* **Martingale** → $(\Omega, \mathcal{F}, P)$

& $\{\mathcal{F}_n\}_{n=0}^{N}$ → filtration & $\mathcal{F}_N = \mathcal{F}$

Defⁿ → A r.v. X is integrable if $E[X] < \infty$

Defⁿ → given a $\{\mathcal{F}_n\}$ a filtration, a seq. of r.v.s $\{X_n\}$ is **adapted** if X_n is $\mathcal{F}_n$-m'ble $\forall n$.

Defⁿ **Martingale** → A martingale $\{M_n\}$ is a seq. of r.v.s s.t.

a) M_n is integrable $\forall n$.
b) $\{M_n\}$ is adapted to $\{\mathcal{F}_n\}$
c) $E[M_{n+1} | \mathcal{F}_n] = M_n \quad \forall n.$

If (c) : $E[M_{n+1} | \mathcal{F}_n] \geq M_n \ \forall n$, then ⇒ $\{M_n\}$ is sub-martingale

If (c) : $E[M_{n+1} / \mathcal{F}_n] \leq M_n \ \forall n$, ⇒ $\{M_n\}$ is called super-martingale.

or $\{M_n, \mathcal{F}_n\}$ is a martingale or we specify prob. measure also.

In prev. ex → In N-period Binomial model we see $\left\{\frac{S_n}{(1+r)^n}\right\}$ is a $\tilde{P}$-martingale.

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* Properties → (Martingale)

1) $E[M_{n+1} | \mathcal{F}_n] = M_n \quad \forall n.$

So, $E[M_{n+2} | \mathcal{F}_{n+1}] = M_{n+1}$

$\Rightarrow E\{E[M_{n+2} | \mathcal{F}_{n+1}] | \mathcal{F}_n\} = E[M_{n+1} | \mathcal{F}_n]$

$\Rightarrow \quad E[M_{n+2} | \mathcal{F}_n] = M_n$

In general $\quad E[M_{n+m} | \mathcal{F}_n] = M_n. \quad$ → m step ahead best predictor.

2) $E[M_{n+1} | \mathcal{F}_n] = M_n$

$E[E[M_{n+1} | \mathcal{F}_n]] = E[M_n] \quad \forall n$

$\Rightarrow \quad E[M_{n+1}] = E[M_n] \quad \forall n$

⇒ Martingales have const. expectation over all time periods.

ie $E[M_N] = E[M_{N-1}] = \cdots = E[M_2] = E[M_1] = E[M_0]$

3) Jensen Ineq.:

ϕ → convex & $\{M_n\}$ → martingale

$\{\phi(M_n)\}$ will be a sub-Martingale.

4) Supp. X r.v. on $(\Omega, \mathcal{F}, P)$ with $E[|X|] < \infty$ + given $\{\mathcal{F}_n\}$ fil…

& let $M_n = E[X|\mathcal{F}_n]$ $\forall n$.

then M_n is a martingale.

Proof → verify 1st 2 prop. (easily). measurability → immedi… integrability → Jensen in…

3 → $E[M_{n+1}|\mathcal{F}_n] = E\big[E[X|\mathcal{F}_{n+1}]\,\big|\,\mathcal{F}_n\big] = E[X|\mathcal{F}_n] = M_n$ (tower prop.)

⇒ Martingale.

★ **Lecture-23.** (Risk Neutral Pricing of Euro. derivatives in Binomial Model).

★ **Binomial Model →**

We have a riskfree asset ⇒ $S_0 \longrightarrow (1+r)S_0$

" " " Risky asset ⇒ $S_0 \xrightarrow{p} uS_0$, $\quad S_0 \xrightarrow{q} dS_0$.

$P = \{p, q\}$ → real world prob.

Define $\tilde{p} = \dfrac{(1+r)-d}{u-d} = 1-\tilde{q}$

then $\tilde{P} = \{\tilde{p}, \tilde{q}\}$ → risk-neutral prob.

We saw $\left\{\dfrac{S_n}{(1+r)^n}\right\}$ → discounted stock price process
→ It is a $\tilde{P}$- martingale
→ It is not a P-martingale

★ Theorem 1 → Consider the general N- period binomial model with $0 < d < 1+r < u$. Let the risk neutral

o/w arbitrage occurs

prob. given by $\tilde{p} = \dfrac{1+r-d}{u-d}$ & $\tilde{q} = \dfrac{u-1-r}{u-d}$.

Then under risk neutral measure $\tilde{P}$, the discounted stock price is a martingale.

<u>Proof</u> → Look at $\tilde{E}\left[\dfrac{S_{n+1}}{(1+r)^{n+1}} \middle| \mathcal{F}_n\right] = \tilde{E}\left[\dfrac{S_n}{(1+r)^{n+1}} \cdot \dfrac{S_{n+1}}{S_n} \middle| \mathcal{F}_n\right]$ u/d.

What happens in (n+1)th toss which is independent of previous coin tosses

$$= \frac{S_n}{(1+r)^n} \cdot \tilde{E}\left[\frac{1}{r+1} \frac{S_{n+1}}{S_n} \middle| \mathcal{F}_n\right] = \frac{S_n}{(1+r)^n} \cdot \frac{1}{1+r} \tilde{E}\left[\frac{S_{n+1}}{S_n}\right].$$

$$= \frac{S_n}{(1+r)^n} \cdot \frac{1}{1+r} \cdot (\underbrace{\tilde{p}u + \tilde{q}d}_{1+r}) = \frac{S_n}{(1+r)^n}$$

Hence proved.

Now, in N-periods, ∃ investor he takes (position)

at n, Δ_n shares of stocks & holds it until $(n+1)$ time.

↳ depends on 1st n coin tosses.

$\{\Delta_0, \Delta_1, \Delta_2, \ldots, \Delta_{N-1}\}$ → the portfolio process which is <u>adapted process</u>.

Now,

Initial wealth X_0 of investor, X_n → self financing (no money inserted or extracted) wealth at time n.

Recall

$$\boxed{X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)} \quad (*) \quad ; n = 0, 1, 2, \ldots, N-$$

at time n you have X_n amount & you decided to hold Δ_n

What would be actual rate of growth of investor portfolio??
(If we take at real world prob., it will depend on current portfolio).
(Don't matter in risk-neutral prob.).
(Expected rate of growth of portfolio = riskfree rate).

So, Discounted wealth process is also $\tilde{P}$-martingale.
$\left(\frac{X_n}{(1+r)^n}\right)$ (Exercise).

Theorem 2 → Consider N-period Binomial model.

Let $\{\Delta_0, \Delta_1, ---, \Delta_{N-1}\}$ be an adapted process.
Let X_0 be a real no. & let $\{X_1, X_2, ---, X_N\}$ be generated by (*) (Previous page) then the discounted wealth process i.e. $\left\{\frac{X_n}{(1+r)^n}\right\}$ is martingale under $\tilde{P}$.

Proof → $\tilde{E}\left[\frac{X_{n+1}}{(1+r)^{n+1}} \middle| \mathcal{F}_n\right] \underset{\text{plug } X_{n+1} \text{ by } (*)}{=} \tilde{E}\left[\frac{\Delta_n S_{n+1}}{(1+r)^{n+1}} \middle| \mathcal{F}_n\right] + \tilde{E}\left[\frac{X_n - \Delta_n S_n}{(1+r)^n}\right]$

$= \underset{\mathcal{F}_n\text{-m'ble}}{\Delta_n} \underbrace{\tilde{E}\left[\frac{S_{n+1}}{(1+r)^{n+1}} \middle| \mathcal{F}_n\right]}_{\tilde{P}\text{ - martingle}} + \frac{X_n - \Delta_n S_n}{(1+r)^n}$

$= \Delta_n \cdot \frac{S_n}{(1+r)^n} + \frac{X_n}{(1+r)^n} - \frac{\Delta_n S_n}{(1+r)^n}$

$= \frac{X_n}{(1+r)^n}$ Hence proved.

Corolary Under the conditions of theorem 2, we have

$$\tilde{E}\left[\frac{X_n}{(1+r)^n}\right] = X_0 . \quad \forall n = 0, 1, 2, \ldots, N.$$

Proof $\Rightarrow$ $\because$ "martingales have constant expectation" fact.

Consequences - 1 There can be no-arbitrage in binomial model.

$\because$ If there were arbitrage, then we could begin with $X_0 = 0$ & find a portfolio process whose wealth process $\{X_1, \ldots, X_N\}$ satisfies $X_N(\omega) \geq 0$ $\forall \omega \in \Omega$. & $X_N(\bar{\omega}) > 0$ for atleast $1 \ \bar{\omega} \in \Omega$.

But then we would have a scenario

$$X_0 = 0 \quad \& \quad \tilde{E}\left[\frac{X_N}{(1+r)^N}\right] > 0 \quad \Rightarrow\Leftarrow \text{ Contrad.}$$

- Risk neutral prob. measure ($\tilde{P}$)
  - $\rightarrow$ agrees with P on what is possible & what's not. (Both p & $q > 0$ & $\tilde{p}$ & $\tilde{q} > 0$).
  - $\rightarrow$ under $\tilde{P}$, the discounted price processes of primary assets are martingales. (Risky & Riskfree asset).

So, ie $\left\{\frac{(1+r)^n}{(1+r)^n}\right\}$ & $\left\{\frac{S_n}{(1+r)^n}\right\}$ are martingales.

$\frac{(1+r)^n}{(1+r)^n} = 1$

$\rightarrow$ 1st Fundamental theo. of asset pricing

So, whatever model you assume, your prob. measure should be risk neutral, $\because$ only in this measure, Prices are martingales & iff they are martingale there is no arbitrage & your model should not have arbitrage.

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Consequence-2 | Risk Neutral pricing formula.

In N-period Bin. model.

V_N = is a $\mathcal{F}_N$ ($=\mathcal{F}$)-measurable r.v.
↳ (derivative).

Recall → $\exists X_0$ & $\{\Delta_0, \Delta_1, \dots, \Delta_{N-1}\}$ s.t. (self financing wealth process) $X_N(\omega) = V_N(\omega)$ $\forall \omega \in \Omega$

If we show this for all ω, then its called model completeness.

We saw $\left\{\frac{X_n}{(1+r)^n}\right\}$ is $\tilde{P}$-martingale.

$$\frac{V_n}{(1+r)^n} = \frac{X_n}{(1+r)^n} = \tilde{E}\left[\frac{X_N}{(1+r)^N} \Big| \mathcal{F}_n\right] = \tilde{E}\left[\frac{V_N}{(1+r)^N} \Big| \mathcal{F}_n\right]$$

X_n → ~~any~~ wealth that we need to hedge ~~that~~ Posit^n from n to N.

So, X_n → no arbi^n price at n. & hence equal to V_n.

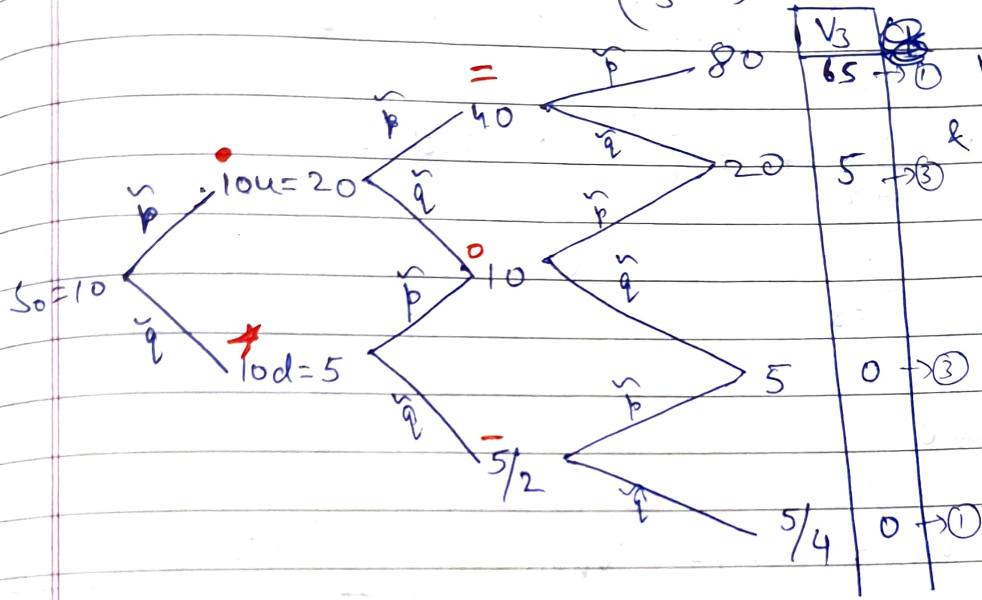
OR $$\boxed{V_n = \tilde{E}\left[\frac{V_N}{(1+r)^{N-n}} \Big| \mathcal{F}_n\right]}, \quad n = 0, 1, \dots, N$$

→ Risk neutral pricing formula.
→ This formula generates same V_n's as by algo we saw earlier before (18th lect).

ex) 3-period Bin. model. $u=2$, $d=\frac{1}{2}$, $r=0.1$

& $S_0=10$ & $K=15$.

Let Derivative $V_3=(S_3-K)^+$ → Euro-Call on stock



$\tilde{p}=0.4$
& $\tilde{q}=0.6$

$$V_0=\frac{1}{(1+r)^3}\cdot\tilde{E}[V_3/\mathcal{F}_0]=\frac{1}{(1+r)^3}\cdot\tilde{E}[V_3]=\frac{1}{(1+r)^3}\left[65\times\tilde{p}^3+5\times\tilde{p}^2\tilde{q}+5\times\tilde{p}^2\tilde{q}+5\times\tilde{p}^2\tilde{q}\right].$$

⇒

$V_0=4.2074$

Now,

$$V_1(\omega_1)=\frac{1}{(1+r)^2}\cdot\tilde{E}[V_3|\mathcal{F}_1](\omega_1).$$

• $V_1(H)=\frac{1}{(1+r)^2}\cdot\left[65\cdot\tilde{p}^2+5\,\tilde{p}\tilde{q}+5\cdot\tilde{p}\tilde{q}\right]=10.5285$

$\star$ $V_1(T)=\frac{1}{(1+r)^2}\left[5\cdot\tilde{p}^2\right]=0.6612$

$$V_2(\omega_1, \omega_2) = \frac{1}{1+r} \cdot \tilde{E}\left[V_3 \mid \mathcal{F}_2\right](\omega_1 \omega_2).$$

$$= V_2(HH) = \frac{1}{1+r} \cdot (65\tilde{p} + 5\tilde{q}). \quad = 26.3636.$$

- $V_2(HT) = \frac{1}{1+r} \cdot (5 \cdot \tilde{p}) = 1.8182$
- $V_2(TH) = \frac{1}{1+r} \cdot (5 \cdot \tilde{p}) = 1.8182$
- $V_2(TT) = \frac{1}{1+r} \cdot (0) = 0.$

Now, we know

$$\Delta_n(\omega_1 --- \omega_n) = \frac{V_{n+1}(\omega_1 -- \omega_n H) - V_{n+1}(\omega_1 -- \omega_n T)}{S_{n+1}(\omega_1 -- \omega_n H) - S_{n+1}(\omega_1 -- \omega_n T)}.$$

Now,

V_3 can be anything.

ex → $V_3 = \max\limits_{0 \le i \le 3} S_i - \min\limits_{0 \le i \le 3} S_i$ → Path dependent payoff.

"$\mathcal{F}_3$-mble (exotic options)"

Find via conditional expectation of (?). & get $\underline{V_0}$.

(Vanilla option don't have path dependent payoff, only on final S_3 value)

$$\boxed{V_n = \tilde{E}\left[\frac{V_N}{(1+r)^{N-n}} \,\middle|\, \mathcal{F}_n\right]} \quad ; \; n = 0, 1, 2, \dots, N.$$

↳ Risk

Lecture-24. (Actual & Risk Nutre Prob., Markov Process, Amer. Opt.).

Rel^n.

Binomial Asset Pricing Model →

2 Prob. measures →

$P = \{p, q\}$ → actual (real world) prob measure
$\tilde{P} = \{\tilde{p}, \tilde{q}\}$ → risk neutral prob measure.

Relationship b/w P & $\tilde{P}$??

Generally we have Ω - finite sample space.

Assume 2 Prob. mesures $P(\omega) > 0$ & $\tilde{P}(\omega) > 0$. $\forall \omega \in \Omega$

Define $Z(\omega) = \dfrac{\tilde{P}(\omega)}{P(\omega)}$ ← Radon Nykodym Derivative. (R N derivative) of $\tilde{P}$ w.r.t. P.

(P → given, $\tilde{P}$ → constructed)
connected via Z.

Properties of Z →

(a) $P(Z > 0) = 1$ (Def^n).

(b) $E[Z] = 1$ $\left(E[Z] = \sum_{\omega \in \Omega} Z(\omega) \cdot P(\omega) = \sum_{\omega \in \Omega} \tilde{P}(\omega) = 1 \right)$

under measure P (not $\tilde{P}$).

Result → For any r.v. Y, we have $\tilde{E}[Y] = E[Z \cdot Y]$.

(easy to Prove).

process $\{Z_n\}$ is defined by

$$\boxed{Z_n = E[Z / \mathcal{F}_n]} \; ; \; n = 0, 1, 2, \ldots N.$$

In particular $\underbrace{Z_N = Z}_{\text{complete filtration available.}}$ & $\underbrace{Z_0 = 1}_{(\because E[Z|\mathcal{F}_0] = E[Z] = 1)}$

Observe $\rightarrow$ $\{Z_n\}$ is a P-martingale.

Now,

Lemma (Cond. Expect. Connection): Assume the cond^{n}s in the def^n above.

Let m & n be fixed s.t. $0 \le n \le m \le N$

Let Y be $\mathcal{F}_m$-measurable.

So,

a) $\tilde{E}[Y] = E[Z_m Y]$

b) $\tilde{E}[Y | \mathcal{F}_n] = \frac{1}{Z_n} E[Z_m Y | \mathcal{F}_n]$ $\quad n=0$

Proof $\rightarrow$ $\tilde{E}[Y] = E[ZY]$ $\rightarrow$ earlier proved. ($Y \rightarrow \mathcal{F}_m$-m-ble $\Rightarrow \mathcal{F}_N$-m-ble)

$= E[E[ZY | \mathcal{F}_m]]$ $\rightarrow$ Tower property.

$= E[Y \, E[Z | \mathcal{F}_m]]$

$= E[Z_m Y]$ $\rightarrow$ Z_m defined above $\{Z_n\}$ process.

b) Use Characterization property $<$ measurability, P.A

We show easily $\underline{\underline{\text{RHS}}}$ is $\mathcal{F}_n$-m'ble [$\because Z_n \rightarrow \mathcal{F}_n$ m'ble & $E[-/\mathcal{F}_n] \rightarrow$ "]

Secondly P.A. prop.
To check P.A. property $\rightarrow$

for $A \in \mathcal{F}_n$, & consider RHS.

complete thing $\mathcal{F}_n$ m'ble.

$$\tilde{E}\Big[\underbrace{\frac{1}{Z_n} E[Z_m Y/\mathcal{F}_n]}_{Y \downarrow \mathcal{F}_n \text{ m'ble}} \cdot \underset{\mathcal{F}_n \text{ m'ble}}{I_A}\Big] = E[E[Z_m Y/\mathcal{F}_n] \cdot I_A]$$

Using Part ① Result (mult. Y by Z_n).

$$= E[E[Z_m Y I_A / \mathcal{F}_n]]$$

Iterative Expect

$$= E[\underset{\mathcal{F}_m \text{ m'ble}}{Z_m Y I_A}]$$

Part ②

$$= \tilde{E}[Y I_A]$$

Using characterization show the rest.

Implication $\rightarrow$ Risk-Neutral pricing formula $\rightarrow$

$V_N \rightarrow$ derivative

We know $V_n = \tilde{E}\left[\frac{V_N}{(1+r)^{N-n}} \Big| \mathcal{F}_n\right] = \frac{1}{Z_n} E\left[\frac{Z_N V_N}{(1+r)^{N-n}} \Big| \mathcal{F}_n\right]$

$\forall n = 0, 1, \ldots, N$

So, we can use real world prob. to find prices.
but we need to know Z_n (R-N process for that).

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• Markov Process →

Defⁿ: N- period binomial model

Let $\{X_0, X_1, \ldots, X_N\}$ be adapted process.

If for every n b/w $(0 \,\&\, N-1)$ & for every funcⁿ $f(x)$, $\exists$ another funcⁿ $g(x)$ (depend on n & f).

s.t.

$$E\left[f(X_{n+1}) \,\middle|\, \underset{\substack{\uparrow \\ \text{Complete info till } n}}{\mathcal{F}_n}\right] = g(\underset{\text{info at time 'n'}}{X_n})$$

then we say $\{X_0, X_1, \ldots, X_N\}$ is a Markov process.

* **Independence Lemma** :→ (Tool to verify given process is Markov or not)

Consider N-period binomial asset pricing model. & let n be fixed s.t. $0 \le n \le N$

Suppose that $X^1, X^2, X^3, \ldots, X^K$ are $\mathcal{F}_n$-m'ble.

& $Y^1, Y^2, \ldots, Y^L$ are independent of $\mathcal{F}_n$.

Let $f(x^1, x^2, \ldots, x^K, y^1, y^2, \ldots, y^L)$ be a funcⁿ of $K+L$ variables.

& define $g(x^1, x^2, \ldots, x^K) = E[f(x^1, x^2, \ldots, x^K, Y^1, Y^2, \ldots, Y^L)]$

Then, $E[f(X^1, X^2, \ldots, X^K, Y^1, Y^2, \ldots, Y^L) \mid \mathcal{F}_n] = g(X^1, X^2, \ldots, X^K)$.

$(n+1)^{th}$ result ind. of $\mathcal{F}_n$

write $S_{n+1} = S_n \frac{S_{n+1}}{S_n}$ $\{u$; w.p. p; d; w.p. q

Eg:→ $\{S_n\}$ is a Markov process

$$E[f(S_{n+1}) | \mathcal{F}_n] = E[f(XY) | \mathcal{F}_n].$$

By Independence Lemma (bec. we have expressed our cond. exp in terms of X → m'ble & Y independent $(n+1)^{th}$ toss result → (ind. of $\mathcal{F}_n$))

$$g(x) = E[f(x_0 \cdot Y)] = p \cdot f(xu) + q f(xd)$$

then $E[f(S_{n+1}) | \mathcal{F}_n] = g(S_n) \Rightarrow \{S_n\}$ is markov. Hence proved

Note: a) $\{S_n\}$ is Markov ~~irresp~~ under both P & $\tilde{P}$.

b) "One step" property can be extended easily to "multistep" property.

Implication → N - period binomial model.

Let $V_N = v_N(S_N)$ → Markov Process.

random derivative (only path independent ~~option~~ derivative

(if path dependent then express V_N in func of some Markov process)

then, $V_{N-1} = \tilde{E}\left[\frac{v_N(S_N)}{1+r} \middle| \mathcal{F}_{N-1}\right]$

$= v_{N-1}(S_{N-1})$. (Markov Process).

$$\Rightarrow v_{N-1}(s) = \frac{1}{1+r}\left[\tilde{p}\, v_N(us) + \tilde{q}\, v_N(d \cdot s)\right]$$

In - general → (Multistep Markov process).

$$V_n = \tilde{E}\left[\frac{v_N(S_N)}{(1+r)^{N-n}} \middle| \mathcal{F}_n\right] = v_n(S_n);\ 0 \le n \le N$$

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★ Markov-based Recursive Algorithm →

Have Euro. derivative $f(S_N)$ → payoff.

Algo. will be →

$$\boxed{\begin{aligned} & v_N(s) = f(s) \\ \& \quad & v_n(s) = \frac{1}{1+r}\left[\tilde{p}\, v_{n+1}(us) + \tilde{q}\, v_{n+1}(ds)\right] \end{aligned}}$$

$n = N-1, N-2, \ldots, 1,$

& Delta hedging strategy →

$$\Delta_n = \frac{v_{n+1}(u \cdot S_n) - v_{n+1}(d S_n)}{(u-d)S_n}.$$

★ American Derivatives → ("Non-dividend paying stocks")

American Call "equivalent" to European Call

& Not true with American Put.

"Path-Independent" Versions Only.

↳ ie Payoff $= g(S_n)$ at time n if exercised at time n.

Here we have to consider both

1) Amount to hedge if not exercised now.
2) Amount to payoff if exercised now

So, we can modify above Recursive algo,

American Algorithm →

$$v_N(s) = g(x)$$

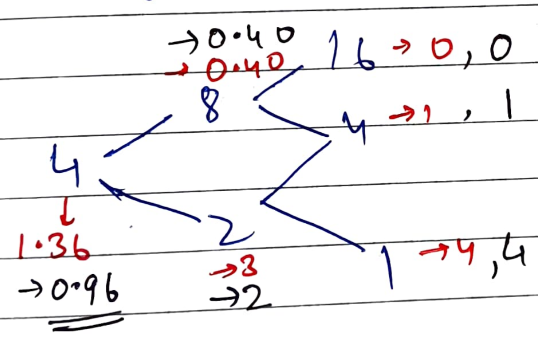
$$v_n(s) = \max\Big\{ \underbrace{g(x)}_{\substack{\downarrow \\ \text{to payoff} \\ \text{(exercised)}}},\ \underbrace{\frac{1}{1+r}\left[\tilde{p}\, v_{n+1}(s \cdot u) + \tilde{q}\, v_{n+1}(s \cdot d)\right]}_{\text{hedge amount (not exercised)}} \Big\}$$

$n = N-1, N-2, \ldots, 0$

Example → American Put in 2-period binomial model.

$r = \frac{1}{4}$, $u = 2 = \frac{1}{d}$ & $\Rightarrow \tilde{p} = \tilde{q} = \frac{1}{2}$, strike $= 5$ & $S_0 = 4$

$$g(S_n) = (5 - S_n)^+$$



P → Prices American Put.

P → European Put prices
↳ remove ~~g(x) fro~~ max from algo to find

$v_2(16) = 0$, $v_2(4) = 1$, $v_2(1) = 4$

$$v_1(8) = \max\Big\{ \underbrace{(5-8)^+}_{0},\ \frac{2}{5}[0+1] \Big\} = 0.40$$

$$v_1(2) = \max\Big\{ \underbrace{(5-2)^+}_{3},\ 2 \Big\} = 3.$$

$$v_0(4) = \max\Big\{ 1,\ \frac{2}{5}[0.40+3] \Big\} = 1.36.$$

(Long posⁿ holder of option)
~~Long~~ option

American Put.

- Values also tell you when to exercise
  ↳ at time ① & down node we get max value of 3.
  ↳ If I don't exercise, then I will get random ~~pos~~ payoff of $\frac{1}{1+r}\left(\frac{1}{2}\times 1 + \frac{1}{2}\times 4\right) < 3.$
  ↳ also $v_1(2) = \max\{3, 2\} = 3$

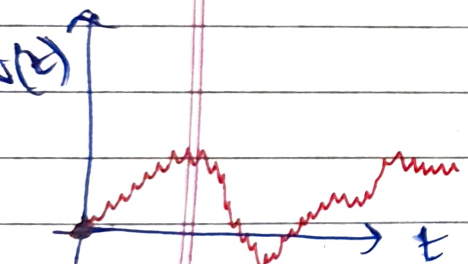
So, whenever we get max. from 1st quantity, we get optimal exercise time at those nodes.

∴ if you don't exercise there, short posⁿ party only need 2 to hedge & he will keep ~~there~~ 1 with itself & it's in advantage.

* Brownian Motion (or Wiener Process) → (BM)

Let $(\Omega, \mathcal{F}, P)$ be a prob. space. A stoch. proc
$\{W(t); t \geq 0\}$ with values in $\mathbb{R}$ is a B.M if follo
4 foll

a) $W(0) = 0$ a.s. (almost surely) ie w.p. = 1 its true
b) the sample paths $t \to W(t)$ are continous a.s.
c) For any $0 = t_0 \leq t_1 \leq \cdots \leq t_m$, the increments
ie $W(t_1) - W(t_0)$, $W(t_2) - W(t_1)$, ----, $W(t_m) - W(t_m$
are independent.
→ ind. of F_s
d) For any $0 \leq s \leq t$, the increment $W(t) - W(s)$
a normal distn with mean = 0 & Var = $t - s$ (length of int



Sample path

- $W(0.25) \sim N(0, 0.25)$
- $W(7) - W(3) \sim N(0, 4)$
- $P(W(5) > 5)$ = via Φ

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* Equivalent characterization of BM.

a) Same as : , b) Same as

c) $\because$ $W(t_1)-W(t_0), \dots, W(t_m)-W(t_{m-1})$ are ind. & Normal

$\Rightarrow$ $W(t_1), W(t_2), \dots, W(t_m)$ are jointly normally distributed &

$\underset{\sim}{W} = (W(t_1), \dots, W(t_m))$ has zero vector as its mean vector

& for $0 \le s < t$; $Cov(W(s), W(t)) = E[W(s) \cdot W(t)] = E[\underbrace{W(s)(W(t)-W(s))}_{\text{independent}} + W(s)^2]$

$= \cancel{E[W(s)]}^{\to 0} \cdot E[W(t)-W(s)] + Var(W(s))$

$= \underline{\underline{s}}$

$\therefore$ Variance-Cov. Matrix $C = \begin{bmatrix} t_1 & t_1 & \dots & t_1 \\ t_1 & t_2 & t_2 \dots & t_2 \\ t_1 & t_2 & t_3 \dots & t_3 \\ \vdots & & & \\ t_1 & t_2 & t_3 & \dots\ t_m \end{bmatrix}$

So, c) is $\underset{\sim}{W}$ as 0 mean vector & C as

Defn (Filtration for BM): Let $\{W(t);\ t \ge 0\}$ be a B.M defined on $(\Omega, \mathcal{F}, P)$.

A filtration for BM is collection of σ-fields $\{\mathcal{F}(t);\ t \ge 0\}$ satisfying 3 $\to$

$\to$ Actually true for all stoch. Proc.

✓a) (Information accumulation) i.e. for $0 \le s < t$, every set in $\mathcal{F}(s)$ is also in $\mathcal{F}(t)$.

b) (Adaptivity) i.e. for each $t \ge 0$, $W(t)$ is $\mathcal{F}(t)$-m'ble

c) (Independence of future increment) i.e for $0 \le t < u$, then the increment $W(u) - W(t)$ is independent of $\underline{\underline{\mathcal{F}(t)}}$.

* Any adapted process have a, b, c

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Q) How to construct such a filtration?

I Two possibilities (ways) → (i) $\{\mathcal{F}(t)\} = \{\mathcal{F}^W(t)\}$ → filtration generated by B.M.

where $\mathcal{F}^W(t) = \sigma(\{W_s ; 0 \le s \le t\})$.

(ii) Have general filtration with a, b & c

$\{\mathcal{F}(t)\} = \{\mathcal{F}^{W,(X,Y \ldots)}(t)\}$

* Properties of B.M. → 1) Martingales

R Q) B.M. is a Martingale.

Proof: Let $0 \le s \le t$ be given, then

$$E[W(t) \mid \mathcal{F}(s)] = E[(W(t) - W(s)) + W(s) \mid \mathcal{F}(s)]$$

$$= \underbrace{E[W(t) - W(s)]}_{\to 0} + W(s)$$

$$= W(s)$$

Note: BM that we consider is also called "standard" B.M.

Notation: $W(t)$ will be denoted by W_t (for ease).

* Martingale-2 → $\{W_t^2 - t\}$ is a Martingale.

D Proof Hint → Write $W_t^2 = (W_t - W_s)^2 + 2(W_t - W_s)W_s + W_s^2$.

* Martingale-3 → (Exponential Martingale)

For σ → constant, $\{e^{\sigma W_t - \frac{1}{2}\sigma^2 t}\}$ is a martingale

Hint: 1) $\sigma W_t - \frac{1}{2}\sigma^2 t = \sigma(W_t - W_s) + \left(\sigma(W_s) - \frac{1}{2}\sigma^2 t\right)$

2) MGF of a ~~normal~~ $N(0, t-s)$ r.v.

1) Sample Paths →

Def^n → The (first-order) variation of a func^n $f:[0,T]\to\mathbb{R}$ is defined to be

$$FV_T(f) = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|$$

where $\Pi = \{t_0, t_1, \dots, t_n\}$ & $0 = t_0 < t_1 < \dots < t_n = T$.

$\|\Pi\|$ → max length b/w any 2 partitions.

Note: If f is such that f' exists, then by M.V.T,

$$FV_T(f) = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} |f'(t_j^*)| \cdot (t_{j+1} - t_j). \qquad t_j^* \in [t_j, t_{j+1}]$$

$$FV_T(f) = \int_0^T |f'(t)|\, dt$$

Def^n → Quadratic Variation → Everything same except →

$$[f,f](T) = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} [f(t_{j+1}) - f(t_j)]^2$$

Note: If f has continuous derivative, then $[f,f](T) = 0$

So property is → Nowhere differentiability property of BM →

With prob. one, the BM $\{W_t; t \geq 0\}$ is non-differentiable at any $t \geq 0$.
(i.e. Samplepaths are non-differentiable)

i.e. For a Brownian Path $W(t)$, there is no value of t for which $\dfrac{dW(t)}{dt}$ is defined.

Proof → Omit.

Theorem → Let $W = \{W_t ; t \geq 0\}$ be a B.M. Then,

$$[W,W](T) = T \quad \forall\ T \geq 0 \quad \text{a.s.}$$

Proof → Let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0,T]$

& Define $Q_\Pi = \sum_{j=0}^{n-1} \left(W_{t_{j+1}} - W_{t_j}\right)^2$

W.T.S. → $Q_\Pi \to T$ as $\|\Pi\| \to 0$

We will show $E[Q_\Pi] = T$ & $Var(Q_\Pi) \to 0$ as $\|\Pi\| \to 0$

(L^2 conv.)

$$E[Q_\Pi] = \sum_{j=0}^{n-1} (t_{j+1} - t_j) = T$$

Now, $Var\left((W_{t_{j+1}} - W_{t_j})^2\right) = E\left[(W_{t_{j+1}} - W_{t_j})^2 - (t_{j+1} - t_j)\right]^2$

(use $X \sim N(0,\sigma^2)$ $\Rightarrow E[X^4] = 3\sigma^4$)

$$= E\left[(W_{t_{j+1}} - W_{t_j})^4\right] - 2(t_{j+1} - t_j)E\left[(W_{t_{j+1}} - W_{t_j})^2\right] + (t_{j+1} - t_j)^2$$

$$= 3(t_{j+1} - t_j)^2 - (t_{j+1} - t_j)^2$$

$$= 2(t_{j+1} - t_j)^2$$

So, $Var(Q_\Pi) = \sum_{j=0}^{n-1} Var\left[(W_{t_{j+1}} - W_{t_j})^2\right] = \sum_j 2(t_{j+1} - t_j)^2$

$$\leq \sum 2\|\Pi\|(t_{j+1} - t_j)$$

$$= 2T\|\Pi\| \to 0 \text{ as } \|\Pi\| \to 0$$

Hence proved.

We write this informally as

$$\boxed{dW(t) \cdot dW(t) = dt = 1 \cdot dt}$$

↳ Brownian motion accumulates quadratic variation at rate 1 per unit time.

Remarks: 1) $\lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} (W_{t_{j+1}} - W_{t_j})(t_{j+1} - t_j) = 0$.

↳ written as $\boxed{dW_t \cdot dt = 0}$

& 2) $\lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} (t_{j+1} - t_j)^2 = 0$ ie $\boxed{dt \cdot dt = 0}$

Result: a) $FV_T(W)$ is infinite a.s.

Hint: $\sum (W_{t_{j+1}} - W_{t_j})^2 \leq \max_{0 \leq k \leq n-1} |W_{t_{k+1}} - W_{t_k}| \cdot \sum |W_{t_{j+1}} - W_{t_j}|$

b) Cubic variation / any other higher order variation $= 0$

ie $\lim_{\|\Pi\| \to 0} \sum (|W_{t_{j+1}} - W_{t_j}|)^3 = 0$.

Result: Levy's Characterization of B.M.:

Let $\{M_t\}$ be a Martingale relative to a filtration $\{\mathcal{F}_t\}$.

Assume $M_0 = 0$ & M_t has continuous paths. & $[M, M](t) = t$ $\forall t \geq 0$, then $\{M_t\}$ is a B.M.

3) Markov Property →

Result → Let $\{W_t\}$ be a B.M. & $\{\mathcal{F}_t\}$ be a filtration for BM, then $\{W_t\}$ is a Markov Process.

Proof → W.T.S. → whenever $0 < s < t$, & f → a Borel m'ble there is another Borel m'ble func^n g, s.t.

$$E[f(W_t) | \mathcal{F}_s] = g(W_s).$$

Write → $E[f(W_t)|\mathcal{F}_s] = E[f(\underbrace{W_t - W_s}_{\substack{\text{one r.v.} \\ \downarrow \\ \text{ind. w.r.t } \mathcal{F}_s}} + \underbrace{W_s}_{\substack{\text{other r.v.} \\ \downarrow \\ \text{m'ble w.r.t } \mathcal{F}_s}}) / \mathcal{F}_s] = g(W_s)$

Now, Apply Independence Lemma →

Def^n → $g(x) = E[f(W_t - W_s + x)]$

$$= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(w+x)\, e^{\frac{-w^2}{2(t-s)}}\, dw$$

(∵ W_t is N

⇒ Markov

ie To evaluate $E[(\text{func}^n \text{ of BM}) | \text{filtration till } s]$ → just know what is value at

★ Lec-28   Ito's Integral (Stochastic Calculus).

★ Ito's Integral for Simple Integrands →

Want ~~dot~~ make sense of $\int_0^T \Delta_t\, dW_t$

$\{W_t ; t \geq 0\}$ → BM

& $\{\mathcal{F}_t ; t \geq 0\}$ → Filtration for BM

& $\{\Delta_t\}$ → adapted to

In ordinary calculus $\int_0^T \Delta(t)\, dg(t) = \int_0^T \Delta(t)\, g'(t) \cdot dt$ if g is diff.

↳ Riemann integral.

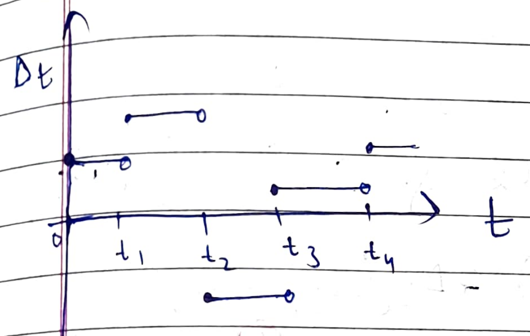
But here W_t is non-differentiable.

So Construction of Integral → We have interval $[0,T]$ & its partition

$\Pi = \{t_0, t_1, \dots, t_n\}$

s.t. $0 = t_0 \le t_1 \le t_2 \le \dots \le t_n = T$

- Also assume Δ_t is constant on each subinterval ie $[t_j, t_{j+1})$
- Such a process is called a simple process.



Now, let's see what is interplay b/w Δ_t & W_t.

- Regard 1) $W_t \to$ as the price / share of some asset at time "t".
(won't be true $W_t \sim N(\ )$ but Price is never -ve).

2) $t_0, t_1, \dots t_n \to$ trading dates.

3) $\Delta_{t_0}, \Delta_{t_1}, \dots \Delta_{t_n} \to$ positions taken (no. of shares you hold) at each trading date

So, gain from trading?

$$I_t = \begin{cases} \Delta_{t_0}(W_t - \underset{0}{W_{t_0}}) = \Delta_0 W_t & \to \forall\ 0 \le t \le t_1 \\ \underbrace{\Delta_0 W_{t_1}}_{\text{gain from } [0-t_1)} + \Delta_{t_1}(W_t - W_{t_1}) & \to \forall\ t_1 \le t \le t_2 \\ \Delta_0 W_{t_1} + \Delta_{t_1}(W_{t_2} - W_{t_1}) + \Delta_{t_2}(W_t - W_{t_2}) & \to \forall\ t_2 \le t \le t_3 \\ \vdots \end{cases}$$

In general; If $t_k \le t \le t_{k+1}$, then my IB

$$\boxed{\underset{[I(t)]}{I_t} = \sum_{j=0}^{k-1} \Delta_{t_j}(W_{t_{j+1}} - W_{t_j}) + \Delta_{t_k}(W_t - W_{t_k})}$$ ☆

→ Ito Integral of Simple Process $\int \Delta\, dW$