

SUPPLEMENTARY FILE

In this supplementary file, Section S.I gives the proofs of theoretical results; Section S.II introduces the resource-limit pair (RP) detection algorithm; Section S.III provides the critical set of RPs (CRP) detection algorithm; Section S.IV shows the partial deadlock detection algorithm for S⁴PR where the initial marking does not change; Section S.V analyzes the existing work via case studies; and Section S.VI presents all CRPs and partial deadlocks of the S⁴PR in Fig. 8 of the paper.

S.I. PROOFS OF THEORETICAL RESULTS

In this section, we prove all theoretical results.

Proof of Property 1:

In an S⁴PR, since $\mathcal{N} = (\{p_{0i}\} \cup P_{Ai}, T_i, F_i, W_i, M_0)$ is a strongly connected state machine, $\forall t_i \in T_i$:

$$|t_i \cap (P_{Ai} \cup \{p_{0i}\})| = |t_i \cap (P_{Ai} \cup \{p_{0i}\})| = 1.$$

According to Definition 5, $\forall t \in T$:

$$|t \cap (P_A \cup \{p_0\})| = |t \cap (P_A \cup \{p_0\})| = 1,$$

i.e.,

$$|t \cap P_A| \leq 1 \text{ and } |t \cap P_A| \leq 1. \quad \blacksquare$$

Proof of Property 2:

$\forall M \in R(M_0), A^T X = M - M_0$. Hence,

$$(A^T X)^T \pi_r = (M - M_0)^T \pi_r,$$

i.e.,

$$X^T (A \pi_r) = (M - M_0)^T \pi_r.$$

By $A \pi_r = \mathbf{0}$,

$$(M - M_0)^T \pi_r = \mathbf{0},$$

i.e.,

$$M^T \pi_r = M_0^T \pi_r.$$

Thus,

$$\sum_{p \in \|\pi_r\|} (M(p) \pi_r(p)) = \sum_{p \in \|\pi_r\|} (M_0(p) \pi_r(p)),$$

i.e.,

$$\sum_{p \in \|\pi_r\|} (M(p) \pi_r(p)) = M_0(r) \pi_r(r) + \sum_{p' \in \Pi_r} (M_0(p') \pi_r(p')).$$

Since $\Pi_r \subseteq P_A$ and $\forall p_i \in P_A: M_0(p_i) = 0$,

$$\sum_{p' \in \Pi_r} (M_0(p') \pi_r(p')) = 0.$$

Hence,

$$\sum_{p \in \|\pi_r\|} (M(p) \pi_r(p)) = M_0(r) \pi_r(r).$$

Since $\pi_r(r) = 1$,

$$\sum_{p \in \|\pi_r\|} (M(p) \pi_r(p)) = M_0(r). \quad \blacksquare$$

Proof of Lemma 1:

Suppose that $p \notin P_c$. According to Definition 8, $\exists t \in p^*$:

$$\forall r \in P_R \text{ and } r \notin t,$$

i.e., $t \cap P_R = \emptyset$. Since $p \in P_M$,

$$p \in P_A \text{ and } M(p) \neq 0.$$

According to Property 1,

$$|t \cap (P_A \cup P_0)| = 1.$$

By $t \in p^*$ and $p \in P_A$, we obtain that $t = \{p\}$. Since $M(p) \neq 0$, t is enabled at M . Hence, $p \in P_c$. \blacksquare

Proof of Corollary 1:

Since M is a partial deadlock, according to Definition 7, $\forall p \in P_M$ and $\forall t \in p^*$ such that t is not enabled at $\forall M' \in R(M)$.

According to Lemma 1, $p \in P_c$. \blacksquare

Proof of Theorem 1:

Sufficiency:

If $\exists r^w \in R_{ij}: M(r) \leq w$, according to Definition 9, $\exists r \in t_j \cap P_R: M(r) \leq W(r, t_j) - 1$, i.e., $M(r) < W(r, t_j)$. Hence, t_j is not enabled at M .

Necessity:

By $p_i \in P_M$ and $P_M = \{p | p \in P_A, M(p) \neq 0\}$, we have that $p_i \in P_A$ and $M(p_i) \neq 0$. By $|t_j \cap (P_A \cup P_0)| = 1$, $p_i \in t_j$, and $p_i \in P_A$, we derive that $t_j \cap P_0 = \emptyset$. Since $M(p_i) \neq 0$ and t_j is not enabled at M , $\exists r \in P_R$:

$$r \in t_j \text{ and } M(r) < W(r, t_j),$$

i.e.,

$$r \in P_R \cap t_j \text{ and } M(r) \leq W(r, t_j) - 1.$$

Since $p_i \in P_M$, according to Corollary 1, $p_i \in P_c$. According to Definition 9, $R_{ij} = \{r^w | r \in t_j \cap P_R, w = W(r, t_j) - 1\}$. Hence, $\exists r^w \in R_{ij}: M(r) \leq w$. \blacksquare

Proof of Corollary 2:

Since M is a partial deadlock, $\forall p_i \in P_M$ and $\forall t_j \in p_i^*$ such that t_j is not enabled at M . Since t_j is not enabled at M , according to Theorem 1, $\exists r^w \in R_{ij}: M(r) \leq w$. \blacksquare

Proof of Theorem 2:

Sufficiency:

If $\Omega_i = \langle P_{Ri}, R_i \rangle$ is an RP of p_i , according to Definitions 9 and 10,

$$R_i \subseteq \bigcup_{j \in J} R_{ij} \text{ and}$$

$$R_{ij} = \{r^y | r' \in t_j \cap P_R, y = W(r', t_j) - 1\},$$

where $J = \{j | t_j \in p_i^*\}$ and $\forall j \in J: |R_i \cap R_{ij}| = 1$. Hence, $\forall t_j \in p_i^*$, $\exists r' \in t_j \cap P_R$:

$$y = W(r', t_j) - 1 \text{ and } r^y \in R_i \cap R_{ij}.$$

Since $\forall r^w \in R_i: M(r) \leq w$,

$$M(r') \leq y.$$

Hence, $\forall t_j \in p_i^*$, $\exists r^y \in R_{ij}: M(r') \leq y$. According to Theorem 1, t_j is not enabled at M .

Necessity:

Since $\forall t_j \in p_i^*$ is not enabled at M , according to Theorem 1,

$$\exists r^w \in R_{ij}: M(r) \leq w.$$

Since $J = \{j | t_j \in p_i^*\}$, $\forall j \in J$ satisfies that $\exists r^w \in R_{ij}: M(r) \leq w$. Hence, $\exists R_i \subseteq \bigcup_{j \in J} R_{ij}$ and $|R_i \cap R_{ij}| = 1$ such that $\forall r^w \in R_i$ satisfies that $M(r) \leq w$. By $p_i \in P_M$, $p_i \in P_A$. According to Definition 5, $\exists r \in P_R: p_i \in \|\pi_r\|$. Thus, there exists $P_{Ri} = \{r' \in P_R, p_i \in \|\pi_{r'}\|\}$. According to Definition 10, $\exists \Omega_i = \langle P_{Ri}, R_i \rangle: \forall r^w \in R_i$ and $M(r) \leq w$. \blacksquare

Proof of Corollary 3:

Since M is a partial deadlock, according to Corollary 2, $\forall p_i \in P_M$ and $\forall t_j \in p_i^*: \exists r^w \in R_{ij}$ and $M(r) \leq w$. By $J = \{j | t_j \in p_i^*\}$, we have that $\forall p_i \in P_M$ and $\forall j \in J: \exists r^w \in R_{ij}$ and $M(r) \leq w$. Hence, $\forall p_i \in P_M$, $\exists R_i \subseteq \bigcup_{j \in J} R_{ij}$ and $|R_i \cap R_{ij}| = 1$ such that $\forall r^w \in R_i$ satisfies that $M(r) \leq w$. Since $\exists r \in P_R: p_i \in \|\pi_r\|$, there exists $P_{Ri} = \{r' \in P_R, p_i \in \|\pi_{r'}\|\}$. According to Definition 10, $\forall p_i \in P_M$, $\exists \Omega_i = \langle P_{Ri}, R_i \rangle: \forall r^w \in R_i$ and $M(r) \leq w$. \blacksquare

Proof of Corollary 4:

According to Corollary 3, it can be easily proved. ■

Proof of Theorem 3:

Sufficiency:

Suppose that $\hat{\Omega}$ is a CRP at M . Since $\hat{\Omega}$ is a CRP at M , according to Definition 11, $\forall p_i \in P_M, \exists \Omega_i = \langle P_{R_i}, R_i \rangle: \forall r^w \in R_i$ and $M(r) \leq w$. According to Theorem 2, $\forall t_j \in p_i^*$ is not enabled at M . Hence, $\forall t' \in T$, if $M[t']M'$, there are two cases.

i) When $t' \notin T_r$, according to Property 3, $\forall p_k \in \Pi_r: p_k \notin t^* \cup t'$. Hence,

$$M(p_k) = M'(p_k).$$

ii) When $t' \in T_r$, since $\forall p_i \in P_M$ and $\forall t_j \in p_i^*$ is not enabled at M , we have that $\forall p_k \in \Pi_r: t' \notin p_k^*$. By $M[t']M'$,

$$M'(p_k) \geq M(p_k).$$

Then,

$$\sum_{p \in \{p | p \in \Pi_r \cap P_M\}} (M'(p) \pi_r(p)) \geq \sum_{p \in \{p | p \in \Pi_r \cap P_M\}} (M(p) \pi_r(p)). \quad (S1)$$

According to Condition 2) of Definition 11,

$$\sum_{p \in \{p | p \in \Pi_r \cap P_M\}} (M(p) \pi_r(p)) \geq M_0(r) - w.$$

By (S1),

$$\sum_{p \in \{p | p \in \Pi_r \cap P_M\}} (M'(p) \pi_r(p)) \geq M_0(r) - w. \quad (S2)$$

Since $M' \in R(M_0)$, according to Property 2,

$$\sum_{p_v \in \|\pi_r\|} (M'(p_v) \pi_r(p_v)) = M_0(r).$$

By $\Pi_r = \|\pi_r\| \setminus \{r\}$,

$$M'(r) \pi_r(r) + \sum_{p_z \in \Pi_r} (M'(p_z) \pi_r(p_z)) = M_0(r).$$

Since $\pi_r(r) = 1$,

$$M'(r) + \sum_{p_z \in \Pi_r} (M'(p_z) \pi_r(p_z)) = M_0(r).$$

By $\Pi_r = (\Pi_r \cap P_M) \cup (\Pi_r \setminus P_M)$,

$$\begin{aligned} M'(r) + \sum_{p' \in \{p' | p' \in \Pi_r \setminus P_M\}} (M'(p') \pi_r(p')) + \\ \sum_{p \in \{p | p \in \Pi_r \cap P_M\}} (M'(p) \pi_r(p)) = M_0(r), \end{aligned}$$

i.e.,

$$\begin{aligned} \sum_{p \in \{p | p \in \Pi_r \cap P_M\}} (M'(p) \pi_r(p)) = \\ M_0(r) - \sum_{p' \in \{p' | p' \in \Pi_r \setminus P_M\}} (M'(p') \pi_r(p')) - M'(r). \end{aligned}$$

By (S2),

$$M_0(r) - \sum_{p' \in \{p' | p' \in \Pi_r \setminus P_M\}} (M'(p') \pi_r(p')) - M'(r) \geq M_0(r) - w,$$

i.e.,

$$M'(r) \leq w - \sum_{p' \in \{p' | p' \in \Pi_r \setminus P_M\}} (M'(p') \pi_r(p')).$$

Since

$$\sum_{p' \in \{p' | p' \in \Pi_r \setminus P_M\}} (M'(p') \pi_r(p')) \geq 0,$$

we have that

$$M'(r) \leq w.$$

According to Theorem 2, $\forall t_j \in p_i^*, t_j$ is not enabled at M' . By analogy, $\forall M'' \in R(M)$, t_j is not enabled at M'' . Hence, M is a partial deadlock.

Necessity:

Suppose that M is a partial deadlock. According to Corollary 3, $\forall p_i \in P_M, \exists \Omega_i = \langle P_{R_i}, R_i \rangle:$

$$\forall r^w \in R_i \text{ and } M(r) \leq w.$$

Let $\hat{\Omega}$ be the set of these Ω_i . Hence, $P_M = \{p_i | \Omega_i \in \hat{\Omega}\}$, and $\forall p_i \in P_M, \exists \Omega_i \in \hat{\Omega}:$

$$\forall r^w \in R_i \text{ and } M(r) \leq w.$$

Thus, $\hat{\Omega}$ satisfies Condition 1) of Definition 11. Suppose that

$$\sum_{p \in \{p | p \in \Pi_r \cap P_M\}} (M(p) \pi_r(p)) < M_0(r) - w.$$

According to Property 2,

$$\begin{aligned} M(r) \pi_r(r) + \sum_{p' \in \{p' | p' \in \Pi_r \setminus P_M\}} (M(p') \pi_r(p')) + \\ \sum_{p \in \{p | p \in \Pi_r \cap P_M\}} (M(p) \pi_r(p)) = M_0(r). \end{aligned}$$

Since $\pi_r(r) = 1$,

$$\begin{aligned} \sum_{p \in \{p | p \in \Pi_r \cap P_M\}} (M(p) \pi_r(p)) \\ = M_0(r) - M(r) - \sum_{p' \in \{p' | p' \in \Pi_r \setminus P_M\}} (M(p') \pi_r(p')). \end{aligned}$$

Hence,

$$M_0(r) - M(r) - \sum_{p' \in \{p' | p' \in \Pi_r \setminus P_M\}} (M(p') \pi_r(p')) < M_0(r) - w,$$

i.e.,

$$\sum_{p' \in \{p' | p' \in \Pi_r \setminus P_M\}} (M(p') \pi_r(p')) + M(r) > w. \quad (S3)$$

Since

$$\forall p' \in \{p' | p' \in \Pi_r \setminus P_M\}: p' \notin P_M,$$

we have that $M(p') = 0$, i.e.,

$$\sum_{p' \in \{p' | p' \in \Pi_r \setminus P_M\}} (M(p') \pi_r(p')) = 0.$$

By (S3),

$$M(r) > w.$$

It contradicts $M(r) \leq w$. Hence,

$$\sum_{p \in \{p | p \in \Pi_r \cap P_M\}} (M(p) \pi_r(p)) \geq M_0(r) - w.$$

Thus, $\hat{\Omega}$ satisfies Condition 2) of Definition 11. Hence, $\hat{\Omega}$ is a CRP at M . ■

Proof of Theorem 4:

Since $\hat{\Omega}$ is a CRP at M , according to Definition 11, $P_M = \{p_i | \Omega_i \in \hat{\Omega}\}$. Suppose that $\exists r \in R: r \notin R'$. Since

$$R' = \{P_{R_i} | \Omega_i \in \hat{\Omega}, \Omega_i = \langle P_{R_i}, R_i \rangle\},$$

we obtain that $\forall \Omega_i \in \hat{\Omega}:$

$$r \notin P_{R_i},$$

where $\Omega_i = \langle P_{R_i}, R_i \rangle$ and $P_{R_i} = \{r' | r' \in P_R, p_i \in \|\pi_r\|\}$. Hence,

$\forall p_i \in \{p_i | \Omega_i \in \hat{\Omega}\}:$

$$p_i \notin \|\pi_r\|.$$

By $P_M = \{p_i | \Omega_i \in \hat{\Omega}\}$, we have that $\forall p_i \in P_M:$

$$p_i \notin \|\pi_r\|,$$

i.e., $\forall p_j \in \|\pi_r\| \cap P_A:$

$$M(p_j) = 0.$$

According to Property 2,

$$M(r) \pi_r(r) + \sum_{p_j \in \{p_j | p_j \in \|\pi_r\| \cap P_A\}} (M(p_j) \pi_r(p_j)) = M_0(r).$$

By $\pi_r(r) = 1$ and $\forall p_j \in \|\pi_r\| \cap P_A: M(p_j) = 0$, we have that:

$$M(r) = M_0(r).$$

Since $r \in R$ and $R = \{r' | r' \in R_i, \Omega_i \in \hat{\Omega}, \Omega_i = \langle P_{R_i}, R_i \rangle\}$, according to Definition 11,

$$M(r) < w.$$

According to Definitions 9 and 10,

$$w = W(r, t_j) - 1,$$

i.e.,

$$M(r) < W(r, t_j).$$

Since $\forall r' \in P_R: M_0(r') \geq \max\{W(r', t) | t \in T, (r', t) \in F\}$, we have that:

$$M_0(r) \geq W(r, t_j).$$

Hence, $M(r) < M_0(r)$. This contradicts that $M(r) = M_0(r)$. Hence, $\forall r \in R: r \in R'$. ■

S.II. RP Detection Algorithm

In this section, Algorithm S1 is proposed to obtain $\tilde{\Omega}$. Particularly, for any activity place p_i , the method determines whether it is a critical place according to Definition 8 (lines

Algorithm S1: Detection of $\tilde{\Omega}$ Input: S⁴PR $\mathcal{A}=(P_0 \cup P_A \cup P_R, T, F, W, M_0)$ Output: $\tilde{\Omega}$

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1  Let  $\tilde{\Omega}=\emptyset$ ;
2  For  $\forall p_i \in P_A$ , do
3    Let  $R_i'=\emptyset$ ;
4    For  $\forall t_j \in T$ , do
5      If  $t_j \in p_i^*$ , then
6        Let  $a=1$  and  $R_i''=\emptyset$ ;
7        For  $\forall r \in P_R$ , do
8          If  $r \in t_j$ , then
9            If  $R_i'=\emptyset$ , then
10              $R_{i-a}=\{r^w\}$ , where  $w=W(r, t_j)-1$ ;
11              $R_i''=R_i'' \cup \{R_{i-a}\}$ ;
12              $a=a+1$ ;
13            Else
14              For  $\forall R \in R_i'$ , do
15                If  $\exists r^w \in R$ :  $w \geq w$ , where  $w=W(r, t_j)-1$ , then
16                   $R=R \setminus \{r^w\}$ ;
17                End
18                 $R_{i-a}=R \cup \{r^w\}$ , where  $w=W(r, t_j)-1$ ;
19                 $R_i''=R_i'' \cup \{R_{i-a}\}$ ;
20                 $a=a+1$ ;
21              End
22            End
23          End
24        End
25         $R_i'=R_i''$  and  $T=T \setminus \{t_j\}$ ;
26      End
27    End
28    If  $R_i' \neq \emptyset$ , then
29      Let  $\tilde{\Omega}_i=P_{R_i}=\emptyset$  and  $b=1$ ;
30      For  $\forall r \in P_R$ , do
31        If  $p_i \in \|\pi_r\|$ , then
32           $P_{R_i}=P_{R_i} \cup \{r\}$ ;
33        End
34      End
35      For  $\forall R_i \in R_i'$ , do
36         $\tilde{\Omega}_i=\tilde{\Omega}_i \cup \{\Omega_{i-b}\}$ , where  $\Omega_{i-b}=\langle P_{R_i}, R_i \rangle$ ;
37         $b=b+1$ ;
38      End
39       $\tilde{\Omega}=\tilde{\Omega} \cup \tilde{\Omega}_i$ ;
40    End
41  End
42  Return  $\tilde{\Omega}$ 
43  End

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4-8 of Algorithm S1). If the above is fulfilled, the sets P_{R_i} (lines 9-27) and R_i (lines 28-34) are computed according to Definition 10. From lines 35-38, for any critical place p_i , the number of elements in $\tilde{\Omega}_i$ is the same as that of R_i' . Via line 19, elements in R_i' are the same as those in R_i'' . Next, the relationship between the structure of an S⁴PR and the number of elements in R_i'' is analyzed. Given an S⁴PR, the maximum number of input resource places for transitions is $\bar{n}=\text{Max}(\{x \mid x=|t \cap P_R|, t \in T\})$, and the maximum number of output transitions for activity places is $\bar{n}=\text{Max}(\{y \mid y=|p^* \cap T|, p \in P_A\})$. From Algorithm S1, when executing the for-loop in lines 7-24 each time, lines 8-23 repeat at most \bar{n} times. When the for-loop in lines 7-24 is executed for the first time, due to $R_i'=\emptyset$, R_i'' is calculated through lines 9-12. Since lines 8-23 repeat at most \bar{n} times, lines 9-12 are executed at most \bar{n} times. As a result, there are at most \bar{n} elements in the obtained R_i'' . By line 25, let $R_i'=R_i''$. Now, there are at most

\bar{n} elements in R_i' . Hence, when the for-loop in lines 7-24 is executed for the second time, $R_i' \neq \emptyset$ and R_i'' is calculated through lines 13-21. Since there are at most \bar{n} elements in R_i' , at most \bar{n} elements in R_i'' are obtained by each execution of the for-loop in lines 13-21. Since lines 8-23 repeat at most \bar{n} times, the for-loop in lines 13-21 is also executed at most \bar{n} times. Hence, there are at most \bar{n}^2 elements in R_i'' computed by the second execution of lines 7-24. For activity place p_i , there are at most \bar{n} output transitions. Hence, according to lines 4-27, the for-loop in lines 7-24 is executed at most \bar{n} times. As a result, the number of elements in R_i'' is no more than $\bar{n}^{\bar{n}}$. Thus, there are at most $\bar{n}^{\bar{n}}$ elements in $\tilde{\Omega}_i$. Since there are at most $|P_A|$ critical places in \mathcal{A} , from line 33, there are no more than $|P_A| \bar{n}^{\bar{n}}$ elements in $\tilde{\Omega}$, i.e., there are at most $|P_A| \bar{n}^{\bar{n}}$ RPs. For an AMS modeled by an S⁴PR, \bar{n} represents the maximum number of resource types required for a single-step processing operation within the system, and \bar{n} denotes the maximum number of parallel processes when machining the same type of part. For example, in the AMS modeled by the S⁴PR in Fig. 4, the processing operation represented by t_3 requires two types of resources (r_2 and r_3), while the other operations require only one type. Hence, we have that $\bar{n}=2$. In addition, the job type represented in the left subnet allows two parallel sequential processes to process the same type of part, while the job type shown in the right subnet can only be completed by one sequential process. Hence, we obtain that $\bar{n}=2$. Notice that the number of RPs is $|P_A| \bar{n}^{\bar{n}}$ only if there are \bar{n} input resource places for each transition and \bar{n} output transitions for each activity place. In general, the number of RPs is much smaller than $|P_A| \bar{n}^{\bar{n}}$. For example, for the S⁴PR in Fig. 4, $|P_A|=7$ and $\bar{n}=\bar{n}=2$. Hence, $|P_A| \bar{n}^{\bar{n}}=28$. However, from Table IV, there are only 6 RPs in this S⁴PR.

Let us now estimate the time complexity of Algorithm S1. By lines 4-27, we have that the time complexity of finding P_{R_i} is $O(|T||P_R| \bar{n}^{\bar{n}})$, where $|T|$ and $|P_R|$ are the number of transitions and resource places of \mathcal{A} , respectively. From lines 28-34, we obtain that the time complexity of finding R_i is $O(|P_R|)$. Next, $\tilde{\Omega}_i$ can be obtained (lines 35-39), where the time complexity is $O(\bar{n}^{\bar{n}})$. For each activity place p_i , we repeat the above steps and obtain $\tilde{\Omega}_i$ of \mathcal{A} (lines 1-35). Hence, the total time complexity of Algorithm S1 is $O(|P_A||T||P_R| \bar{n}^{\bar{n}})$.

The following example visually illustrates the process of finding RPs in an S⁴PR through Algorithm S1.

Example S1: For S⁴PR in Fig. 4, $P_A=\{p_2, p_3, p_4, p_5, p_7, p_8, p_9\}$, $P_R=\{r_1, r_2, r_3, r_4\}$, and $T=\{t_1, t_2, \dots, t_{10}\}$. Starting from line 1 of Algorithm S1, $\tilde{\Omega}=\emptyset$. For $p_2 \in P_A$, $p_2^*=\{t_2, t_3\}$. For $t_2 \in p_2^*$, from lines 3 and 6,

$$R_2'=R_2''=\emptyset \text{ and } a=1.$$

From Fig. 4, $t_2 \cap P_R=\{r_2\}$. Since $R_2'=\emptyset$ and $W(r_2, t_2)-1=1$, according to lines 8-12,

$$R_{2-1}=\{r_2^1\}, R_2''=\{R_{2-1}\}, \text{ and } a=2.$$

With line 25,

$$R_2'=R_2''=\{R_{2-1}\}=\{\{r_2^1\}\} \text{ and } T=T \setminus \{t_2\}.$$

For $t_3 \in p_2^*$, from line 6,

$$R_2'' = \emptyset \text{ and } a=1.$$

From Fig. 4, $t_3 \cap P_R = \{r_2, r_3\}$. For $r_2 \in t_3 \cap P_R$, since $w = W(r_2, t_3) - 1 = 0$ and $\exists R = R_{2-1} \in R_2'$: $r_2^y = r_2^1 \in R_{2-1}$ and $y > w$, from lines 13-17, $R = \emptyset$. By lines 18-21,

$$R_{2-1} = R \cup \{r_2^0\} = \{r_2^0\}, R_2'' = \{R_{2-1}\} = \{R_{2-1}\}, \text{ and } a=2.$$

For $r_3 \in t_3 \cap P_R$, since $W(r_3, t_3) - 1 = 1$ and $R_2' = \{\{r_2^1\}\}$, according to lines 18-21,

$$R_{2-2} = \{r_2^1\} \cup \{r_3^1\} = \{r_2^1, r_3^1\},$$

$$R_2'' = R_2'' \cup R_{2-2} = \{\{r_2^0\}, \{r_2^1, r_3^1\}\}, \text{ and } a=3.$$

Via line 25,

$$R_2' = R_2'' = \{\{r_2^0\}, \{r_2^1, r_3^1\}\} \text{ and } T = T \setminus \{t_3\}.$$

Then, $\forall t \in T: t \notin p_2$. By line 29,

$$\tilde{\Omega}_2 = P_{R_2} = \emptyset \text{ and } b=1.$$

Since $p_2 \in \|\pi_1\|$, from line 30-34,

$$P_{R_2} = \{r_1\}.$$

Since $R_2' = \{\{r_2^0\}, \{r_2^1, r_3^1\}\}$, from lines 35-38,

$$\tilde{\Omega}_2 = \{\Omega_{2-1}, \Omega_{2-2}\},$$

where $\Omega_{2-1} = \langle P_{R_2}, R_{2-1} \rangle$ and $\Omega_{2-2} = \langle P_{R_2}, R_{2-2} \rangle$. Via line 39,

$$\tilde{\Omega} = \{\Omega_{2-1}, \Omega_{2-2}\},$$

where $\Omega_{2-1} = \langle \{r_1\}, \{r_2^0\} \rangle$ and $\Omega_{2-2} = \langle \{r_1\}, \{r_2^1, r_3^1\} \rangle$. For p_3 - p_5 and p_7 - p_9 , we repeat the above steps. Finally, we obtain

$$\tilde{\Omega} = \{\Omega_{2-1}, \Omega_{2-2}, \Omega_{3-1}, \Omega_{4-1}, \Omega_{7-1}, \Omega_{8-1}\},$$

where $\Omega_{3-1} = \langle \{r_2\}, \{r_4^0\} \rangle$, $\Omega_{4-1} = \langle \{r_2, r_3\}, \{r_4^0\} \rangle$, $\Omega_{7-1} = \langle \{r_4\}, \{r_3^0\} \rangle$, and $\Omega_{8-1} = \langle \{r_3\}, \{r_1^0\} \rangle$.

S.III. CRP Detection Algorithm

In this section, Algorithm S2 is introduced to detect all the CRPs. To be specific, for each RP Ω_i in $\tilde{\Omega}$, we need to determine if there is a CRP $\hat{\Omega}$ such that $\Omega_i \in \hat{\Omega}$ (lines 2-6 of Algorithm S2). For each subset $\hat{\Omega}$ of $\tilde{\Omega}$, if $\hat{\Omega}$ is a CRP, it satisfies $\forall r \in R: r \in R'$ according to Theorem 4. Hence, for each $\hat{\Omega}$, we need to compute R and R' and determine whether $\exists r \in R$ such that $r \notin R'$ (lines 1 and 2 of Function S1). If $\exists r \in R$ such that $r \notin R'$, since $R' = \{P_{R_i} | \Omega_i \in \hat{\Omega}, \Omega_i = \langle P_{R_i}, R_i \rangle\}$, we need to find a RP $\Omega_j = \langle P_{R_j}, R_j \rangle$ such that $r \in P_{R_j}$ and update

Algorithm S2: Detection of $\tilde{\Omega}$

```

Input:  $\tilde{\Omega}$ 
Output:  $\bar{\Omega}$ 
1  Let  $\bar{\Omega} = \emptyset$ ;
2  For  $\forall \Omega_i \in \tilde{\Omega}$ , where  $\Omega_i = \langle P_{R_i}, R_i \rangle$ , do
3    Let  $\bar{\Omega}' = \{\{\Omega_i\}\}$  and  $\bar{\Omega}_i = \emptyset$ ;
4    For  $\forall \hat{\Omega} \in \bar{\Omega}'$ , do
5       $[\bar{\Omega}_i, \bar{\Omega}] = F_{s1}(\hat{\Omega}, \tilde{\Omega}, \bar{\Omega}', \bar{\Omega}_i)$ ;
6    End
7  For  $\forall \hat{\Omega} \in \bar{\Omega}_i$ , do
8     $[\bar{\Omega}_i] = F_{s2}(\hat{\Omega}, \bar{\Omega}_i, \tilde{\Omega})$ ;
9  End
10  $\bar{\Omega} = \bar{\Omega} \cup \bar{\Omega}_i$ ;
11  $\tilde{\Omega} = \tilde{\Omega} \setminus \{\Omega_i\}$ ;
12  $[\tilde{\Omega}] = F_{s3}(P_{R_i}, \tilde{\Omega}, P_A)$ 
13 End
14 Return  $\bar{\Omega}$ 
15 End

```

Function S1: $[\bar{\Omega}_i, \bar{\Omega}] = F_{s1}(\hat{\Omega}, \tilde{\Omega}, \bar{\Omega}', \bar{\Omega}_i)$

```

Input:  $\hat{\Omega}, \tilde{\Omega}, \bar{\Omega}', \bar{\Omega}_i$ 
Output:  $\bar{\Omega}_i, \bar{\Omega}'$ 
Let  $R = \{r | r^w \in R_k, \Omega_k \in \hat{\Omega}, \Omega_k = \langle P_{R_k}, R_k \rangle\}$ ,
1   $R' = \{r | r \in P_{R_k}, \Omega_k \in \hat{\Omega}, \Omega_k = \langle P_{R_k}, R_k \rangle\}$ ,
    $\bar{\Omega} = \bar{\Omega} \setminus \{\hat{\Omega}\}$ , and  $\bar{\Omega}' = \bar{\Omega}' \setminus \{\hat{\Omega}\}$ ;
2  If  $\exists r \in R \setminus R'$ , then
3    For  $\forall \Omega_j \in \tilde{\Omega}'$ , where  $\Omega_j = \langle P_{R_j}, R_j \rangle$ , do
4      If  $r \in P_{R_j}$ , then
5         $\hat{\Omega}' = \hat{\Omega} \cup \{\Omega_j\}$  and  $\bar{\Omega}' = \bar{\Omega}' \cup \{\hat{\Omega}'\}$ ;
6      End
7    End
8  Else
9     $\bar{\Omega}_i = \bar{\Omega}_i \cup \{\hat{\Omega}\}$ ;
10 End
11 Return  $\bar{\Omega}_i$  and  $\bar{\Omega}'$ ;
12 End

```

Function S2: $[\bar{\Omega}_i] = F_{s2}(\hat{\Omega}, \bar{\Omega}_i, \tilde{\Omega})$

```

Input:  $\hat{\Omega}, \bar{\Omega}_i, \tilde{\Omega}$ 
Output:  $\bar{\Omega}_i$ 
Let  $R' = \{r | r \in P_{R_k}, \Omega_k \in \hat{\Omega}, \Omega_k = \langle P_{R_k}, R_k \rangle\}$  and
1   $P_K = \{p_k | \Omega_k \in \hat{\Omega}\}$ ;
2  For  $\forall \Omega_z \in \tilde{\Omega} \setminus \hat{\Omega}$ , do
3    If  $\forall r^w \in R_z: r \in R'$  and  $p_z \notin P_K$ , then
4      Let  $\hat{\Omega}' = \hat{\Omega} \cup \{\Omega_z\}$  and  $\bar{\Omega}_i = \bar{\Omega}_i \cup \{\hat{\Omega}'\}$ ;
5    End
6  End
7  Return  $\bar{\Omega}_i$ ;
8  End

```

Function S3: $[\tilde{\Omega}] = F_{s3}(P_{R_i}, \tilde{\Omega}, P_A)$

```

Input:  $P_{R_i}, \tilde{\Omega}, P_A$ 
Output:  $\tilde{\Omega}$ 
1  For  $\forall r \in P_{R_i}$ , do
2    If  $\forall \Omega_j \in \tilde{\Omega}: p_j \notin \|\pi_r\|$ , then
3      For  $\forall \Omega_u \in \tilde{\Omega}$ , where  $r^w \in R_u$ , do
4         $\tilde{\Omega} = \tilde{\Omega} \setminus \{\Omega_u\}$ ;
5      End
6    End
7  End
8  Return  $\tilde{\Omega}$ ;
9  End

```

$\hat{\Omega}$ (lines 2-7 of Function S1); otherwise, if $\forall r \in R: r \in R'$, $\hat{\Omega}$ is a CRP (lines 8-10 of Function S1). For each CRP $\hat{\Omega}$ obtained through the above process, if $\exists \Omega_z \in \tilde{\Omega}$ and $\Omega_z \notin \hat{\Omega}$: $\forall r^w \in R_z$ and $r \in R'$, we have that $\hat{\Omega}' = \hat{\Omega} \cup \{\Omega_z\}$ is a CRP (lines 7-9 of Algorithm S2 and Function S2). Then, we obtain all CRPs that contain Ω_i . Subsequently, we can remove Ω_i from $\tilde{\Omega}$ (line 11 of Algorithm S2). For each $r \in P_{R_i}$, if $\forall \Omega_j \in \tilde{\Omega}: p_j \notin \|\pi_r\|$, we have that $\forall \hat{\Omega}' \subseteq \tilde{\Omega}: r \notin \{P_{R_k} | \Omega_k \in \hat{\Omega}', \Omega_k = \langle P_{R_k}, R_k \rangle\}$. For each $\Omega_u \in \tilde{\Omega}$, if $r^w \in R_u$, according to Theorem 4, there is no CRP containing Ω_u . Hence, Ω_u can be removed from $\tilde{\Omega}$ (Function S3). For each RP, we repeat the above process, all CRPs can be obtained (lines 1-4 of Algorithm S2). $\forall \Omega_i \in \tilde{\Omega}$, $\bar{\Omega}_i$ represents the set of CRPs that contains Ω_i , where the number of elements in $\bar{\Omega}_i$ is no more than $|\tilde{\Omega}| - 1$, where $|\tilde{\Omega}|$ is the number of elements in $\tilde{\Omega}$. From

line 10 of Algorithm S2,

$$\bar{\Omega} = \bigcup_{i \in \{i | \Omega_i \in \bar{\Omega}\}} \bar{\Omega}_i.$$

Since there are at most $|\bar{\Omega}|$ elements in $\bar{\Omega}$, the number of CRPs is no greater than $|\bar{\Omega}|(|\bar{\Omega}|-1)$. According to Algorithm S1, $|\bar{\Omega}| \leq |P_A| \bar{n}^{\bar{n}}$.

Next, we evaluate the time complexity of Algorithm S2. From line 1 of Function S1, $\bar{\Omega} \leftarrow \bar{\Omega} \setminus \{\hat{\Omega}\}$, i.e., $|\bar{\Omega}| \leq |\bar{\Omega}|$. Hence, the time complexity of Function S1 is $O(|\bar{\Omega}|)$. In the worst case, $\forall \Omega_j, \Omega_k \in \bar{\Omega} : \hat{\Omega} = \{\Omega_j, \Omega_k\}$ is a CRP. In this case, Function S1 needs to be repeated $|\bar{\Omega}|$ times and there are $|\bar{\Omega}|-1$ elements in $\bar{\Omega}_i$. Thus, the complexity of lines 4-6 of Algorithm S2 is $O(|\bar{\Omega}|^2)$. According to line 2 of Function S2, $\Omega_z \in \bar{\Omega} \setminus \hat{\Omega}$, i.e., $|\Omega_z| \leq |\bar{\Omega}|$. Hence, the time complexity of Function S2 is $O(|\bar{\Omega}|)$. Since there $|\bar{\Omega}|-1$ elements in $\bar{\Omega}_i$, the time complexity of lines 7-9 of Algorithm S2 is $O(|\bar{\Omega}|^2)$. For Function S3, since $P_{R'} \subseteq P_R$ and $\Omega_u \in \bar{\Omega}$, the time complexity of Function S3 is $O(|P_R||\bar{\Omega}|)$. Thus, the time complexity of lines 3-12 of Algorithm S2 is $O(|\bar{\Omega}|^2 + |P_R||\bar{\Omega}|)$. From lines 2-13, the above steps need to be repeated $|\bar{\Omega}|$ times. Hence, the total time complexity of Algorithm S2 is $O(|\bar{\Omega}|^3 + |P_R||\bar{\Omega}|^2)$.

Let us illustrate Algorithm S2 by the following example.

Example S2: In Fig. 4, according to Example S2, $\bar{\Omega} = \{\Omega_{2-1}, \Omega_{2-2}, \Omega_{3-1}, \Omega_{4-1}, \Omega_{7-1}, \Omega_{8-1}\}$ can be obtained via Algorithm S1. From line 1 of Algorithm S2, $\bar{\Omega} = \emptyset$. For $\Omega_{2-1} \in \bar{\Omega}$, where $\Omega_{2-1} = \langle P_{R2}, R_{2-1} \rangle = \langle \{r_1\}, \{r_2^0\} \rangle$, from line 3,

$$\bar{\Omega} \leftarrow \{\{\Omega_{2-1}\}\} \text{ and } \bar{\Omega}_2 = \emptyset.$$

Since $\{\Omega_{2-1}\} \in \bar{\Omega}'$, from lines 4-5 and line 1 of Function S1,

$$\hat{\Omega} = \{\Omega_{2-1}\}, R = \{r_2\}, R' = \{r_1\},$$

$$\bar{\Omega} \leftarrow \{\Omega_{2-2}, \Omega_{3-1}, \Omega_{4-1}, \Omega_{7-1}, \Omega_{8-1}\}, \text{ and } \bar{\Omega}' = \emptyset.$$

Since $\exists r_2 \in R \setminus R'$ and $\Omega_{3-1}, \Omega_{4-1} \in \bar{\Omega}' : r_2 \in P_{R3}$ and $r_2 \in P_{R4}$, from line 5 of Function S1,

$$\bar{\Omega} \leftarrow \{\{\Omega_{2-1}, \Omega_{3-1}\}, \{\Omega_{2-1}, \Omega_{4-1}\}\}.$$

Since $\{\Omega_{2-1}, \Omega_{3-1}\} \in \bar{\Omega}'$, from lines 4-5 and line 1 of Function S1,

$$\hat{\Omega} = \{\Omega_{2-1}, \Omega_{3-1}\}, R = \{r_2, r_4\}, R' = \{r_1, r_2\},$$

$$\bar{\Omega} \leftarrow \{\Omega_{2-2}, \Omega_{4-1}, \Omega_{7-1}, \Omega_{8-1}\}, \text{ and}$$

$$\bar{\Omega} \leftarrow \{\{\Omega_{2-1}, \Omega_{4-1}\}\}.$$

Since $\exists r_4 \in R \setminus R'$ and $\Omega_{7-1} \in \bar{\Omega}' : r_4 \in P_{R7}$, from line 5 of Function S1,

$$\bar{\Omega} \leftarrow \{\{\Omega_{2-1}, \Omega_{3-1}, \Omega_{7-1}\}, \{\Omega_{2-1}, \Omega_{4-1}\}\}.$$

Since $\{\Omega_{2-1}, \Omega_{3-1}, \Omega_{7-1}\} \in \bar{\Omega}'$, from lines 4-5 and line 1 of Function S1,

$$\hat{\Omega} = \{\Omega_{2-1}, \Omega_{3-1}, \Omega_{7-1}\}, R = \{r_2, r_3, r_4\}, R' = \{r_1, r_2, r_4\},$$

$$\bar{\Omega} \leftarrow \{\Omega_{2-2}, \Omega_{4-1}, \Omega_{8-1}\}, \text{ and } \bar{\Omega} \leftarrow \{\{\Omega_{2-1}, \Omega_{4-1}\}\}.$$

Since $\exists r_3 \in R \setminus R'$ and $\Omega_{4-1}, \Omega_{8-1} \in \bar{\Omega}' : r_3 \in P_{R4}$ and $r_3 \in P_{R8}$, from line 5 of Function S1,

$$\bar{\Omega} \leftarrow \{\{\Omega_{2-1}, \Omega_{3-1}, \Omega_{4-1}, \Omega_{7-1}\},$$

$$\{\Omega_{2-1}, \Omega_{3-1}, \Omega_{7-1}, \Omega_{8-1}\}, \{\Omega_{2-1}, \Omega_{4-1}\}\}.$$

Since $\{\Omega_{2-1}, \Omega_{3-1}, \Omega_{4-1}, \Omega_{7-1}\} \in \bar{\Omega}'$, from lines 4-5 and line 1 of Function S1,

$$\hat{\Omega} = \{\Omega_{2-1}, \Omega_{3-1}, \Omega_{4-1}, \Omega_{7-1}\}, R = \{r_2, r_3, r_4\},$$

$$R' = \{r_1, r_2, r_3, r_4\}, \bar{\Omega} \leftarrow \{\Omega_{2-2}, \Omega_{8-1}\}, \text{ and}$$

$$\bar{\Omega} \leftarrow \{\{\Omega_{2-1}, \Omega_{3-1}, \Omega_{4-1}, \Omega_{8-1}\}, \{\Omega_{2-1}, \Omega_{4-1}\}\}.$$

Since $R \setminus R' = \emptyset$, from line 9 of Function S1,

$$\bar{\Omega}_2 = \{\{\Omega_{2-1}, \Omega_{3-1}, \Omega_{4-1}, \Omega_{7-1}\}\}.$$

Since $\{\Omega_{2-1}, \Omega_{3-1}, \Omega_{7-1}, \Omega_{8-1}\} \in \bar{\Omega}'$, from lines 4-5 and line 1 of Function S1,

$$\hat{\Omega} = \{\Omega_{2-1}, \Omega_{3-1}, \Omega_{7-1}, \Omega_{8-1}\}, R = \{r_1, r_2, r_3, r_4\},$$

$$R' = \{r_1, r_2, r_3, r_4\}, \bar{\Omega} \leftarrow \{\Omega_{2-2}, \Omega_{4-1}\}, \text{ and}$$

$$\bar{\Omega} \leftarrow \{\{\Omega_{2-1}, \Omega_{4-1}\}\}.$$

Since $R \setminus R' = \emptyset$, from line 9 of Function S1,

$$\bar{\Omega}_2 = \{\{\Omega_{2-1}, \Omega_{3-1}, \Omega_{4-1}, \Omega_{7-1}\}, \{\Omega_{2-1}, \Omega_{3-1}, \Omega_{7-1}, \Omega_{8-1}\}\}.$$

Since $\{\Omega_{2-1}, \Omega_{4-1}\} \in \bar{\Omega}'$, from lines 4-5 and line 1 of Function S1,

$$\hat{\Omega} = \{\Omega_{2-1}, \Omega_{4-1}\}, R = \{r_2, r_4\}, R' = \{r_1, r_2, r_3\},$$

$$\bar{\Omega} \leftarrow \{\Omega_{2-2}, \Omega_{3-1}, \Omega_{7-1}, \Omega_{8-1}\}, \text{ and } \bar{\Omega}' = \emptyset.$$

Since $\exists r_4 \in R \setminus R'$ and $\Omega_{7-1} \in \bar{\Omega}' : r_4 \in P_{R7}$, from line 5 of Function S1,

$$\bar{\Omega} \leftarrow \{\{\Omega_{2-1}, \Omega_{4-1}, \Omega_{7-1}\}\}.$$

Since $\{\Omega_{2-1}, \Omega_{4-1}, \Omega_{7-1}\} \in \bar{\Omega}'$, from lines 4-5 and line 1 of Function S1,

$$\hat{\Omega} = \{\Omega_{2-1}, \Omega_{4-1}, \Omega_{7-1}\}, R = \{r_2, r_3, r_4\}, R' = \{r_1, r_2, r_3, r_4\},$$

$$\bar{\Omega} \leftarrow \{\Omega_{2-2}, \Omega_{3-1}, \Omega_{8-1}\}, \text{ and } \bar{\Omega}' = \emptyset.$$

Since $R \setminus R' = \emptyset$, from line 9 of Function S1,

$$\bar{\Omega}_2 = \{\{\Omega_{2-1}, \Omega_{3-1}, \Omega_{4-1}, \Omega_{7-1}\},$$

$$\{\Omega_{2-1}, \Omega_{3-1}, \Omega_{7-1}, \Omega_{8-1}\}, \{\Omega_{2-1}, \Omega_{4-1}, \Omega_{7-1}\}\}.$$

Since $\{\Omega_{2-1}, \Omega_{3-1}, \Omega_{4-1}, \Omega_{7-1}\} \in \bar{\Omega}_2$, from line 1 of Function S2,

$$\hat{\Omega} = \{\Omega_{2-1}, \Omega_{3-1}, \Omega_{4-1}, \Omega_{7-1}\},$$

$$R' = \{r_1, r_2, r_3, r_4\}, \text{ and } P_K = \{p_2, p_3, p_4, p_7\}.$$

Since $\Omega_{8-1} \in \bar{\Omega} \setminus \hat{\Omega}$ and $\forall r^{pw} \in R_8 : r \in R'$ and $p_8 \notin P_K$, from lines 2-6 of Function S2,

$$\bar{\Omega}_2 = \bar{\Omega}_2 \cup \{\{\Omega_{2-1}, \Omega_{3-1}, \Omega_{4-1}, \Omega_{7-1}, \Omega_{8-1}\}\}.$$

Since $\{\Omega_{2-1}, \Omega_{4-1}, \Omega_{7-1}\} \in \bar{\Omega}_2$, from line 1 of Function S2,

$$\hat{\Omega} = \{\Omega_{2-1}, \Omega_{4-1}, \Omega_{7-1}\},$$

$$R' = \{r_1, r_2, r_3, r_4\}, \text{ and } P_K = \{p_2, p_4, p_7\}.$$

Since $\exists \Omega_{8-1} \in \bar{\Omega} \setminus \hat{\Omega}$ and $\forall r^{pw} \in R_8 : r \in R'$ and $p_8 \notin P_K$, from lines 2-6 of Function S2,

$$\bar{\Omega}_2 = \bar{\Omega}_2 \cup \{\{\Omega_{2-1}, \Omega_{4-1}, \Omega_{7-1}, \Omega_{8-1}\}\}.$$

From lines 10-11,

$$\bar{\Omega} = \{\{\Omega_{2-1}, \Omega_{3-1}, \Omega_{4-1}, \Omega_{7-1}, \Omega_{8-1}\}, \{\Omega_{2-1}, \Omega_{3-1}, \Omega_{4-1}, \Omega_{7-1}\},$$

$$\{\Omega_{2-1}, \Omega_{3-1}, \Omega_{7-1}, \Omega_{8-1}\}, \{\Omega_{2-1}, \Omega_{4-1}, \Omega_{7-1}, \Omega_{8-1}\},$$

$$\{\Omega_{2-1}, \Omega_{4-1}, \Omega_{7-1}\}\} \text{ and}$$

$$\bar{\Omega} = \{\Omega_{2-2}, \Omega_{3-1}, \Omega_{4-1}, \Omega_{7-1}, \Omega_{8-1}\}.$$

For each $\hat{\Omega} \in \bar{\Omega}$, we repeat the above steps. Finally, we obtain CRPs as shown in Table S1.

S.IV. Partial Deadlock Detection Algorithm for S^4PR with Invariant Initial Markings

In this section, for an S^4PR in which its initial marking never changes, a partial deadlock detection algorithm is given as shown in Algorithm S3.

Table S1
CRPs of S⁴PR in FIG. 4

$\hat{\Omega}_1$	$\{\Omega_{4-1}, \Omega_{7-1}\}$
$\hat{\Omega}_2$	$\{\Omega_{2-1}, \Omega_{4-1}, \Omega_{7-1}\}$
$\hat{\Omega}_3$	$\{\Omega_{2-2}, \Omega_{4-1}, \Omega_{7-1}\}$
$\hat{\Omega}_4$	$\{\Omega_{3-1}, \Omega_{4-1}, \Omega_{7-1}\}$
$\hat{\Omega}_5$	$\{\Omega_{2-1}, \Omega_{3-1}, \Omega_{4-1}, \Omega_{7-1}\}$
$\hat{\Omega}_6$	$\{\Omega_{2-1}, \Omega_{3-1}, \Omega_{7-1}, \Omega_{8-1}\}$
$\hat{\Omega}_7$	$\{\Omega_{2-1}, \Omega_{4-1}, \Omega_{7-1}, \Omega_{8-1}\}$
$\hat{\Omega}_8$	$\{\Omega_{2-2}, \Omega_{3-1}, \Omega_{4-1}, \Omega_{7-1}\}$
$\hat{\Omega}_9$	$\{\Omega_{2-2}, \Omega_{3-1}, \Omega_{7-1}, \Omega_{8-1}\}$
$\hat{\Omega}_{10}$	$\{\Omega_{2-2}, \Omega_{4-1}, \Omega_{7-1}, \Omega_{8-1}\}$
$\hat{\Omega}_{11}$	$\{\Omega_{2-1}, \Omega_{3-1}, \Omega_{4-1}, \Omega_{7-1}, \Omega_{8-1}\}$
$\hat{\Omega}_{12}$	$\{\Omega_{2-2}, \Omega_{3-1}, \Omega_{4-1}, \Omega_{7-1}, \Omega_{8-1}\}$

Algorithm S3: Partial Deadlock Detection for S⁴PR with

Invariant Initial Markings

Input: $\hat{\Omega}$ and M_0

Output: Set of partial deadlocks M_D

Let $\bar{\Omega} = M_D = \emptyset$;

For $\forall \Omega_i \in \hat{\Omega}$, where $\Omega_i = \langle P_{R_i}, R_i \rangle$, **do**

Let $\bar{\Omega}' = \{\Omega_i\}$ and $\bar{\Omega}_i = \emptyset$;

For $\forall \hat{\Omega} \in \hat{\Omega}$, **do**

$[\bar{\Omega}_i, \bar{\Omega}] = F_{s1}(\hat{\Omega}, \bar{\Omega}, \bar{\Omega}', \bar{\Omega}_i)$;

End

For $\forall \hat{\Omega} \in \bar{\Omega}_i$, **do**

$[I, M_D] = F_1(\hat{\Omega}, M_0, M_D)$;

If $I=1$, **then**

$[\bar{\Omega}_i] = F_{s2}(\hat{\Omega}, \bar{\Omega}_i, \bar{\Omega}_i)$;

End

End

$\bar{\Omega} = \bar{\Omega} \cup \bar{\Omega}_i$;

$\bar{\Omega} = \bar{\Omega} \setminus \{\Omega_i\}$;

$[\bar{\Omega}] = F_{s3}(P_{R_i}, \bar{\Omega}, P_A)$

End

Return M_D

End

S.V. ANALYSIS OF EXISTING WORK

In this section, we analyze the existing structure-based approaches. Since this work focuses on deadlock detection for S⁴PR, we only analyze structure-based methods that can handle the deadlock detection problems in S⁴PR. From Table I, perfect activity-circuits (PA-circuit) [25], CTR-circuits [35], and siphons [28] can characterize deadlocks in S⁴PR. Next, the performance of these methods on the problem of deadlock detection in S⁴PR is analyzed through case studies.

A. PA-circuit-based method

First, let's review the basic concepts in [25].

Definition S1: A path $\alpha = pt$ is called a single activity-path (SA-path) if $p \in P_A$ and $t \in T$. Moreover, if $p \in \|\pi_r\|$, then we say that SA-path $\alpha = pt$ is with respect to (w.r.t.) r .

Given SA-paths $\alpha_1 = p_1t_1$ w.r.t. r_1 and $\alpha_2 = p_2t_2$ w.r.t. r_2 . If $r_2 \in {}^*t_1$, then α_1 is reachable from α_2 , denoted by $\alpha_1 \leftarrow \alpha_2$.

Definition S2: A sequence of SA-path $\beta = \alpha_1 \alpha_2 \dots \alpha_k$ is called activity-chain if a) $\forall i \in \mathbb{N}_k^+$ such that $\alpha_i = p_it_i$ is a SA-path w.r.t. r_i ; and b) $\forall j \in \mathbb{N}_{k-1}^+$ such that $\alpha_j \leftarrow \alpha_{j+1}$, where $R_\beta = \{r_i | i \in \mathbb{N}_k^+\}$, $P_\beta = \{p_i | i \in \mathbb{N}_k^+\}$, and $T_\beta = \{t_i | i \in \mathbb{N}_k^+\}$.

Definition S3: Let $\beta = \alpha_1 \alpha_2 \dots \alpha_k$ be an activity-chain. If

$\alpha_k \leftarrow \alpha_1$, then β is called an activity-circuit. β is called a perfect activity-circuit (PA-circuit) if $T_\beta = T_{\beta'}$, where $T_{\beta'} = \{t' | t \in T_\beta, p \in {}^*t \cap P_A, t' \in p^*\}$.

Definition S4: A PA-circuit β is saturated at a marking M if a) $\forall p \in P_\beta: M(p) \geq 1$; and b) $\forall r \in R_\beta: \min\{W(r, t) | t \in r^* \cap T_\beta\} - 1 \geq M_0(r) - \sum_{p \in P_\beta \cap \|\pi_r\|} \pi_r(p)M(p)$.

From [25], there is an equivalence relationship between deadlocks in S⁴PR and the saturation of PA-circuits. However, in some S⁴PR, for any deadlock M , there is no PA-circuit β such that β is saturated at M . In this case, PA-circuit-based methods cannot be used to detect deadlocks. Now, we demonstrate it with the S⁴PR shown in Fig. 6, where $M_0 = 4p_1 + 4p_6 + 2r_1 + r_2$. In this net, $P_A = \{p_2, p_3, p_4, p_5\}$ and $P_R = \{r_1, r_2\}$. According to Definition S1, there are four SA-paths, i.e., $\alpha_1 = p_2t_2$ w.r.t. r_1 , $\alpha_2 = p_3t_3$ w.r.t. r_1 , $\alpha_3 = p_4t_4$ w.r.t. r_1 and r_2 , and $\alpha_4 = p_5t_6$ w.r.t. r_2 . By

$$r_1 \in {}^*t_2,$$

we have that

$$\alpha_1 \leftarrow \alpha_2 \text{ and } \alpha_1 \leftarrow \alpha_3.$$

Since

$$r_2 \in {}^*t_3,$$

we obtain that

$$\alpha_2 \leftarrow \alpha_3 \text{ and } \alpha_2 \leftarrow \alpha_4.$$

According to Definition S2, there are six activity-chains, i.e., $\beta_1 = \alpha_1 \alpha_2$, $\beta_2 = \alpha_1 \alpha_3$, $\beta_3 = \alpha_2 \alpha_3$, $\beta_4 = \alpha_2 \alpha_4$, $\beta_5 = \alpha_1 \alpha_2 \alpha_3$, and $\beta_6 = \alpha_1 \alpha_2 \alpha_4$. Since α_1 cannot reach α_2 , α_3 , and α_4 , according to Definition S3, β_1 , β_2 , β_5 , and β_6 are not activity-circuits. Similarly, since α_2 cannot reach α_3 and α_4 , β_3 and β_4 are not activity-circuits. Thus, there is no activity-circuit in this net. Hence, for any marking M , according to Definitions S3 and S4, there is no PA-circuit is saturated at M . However, there is a partial deadlock $M = 2p_1 + 2p_2 + 4p_6 + r_2$. Thus, for this S⁴PR, the methods based on PA-circuits cannot detect it.

B. CTR-circuit based method

In [35], a circuit is a path in which the first and last nodes are identical, while the others are different. Then, some basic concepts in [35] are reviewed.

Definition S5: Let M be a deadlock. t is called a critical transition at M if $p \in {}^*t \cap P_A: M(p) > 0$. Let T_M denote the set of critical transitions at M .

Definition S6: Let M be a deadlock. A circuit is called a CTR-circuit at M if it contains only a set of critical transitions and a set of resources such that $\exists t \in \theta$ and $p \in {}^*t: M(p) = 0$ and $\forall r \in \theta: M(r) < W(r, t)$, where $t \in T_M$.

For the S⁴PR shown in Fig. 6, $M = 2p_1 + 2p_2 + 4p_6 + r_2$ is a partial deadlock. By $p_2 \in {}^*t_2 \cap P_A$ and $M(p_2) > 0$, $T_M = \{t_2\}$ according to Definition S5. From Definition S6, if there is a CTR-circuit at M , it only contains one transition, i.e., t_2 . However, there is no such circuit in the S⁴PR. Hence, there is no CTR-circuit at M . Thus, CTR-based method proposed in [35] cannot handle the partial deadlock detection problem of S⁴PR shown in Fig. 6.

C. Siphon-based method

Definition S7 ^[28]: $S \subseteq P$ is called a siphon if ${}^*S \subseteq S^*$, where ${}^*S = \{t | t \in {}^*p, p \in S\}$ and $S^* = \{t | t \in p^*, p \in S\}$.

Definition S8 ^[28]: Siphon S is marked at a marking M if $M(S) > 0$; otherwise, S is unmarked at M , where $M(S) = \sum_{p \in S} M(p)$.

In the existing siphon-based deadlock control strategies, deadlocks are obtained by solving an integer program. Specifically, for each M that satisfies

$$M = M_0 + A^T X \quad (S.1)$$

where $X > 0$ and $M > 0$, if there exists a siphon that is unmarked at M , then M is a deadlock. Notice that the obtained markings by solving (S.1) may not be reachable. For example, in the

S^4PR in Fig. 7, $S = \{r_1, r_2, p_4, p_7\}$ is a siphon. Suppose that $M_0 = 4p_1 + 4p_8 + r_1 + r_2$. By solving (S.1), we have that there are four markings, i.e., $M_1 = 2p_1 + p_2 + p_3 + 4p_8$, $M_2 = 3p_1 + p_2 + p_5 + 3p_8$, $M_3 = 4p_1 + p_5 + p_6 + 2p_8$, and $M_4 = 3p_1 + p_3 + p_6 + 3p_8$ such that S is unmarked at them. However, among them, M_4 is not reachable from M_0 . Thus, the methods based on siphons cannot determine the set that only contains reachable deadlocks.

S.VI. CRPs AND PARTIAL DEADLOCKS OF S^4PR IN FIG. 8

Table S2 shows all the partial deadlocks and CRPs of the S^4PR shown in Fig. 8 of this paper.

TABLE S2
CRPs AND PARTIAL DEADLOCKS OF S^4PR IN FIG. 8.

CRPs	Partial deadlocks
$\hat{\Omega}_1 = \{\Omega_{1-1}, \Omega_{15-1}\}$	$M_1 = p_1 + 9p_8 + 10p_{12} + 2p_{15} + 8p_{20} + 3p_{22} + p_{23} + 3p_{24} + 3p_{25}$
$\hat{\Omega}_2 = \{\Omega_{1-1}, \Omega_{14-1}, \Omega_{15-1}\}$	$M_2 = p_1 + 9p_8 + 10p_{12} + p_{14} + 2p_{15} + 7p_{20} + p_{22} + p_{23} + 2p_{24} + 3p_{25}$
$\hat{\Omega}_3 = \{\Omega_{1-1}, \Omega_{15-1}, \Omega_{16-1}\}$	
$\hat{\Omega}_4 = \{\Omega_{1-1}, \Omega_{2-1}, \Omega_{14-1}, \Omega_{15-1}\}$	
$\hat{\Omega}_5 = \{\Omega_{1-1}, \Omega_{3-1}, \Omega_{14-1}, \Omega_{15-1}\}$	$M_3 = p_1 + p_3 + 8p_8 + 10p_{12} + p_{14} + 2p_{15} + 7p_{20} + p_{22} + p_{23} + p_{24} + 3p_{25}$
	$M_4 = p_1 + 2p_3 + 7p_8 + 10p_{12} + p_{14} + 2p_{15} + 7p_{20} + p_{22} + p_{23} + 3p_{25}$
$\hat{\Omega}_6 = \{\Omega_{1-1}, \Omega_{6-1}, \Omega_{14-1}, \Omega_{15-1}\}$	
$\hat{\Omega}_7 = \{\Omega_{1-1}, \Omega_{7-1}, \Omega_{14-1}, \Omega_{15-1}\}$	$M_5 = p_1 + 2p_7 + 7p_8 + 10p_{12} + p_{14} + 2p_{15} + 7p_{20} + p_{22} + p_{23} + 3p_{25}$
	$M_6 = p_1 + p_7 + 8p_8 + 10p_{12} + p_{14} + 2p_{15} + 7p_{20} + p_{22} + p_{23} + p_{24} + 3p_{25}$
$\hat{\Omega}_8 = \{\Omega_{1-1}, \Omega_{10-1}, \Omega_{14-1}, \Omega_{15-1}\}$	
$\hat{\Omega}_9 = \{\Omega_{1-1}, \Omega_{13-1}, \Omega_{14-1}, \Omega_{15-1}\}$	
$\hat{\Omega}_{10} = \{\Omega_{1-1}, \Omega_{3-1}, \Omega_{6-1}, \Omega_{14-1}, \Omega_{15-1}\}$	$M_7 = p_1 + 2p_3 + p_6 + 6p_8 + 10p_{12} + p_{14} + p_{15} + 8p_{20} + p_{22} + p_{23} + 3p_{25}$
$\hat{\Omega}_{11} = \{\Omega_{1-1}, \Omega_{3-1}, \Omega_{6-1}, \Omega_{7-1}, \Omega_{14-1}, \Omega_{15-1}\}$	$M_8 = p_1 + p_3 + p_6 + p_7 + 6p_8 + 10p_{12} + p_{14} + p_{15} + 8p_{20} + p_{22} + p_{23} + 3p_{25}$
$\hat{\Omega}_{12} = \{\Omega_{1-1}, \Omega_{3-1}, \Omega_{6-1}, \Omega_{7-1}, \Omega_{10-1}, \Omega_{14-1}, \Omega_{15-1}\}$	$M_9 = p_1 + p_3 + p_6 + p_7 + 6p_8 + p_{10} + 9p_{12} + p_{14} + p_{15} + 8p_{20} + p_{22} + p_{23} + p_{25}$
$\hat{\Omega}_{13} = \{\Omega_{1-1}, \Omega_{3-1}, \Omega_{4-2}, \Omega_{6-1}, \Omega_{7-1}, \Omega_{10-1}, \Omega_{14-1}, \Omega_{15-1}\}$	
$\hat{\Omega}_{14} = \{\Omega_{1-1}, \Omega_{3-1}, \Omega_{6-1}, \Omega_{7-1}, \Omega_{10-1}, \Omega_{13-1}, \Omega_{14-1}, \Omega_{15-1}\}$	
$\hat{\Omega}_{15} = \{\Omega_{1-1}, \Omega_{3-1}, \Omega_{6-1}, \Omega_{7-1}, \Omega_{10-1}, \Omega_{14-1}, \Omega_{15-1}, \Omega_{16-1}\}$	
$\hat{\Omega}_{16} = \{\Omega_{1-1}, \Omega_{3-1}, \Omega_{6-1}, \Omega_{7-1}, \Omega_{13-1}, \Omega_{14-1}, \Omega_{15-1}\}$	
$\hat{\Omega}_{17} = \{\Omega_{1-1}, \Omega_{3-1}, \Omega_{6-1}, \Omega_{7-1}, \Omega_{14-1}, \Omega_{15-1}, \Omega_{16-1}\}$	
$\hat{\Omega}_{18} = \{\Omega_{1-1}, \Omega_{3-1}, \Omega_{6-1}, \Omega_{10-1}, \Omega_{14-1}, \Omega_{15-1}\}$	$M_{10} = p_1 + 2p_3 + p_6 + 6p_8 + p_{10} + 9p_{12} + p_{14} + p_{15} + 8p_{20} + p_{22} + p_{23} + p_{25}$
$\hat{\Omega}_{19} = \{\Omega_{1-1}, \Omega_{3-1}, \Omega_{4-2}, \Omega_{6-1}, \Omega_{10-1}, \Omega_{14-1}, \Omega_{15-1}\}$	
$\hat{\Omega}_{20} = \{\Omega_{1-1}, \Omega_{3-1}, \Omega_{6-1}, \Omega_{10-1}, \Omega_{13-1}, \Omega_{14-1}, \Omega_{15-1}\}$	
$\hat{\Omega}_{21} = \{\Omega_{1-1}, \Omega_{3-1}, \Omega_{6-1}, \Omega_{10-1}, \Omega_{14-1}, \Omega_{15-1}, \Omega_{16-1}\}$	
$\hat{\Omega}_{22} = \{\Omega_{1-1}, \Omega_{3-1}, \Omega_{6-1}, \Omega_{13-1}, \Omega_{14-1}, \Omega_{15-1}\}$	
$\hat{\Omega}_{23} = \{\Omega_{1-1}, \Omega_{3-1}, \Omega_{6-1}, \Omega_{10-1}, \Omega_{14-1}, \Omega_{15-1}, \Omega_{16-1}\}$	
$\hat{\Omega}_{24} = \{\Omega_{1-1}, \Omega_{3-1}, \Omega_{7-1}, \Omega_{14-1}, \Omega_{15-1}\}$	$M_{11} = p_1 + p_3 + p_7 + 7p_8 + 10p_{12} + p_{14} + 2p_{15} + 7p_{20} + p_{22} + p_{23} + 3p_{25}$
$\hat{\Omega}_{25} = \{\Omega_{1-1}, \Omega_{2-1}, \Omega_{3-1}, \Omega_{7-1}, \Omega_{14-1}, \Omega_{15-1}\}$	$M_{12} = p_1 + p_2 + p_3 + p_7 + 6p_8 + 10p_{12} + p_{14} + p_{15} + 8p_{20} + p_{22} + p_{23} + 3p_{25}$
$\hat{\Omega}_{26} = \{\Omega_{1-1}, \Omega_{3-1}, \Omega_{7-1}, \Omega_{10-1}, \Omega_{14-1}, \Omega_{15-1}\}$	$M_{13} = p_1 + p_3 + p_7 + 7p_8 + p_{10} + 9p_{12} + p_{14} + 2p_{15} + 7p_{20} + p_{22} + p_{23} + p_{25}$
$\hat{\Omega}_{27} = \{\Omega_{1-1}, \Omega_{3-1}, \Omega_{4-2}, \Omega_{7-1}, \Omega_{10-1}, \Omega_{14-1}, \Omega_{15-1}\}$	
$\hat{\Omega}_{28} = \{\Omega_{1-1}, \Omega_{3-1}, \Omega_{7-1}, \Omega_{10-1}, \Omega_{13-1}, \Omega_{14-1}, \Omega_{15-1}\}$	
$\hat{\Omega}_{29} = \{\Omega_{1-1}, \Omega_{3-1}, \Omega_{7-1}, \Omega_{10-1}, \Omega_{14-1}, \Omega_{15-1}, \Omega_{16-1}\}$	
$\hat{\Omega}_{30} = \{\Omega_{1-1}, \Omega_{3-1}, \Omega_{7-1}, \Omega_{13-1}, \Omega_{14-1}, \Omega_{15-1}\}$	
$\hat{\Omega}_{31} = \{\Omega_{1-1}, \Omega_{3-1}, \Omega_{7-1}, \Omega_{14-1}, \Omega_{15-1}, \Omega_{16-1}\}$	
$\hat{\Omega}_{32} = \{\Omega_{1-1}, \Omega_{3-1}, \Omega_{10-1}, \Omega_{14-1}, \Omega_{15-1}\}$	$M_{14} = p_1 + 2p_3 + 7p_8 + p_{10} + 9p_{12} + p_{14} + 2p_{15} + 7p_{20} + p_{22} + p_{23} + p_{25}$
$\hat{\Omega}_{33} = \{\Omega_{1-1}, \Omega_{2-1}, \Omega_{3-1}, \Omega_{10-1}, \Omega_{14-1}, \Omega_{15-1}\}$	$M_{15} = p_1 + p_2 + 2p_3 + 6p_8 + p_{10} + 9p_{12} + p_{14} + p_{15} + 8p_{20} + p_{22} + p_{23} + p_{25}$
$\hat{\Omega}_{34} = \{\Omega_{1-1}, \Omega_{2-1}, \Omega_{3-1}, \Omega_{7-1}, \Omega_{10-1}, \Omega_{14-1}, \Omega_{15-1}\}$	$M_{16} = p_1 + p_2 + p_3 + p_7 + 6p_8 + p_{10} + 9p_{12} + p_{14} + p_{15} + 8p_{20} + p_{22} + p_{23} + p_{25}$
$\hat{\Omega}_{35} = \{\Omega_{1-1}, \Omega_{3-1}, \Omega_{4-2}, \Omega_{10-1}, \Omega_{14-1}, \Omega_{15-1}\}$	
$\hat{\Omega}_{36} = \{\Omega_{1-1}, \Omega_{3-1}, \Omega_{10-1}, \Omega_{13-1}, \Omega_{14-1}, \Omega_{15-1}\}$	

$\hat{\Omega}_{193}=\{\Omega_{3-1}, \Omega_{6-1}, \Omega_{13-1}\}$	$M_{115}=3p_3+2p_6+5p_8+10p_{12}+p_{13}+p_{18}+p_{19}+9p_{20}+p_{22}+3p_{25}$
$\hat{\Omega}_{194}=\{\Omega_{3-1}, \Omega_{4-1}, \Omega_{6-1}, \Omega_{13-1}\}$	
$\hat{\Omega}_{195}=\{\Omega_{3-1}, \Omega_{6-1}, \Omega_{7-1}, \Omega_{13-1}\}$	$M_{116}=2p_3+2p_6+p_7+5p_8+10p_{12}+p_{13}+p_{18}+p_{19}+9p_{20}+p_{22}+3p_{25}$
	$M_{117}=p_3+2p_6+2p_7+5p_8+10p_{12}+p_{13}+p_{18}+p_{19}+9p_{20}+p_{22}+3p_{25}$
$\hat{\Omega}_{196}=\{\Omega_{3-1}, \Omega_{6-1}, \Omega_{7-1}, \Omega_{10-1}, \Omega_{13-1}\}$	$M_{118}=2p_3+2p_6+p_7+5p_8+p_{10}+9p_{12}+p_{13}+p_{18}+p_{19}+9p_{20}+p_{22}+p_{25}$
	$M_{119}=p_3+2p_6+2p_7+5p_8+p_{10}+9p_{12}+p_{13}+p_{18}+p_{19}+9p_{20}+p_{22}+p_{25}$
$\hat{\Omega}_{197}=\{\Omega_{3-1}, \Omega_{4-2}, \Omega_{6-1}, \Omega_{7-1}, \Omega_{10-1}, \Omega_{13-1}\}$	
$\hat{\Omega}_{198}=\{\Omega_{3-1}, \Omega_{6-1}, \Omega_{10-1}, \Omega_{13-1}\}$	$M_{120}=3p_3+2p_6+5p_8+p_{10}+9p_{12}+p_{13}+p_{18}+p_{19}+9p_{20}+p_{22}+p_{25}$
$\hat{\Omega}_{199}=\{\Omega_{3-1}, \Omega_{4-2}, \Omega_{6-1}, \Omega_{10-1}, \Omega_{13-1}\}$	
$\hat{\Omega}_{200}=\{\Omega_{6-1}, \Omega_{7-1}, \Omega_{13-1}\}$	$M_{121}=2p_6+3p_7+5p_8+10p_{12}+p_{13}+p_{18}+p_{19}+9p_{20}+p_{22}+3p_{25}$
$\hat{\Omega}_{201}=\{\Omega_{6-1}, \Omega_{7-1}, \Omega_{10-1}, \Omega_{13-1}\}$	$M_{122}=2p_6+3p_7+5p_8+p_{10}+9p_{12}+p_{13}+p_{18}+p_{19}+9p_{20}+p_{22}+p_{25}$
$\hat{\Omega}_{202}=\{\Omega_{4-1}, \Omega_{6-1}, \Omega_{7-1}, \Omega_{10-1}, \Omega_{13-1}\}$	
$\hat{\Omega}_{203}=\{\Omega_{4-2}, \Omega_{6-1}, \Omega_{7-1}, \Omega_{10-1}, \Omega_{13-1}\}$	
$\hat{\Omega}_{204}=\{\Omega_{4-1}, \Omega_{6-1}, \Omega_{7-1}, \Omega_{13-1}\}$	
$\hat{\Omega}_{205}=\{\Omega_{3-1}, \Omega_{4-2}, \Omega_{6-1}, \Omega_{10-1}\}$	
$\hat{\Omega}_{206}=\{\Omega_{4-2}, \Omega_{6-1}, \Omega_{7-1}, \Omega_{10-1}\}$	
$\hat{\Omega}_{207}=\{\Omega_{3-1}, \Omega_{4-1}, \Omega_{13-1}\}$	
$\hat{\Omega}_{208}=\{\Omega_{3-1}, \Omega_{4-2}, \Omega_{10-1}\}$	
$\hat{\Omega}_{209}=\{\Omega_{3-1}, \Omega_{13-1}\}$	$M_{123}=3p_3+7p_8+10p_{12}+p_{13}+p_{18}+p_{19}+9p_{20}+2p_{21}+p_{22}+3p_{25}$
$\hat{\Omega}_{210}=\{\Omega_{3-1}, \Omega_{7-1}, \Omega_{13-1}\}$	$M_{124}=2p_3+p_7+7p_8+10p_{12}+p_{13}+p_{18}+p_{19}+9p_{20}+2p_{21}+p_{22}+3p_{25}$
	$M_{125}=p_3+2p_7+7p_8+10p_{12}+p_{13}+p_{18}+p_{19}+9p_{20}+2p_{21}+p_{22}+3p_{25}$
$\hat{\Omega}_{211}=\{\Omega_{3-1}, \Omega_{7-1}, \Omega_{10-1}, \Omega_{13-1}\}$	$M_{126}=2p_3+p_7+7p_8+p_{10}+9p_{12}+p_{13}+p_{18}+p_{19}+9p_{20}+2p_{21}+p_{22}+p_{25}$
	$M_{127}=p_3+2p_7+7p_8+p_{10}+9p_{12}+p_{13}+p_{18}+p_{19}+9p_{20}+2p_{21}+p_{22}+p_{25}$
$\hat{\Omega}_{212}=\{\Omega_{3-1}, \Omega_{10-1}, \Omega_{13-1}\}$	$M_{128}=3p_3+7p_8+p_{10}+9p_{12}+p_{13}+p_{18}+p_{19}+9p_{20}+2p_{21}+p_{22}+p_{25}$
$\hat{\Omega}_{213}=\{\Omega_{4-1}, \Omega_{7-1}, \Omega_{13-1}\}$	
$\hat{\Omega}_{214}=\{\Omega_{4-2}, \Omega_{7-1}, \Omega_{10-1}\}$	
$\hat{\Omega}_{215}=\{\Omega_{7-1}, \Omega_{13-1}\}$	$M_{129}=3p_7+7p_8+10p_{12}+p_{13}+p_{18}+p_{19}+9p_{20}+2p_{21}+p_{22}+3p_{25}$
$\hat{\Omega}_{216}=\{\Omega_{7-1}, \Omega_{10-1}, \Omega_{13-1}\}$	$M_{130}=3p_7+7p_8+p_{10}+9p_{12}+p_{13}+p_{18}+p_{19}+9p_{20}+2p_{21}+p_{22}+p_{25}$