AdS-CFT correspondence and its implication for condensed matter systems

Suyash Singh, 2018PH1053

1 Problem statement:

Show how ADS-CFT correspondence relates a gravity theory in d + 1 dimensional in ADS space -time to the d-dimensional field theory of a correlated system living on its boundary.
Now illustrate with the help of a condensed matter system how this method can be used to calculate correlation or response functions in such real condensed matter system.

2 Solution:

Before explaining the AdS-CFT correspondence, let us consider the meaning of the two parts individually first.

2.1 Anti-deSitter spacetime (AdS spacetime)

The anti-deSitter spacetime is the solution to the Einstein field equations in a vaccum, and with negative cosmological constant Λ [1]. In suitable coordinates, the AdS metric can be expressed as:

$$ds^2 = \frac{L^2}{r^2}(-dt^2 + dr^2 + dx^i dx^i) \tag{1}$$

where i runs over the 'spatial' dimensions.
It leads to the following non-vanishing Christoffel symbols:

$$\Gamma^x_{xr} = \Gamma^x_{rx} = -r^{-1} \qquad \Gamma^r_{xx} = r^{-1} \qquad \Gamma^r_{rr} = -r^{-1}$$

which eventually leads us to the following radius of curvature:

$$R = \frac{-2}{L^2} \tag{2}$$

which is negative. Note that when we slice this space-time up at a particular $r = r_0$, we get the metric:

$$ds^2 = \frac{L^2}{r_0^2}(-dt^2 + dx^i dx^i). \tag{3}$$

This represents a flat Minkowski space-time, with scaled coordinates. Therefore, the anti-deSitter space of d+1 dimensions can be thought of as a collection of flat Minkowski spaces of d dimensions. This fact is crucial for the correspondence since our quantum field theories are constructed on flat Minkowski spaces.
Traveling along r direction in anti-deSitter space, the length of line segment to go from r1 to r2, x coordinates remain unchanged, is:

$$\begin{aligned} \Delta_s &= \int_{r1}^{r2} \sqrt{g_{\mu\nu} \frac{dx^\mu}{dr} \frac{dx^\nu}{dr}} \, dr \\ &= L \ln(\frac{r2}{r1}) \end{aligned} \tag{4}$$

As $r1 \to 0$, this length becomes infinite irrespective of r2. Due to this reason, we identify $r = 0$ with the boundary of this space-time, $r \to \infty$ as the horizon.

2.2 Conformal transformations

In general, quantum field theories are set to obey the symmetries of the Poincare group: space-time translations, Lorentz boosts and rotations. While studying quantum phase transitions, the fixed points of these transitions obey these symmetries. But they also remain invariant under conformal transformations, which are defined by:

$$d\tilde{s}^2 = \Omega(x) ds^2 \tag{5}$$

where $\Omega(x)$ leads to a local change of scales. These transformations preserve angles between vectors. A special case of conformal transformations called dilatations occur when $\Omega(x) = \lambda$, Ω becomes independent of space-time. These are also called scale transformations.
Therefore, the study of such quantum critical behavior motivates us to look for theories that satisfy all of these symmetries. One metric that fulfills these requirements, in 1 dimension greater than that in which the field theory exists, is:

$$ds^2 = \frac{L^2}{r^2}(-dt^2 + dr^2 + dx^i dx^i) \tag{6}$$

It's easy to see that $(t, x^i, r) \to (\lambda t, \lambda x^i, \lambda r)$ leaves this metric unchanged. This is nothing but the metric for AdS spacetime in (1)! Once we identify (t, x^i) as the parameters of the d dimensional space where the field theory lies and r as the extra coordinate running from $r = 0$ 'boundary' to $r = \infty$ 'horizon', we are now in a position to establish the AdS-CFT correspondence.

To this extent, the duality between the two theories was conjectured by Maldacena [6], but he did not specify the way the two theories should be mapped. We will try to understand the mapping proposed by Gubser, Klebanov and Polyakov [3] and by Witten [2], but first we will take a digression into understanding the role of partition function and linear response theory in understanding behavior of condensed matter systems.

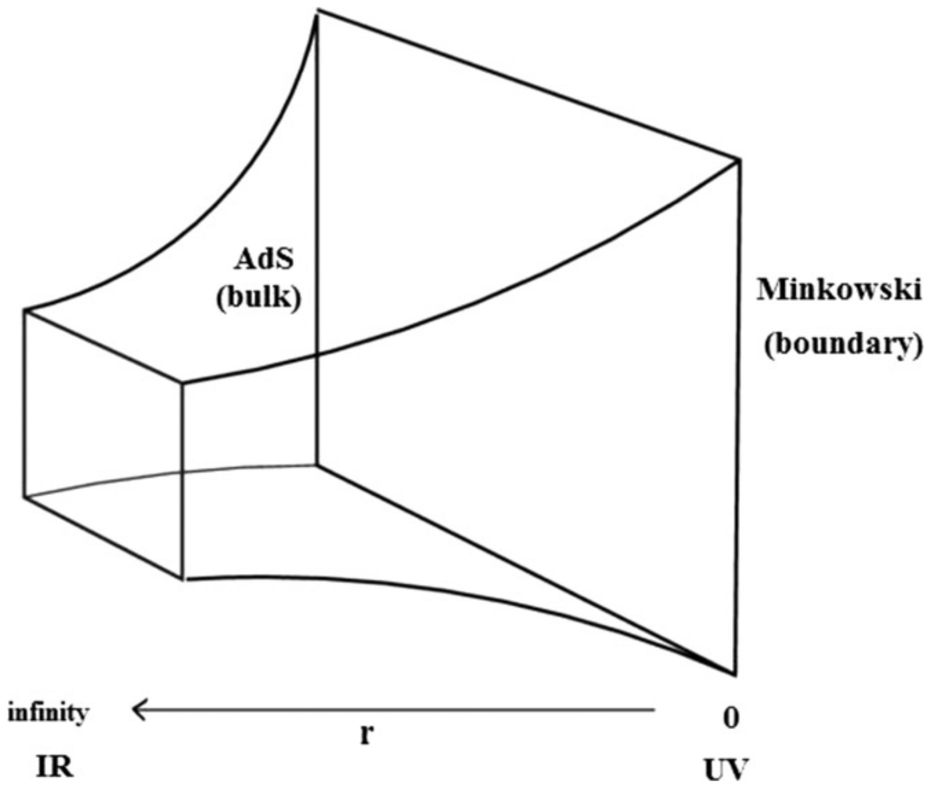

Figure 1: The field theory resides on the Minkowski boundary of AdS spacetime [7]

2.3 Partition function and linear response

In statistical mechanics, the partition function Z is defined by:

$$Z = \exp\left[-\beta F\right] \tag{7}$$

where F is the free energy of the system, and $\beta = 1/k_B T$. From Z, in principle, we can obtain all the information about the equilibrium properties of the system. We are interested in calculating Z when we apply some external perturbation to the condensed matter system that we wish to study, so that the response can help us understand the physical properties of the system.
When an external field, say J(x), is applied to a system via coupling to a local operator, labelled $\hat{O}$, the partition function is given by:

$$Z = tre^{-\beta[H_0 - \int \hat{O} J(x)\, d\vec{x}]} \tag{8}$$

where we term J(x) as the source, and $\hat{O}$ as the response. H_0 describes the unperturbed equilibrium state Hamiltonian. As per linear response theory, we can find the n point correlation functions of the local operator $\hat{O}$ by differentiation the natural logarithm of Z by the source function at different points, and then taking vanishing source strength. For example, the 2-point correlation function can be calculated as:

$$\ll O(\hat{x})O(\hat{y}) \gg = \frac{\delta}{\delta J(x)}\frac{\delta}{\delta J(y)} \ln Z|_{J=0} \tag{9}$$

These functions further help us calculate many of the static and dynamical properties of the system, including transport coefficients. Therefore, calculating Z for critical systems becomes crucial to characterizing their behavior. Let us see how AdS/CFT correspondence becomes a powerful tool in this regard.

2.4 AdS-CFT Correspondence: GPKW master rule

The GPKW master rule, named after the people who first proposed it: Gubser, Klebanov, Polyakov and Witten, precisely maps AdS and CFT quantities in the following way:

$$\langle \exp\left(\int_{boundary} d^d x \phi_0 \hat{O}\right)\rangle = e^{-S[\phi]} \tag{10}$$

Here $\phi(x,r)$ is a dynamical gauge field in the AdS space which at the boundary tends to the source function of the CFT: $\phi(x,r)|_{r=0} \to J(x)$. Therefore, the left hand side becomes the generating functional for the correlators in the CFT, which exists at the boundary of the AdS space-time.
The right hand side of (10) is the exponent of the action evaluated classically by solving for ϕ in the bulk of the AdS space-time with the boundary condition $\phi|_{r=0} = \phi_0$. In this manner, the gravity theory in d+1 dimension (r is the extra dimension) is related to the d-dimensional conformal field theories of a correlated system on this boundary ($r = 0$).
There is a ϕ corresponding to every $\hat{O}$ in this dual field theory. For example, $T_{\mu\nu}$ is considered to be the response of a local QFT to a local change in metric $g_{\mu\nu}$ in the bulk AdS space-time. Similarly, electromagnetic gauge fields in the bulk correspond to currents in the boundary theory.
All of the connections and properties of the AdS-CFT correspondence, albeit interesting, are **not** what make it extremely powerful. The main utility of the correspondence is in the fact that it is a strong-weak coupling duality. So, any process of energy scale E in the boundary will have a corresponding process of scale $1/E$ in the bulk[1]. This allows us to relate strongly coupled (strongly interacting) systems in the boundary CFT to weakly interacting gravity theory in the bulk: thereby allowing the use of perturbation theory.

2.5 Effect of finite temperature and chemical potential

Results stated so far are for zero temperature field theories with zero chemical potential. To treat non-zero T and μ, we need to modify our bulk theory to

[1] A rigorous explanation of this strong-weak coupling requires understanding of the renormalization group (RG) and RG flows, which the author does not have.

account for the broken symmetries.
Finite temperature or finite chemical chemical potential are said to break the scaling invariance of space-time. However, we will require that the bulk theory be asymptotically AdS (i.e. that it recovers scaling invariance as $r \to 0$). A suitable metric which satisfies this property and obeys Einstein's field equations is:

$$ds^2 = \frac{L^2}{r^2}(-f(r)dt^2 + \frac{dr^2}{f(r)} + dx^i dx^i) \tag{11}$$

with

$$f(r) = 1 - (\frac{r}{r_+})^d \tag{12}$$

where r_+ is some positive constant. We clearly see that as $r \to 0$, $f(r) \to 1$: which when substituted in (11) leads us back to AdS metric of (1).
To examine the space-time defined by (11) more closely, consider a light ray travelling along the r direction: $(ds^2 = 0,\ dx^i = 0)$

$$\frac{dt}{dr} = \pm\frac{1}{f(r)^2} \tag{13}$$

Now, as $r \to r_+$, $f(r) \to 0$. This would lead RHS of (13) to tend to infinity, making the escape of even light from $r = r_+$ impossible. This motivates us to identify $r = r_+$ as the horizon of a black hole! Events at $r > r_+$ won't influence the boundary at $r = 0$ in any way, because they lie behind the event horizon of the black hole.
To relate this black hole (defined so far by its radius r_+) to the finite temperature physics of the condensed matter system, consider an analytic continuation to imaginary time $t = \iota\tau$. (Note that imaginary time formalism is the basis for purely field theoretic treatment of finite temperature many-body physics as well: as in the Matsubara formalism where the partition function at temperature T is defined in terms of a path integral of imaginary time over a periodic time path.) Our metric becomes:

$$ds_E^2 = \frac{L^2}{r^2}(f(r)d\tau^2 + \frac{dr^2}{f(r)} + dx^i dx^i) \tag{14}$$

Near $r = r_+$, we can expand f(r) in a Taylor series up to 1st order. Noting that $f(r = r_+) = 0$, our expansion becomes:

$$f(r) \approx f'(r_+)(r - r_+) \tag{15}$$

Putting this into (14), we obtain in the vicinity of the horizon:

$$ds_E^2 \approx \frac{f'(r_+)(r-r_+)d\tau^2}{r_+^2} f(r)d\tau^2 + \frac{dr^2}{f'(r_+)(r-r_+)r_+^2} + \frac{dx^i dx^i}{r_+^2} \tag{16}$$

Notice that there is a singularity at $r - r_+ = 0$. This is a coordinate singularity which can be removed by the following transformation for $t, r \to \rho, \phi$:

$$d\rho^2 = \frac{dr^2}{f'(r_+)(r - r_+)r_+^2} \tag{17}$$

$$\rho = \frac{2}{r_+\sqrt{f'(r_+)}} \tag{18}$$

$$\phi = \beta\tau \tag{19}$$

Substituting these in (16), and ignoring dx^i terms since they were not causing us a problem anyway, we get:

$$ds_E^2 = d\rho^2 + \rho^2 d\phi^2 \tag{20}$$

where we identify $\beta = f'(r_+)/2$. Using (12), we calculate and obtain $f'(r+) = d/r_+$.
The metric defined in (20) is exactly of the form of polar coordinates on a sphere. It is well defined if ϕ has a period of 2π like in a sphere. Imposing this condition:

$$\phi = \beta\tau \to \beta\tau + 2\pi \qquad\qquad \tau \to \tau + \frac{2\pi}{\beta}$$

Identify the inverse of this period as the Hawking temperature, and substitute value of β to get:

$$T = \frac{d}{4\pi r_+} \tag{21}$$

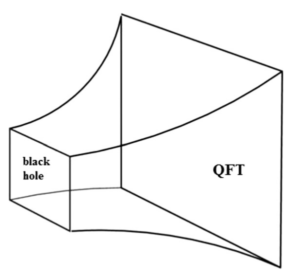

Figure 2: A QFT with finite temperature T is coupled to black-hole of same Hawking temperature[7]

The Hawking temperature (21) of the black hole is mapped to the temperature of the field theory on the boundary. We can see how, as $T \to 0$, $r_+ \to \infty$, which is what we took as the horizon for our zero temperature formalism ($r \to \infty$). Within this scheme, the entropy of the system living on the boundary is equal to the entropy of the black hole defined by the aforementioned Hawking temperature. Therefore, finding the entropy of a strongly correlated system becomes equivalent to finding the area of the black hole horizon [2]for the dual weakly coupled gravity theory, a much easier task!

For finite chemical potential, we first notice that in the boundary theory, μ has the effect of fixing the number of particles (usually charged e.g. electrons). This becomes equivalent to conserving the total charge of the boundary theory (since its a product of number of charges with the unit charge). As per GPKW scheme, the conserved current J^μ in the boundary corresponds to gauge fields A^μ in the bulk.
It can be shown that the effect of this ultimately boils down to having a charge on the black hole Q which is proportional to the chemical potential [7]. However, proving this is beyond the scope of the author's abilities.

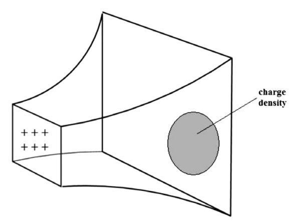

Figure 3: Chemical potential in boundary QFT leads to charged black holes[7]

2.6 AdS/CFT in Condensed matter systems

Even the simplest finite temperature calculations using AdS/CFT correspondence (such as derivation of shear viscosity of relativistic fluids or re-derivation of Ohm's Law) require solving general relativistic systems with black holes and gravitational waves. These are beyond the abilities of the author. Therefore, summaries of a few applications of AdS/CFT correspondence will be given:

- **Viscosity of hydrodynamic fluids [5]:** Relativistic form of Navier-Stokes equation is equivalent to expressing the conservation of the energy-momentum tensor:
$$d_\mu T^{\mu\nu} = 0 \tag{22}$$
The linear response Kubo relation in this case expresses the shear viscosity in terms of the absorptive part of the spatially transversal T_{xy} energy momentum propagator. As per GPKW dictionary, $T^{\mu\nu}$ in boundary theory corresponds to $g^{\mu\nu}$ in the bulk.
The transversal components T_{xy} dualize to gravitational waves in the bulk, which have to be solved keeping in mind the boundary condition and the AdS Schwarzchild blackhole that accounts for finite temperatures. Eventually, the viscosity turns out to be proportional to the area of the event

[2]This is because the entropy of the black hole is proportional to its area

horizon of this black hole, giving us the final relation (that agrees with the famous result):

$$\frac{\eta}{s} = \frac{1}{4\pi}\frac{\hbar}{k_B} \tag{23}$$

- **Holographic superconductors [4]:** This model provides one of the few theoretical explanations of superconductivity apart from the BCS theory. It involves pair fermionic formation at T_C like in BCS theory, but the superconducting phase here consists of a non-Fermi liquid state where Cooper mechanism is not active.
  The charged AdS black hole serves as the dual for the normal metal phase of the superconductor. At low temperatures, this black hole doesn't remain stable as the ground state. To describe superconducting phase, we include in the Einstein–Maxwell action of the bulk theory additional complex scalar charged fields.
  It is expected that this model of the superconductor might shed light on the properties of high-temperature superconductors that lie beyond the reach of BCS theory.

References

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