Discrete Structures (Monsoon 2021)

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Mathematical Induction

Peano's postulates on set of natural numbers N

Let N be the set of natural numbers, $N = \{1, 2, 3, \ldots, n, \ldots\}$. For the nonempty set N of natural numbers:

- **Postulate 1.** $1 \in N$, that is, 1 is a natural number.
- **Postulate 2.** For each $n \in N$, there exists a unique natural number $n^+ \in N$, called the *successor* of n $[n^+ = n + 1]$.
- **Postulate 3.** 1 is not the successor of any natural number, that is, there is NO $n \in N$ for which $n^+ = 1$.
- **Postulate 4.** If $m, n \in N$ and $m^+ = n^+$, then $m = n$, that is, each natural number, if it is a successor, is the successor of a unique natural number.
- **Postulate 5.** If $K \subseteq N$ such that $1 \in K$ and $n \in K \Rightarrow n^+ \in K$, then $K = N$.

Deduction 1. Every element $n(\neq 1)$ is the successor of some other element of N.

Deduction 2. $m^+ \neq m, \forall m \in N$.

Order relations in the system of natural numbers

1. **[Law of Trichotomy]** If $m, n \in N$, any one of the following must hold:
   (i) $m > n$, (ii) $m = n$, (iii) $m < n$
2. **[Law of Transitivity]** If $m, n, p \in N$, then $m > n$ and $n > p$ $\Rightarrow m > p$.
3. **[Monotone Law of Addition]** If $m, n, p \in N$, then $m > n$ $\Rightarrow m + p > n + p$.
4. **[Monotone Law of Multiplication]** If $m, n, p \in N$, then $m > n$ $\Rightarrow mp > np$.

Mathematical Induction

First Principle of Mathematical Induction (Weak Induction)

For a given statement $P(n)$ involving a natural number n, if we can show that:

1. The statement $P(n)$ is true for $n = n_0$; and
2. The statement $P(n)$ is true for $n = k + 1$, assuming that $P(n)$ is true for $n = k$, $(k \geq n_0)$,

then we can conclude that $P(n)$ is for all natural numbers $n \geq n_0$.
(1) is referred to as the ***basis of induction*** and (2) is usually referred to as the ***induction step***.
The assumption that the statement is true for $n = k$ in (2) is usually referred to as the ***induction hypothesis***.

Second Principle of Mathematical Induction (Strong Induction)

For a given statement $P(n)$ involving a natural number n, if we can show that:

1. The statement $P(n)$ is true for $n = n_0$; and
2. The statement $P(n)$ is true for $n = k + 1$, assuming that $P(n)$ is true for $n_0 \leq k \leq n$,

then we can conclude that $P(n)$ is for all natural numbers $n \geq n_0$.
(1) is referred to as the ***basis of induction*** and (2) is usually referred to as the ***induction step***.
The assumption that the statement is true for $n = k$ in (2) is usually referred to as the ***induction hypothesis***.

Mathematical Induction

Problem: Using the mathematical induction, show that $(10^{n+1} + 10^n + 1)$ is divisible by 3 for a positive integer n.

Solution: Let "$P(n)$: $10^{n+1} + 10^n + 1$ be divisible by 3" be a statement.

- **[Basis Step.]** Here $n_0 = 1$. Then, $P(1) = 10^2 + 10^1 + 1 = 111$, which is divisible by 3. Thus, the statement $P(1)$ is true for $n = n_0 = 1$.
- **[Induction Step.]** Consider

$$\begin{aligned} P(k+1) - P(k) : \quad & (10^{k+2} + 10^{k+1} + 1) - (10^{k+1} + 10^k + 1) \\ = \quad & 10^{k+2} - 10^k = 10^k(10^2 - 1) \\ = \quad & 10^k.99 = 3(33.10^k) = 3.p, \textit{ say} \end{aligned}$$

where $p = 33.10^k$. Thus, $P(k+1) - P(k)$ is divisible by 3. Hence, $P(k+1)$ is divisible by 3, if $P(k)$ be so (by **Induction Hypothesis**). By the first principle of mathematical induction, it follows that $P(n)$ is true for all $n \in N$.

Problem: Let $\alpha = \frac{1+\sqrt{5}}{2}$. Then, show that $\alpha^{n-2} < F_n < \alpha^{n-1}$, where $n \geq 3$ and F_n is the n^{th} Fibonacci number.

Solution: Note that $\alpha = \frac{1+\sqrt{5}}{2}$ is a solution of the equation

$$x^2 = x + 1.$$

So,

$$\alpha^2 = \alpha + 1.$$

The Fibonacci sequence is defined as follows:

$$\begin{aligned} F_1 &= 1 \\ F_2 &= 1 \\ F_{k+1} &= F_k + F_{k-1}, k \geq 2. \end{aligned}$$

Mathematical Induction

Let $P(n) : \alpha^{n-2} < F_n$, where $n \geq 3$, be a statement.

- **[Basis Step.]** Since the induction step uses the recurrence relation: $F_{k+1} = F_k + F_{k-1}$, the basis step involves verifying that both $P(3)$ and $P(4)$ are true.
  1. To show that $P(3)$ is true: when $n = 3$,
  $$\alpha^{n-2} = \alpha = \frac{1+\sqrt{5}}{2} < \frac{1+3}{2} = 2 = F_3.$$
  So, $P(3)$ is true.
  2. To show that $P(4)$ is true: when $n = 4$,
  $$\alpha^{n-2} = \alpha^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4}$$
  $$= \frac{3+\sqrt{5}}{2} < \frac{3+3}{2} = 3 = F_4.$$
  So, $P(4)$ is true.

Mathematical Induction

- **[Induction Step.]** Assume $P(3), P(4), \cdots, P(k)$ are true; that is, assume $\alpha^{i-2} < F_i$, for $5 \leq i \leq k$. We must show that $P(k+1)$ is true; that is, $\alpha^{k-1} < F_{k+1}$. We have,

$$\alpha^2 = \alpha + 1$$

since $\alpha = \frac{1+\sqrt{5}}{2}$ is a root of the equation $x^2 = x + 1$.
Then,

$$\alpha^{k-3}(\alpha^2) = \alpha^{k-3}(\alpha + 1)$$
$$\Rightarrow \alpha^{k-1} = \alpha^{k-2} + \alpha^{k-3}, \text{ since } k-3 \geq 2$$
$$< F_k + F_{k-1}, \text{ by the Induction Hypothesis}$$
$$= F_{k+1}, \text{ by the recurrence relation}$$

So, $P(k+1)$ is true. Thus, by the Second Principle of Mathematical Induction (Strong Induction), $\alpha^{n-2} < F_n$, for every $n \geq 3$.

Mathematical Induction

Problem: Suppose a post office sells only 2 Rs. and 3 Rs. stamps. Show that any postage of 2 Rs. or 3 Rs. can be paid using only these stamps.

Solution: Construct a statement as follows:
$P(n) : \forall n \geq 2, \exists m_2, m_3 (\geq 0)$ such that $n = m_2 * 2 + m_3 * 3$
that is, $P(n) : \forall n [n \geq 2 \Rightarrow \exists m_2, m_3 (\geq 0)$ such that $n = m_2 * 2 + m_3 * 3$

- **[Basis Step.]** $n = 2$
  Then, $2 = 1 * 2 + 0 * 3$, when $m_2 = 1$ and $m_3 = 0$.
  Thus, $P(2)$ is true.
- **[Induction Step.]**
  Induction Hypothesis]: Assume that $P(n)$ is true for some $n = k$, $k > 2$.
  Required to Prove (RTP): $P(k+1)$ is true.
  By hypothesis,
  $$k = m_2 * 2 + m_3 * 3$$

Mathematical Induction

- **[Induction Step (Continued...).]** Then,

$$k + 1 = m_2 * 2 + m_3 * 3 + 1$$

$$= (m_2 - 1) * 2 + (m_3 + 1) * 3, \text{ for } m_2 \neq 0$$

$$OR$$

$$= (m_2 + 2) * 2 + (m_3 - 1) * 3, \text{ for } m_3 \neq 0$$

Thus, $P(k + 1)$ holds.
Since $P(n)$ is true for $n = 2$, so it holds for $n = 2 + 1 = 3$, $n = 3 + 1 = 4$, and so on.
Therefore, $P(n)$ is true for all $n \geq 2$.

Well-Ordering Principle

The set of natural numbers, N, is well-ordered, that is, every non-empty subset of N has a least element.

1. N has least element $1 \in N$.
2. Z, the set of all integers, $Z = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$ is not well-ordered, because it has no least element, that is, it has no lower bound.

Principle of Mathematical Induction

Let $S \subseteq N$ such that

1. $1 \in S$, and
2. $t \in S$ implies $t + 1 \in S$, for $t \in N$, then $S = N$.