

Assignment-05

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Download all python codes from

<https://github.com/suyogtangade/Assignment5.git>

and latex-tikz codes from

<https://github.com/suyogtangade/Assignment5.git>

Question taken from

https://github.com/gadepall/ncert/blob/main/linalg/quadratic_forms/gvv_ncert_quadratic_forms.pdf—Q.no.2.31

1 QUADRATIC FORMS.PDF-Q.NO.2.31

Find the equation of the hyperbola with foci $\begin{pmatrix} 0 \\ \pm 3 \end{pmatrix}$ and length of the latus rectum 36.

2 SOLUTION

The equation of a conic with directrix $\mathbf{n}^T \mathbf{x} = c$, eccentricity e and focus \mathbf{F} is given by

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.1)$$

The eccentricity of the conic is given by

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \quad (2.0.2)$$

Definition 1 (Latus rectum). *The latus rectum of a conic section is the chord (line segment) that passes through the focus, is perpendicular to the major axis and has both endpoints on the curve.*

For $|\mathbf{V}| \neq 0$, the lengths of semi-major and semi-minor axes of the conic are

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}}, \sqrt{\left| \frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2} \right|} \quad (2.0.3)$$

The equation latus rectum of the conic in (2.0.1) is given by

$$\mathbf{n}^T (\mathbf{x} - \mathbf{F}) = 0 \quad (2.0.4)$$

For $|\mathbf{V}| \neq 0$, the length of latus rectum (LLR) of the conic in (2.0.1) is given by

$$LLR = \frac{2 \left| \frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2} \right|}{\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}}} \quad (2.0.5)$$

Proof. Using (2.0.3), we can write

$$\mathbf{F} = \mathbf{c} \pm \left(\sqrt{\frac{(\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f)(\lambda_2 - \lambda_1)}{\lambda_1 \lambda_2}} \right) \mathbf{p}_1 \quad (2.0.6)$$

Given, length of latus rectum is 36 and foci are $\begin{pmatrix} 0 \\ \pm 12 \end{pmatrix}$. Let us consider $\begin{pmatrix} 0 \\ 12 \end{pmatrix}$ for solving the problem.

$$\mathbf{F} = \begin{pmatrix} 0 \\ 12 \end{pmatrix} \Rightarrow \|\mathbf{F}\| = 12 \quad (2.0.7)$$

Let $\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = \alpha$. From (2.0.3), (2.0.2), (2.0.5)

$$\sqrt{\frac{\alpha}{\lambda_1}} \sqrt{1 - \frac{\lambda_1}{\lambda_2}} = 12 \quad (2.0.8)$$

$$\frac{2 \left(\frac{-\alpha}{\lambda_2} \right)}{\sqrt{\frac{\alpha}{\lambda_1}}} = 36 \quad (2.0.9)$$

Dividing (2.0.8) by (2.0.9) gives

$$\frac{\lambda_1}{\lambda_2} = -3 \quad (2.0.10)$$

$$\Rightarrow e = 2 \quad (2.0.11)$$

$$\Rightarrow \sqrt{\frac{\alpha}{\lambda_1}} = 6 \quad (2.0.12)$$

The associated directrix is perpendicular to the y-axis and passes through the point

$$\begin{pmatrix} 0 \\ \sqrt{\frac{\alpha}{e^2 \lambda_1}} \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \quad (2.0.13)$$

Hence, its equation is

$$(0 \ 1) \left(\mathbf{x} - \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right) = 0 \quad (2.0.14)$$

$$\Rightarrow (0 \ 1) \mathbf{x} = 3 \quad (2.0.15)$$

Comparing it with $\mathbf{n}^\top \mathbf{x} = c$

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, c = 3 \Rightarrow \|\mathbf{n}\| = 1 \quad (2.0.16)$$

Calculating \mathbf{V}, \mathbf{u} and f ,

$$\mathbf{V} = 1^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 2^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \quad (2.0.17)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \quad (2.0.18)$$

$$\mathbf{u} = 3(2^2) \begin{pmatrix} 0 \\ 1 \end{pmatrix} - 1^2 \begin{pmatrix} 0 \\ 12 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.0.19)$$

$$f = 1^2(12^2) - 3^2(2^2) = 108 \quad (2.0.20)$$

Hence, the required equation is

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \mathbf{x} + 108 = 0 \quad (2.0.21)$$

Also, from (2.0.4), the equations of latus rectum is

$$(0 \ 1) \left(\mathbf{x} - \begin{pmatrix} 0 \\ 12 \end{pmatrix} \right) = 0 \quad (2.0.22)$$

$$\Rightarrow (0 \ 1) \mathbf{x} = 12 \quad (2.0.23)$$

Similarly, the equations of directrix and latus rectum associated with $\begin{pmatrix} 0 \\ -12 \end{pmatrix}$ are given by

$$(0 \ 1) \mathbf{x} = -3 \quad (2.0.24)$$

$$(0 \ 1) \mathbf{x} = -12 \quad (2.0.25)$$

The plot of the hyperbola is given in the fig 2.1

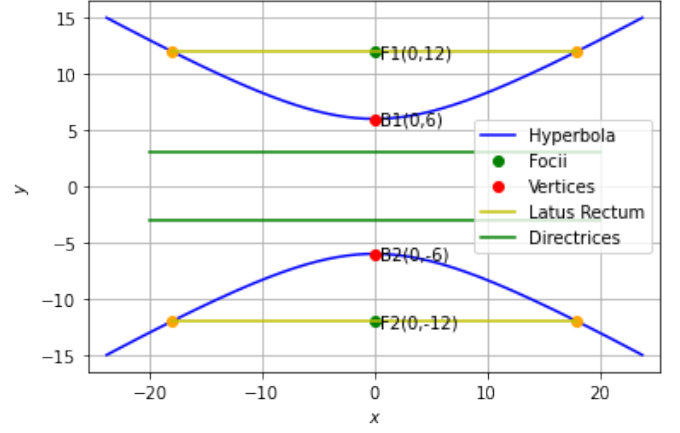


Fig. 2.1: Hyperbola