Sigma(i\omega_n) = \text{---}\!\leftarrow\!\text{---} = n_i \sum_{\vec{k}'} |u_k(\vec{k}-\vec{k}')|^2 \, G(\vec{k}', i\omega_n)$$

$$= n_i \sum_{\vec{k}'} |u(\vec{k}-\vec{k}')|^2 \frac{1}{i\omega_n - \epsilon_{\vec{k}'}}$$

$$\sum_{k'} \rightarrow \int \frac{d\Omega_{\vec{k}'}}{4\pi} \, d\epsilon' \, \underset{\text{D.O.S.}}{\underbrace{N(\epsilon')}}$$

[Replace int by momentum with integral over solid angle & energy

$$\Sigma(i\omega_n) = n_i u_0^2 \int d\epsilon \, N(\epsilon) \frac{1}{i\omega_n - \epsilon}$$

$$\frac{u_0^2}{d} = \int \frac{d\Omega_{\vec{k}'}}{4\pi} |u(\vec{k}-\vec{k}')|^2 = \frac{1}{2}\int_{-1}^{1} d\cos\theta \, |u(\theta)|^2$$

angular average of squared scattering amplitude

To Good approximation → this can be calc. by replacing energy dependent D.O.S. by value at Fermi energy.

$$\therefore \Sigma(i\omega_n) - n_i u_0^2 N(0) \int_{-\infty}^{\infty} d\epsilon \frac{1}{i\omega_n - \epsilon} = \frac{-i}{2\tau} \operatorname{sgn}(\omega_n)$$

where $\frac{1}{\tau} = 2\pi n_i u_0^2$ → e^- scattering rate

Note: limits of integration extended to infinity → approximation that neglects terms of order $\frac{1}{\epsilon_F \tau}$.

$$\therefore \; G(\vec{k}, z) = \frac{1}{z - \epsilon_{\vec{k}} + \frac{i}{2\tau}\,\mathrm{sign}(\mathrm{Im}\, z)}$$

Spectral fn. $A(\vec{k}, \omega) = \frac{1}{\pi} \mathrm{Im}\, G(\vec{k}, \omega - i\delta)$

$$= \underbrace{\frac{1}{\pi} \frac{(2\tau)^{-1}}{(\omega - \epsilon_k)^2 + (2\tau)^{-2}}}_{\text{Lorentzian of width}}$$

$\Leftarrow$ $\bar{e}$ of momentum $\vec{k}$ now has lifetime τ.