ie $\{I_t\}$ → process → (Ito Integral process)

So, $I_t = \int_0^t \Delta_u \, dW_u$ → meaning is from ☆ (previous Page)

☆ Properties of Ito Integral process →

We know $\{W_t\}$ is a Martingale & we are trading in Martingale asset

So, $\boxed{\text{Res-1}}$ → Ito Integral defined by ☆ is a Martingale

Proof → Let $0 \le s < t \le T$ be given

Assume s & t are in different subintervals.

ie $\exists\, t_l$ & t_k s.t. $t_l < t_k$ & $s \in [t_l, t_{l+1})$ & $t \in [t_k, t_{k+1})$

Now, $$I_t = \underbrace{\sum_{j=0}^{l-1} \Delta_{t_j}(W_{t_{j+1}} - W_{t_j})}_{1} + \underbrace{\Delta_{t_l}(W_{t_{l+1}} - W_{t_l})}_{2}$$

$$+ \underbrace{\sum_{j=l+1}^{k-1} \Delta_{t_j}(W_{t_{j+1}} - W_{t_j})}_{3} + \underbrace{\Delta_{t_k}(W_t - W_{t_k})}_{4}.$$

W.T.S → $E[I_t \mid \mathcal{F}_s] = I_s$.

Terms

1 is $\mathcal{F}_s$ m'ble. → $E[1 \mid \mathcal{F}_s] = 1$

2 $E[\Delta_{t_l}(W_{t_{l+1}} - W_{t_l}) \mid \mathcal{F}_s] = \Delta_{t_l}\big(\underbrace{E[W_{t_{l+1}} \mid \mathcal{F}_s]}_{\text{Martingale}} - W_{t_l}\big)$

$= \Delta_{t_l}[W_s - W_{t_l}]$

~~3~~ From 1 & 2 terms we see that $E[I_t \mid \mathcal{F}_s] = I_s$.

For 3 → $E[\Delta_{t_j}(W_{t_{j+1}} - W_{t_j}) \mid \mathcal{F}_s]$ $\quad t_j \geq t_{l+1} \geq s$

$= E\big[E[\Delta_{t_j}(W_{t_{j+1}} - W_{t_j}) \mid \mathcal{F}_{t_j}] \mid \mathcal{F}_s\big]$ → Tower property.

$= E\left[\Delta_{t_j}\left(\underbrace{E[W_{t_{j+1}}|\mathcal{F}_{t_j}]}_{W_{t_j}\text{ (Martingale)}} - W_{t_j}\right)\Big|\mathcal{F}_s\right] = \cancel{E[\Delta_{t_j}}\ 0.$

$\Rightarrow$ ③ $= 0$ & IIIly ④ $= 0$

Note: $\{I_t\}$ is a martingale & $I_0 = 0$

$\Rightarrow E[I_t] = 0 \quad \forall t \geq 0.$

& $Var(I_t) = E[I_t^2]$

Res-2 (Ito Isometry) $E[I_t^2] = E\left[\int_0^t \Delta_u^2\, du\right]$

Proof: Notation: $D_j = W_{t_{j+1}} - W_{t_j}$; $j = 0, 1, \dots k-1$

$D_k = W_t - W_{t_j}$; $j = k$.

So, $I_t = \sum_{j=0}^{k} \Delta_{t_j} D_j$

So, $I_t^2 = \sum_{j=0}^{k} \Delta_{t_j}^2 D_j^2 + 2\underbrace{\sum_{0 \leq i < j \leq k} \Delta_{t_i} \cdot \Delta_{t_j} \cdot D_i D_j}_{E[\ ] = 0}$

Note: $E[\underbrace{\Delta_{t_i} \Delta_{t_j} \Delta_i D_j}_{\mathcal{F}_{t_j}\text{-m'ble}}] \xrightarrow{\text{ind. of } \mathcal{F}_{t_j}} = E[\Delta_{t_i} \Delta_{t_j} \Delta_i] \cdot \cancel{E[D_j]} = 0$ $[E_x]$

$= 0.$

$\Rightarrow E[I_t^2] = \sum_{j=0}^{k} E[\Delta_{t_j}^2 D_j^2] = \sum_{j=0} E[\Delta_{t_j}^2] \cdot E[D_j^2]$

Ind.

$= \sum_{j=0}^{k-1} E[\Delta_{t_j}^2] \cdot (t_{j+1} - t_j)$

$+ E[\Delta_{t_k}^2](t - t_k)$

Now, $\Delta_{t_j} \to$ Constant in each sub interval & simple integrand

If not write this $\int$ form

$= \sum_{j=0}^{k-1} E\left[\int_{t_j}^{t_{j+1}} \Delta_u^2\, du\right] + E\left[\int_{t_k}^{t} \Delta_u^2\, du\right]$

$$= E\left[\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}} \Delta_u^2\,du + \int_{t_k}^{t}\Delta_u^2\,du\right] = E\left[\int_0^t \Delta_u^2\,du\right]$$

$\boxed{\text{Res-3}}$ The Quadratic Variation (QV) accumulated upto t by the Ito Integral is

$$[I, I](t) = [I, I]_t = \int_0^t \Delta_u^2\,du$$

↳ Vary from path to path

(In BM both Var & QV were same, here I_t → depends on path)

Proof Idea: Take $[t_j, t_{j+1}]$ on which Δ_u is constant.

Choose $t_j = s_0 < s_1 < \dots < s_m = t_{j+1}$

Now, $$\sum_{i=0}^{m-1}\left(I_{s_{i+1}} - I_{s_i}\right)^2 = \Delta_{t_j}^2 \underbrace{\sum_{j=0}^{m-1}\left(W_{s_{i+1}} - W_{s_i}\right)^2}$$

as $\|[t_j, t_{j+1}]\| \to 0$.

this converges to $(t_{j+1} - t_j)$

So, we see that QV in sub interval is $\Delta_{t_j}^2 (t_{j+1} - t_j)$

$$= \int_{t_j}^{t_{j+1}} \Delta_u^2\,du$$

So, for full $[0, t]$ ⇒ sum all these quant

We have seen 3 properties of Ito Integral.

Recall → $dW_t \, dW_t = dt$ $\iff$ ie $[W, W]_t = t$, ; $t \geq 0$.

Now;

Ito Integral $I_t = \int_0^t \Delta_u \, dW_u$ can be written as

$$dI_t = \Delta_t \, dW_t$$

So, $dI_t \, dI_t = \Delta_t^2 \, dW_t \, dW_t = \underbrace{\Delta_t^2}_{} dt$

↳ Ito Integral accumulation rate/unit time (Rec-3)

Notation →

$$I_t = \int_0^t \Delta_u \, dW_u \quad \text{to} \quad dI_t = \Delta_t \, dW_t$$

or $I_t = I_0 + \int_0^t \Delta_u \, dW_u$

differential form (assume $I_0 = 0$)

integral form.

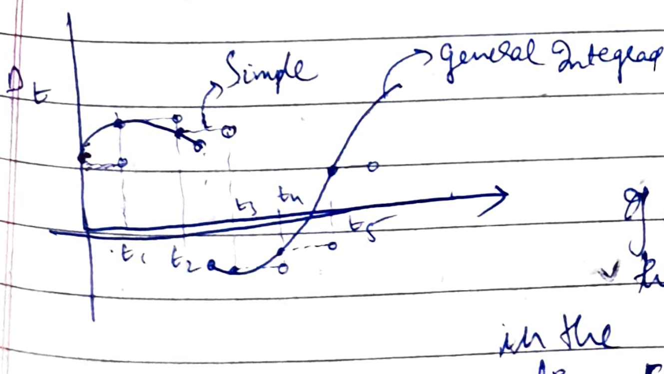
Ito's Integral for General Integrands →

Assumptions → 1) $\{\Delta_t\}$ → is adapted to $\{\mathcal{F}_t\}$

2) Square-Integrability Condⁿ.

ie $E\left[\int_0^T \Delta_t^2 \, dt\right] < \infty$

Idea → approximate $\{\Delta_t\}$ using simple process.



In general it is possible to choose a seq. $\{\Delta_n(t)\}$ of simple processes s.t. as $n \to \infty$, these simple processes converge to Δ_t in the sense that

$$\lim_{n \to \infty} E \int_0^T |\Delta_n(t) - \Delta(t)|^2 \, dt = 0.$$

We define

$$\text{So, } \boxed{I_t = \int_0^t \Delta_u \, dW_u = \lim_{n\to\infty} \int_0^t \Delta_n(u)\, dW(u) \; ; \; 0 \le t \le T}$$

Properties of General Ito Integrals→

Res-8 ⇒ Have $[0, T]$ & $\{\Delta_t\}$ → adapted

& Define I_t as above

a) (Continuity→) As a func^n of upper limit of integration t, the paths of I_t are continuous

b) (Adaptivity) For each t, I_t is $\mathcal{F}_t$-m'ble

c) (Linearity) If $I_t = \int_0^t \Delta_u dW_u$ & $J_t = \int_0^t \Gamma_u dW_u$ then

$$I_t \pm c\cdot J_t = \int_0^t (\Delta_u \pm c\cdot\Gamma_u)\, dW_u \quad \text{for}$$

c → constant

d) (Martingale) $\{I_t\}$ is martingale.

e) (Ito Isometry) $E[I_t^2] = E\left[\int_0^t \Delta_u^2 \, du\right]$

f) (Q.V) $[I, I]_t = \int_0^t \Delta_u^2 \, du$.

Eg → Take $\int_0^T W_t \cdot dW_t$ ⟹ Here we have to approximate W_t via simple integrals

$$\text{Approximation} \quad \Delta_n(t) = \begin{cases} W_0 \; ; \text{ if } 0 \le t \le T/n \\ W_{T/n} \; ; \text{ if } T/n \le t \le 2T/n \\ \vdots \\ W_{\frac{(n-1)T}{n}} \; ; \text{ if } \frac{n-1}{n} T \le t \le T \end{cases}$$

Leftmost pt. in interval ∴ $\Delta_n(t)$ has to be $\mathcal{F}(t)$ adapted

approximating simple integrals

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$$\int_0^T W_t \, dW_t = \lim_{n\to\infty} \int_0^T \Delta_n t \, dW_t$$

$$= \lim_{n\to\infty} \sum_{j=0}^{n-1} W_{j\frac{T}{n}} \left[W_{(j+1)\frac{T}{n}} - W_{jT/n} \right]$$

Now, One can show

$$\sum_{j=0}^{n-1} W_{j\frac{T}{n}} \left[W_{(j+1)\frac{T}{n}} - W_{j\frac{T}{n}} \right] = \frac{1}{2} W_T^2 - \frac{1}{2} \sum_{j=0}^{n-1} \left[W_{(j+1)\frac{T}{n}} - W_{j\frac{T}{n}} \right]^2$$

tends to

Now, as $n \to \infty$ $\to \frac{1}{2} W_T^2 - \frac{1}{2} [W, W]_T$

$$= \frac{1}{2} W_T^2 - \frac{1}{2} T$$

due to Q.V.

$$\Rightarrow \boxed{\int_0^T W_T \cdot dW_T = \frac{1}{2} W_T^2 - \frac{1}{2} T}$$

$\hookrightarrow$ Contrasting to ordinary Calculus. (2nd term coming due to the Q.V. of Brownian motion)

So, $$\int_0^t W_u \, dW_u = \frac{1}{2} W_t^2 - \frac{1}{2} t, \quad \forall t > 0.$$

$$= \frac{1}{2} (W_t^2 - t) \to g_t \text{ is Martingale}$$

Also a Martingale.

2 approaches — risk neutral framework

Lec-31 → (BSM Model)

★ Cont. Time Asset Pricing Theory → (APT)

Recall → Risky asset: $dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t$ — adapted

$\{W_t; t\geq 0\}$, $\{\mathcal{F}_t; t\geq 0\}$, $T>0$, $[0,T]$

Riskfree asset: $dB_t = R_t B_t dt$

- Simple case → α, σ, R → non Random
- Simplest case → " → constants → we will be content with this!

★ Black-Scholes-Merton (BSM) Model → (Corr. to Binomial model in Discrete time)

Risky asset: $dS_t = \alpha S_t dt + \sigma S_t dW_t$, $S_0 > 0$

Riskfree asset → $dB_t = r B_t dt$; $B_0 = 1$

Classical BSM (constant coeffic.)

Equivalently: $S_t = S_0 e^{\sigma W_t + (\alpha - \frac{1}{2}\sigma^2)t}$ → is lognormally r.v.

Ex → Find mean & Var. of S_t.

& $B_t = B_0 e^{rt}$

★ Portfolio Value Evolution →

At t, Δ_t → # shares of stock
→ adapted i.e at time t, Δ_t is known.

At time t, wealth is X_t → $\Delta_t S_t$ in Stock
→ $X_t - \Delta_t S_t$ in money market/Bond

So, wealth evolving → $dX_t = \underbrace{\Delta_t dS_t}_{\text{due to stock}} + r\underbrace{(X_t - \Delta_t S_t)}_{\text{riskfree grows at rate "}r\text{"}} dt$

Analogy to Discrete time → $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$

⇒ $X_{n+1} - X_n = \Delta_n(S_{n+1} - S_n) + r(X_n - \Delta_n S_n)$

So, $dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t)dt$

$= \Delta_t(\alpha S_t dt + \sigma S_t dW_t) + r(X_t - \Delta_t S_t)dt$

$= \underbrace{rX_t dt}_{\text{avg. underlying ROR for portfolio}} + \underbrace{\Delta_t(\alpha - r)(S_t dt)}_{\text{Risk premium for taking risk}} + \underbrace{\Delta_t \sigma S_t dW_t}_{\text{Pure Volatility term}}$

$\alpha \to$ ROR $\to$ risky

$r \to$ " $\to$ riskfree

We consider Discounted values of S & X. (i.e at single time pt.)

So, $d(e^{-rt}S_t) = df(t, S_t)$

Apply Itos formula

$$= f_t(t,S_t)dt + f_x(t,S_t)dS_t + \frac{1}{2}f_{xx}(t,S_t)dS_t dS_t$$

$\boxed{f(t,x) = e^{-rt}x}$

So, $f_t = -re^{-rt}x$

$f_x = e^{-rt}$

$f_{xx} = 0$

$= -re^{-rt}S_t dt + e^{-rt}dS_t \to$ (Put dS_t)

$d(e^{-rt}S_t) = (\alpha - r)(e^{-rt}S_t)dt + \sigma(e^{-rt}S_t)dW_t$

& $d(S_t) = \alpha(S_t)dt + \sigma(S_t)dW_t$

Only change in Mean ROR change $\alpha \to \alpha - r$

& IIIly $d(e^{-rt}X_t) = -re^{-rt}X_t dt + e^{-rt}dX_t \to$ (Put dX_t)

$= \Delta_t(\alpha - r)(e^{-rt}S_t)dt + \Delta_t\sigma(e^{-rt}S_t)dW_t$

$= \Delta_t d(e^{-rt}S_t)$ — (1) $\to$ (avg. ROR vanishes)

Option Value Dynamics $\to$

Consider European Call option with Pay off $(S_T - K)^+$ at T $= \max\{S_T - K, 0\}$

- BSM argue that Price at ~~date~~ t ~~pate~~ depends only on S_t & t

Denote $\to$ $c(t,x)$ = Value of call at t if $S_t = x$

- We know S_t is Sol^n of diffusion type SDE & hence $C(t, S_t) \Rightarrow$ depends only on $\mathcal{F}_t$.

So, Price Process is $\{C(t, S_t)\}$

Portfolio value process / option price.

Now, $dC(t, S_t) = C_t(t, S_t)\,dt + C_x(t, S_t)\,dS_t + \frac{1}{2} C_{xx}(t, S_t)\, dS_t\, dS_t$

$$= \left[C_t(t, S_t) + \alpha S_t C_x(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 C_{xx}(t, S_t)\right] dt$$

$$+ \sigma S_t C_x(t, S_t)\, dW_t$$

Now, $d(e^{-rt} C(t, S_t)) = d f(t, C(t, S_t))$.

$\boxed{f = e^{-rt} \cdot x}$

$$= f_t(t, C(t, S_t))\,dt + f_x(t, C(t, S_t))\,dC(t, S_t)$$

$$+ \cancel{\frac{1}{2} f_{xx}}(t, C(t, S_t)) \cdot dC(t, S_t) \cdot dC(t, S_t)$$

$$= -r e^{-rt} C(t, S_t)\,dt + e^{-rt}\,dC(t, S_t)$$

$$= e^{-rt}\left[-rC(t, S_t) + C_t(t, S_t) + \alpha S_t C_x(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 C_{xx}(t, S_t)\right] dt$$

$$+ e^{-rt} \sigma S_t C_x(t, S_t)\, dW_t \quad (2)$$

Now, How to find position (Δ) & Price (X) $\rightarrow$ Equate the 2 evolutions

Short Hedge :$\rightarrow$ Starts with X_0 & invest in the ~~underlying~~ market assets (risky & riskfree assets) so that $X_t = C(t, S_t)$ $\forall t \in [0, T]$ ($\forall t$ o/w arbitrage).

$$(\Rightarrow) \quad e^{-rt} X_t = e^{-rt} C(t, S_t) \ \forall t$$

Now,

~~One~~ way to ensure this :$\rightarrow$ $\boxed{d(e^{-rt} X_t) = d(e^{-rt} C(t, S_t)) \ \forall t \in [0, T) \ \& \ X_0 = C(0, S_0)}$

Comparing (1) & (2)

Previous page

① = ②

$\Delta_t (\alpha-r) S_t dt + \Delta_t \sigma S_t dW_t$

$=$

$$\left[-rC(t,S_t) + C_t(t,S_t) + \alpha S_t C_x(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 C_{xx}(t,S_t) \right] dt + \sigma S_t C_x(t,S_t)\, dW_t \quad \text{—③}$$

First look at dW_t terms & dt terms →

$$\boxed{\Delta_t = C_x(t,S_t) \quad \forall\, t \in [0,T)}$$

↳ Delta Hedging Rule. → dW_t ✓✓

→ ||ly to Discrete time.

Greeks ↗ "Delta" of the option

Next, dt term

$$(\cancel{\alpha}-r) S_t C_x(t,S_t) = -rC(t,S_t) + C_t(t,S_t) + \cancel{\alpha S_t C_x(t,S_t)} + \frac{1}{2}\sigma^2 S_t^2 C_{xx}(t,S_t)) \quad \forall\, t \in [0,T)$$

$$\Rightarrow rC(t,S_t) = C_t(t,S_t) + rS_t C_x(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 C_{xx}(t,S_t)$$

We seek a continuous funct $C(t,x)$ ie a sol of

$\boxed{\text{BSM P.D.E}}$

ie $$C_t(t,x) + rx\, C_x(t,x) + \frac{1}{2}\sigma^2 x^2 C_{xx}(t,x) = rC(t,x) \quad \forall\, t \in [0,T),\ x \geq 0$$

& satisfies the terminal cond^n

ie $C(T,x) = [x-K]^+$

Suppose we have the sol^n, then we can start with an $X_0 = C(0,S_0)$ & use the hedge $\Delta_t = C_x(t,S_t)$

then

$X_t = C(t,S_t) \ \forall\, t \in [0,T)$.

Taking the limit as t increases to T & X_t & $C(t,S_t)$ are cont. $\Rightarrow X_T = C(T,S_T) = (S_T-K)^+$

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Sol^n ~~to PDF is~~ → to BSM P.D.E is

We also need boundary cond^n at $x=0$ & $x=\infty$

* with $x=0$ in PDE, $\Rightarrow C_t(t,0) = r \cdot C(t,0)$ → ODE

$$\Rightarrow C(t,0) = e^{rt} C(0,0)$$

So for $t=T$, $C(T,0) = 0 \Rightarrow C(0,0) =$

& hence $C(t,0) = 0 \ \forall \ t \in [0,T]$

* with $x=\infty$ in PDE $\Rightarrow [x-k]^+ = [x-k]$

ie $$\lim_{x\to\infty} \left[C(t,x) - \left(x - e^{-r(T-t)} \cdot k\right) \right] = 0 \ \forall \ t \in [0,T]$$

So, Now Sol^n is given by

$$C(t,x) = x \cdot N(d_+(T-t,x)) - k e^{-r(T-t)} N(d_-(T-t,x))$$

; $0 \le t < T$

& $x > 0$

where $$d_\pm(\tau,x) = \frac{1}{\sigma\sqrt{\tau}} \left[\ln\left(\frac{x}{k}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau \right]$$

& N is the CDF of Standard Normal dist.

ie $$N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{u^2}{2}} du = \frac{1}{\sqrt{2\pi}} \int_{-y}^{\infty} e^{-\frac{t^2}{2}} dt$$

We shall write sometimes as

$$BSM(\tau, x; k, r, \sigma) = x N(d_+(\tau,x)) - k e^{-r\tau} N(d_-(\tau,x))$$

↳ BSM. func^n or BSM formula for European Call option price.

Also, Delta hedging: $\boxed{\Delta_t = C_x(t,x) = N(d_+(T-t,x))}$ → [Ex]

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Lec-32 → Greeks, Put-Call Parity, Change of Measure.

Recall→ (BSM-Euro Call) $c(t,x) = x\,N(d_+(T-t,x)) - Ke^{-r(T-t)}\cdot N(d_-(T-t,x))$

$0 \le t \le T$, $x > 0$

where $d_\pm(\tau, x) = \dfrac{1}{\sigma\sqrt{\tau}}\left[\ln\left(\dfrac{x}{K}\right) + \left(r \pm \dfrac{\sigma^2}{2}\right)\tau\right]$

$N(y) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{y} e^{-u^2/2}\,du$

$= BSM(\tau, x; K, r, \sigma)$

Note: $\lim_{t \to T} c(t,x) = (x-K)^+$ & $\lim_{x \downarrow 0} c(t,x) = 0$

Pos^n of stock (Δ_t) for Replication

***Greeks→**

1) Delta: → $c_x(t,x) = N(d_+(T-t,x))$ (+ve)

2) Theta: → $c_t(t,x) = -rKe^{-r(T-t)}N(d_-(T-t,x)) - \dfrac{\sigma x}{2\sqrt{T-t}}N'(d_+(T-t,x))$ < 0

N' → Pdf of Std. Normal

- Delta > 0
- Theta < 0
- Gamma > 0

3) Gamma: → $c_{xx}(t,x) = \dfrac{1}{\sigma x\sqrt{T-t}}N'(d_+(T-t,x)) > 0$

- **Short option Hedge:** → At t, the hedging portfolio value is

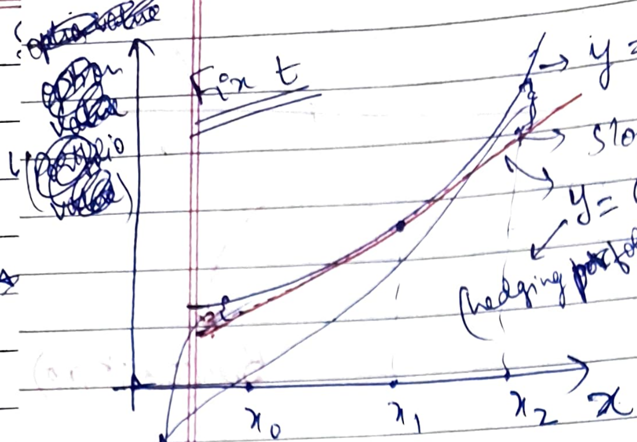
$c = xN(d_+) - Ke^{-r(T-t)}N(d_-)$

→ position in stock is c_x ⇒ Value invested in stock $= x\cdot c_x$
$= x\cdot N(d_+)$

⇒ Amount in Riskfree $= c - xc_x = -Ke^{-r(T-t)}N(d_-)$

- **Long Hedge→** Just the opposite. → Requires $(-c_x)$ shares of stock.
& $Ke^{-r(T-t)}\cdot N(d_-)$ → in riskfree

Now, if we fix a t, then look at curves of Delta & Gamma



(option value) $\rightarrow y = C(t, x)$ At time t,

$\rightarrow$ slope $\Rightarrow C_x$ $\qquad x_1$ = value of current share price

$y = C_x(t, x_1)(x - x_1) + C(t, x_1)$

(hedging portfolio value)

~~hedging~~

~~long option hedge~~ $\qquad C(t, x_1) \qquad x_1 C_x$

So, Portfolio $\Rightarrow$ long option, short stock

& long riskfree

$m = x_1 C_x(t, x_1) - C(t, x_1)$

?? ☹ Not understood

So, Portfolio Value at t is essentially

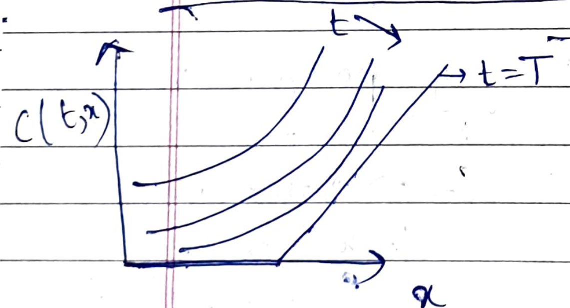
$$C(t, x_1) - x_1 C_x(t, x_1) + m$$

& this value = 0

Now, if stock price were to fall to x_0 instantaneously.

$\rightarrow$ Long option hedge $\rightarrow$ Profit

$\rightarrow$ delta-neutral & long gamma $\rightarrow$ Portfolio above



$\rightarrow$ t fixed.

Greeks used to make portfolio insensitive to sudden changes in stock price or time.

☆ Put-Call Parity $\rightarrow$

Consider Forward Contract $\rightarrow$ In BSM formula only terminal condition changes i.e. $(x-K)^+ \rightarrow (x-K)$

But we will arrive at price of forward with arbitrage arguments. So →

Forward → delivery/ exchange price = K
expiration/maturity = T
Payoff (for long posn holder) = $S_T - K$ (ie buy asset for price K)

Let $f(t,x)$ denote value of the forward contract at $t \in [0,T]$ if the stock price at $t = x$ ie S_t.

Now, Argue : $f(t,x) = x - e^{-r(T-t)} \cdot K$

Arbitrage argument → 1) At 0, sell Forward Cont. for $f(t,S_0) = S_0 - e^{-rT}K$.

2) Setup a Static hedge : purchase 1 stock for S_0.
now → Borrow $e^{-rT}K$ from Bank

So, Net value = 0

3) At T, value of this hedging portfolio is $= \underbrace{S_T - K}_{\text{exactly value of forward contract}} =$ Long posn Payoff.

$\Rightarrow f(t,x) = x - e^{-r(T-t)} \cdot K$. if $S_t = x$

Now, Forward Price → $\text{Forw}(t) = e^{r(T-t)} S_t$

× European Put → Payoff $= [K - S_T]^+$.

But, For any x, $x - K = (x-K)^+ - (K-x)^+$

Now, $P(t,x)$ → price of Europut (with strike = K & maturity = T) at t when $S_t = x$.

$f(x-k) = (x-k)^+ - (k-x)^+$

$\Rightarrow f(T, S_T) = c(T, S_T) - p(T, S_T)$

& Due to No arbitrage $\Rightarrow f(t, x) = c(t, x) - p(t, x)$; $x \geq 0$, $0 \leq t \leq T$

put-call parity (No assumptions)

Now, if we add assumptions →

1) For a constant r, $f(t, x) = x - e^{-r(T-t)} \cdot k$

2) In addition, if stock price follows G.B.M with $\sigma > 0$

then $c(t, x) = x N(d_+) - k e^{-r(T-t)} N(d_-)$

In such case,

$p(t, x) = c(t, x) - f(t, x) = x\left[N(d_+(T-t, x)) - 1\right] - k e^{-r(T-t)} \cdot \left[N(d_-(T-t, x)) - 1\right]$

Risk-Neutral Measure →

Till now, we used replication & no-arbitrage.

Recall : Change of measure idea

$(\Omega, \mathcal{F}, P)$ & r.v. Z & $P(Z > 0) = 1$ & $E[Z] = 1$

then $\tilde{P}(A) = \int_A Z(\omega) \, dP(\omega) \quad \forall A \in \mathcal{F}$

then → P & $\tilde{P}$ are equivalent (both agree on which sets have prob = 0) → Radon Nykodym derivative

→ for a r.v. X, $\tilde{E}[X] = E[XZ]$ & $E[X] = \tilde{E}\left[\frac{X}{Z}\right]$; $Z = \frac{d\tilde{P}}{dP}$

Now, R-N Deri for whole process.

Assume $\{\mathcal{F}_t\} \to$ filtration $0 \le t \le T$ $[0,T]$ $Z = Z_T$

R-N Derivative process $Z_t = E[Z|\mathcal{F}_t]$; $0 \le t \le T$

$\{Z_t\} \to$ martingale (under P)

Recall →

Lemma A → Let t satisfying $0 \le t \le T$ be given & let Y be a $\mathcal{F}_t$-m'ble r.v.

Then $\tilde{E}[Y] = E[YZ_t]$

Proof → $\tilde{E}[Y] = E[YZ] = E[E[YZ|\mathcal{F}_t]]$

$= E[Y E[Z|\mathcal{F}_t]] = E[YZ_t]$

Lemma B → Let s & t be fixed s.t. $0 \le s \le t \le T$ & Y be a $\mathcal{F}_t$-m'ble r.v. Then

$$\tilde{E}[Y|\mathcal{F}_s] = \frac{1}{Z_s} E[YZ_t|\mathcal{F}_s]$$

Proof → RHS is the conditional expectation of Y given $\mathcal{F}_s$ under $\tilde{P}$
So, we just have to show 2 things

i.e 1) RHS is $\mathcal{F}_s$ m'ble & 2) PA Property

1) $\frac{1}{Z_s} E[YZ_t|\mathcal{F}_s] \to$ clearly $\mathcal{F}_s$ m'ble

(ii) PA property (verify for RHS)

For $A \in \mathcal{F}_s$, $\tilde{E}\left[I_A \frac{1}{Z_s} E[YZ_t|\mathcal{F}_s]\right]$ → W.T.S $\tilde{E}[I$

use Lemma A → $= E\left[I_A \cancel{\frac{1}{Z_s}} \cdot E[YZ_t|\mathcal{F}_s] \cancel{Z_s}\right]$

$= E\left[E\left[I_A Y Z_t|\mathcal{F}_s\right]\right]$

$= E\left[I_A Y Z_t\right]$

$= \tilde{E}\left[I_A Y\right]$ □

★ Girsanov Theorem →

★ $\boxed{\text{Lec-33}}$

Recall Lemma A & Lema B. → $\tilde{E}[Y] = E[YZ_t]$; $Y \to \mathcal{F}_T$ m'ble

$\tilde{E}[Y|\mathcal{F}_s] = \frac{1}{Z_s} E[YZ_t|\mathcal{F}_s]$; $s \le t$

In above, $Z_t = E[Z|\mathcal{F}_t]$ & $Z = \frac{d\tilde{P}}{dP}$ & $P(Z>0)=1$ & $E[Z]=1$

& $\tilde{P}(A) = \int_A Z(\omega)\, dP(\omega)$; $A \in \mathcal{F}$ ★

- P & $\tilde{P}$ are equivalent.

★ Girsanove Theorem (1-D) →

Let $\{W_t; 0 \le t \le T\}$ be a BM on $(\Omega, \mathcal{F}, P)$ & let $\{\mathcal{F}_t\}_{t\in[0,T]}$ be a filtration for this BM.

Now let $\theta \in \mathbb{R}$ & define process Z_t →

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$$Z_t = e^{-\theta W_t - \frac{1}{2}\theta^2 t} \quad \& \quad \underbrace{\tilde{W}_t}_{N(\theta t, t)} = \underbrace{W_t}_{N(0,t)} + \theta t$$

Let $Z = Z_T$, then $E[Z] = 1 \Rightarrow P[Z > 0] = 1$
& under $\tilde{P}$ given by ✱,
the process $\{\tilde{W}_t\ 0 \le t \le T\}$ is a BM.

Proof → Use Levy Criteria to show Process is BM.
i.e. QV = 1, Cont. path, Martingale.

⊙ See that $\{\tilde{W}_t\}$ starts at 0 and has continuous paths.

⊙ $[\tilde{W}, \tilde{W}]_t = [W, W]_t = t$ $\quad$ (∵ ~~$\theta t \Rightarrow QV = 0$~~ write in terms of SDE)

⊙ To show $\{\tilde{W}_t\}$ is a martingale under $\tilde{P}$ i.e. $\tilde{P}$-Martingale

observe: $\{Z_t\}$ is a P-martingale. $\quad$ (||ly to S_t).

or By Ito's formula; $dZ_t = -\theta Z_t \, dW_t$ → Only Pure Ito Integral → & Ito Integrals are martingales.

$$Z_t = Z_0 - \theta \underbrace{\int_0^t Z_u \, dW_u}_{\text{Martingale}}$$

So, In particular $E[Z] = E[Z_T] = Z_0 = 1$
(due to Martingale)

So, we have $Z_t = E[Z_T | \mathcal{F}_t] = E[Z | \mathcal{F}_t]$

⇒ This shows $\{Z_t\}$ is a R-N derivative process.
So, Lemma A & B applies ~~as prev~~.

Now, we will show $\{\tilde{W}_t Z_t\}$ is a P-martingale.

$$d(\tilde{W}_t Z_t) = \tilde{W}_t \, dZ_t + Z_t \, d\tilde{W}_t + dZ_t \, d\tilde{W}_t$$

$$= -\tilde{W}_t \theta Z_t \, dW_t + Z_t \, dW_t + Z_t \cancel{\theta \, dt} + \cancel{[dW_t + \theta dt]} [-\theta Z_t dW_t]$$

$$= [-\tilde{W}_t \theta + 1] Z_t \, dW_t$$

$\Rightarrow \{\tilde{W}_t Z_t\}$ is a P martingale

Now, let $0 \le s \le t \le T$

Lemma-B & P-martingale property of $\{\tilde{W}_t Z_t\}$ imply

$$\tilde{E}[\tilde{W}_t | \mathcal{F}_s] = \frac{1}{Z_s} E[\tilde{W}_t Z_t | \mathcal{F}_s] = \frac{1}{Z_s} \tilde{W}_s Z_s = \tilde{W}_s$$

$\Rightarrow \{\tilde{W}_t\}$ is a $\tilde{P}$ martingale.

$\Rightarrow$ " " " BM under $\tilde{P}$.

★ **BSM Model** $\to$ $[0, T]$ T-fixed.

Let $\{W_t; 0 \le t \le T\}$ be a BM on $(\Omega, \mathcal{F}, P)$ &
Let $\{\mathcal{F}_t; 0 \le t \le T\}$ be a Filtration for above BM.

Consider, risky stock price process whose differential is given by

$$dS_t = \alpha S_t \, dt + \sigma S_t \, dW_t \qquad (\sigma \neq 0) \; > 0$$

ie $S_t = S_0 \cdot e^{\sigma W_t + (\alpha - \frac{1}{2}\sigma^2) t}$ $\leftarrow$ GBM

limit of Binomial asset pricing model

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Riskfree asset dynamics → $dB_t = r B_t dt$; $B_0 = 1$
(Bond)

or $S_t = B_0 e^{rt} = e^{rt}$

or $dD_t = -r D_t dt$ or $D_t = e^{-rt}$

discount factor

Look at discounted stock price proc →

$$d(e^{-rt} S_t) = (\alpha - r) e^{-rt} S_t dt + \sigma e^{-rt} S_t dW_t$$

$$= \sigma e^{-rt} S_t \cdot [\theta dt + dW_t]$$

; $\theta = \frac{\alpha - r}{\sigma}$ → market Price of risk. (Sharpe ratio)

Girsanov with θ as above (get $\hat{P}$, Z & $\hat{W}$).

$$= \sigma e^{-rt} S_t d\hat{W}_t$$

⇒ $\{e^{-rt} S_t\}$ is a $\hat{P}$-martingale

Now, $\boxed{\hat{P} \rightarrow \text{risk neutral measure}}$ → a) P & $\hat{P}$ are equivalent
b) $\{e^{-rt} S_t\}$ → $\hat{P}$-martingale

Stock under $\hat{P}$ →

Now, undiscounted price process = ??

$$dS_t = \alpha S_t dt + \sigma S_t dW_t \quad dW_t \rightarrow d\hat{W}_t$$

$$= \alpha S_t dt + \sigma S_t [d\hat{W}_t - \theta dt] \quad \left(\theta = \frac{\alpha - r}{\sigma}\right)$$

$$= r S_t dt + \sigma S_t d\hat{W}_t$$

or $S_t = S_0 e^{\sigma \hat{W}_t + (r - \frac{1}{2}\sigma^2)t}$

So under $\hat{P}$ → S_t → Mean R OR $\alpha \to r$

Value of Portfolio under $\tilde{P}$ →

(i) Initial X_0 & at each time, hold Δ_t position S_t

$$dX_t = \Delta_t \, dS_t + r(X_t - \Delta_t S_t)\,dt$$

(earlier done)

$$= r X_t \, dt + \Delta_t(\alpha - r) S_t \, dt + \Delta_t \sigma S_t$$

$$= r X_t \, dt + \Delta_t \sigma S_t \left[\theta\, dt + dW_t\right] \quad (\theta =$$

Now, $d(e^{-rt}X_t) = \Delta_t \sigma (e^{-rt} S_t)\left[\theta\, dt + dW_t\right] = \cancel{\Delta_t \sigma e^{-rt} S_t}$ $\Delta_t \, d(e^{-rt}$

$$= \Delta_t \sigma (e^{-rt} S_t)\, d\tilde{W}_t$$

$\Rightarrow \{e^{-rt} X_t\}$ is a $\tilde{P}$ martingale.

Pricing Under $\tilde{P}$ →

Generic Euro derivative → V_T → is any $\mathcal{F}_T$-m'ble r
maturity T

eg → $(S_T - K)^+$, $(K - S_T)^+$, $(S_T - K)$, $S_T^m; m > 0$, $\ln(S_T)$ etc.
$\left(\max_{0 \le u \le T} (S_u - K)\right)^+$

To know? : X_0 & $\{\Delta_t ; 0 \le t \le T\}$ to setup a short-hedge.
ie to meet $X_T = V_T$ almost surely.

Assume $\exists$ such a $\{\Delta_t\}$

We know $\{e^{-rt} X_t\}$ → $\tilde{P}$ martingale classmate

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$$\Rightarrow e^{-rt}V_t = \underbrace{e^{-rt}X_t}_{\text{Martingale}} = \tilde{E}[e^{-rT}X_T|\mathcal{F}_t] = \tilde{E}[e^{-rT}V_T|\mathcal{F}_t]$$

X_t = hedge from $t-T$, wealth needed

V_t = Price = X_t

X_T replicates V_T

$$\Rightarrow e^{-rt}V_t = \tilde{E}[e^{-rT}V_T|\mathcal{F}_t]$$

$$\boxed{\text{or } V_t = \tilde{E}[e^{-r(T-t)}V_T|\mathcal{F}_t] \; ; \; 0 \le t \le T}$$

↳ Risk neutral Pricing formula.

* BSM formula for Euro Call option → (No PDEs ☺)

$$V_T = (S_T - K)^+$$

$$V_t = \tilde{E}[e^{-r(T-t)}(S_T-K)^+|\mathcal{F}_t]$$

$\{S_t\}$ → GBM is a Markov process

$\{S_t\}$ → GBM → is a Markov process

$$\Rightarrow V_t = c(t, S_t) \quad (\because \text{def}^n \text{ of Markov process})$$

Now, To compute $c(t,x)$ → Use Independence Lemma.

Note: $S_t = S_0 e^{\sigma \tilde{W}_t + (r-\frac{1}{2}\sigma^2)t}$

$$S_T = S_t e^{\sigma(\tilde{W}_T - \tilde{W}_t) + (r-\frac{1}{2}\sigma^2)(T-t)}$$

$$= S_t e^{-\sigma\sqrt{\tau}\,Y + (r-\frac{\sigma^2}{2})\tau} \quad (\tau = T-t)$$

where $Y = \dfrac{-(\tilde{W}_T - \tilde{W}_t)}{\sqrt{T-t}}$ is standard Normal r.v. ind. of $\mathcal{F}_t$

& τ = Time to expiration $= T-t$

Now, S_t is $\mathcal{F}_t$ m'ble & Y is ind. of $\mathcal{F}_t$

So, $C(t,x) = \hat{E}\left[e^{-r\tau}\left(x\cdot e^{-\sigma\sqrt{\tau}\,Y + \left(r-\frac{\sigma^2}{2}\right)\tau} - K\right)^{+}\right]$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-r\tau}\underbrace{\left(x e^{-\sigma\sqrt{\tau}\,y + \left(r-\frac{\sigma^2}{2}\right)t} - k\right)^{+}}_{\text{This is +ve iff}} e^{-\frac{y^2}{2}} dy$$

$y < d_-(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}\left[\ln\left(\frac{x}{k}\right) + \left(r-\frac{\sigma^2}{2}\right)\tau\right]$

$$\Rightarrow C(t,x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(\tau,x)} e^{-r\tau}\left(x\cdot e^{-\sigma\sqrt{\tau}\,y + \left(r-\frac{\sigma^2}{2}\right)\tau} - k\right) e^{-\frac{1}{2}y^2} dy$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-} x\cdot e^{-\frac{y^2}{2} - \sigma\sqrt{\tau}\,y - \frac{\sigma^2\tau}{2}} dy - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-} e^{-r\tau}\cdot k\, e^{-\frac{1}{2}y^2} dy$$

$$= \frac{x}{\sqrt{2\pi}}\int_{-\infty}^{d_-} e^{-\frac{1}{2}(y+\sigma\sqrt{\tau})^2} dy - k e^{-r\tau} N(d_-(\tau,x))$$

$$= \frac{x}{\sqrt{2\pi}}\int_{-\infty}^{d_- + \sigma\sqrt{\tau}} e^{-\frac{z^2}{2}} dz - k e^{-r\tau} N(d_-(\tau,x))$$

$$= x N(d_+(\tau,x)) - k e^{-r\tau} N(d_-(\tau,x))$$

where; $d_+(\tau,x) = d_-(\tau,x) + \sigma\sqrt{\tau}$

$$= \frac{1}{\sigma\sqrt{\tau}}\left[\ln\left(\frac{x}{k}\right) + \left(r+\frac{\sigma^2}{2}\right)\tau\right]$$

Hence derivation for Euro Call.

[Ex] → ?? $V_T = S_T^m$; $m > 0$ | $V_T = (K - S_T)^{+}$

$V_T = S_T - K$

All this was done under assumption the $\{\Delta_t\}$ existed. Now we will prove that it does exist.

Lec-34

So far → Classical BSM model (one stock & constant parameters)

$$\overset{\text{GBM}}{dS_t} = \alpha S_t dt + \sigma S_t dW_t$$

$$dB_t = r B_t dt$$

V_T → Derivative

Risk Neutral Pricing formula:

$$V_t = \cancel{\tilde{E}[V_T | \mathcal{F}_t]} \; \tilde{E}[e^{-r(T-t)} V_T | \mathcal{F}_t] \quad 0 \le t \le T$$

P → real world Prob. measure

$\tilde{P}$ → Risk neutral < a) eq. to P; b) $\{e^{-rt} S_t\}$ → $\tilde{P}$ martingale.

Q) Relied on fact that $\{\Delta_t\}$ & X_0 (→ initial wealth) exists s.t. $X_T = V_T$

Answered by: Martingal Rep. Theorem (MRT) (1-D) (ie 1 BM).

Res: Let $\{W_t\}_{t\in[0,T]}$ be a BM on $(\Omega, \mathcal{F}, P)$ & let $\{\mathcal{F}_t\}_{t\in[0,T]}$ be the filtration generated by this BM. Let $\{M_t\}_{t\in[0,T]}$ be a martingale w.r.t. this (above) filtration. Then, there exists an adapted process $\{\gamma_t\}_{t\in[0,T]}$ s.t. $M_t = M_0 + \int_0^t \gamma_u dW_u \quad 0 \le t \le T$

Here $\mathcal{F}_t = \mathcal{F}_t^W$ (Not necessarily in Girsanov theorem).

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Corrolary → $\{W_t\}_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, P)$ & $\{\mathcal{F}_t^W\}$ - filtration

then for θ, define $Z_t = e^{-\theta W_t - \frac{1}{2}\theta^2 t}$

& $\tilde{W}_t = W_t + \theta t$

Set $Z = Z_T$, then $E[Z] = 1$ & under $\tilde{P}$ defined by

$\tilde{P}(A) = \int_A Z(\omega)\, dP(\omega)$, $A \in \mathcal{F}$, the process $\{\tilde{W}_t\}$ is a $\tilde{P}$-BM

→ this is Girsanov.

Now, let $\{\tilde{M}_t\}$ be a $\tilde{P}$-martingale. Then $\exists$ an adapted process $\{\tilde{\Gamma}_t\}$ s.t.

$$\tilde{M}_t = \tilde{M}_0 + \int_0^t \tilde{\Gamma}_u \, d\tilde{W}_u \qquad 0 \le t \le T$$

Filtration is still $\{\mathcal{F}_t^W\}$ ✓ & not $\{\mathcal{F}_t^{\tilde{W}}\}$ ✗ → Note.

Now, Hedging with one Stock in BSM →

Assume the filtration to be $\{\mathcal{F}_t^W\}$.

Now, V_T → an $\mathcal{F}_T$ m'ble r.v. (Derivative)

& we have $e^{-rt} V_t = \tilde{E}[e^{-rT} V_T / \mathcal{F}_t]$

i.e. $\{e^{-rt} V_t\}$ is a $\tilde{P}$-martingale

⇒ This has a representation as

$$e^{-rt} V_t = e^{-r \cdot 0} V_0 + \int_0^t \tilde{\Gamma}_u \, d\tilde{W}_u \; ; \quad 0 \le t \le T$$

for some $\{\tilde{\Gamma}_u\}$ adapted