Assignment-05

Suyog Tangade
MD/2020/710

Download all python codes from

https://github.com/suyogtangade/Assignment5.git

and latex-tikz codes from

https://github.com/suyogtangade/Assignment5.git

Question taken from

https://github.com/gadepall/ncert/blob/main/linalg/quadratic_forms/gvv_ncert_quadratic_forms.pdf−Q.no.2.31

1 quadratic forms.pdf-Q.no.2.31

Find the equation of the hyperbola with foci $\begin{pmatrix} 0 \\ \pm 12 \end{pmatrix}$ and length of the latus rectum 36.

2 Solution

The equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity e and focus $\mathbf{F}$ is given by

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{2.0.1}$$

The eccentricity of the conic is given by

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{2.0.2}$$

Definition 1 (Latus rectum)**.** *The latus rectum of a conic section is the chord (line segment) that passes through the focus, is perpendicular to the major axis and has both endpoints on the curve.*

For $|\mathbf{V}| \neq 0$, the lengths of semi-major and semi-minor axes of the conic are

$$\sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}}, \sqrt{\left|\frac{f - \mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u}}{\lambda_2}\right|} \tag{2.0.3}$$

The equation latus rectum of the conic in (2.0.1) is given by

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{F}) = 0 \tag{2.0.4}$$

For $|\mathbf{V}| \neq 0$, the length of latus rectum (LLR) of the conic in (2.0.1) is given by

$$LLR = \frac{2\left|\frac{f - \mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u}}{\lambda_2}\right|}{\sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}}} \tag{2.0.5}$$

Proof. Using (2.0.3), we can write

$$\mathbf{F} = \mathbf{c} \pm \left(\sqrt{\frac{(\mathbf{u}^T \mathbf{V}^{-1}\mathbf{u} - f)(\lambda_2 - \lambda_1)}{\lambda_1 \lambda_2}}\right)\mathbf{p_1} \tag{2.0.6}$$

Given, length of latus rectum is 36 and focii are $\begin{pmatrix} 0 \\ \pm 12 \end{pmatrix}$. Let us consider $\begin{pmatrix} 0 \\ 12 \end{pmatrix}$ for solving the problem.

$$\mathbf{F} = \begin{pmatrix} 0 \\ 12 \end{pmatrix} \Rightarrow \|\mathbf{F}\| = 12 \tag{2.0.7}$$

Let $\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f = \alpha$. From (2.0.3),(2.0.2),(2.0.5)

$$\sqrt{\frac{\alpha}{\lambda_1}}\sqrt{1 - \frac{\lambda_1}{\lambda_2}} = 12 \tag{2.0.8}$$

$$\frac{2\left(\frac{-\alpha}{\lambda_2}\right)}{\sqrt{\frac{\alpha}{\lambda_1}}} = 36 \tag{2.0.9}$$

Dividing (2.0.8) by (2.0.9) gives

$$\frac{\lambda_1}{\lambda_2} = -3 \tag{2.0.10}$$

$$\Rightarrow e = 2 \tag{2.0.11}$$

$$\Rightarrow \sqrt{\frac{\alpha}{\lambda_1}} = 6 \tag{2.0.12}$$

The associated directrix is perpendicular to the y-axis and passes through the point

$$\begin{pmatrix} 0 \\ \sqrt{\frac{\alpha}{e^2 \lambda_1}} \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \tag{2.0.13}$$

Hence, its equation is

$$\begin{pmatrix} 0 & 1 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 0 \\ 3 \end{pmatrix}\right) = 0 \tag{2.0.14}$$

$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = 3 \tag{2.0.15}$$

Comparing it with $\mathbf{n}^\top \mathbf{x} = c$

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, c = 3 \implies \|\mathbf{n}\| = 1 \tag{2.0.16}$$

Calculating $\mathbf{V}, \mathbf{u}$ and f,

$$\mathbf{V} = 1^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 2^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} \tag{2.0.17}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \tag{2.0.18}$$

$$\mathbf{u} = 3(2^2)\begin{pmatrix} 0 \\ 1 \end{pmatrix} - 1^2 \begin{pmatrix} 0 \\ 12 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.19}$$

$$f = 1^2(12^2) - 3^2(2^2) = 108 \tag{2.0.20}$$

Hence, the required equation is

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}\mathbf{x} + 108 = 0 \tag{2.0.21}$$

Also, from (2.0.4), the equations of latus rectum is

$$\begin{pmatrix} 0 & 1 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 0 \\ 12 \end{pmatrix}\right) = 0 \tag{2.0.22}$$

$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = 12 \tag{2.0.23}$$

Similarly, the equations of directrix and latus rectum associated with $\begin{pmatrix} 0 \\ -12 \end{pmatrix}$ are given by

$$\begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = -3 \tag{2.0.24}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = -12 \tag{2.0.25}$$

The plot of the hyperbola is given in the fig 2.1

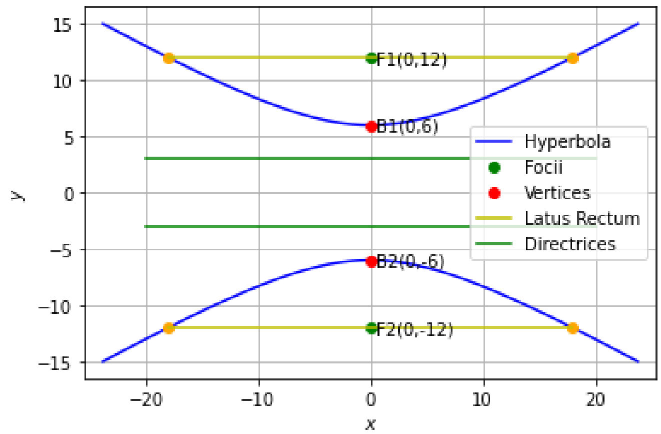

Fig. 2.1: Hyperbola