ALGORITHMS AND DATA STRUCTURES II

Lecture 4

Spanning Tree,
Weighted Graphs,
Prim's and Kruskal's algorithms.

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SPANNING TREE

- Assume you have an **undirected graph**

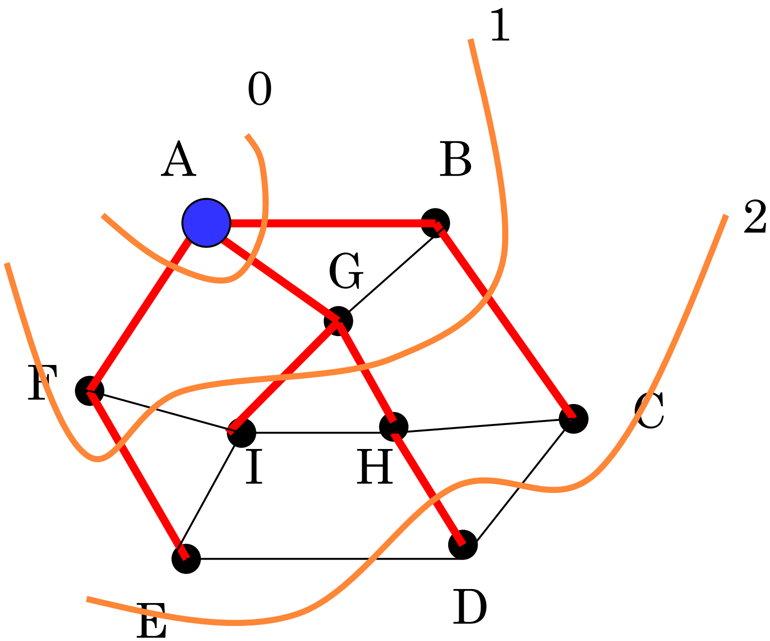
  $$\mathbf{G = (V,E)}$$

- Spanning tree of graph **G** is tree

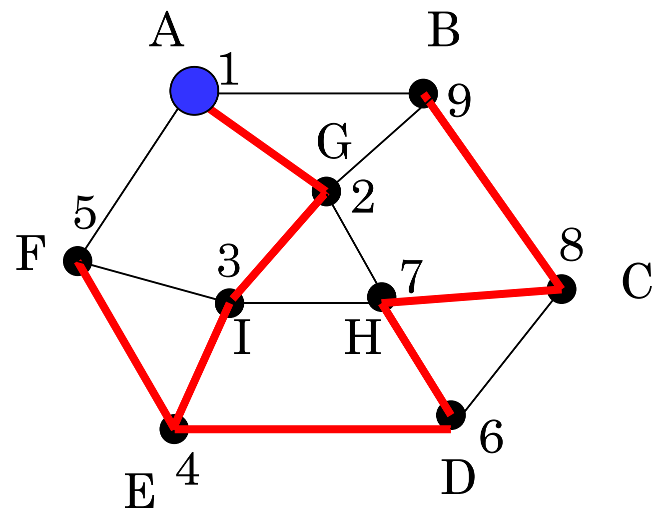
  $$\mathbf{T = (V,E_T \subseteq E, R)}$$

  - Tree has same set of nodes.
  - All tree edges are graph edges.
  - Root of tree is R.
- **Think:** "smallest set of edges needed to connect everything together".

SPANNING TREE



Breadth-first Spanning Tree

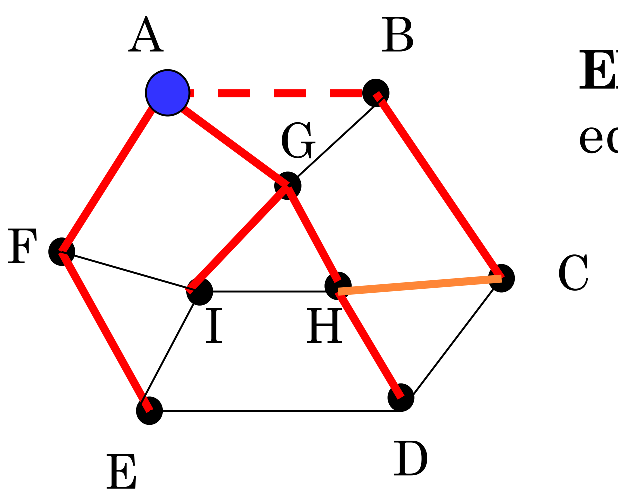


Depth-first spanning tree

SPANNING TREE

Properties:

- In any tree **T** = (**V**,**E**), |**E**|=|**V**| - 1
- For any edge ***e*** in **G** but not in **T**, there is a simple cycle **Y** containing only edge ***e*** and edges in spanning tree.
- Moreover, inserting edge ***e*** into **T** and deleting any edge in **Y** gives another spanning tree **T'**.



EXAMPLE:

edge (**H**,**C**):

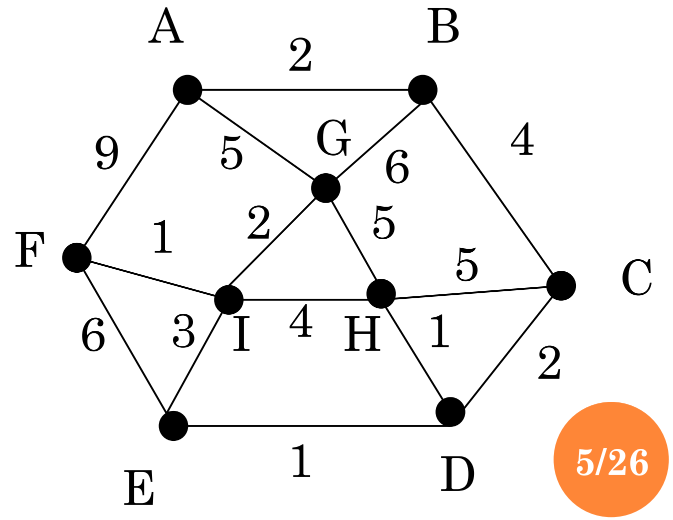
simple cycle is (**H**,**C**,**B**,**A**,**G**,**H**)

adding (**H**,**C**) to **T** and deleting (**A**,**B**)

gives another spanning tree

WEIGHTED GRAPHS

- **Definition:**
  - A **weighted graph** is a graph $\boldsymbol{G}$ $(\boldsymbol{V},\boldsymbol{E})$ with real valued weights assigned to each edge.
  - Equivalently, a weighted graph is a triple $\boldsymbol{G}$ $(\boldsymbol{V},\boldsymbol{E},\boldsymbol{W})$, where $\boldsymbol{V}$ is the set of vertices, $\boldsymbol{E}$ is the set of edges, and $\boldsymbol{W}$ is the set of weights. The weights on edges are also called **distances** or **costs**.



Weighted Graph

WEIGHTED GRAPHS

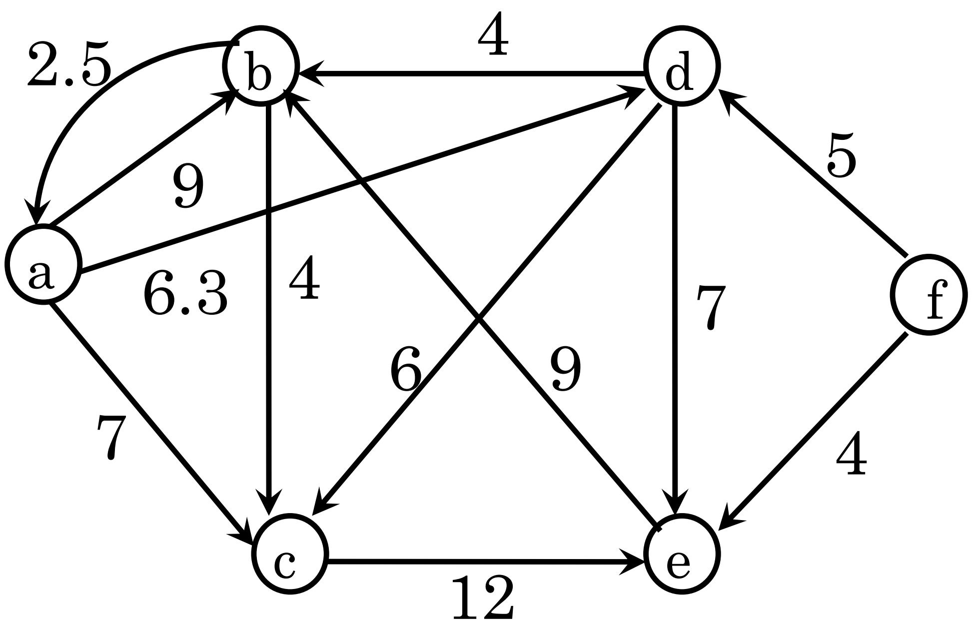
- **Representation:**
  - A weighted graph $\boldsymbol{G}(\boldsymbol{V},\boldsymbol{E},\boldsymbol{W})$ can be represented by a **distance matrix**

$$D_{n\times n} = \begin{bmatrix} d(1,1) & \dots & d(1,n) \\ \dots & \dots & \dots \\ d(n,1) & \dots & d(n,n) \end{bmatrix} \quad n=|V|$$

where $d(i,i) = 0$,
and for $1 \le i \ne j \le n$, if edge $(i,j) \in E$ then $d(i,j)$ is the weight of (i,j), otherwise $d(i,j)$ is infinite ∞ (a sufficiently large number in practice).

WEIGHTED GRAPHS

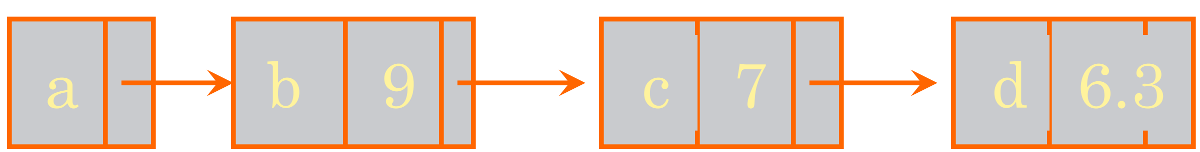
- **Representation:**



Distance Matrix

	a	b	c	d	e	f
a	0	9	7	6.3	∞	∞
b	2.5	0	4	∞	∞	∞
c	∞	∞	0	∞	12	∞
d	∞	4	6	0	7	∞
e	∞	9	∞	∞	0	∞
f	∞	∞	∞	5	4	0

Adjacency list



MINIMUM SPANNING TREE (MST)

- Let $\boldsymbol{T}(\boldsymbol{V'}, \boldsymbol{E'})$ be a spanning tree of a weighted graph $\boldsymbol{G}$ and

$$\boldsymbol{W}(\boldsymbol{T}) = \Sigma_{(\boldsymbol{v},\boldsymbol{w}) \in \boldsymbol{E'}} \boldsymbol{W}(\boldsymbol{v},\boldsymbol{w})$$

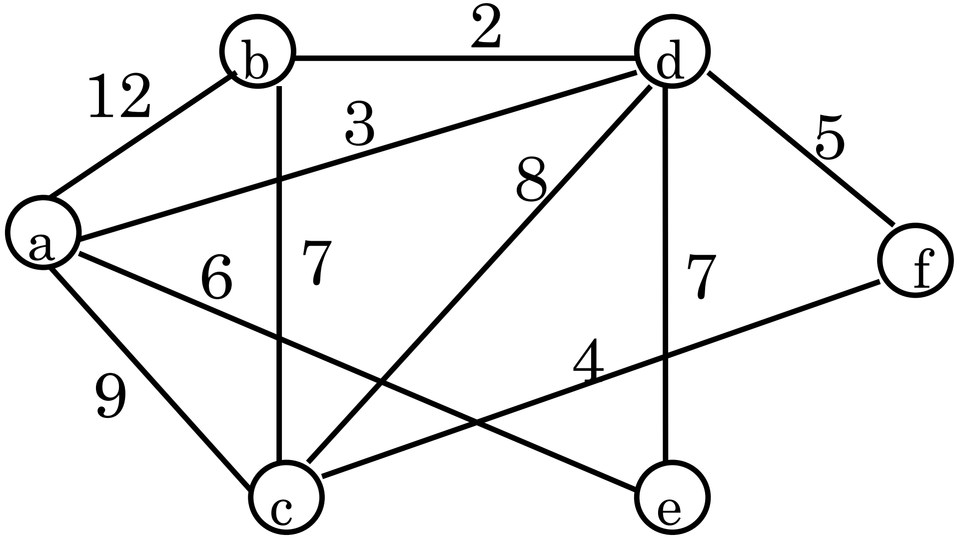
be the sum of weights of edges in $\boldsymbol{T}$, where $\boldsymbol{W}(\boldsymbol{v},\boldsymbol{w})$ denotes the weight of edge $(\boldsymbol{v},\boldsymbol{w})$.

- A **minimum spanning tree** of $\boldsymbol{G}$ is a spanning tree $\boldsymbol{T^m}$ of $\boldsymbol{G}$ such that

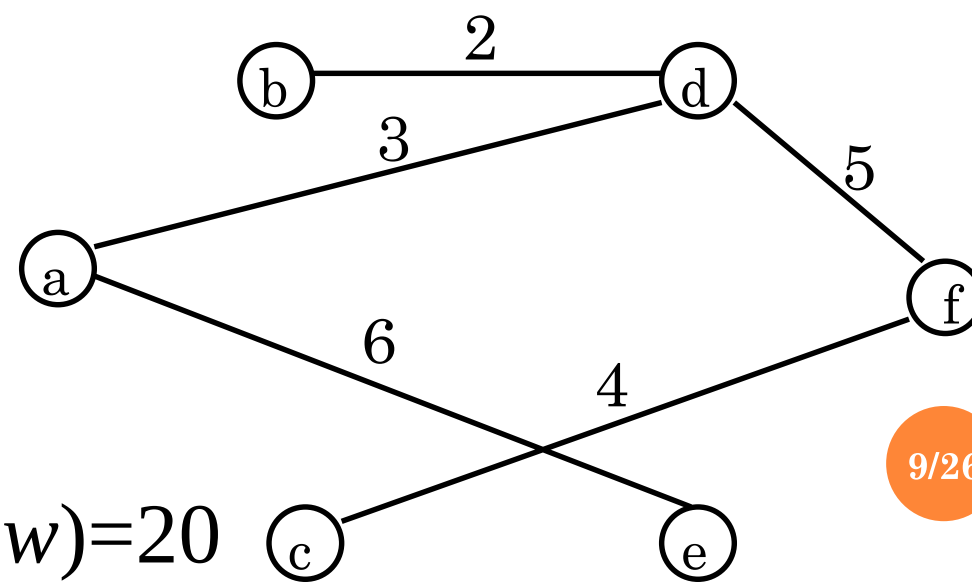
$$\boldsymbol{W}(\boldsymbol{T^m}) = \mathbf{min}_{\mathrm{T}} \{\boldsymbol{W}(\boldsymbol{T})\}.$$

MINIMUM SPANNING TREE (MST)

Weighted Graph G



Minimum Spanning Tree of G



$$W(T^m) = \Sigma_{(v,w) \in E'} W(v,w) = 20$$

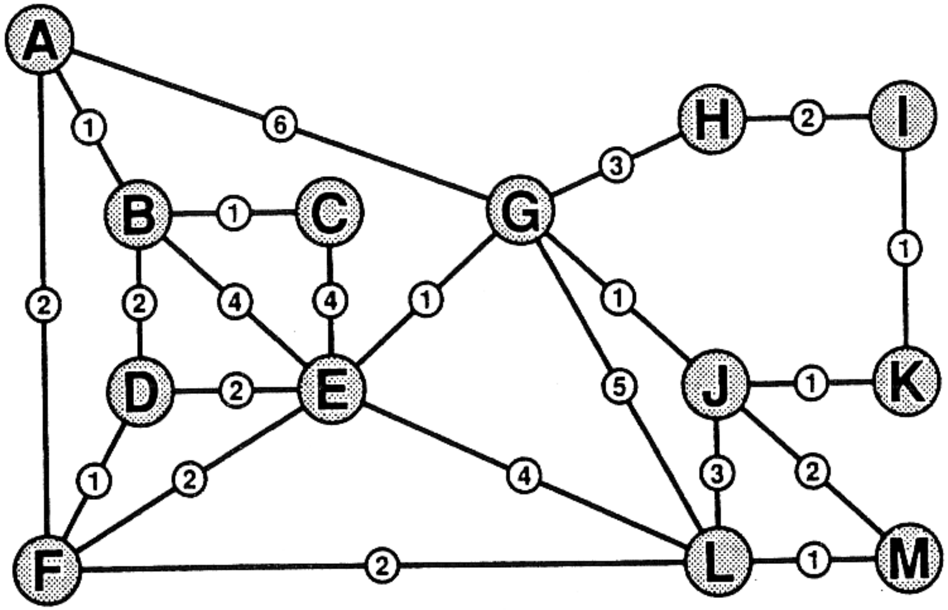
MINIMUM SPANNING TREE (MST)

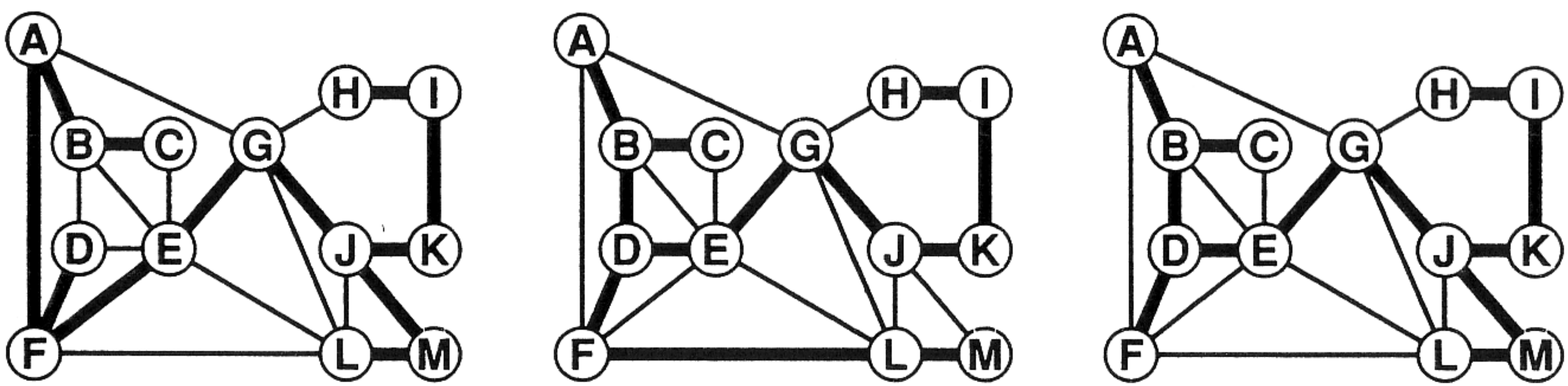
- **Minimum spanning tree** is useful when we attempt to minimize the cost of connecting all the nodes.
- **Applications:**
  - Constructing electric power networks or telephone networks.
  - Making printed circuit boards (PCBs).
  - Etc.

- **Note:** Minimum spanning tree need not to be unique. (simple examples)

MINIMUM SPANNING TREE (MST)

Weighted Graph G





Multiple Minimum Spanning Trees of G

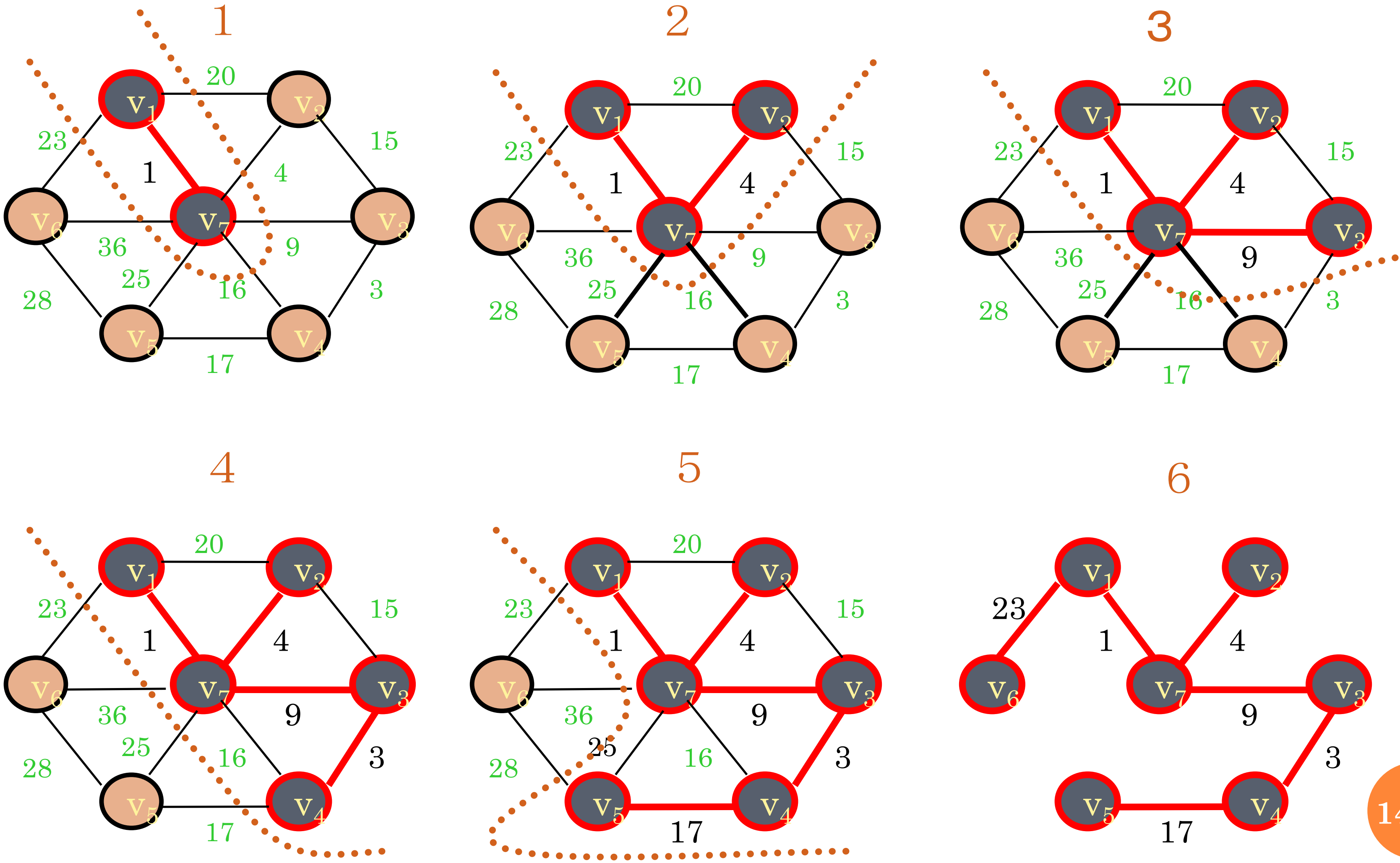
MINIMUM SPANNING TREE (MST)

- **Building MST** – two strategies:
  - **Prim's algorithm** – start with a root node ***s*** and try to grow a tree from ***s*** outward. At each step, add the node that can be attached as cheaply as possible to the partial tree we already have.
  - **Kruskal's algorithm** – start with no edges and successively insert edges from ***E*** in order of increasing cost. If an edge makes cycle when added, skip this edge.

PRIM'S ALGORITHM

1) Pick an arbitrary vertex $\boldsymbol{r}$ of $\boldsymbol{G(V,E)}$ as the root of the minimum spanning tree of $\boldsymbol{G}$. Assume a partial solution (spanning tree) $\boldsymbol{T}$ has been obtained (initially, $\boldsymbol{T} = \{\boldsymbol{r}\}$).
2) Choose an edge $(\boldsymbol{v},\boldsymbol{w})$ such that $\boldsymbol{v} \in \boldsymbol{T}$, $\boldsymbol{w} \in \boldsymbol{V}-\boldsymbol{T}$, and the weight of edge $(\boldsymbol{v},\boldsymbol{w})$ is the minimum among that of edges from the nodes of $\boldsymbol{T}$ to nodes of $\boldsymbol{V}-\boldsymbol{T}$.
3) Add the node $\boldsymbol{w}$ into $\boldsymbol{T}$.
4) Repeat the above 2) and 3) until $\boldsymbol{T} = \boldsymbol{V}$.

PRIM’S ALGORITHM



PRIM'S ALGORITHM

- If the graph is represented by an **adjacency (distance) matrix**, the time complexity of Prim's algorithm is $O(|V|^2)$.

- Prim's algorithm can be made more efficient by maintaining the graph using **adjacency lists** and keeping a **priority queue** of the nodes not in T. Under this implementation, the time complexity of Prim's algorithm is $O((|V|+|E|)\log|V|)$.

PRIM'S ALGORITHM

- Implementation:

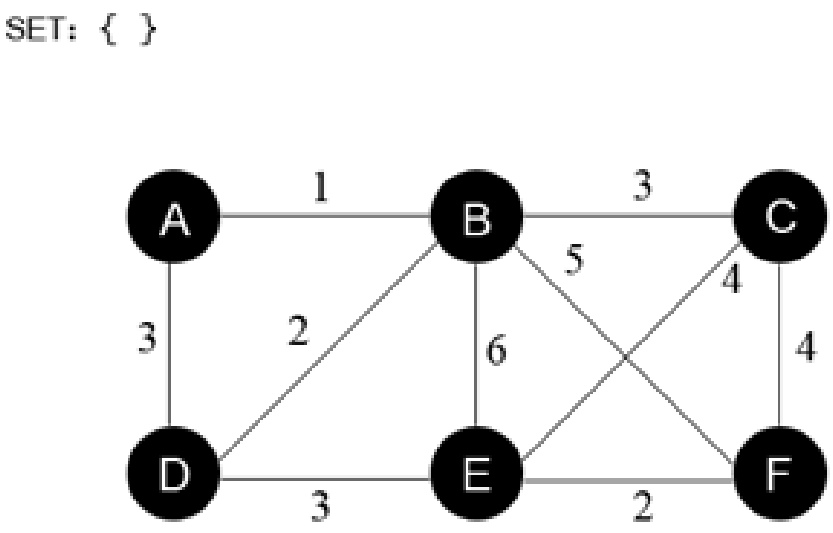
```
def MST-PRIM (G, w, r)
//  Graph G with set of nodes G.V, weight matrix w and
// root node r. MST is the edges set A={(v,v.π), v ∈ V-r}.
    for each u ∈ G.V:
        u.key = ∞
        u.π = NIL
    r.key = 0
    Q = Min-Priority-Queue (G.V)
    while Q ≠ ∅:
        u = Extract-Min (Q)
        for each v ∈ G.Adj[u]:
            if v ∈ Q and w(u,v) < v.key:
                v.π = u
                v.key = w(u,v)
```

PRIM'S ALGORITHM

- Implementation notes:
  - During execution of the algorithm, all nodes that are **NOT** in the **MST**, reside in the **minimum priority queue** based on the *key* attribute.
  - For each node $\boldsymbol{v}$, the attribute $\boldsymbol{v.key}$ is the minimum weight of any edge connecting $\boldsymbol{v}$ to a node in the **MST**.
  - If there is no edge $\boldsymbol{v.key} = \infty$.
  - The attribute $\boldsymbol{v.\pi}$ names the parent of $\boldsymbol{v}$ in the **MST**.

PRIM'S ALGORITHM

- Animated example:



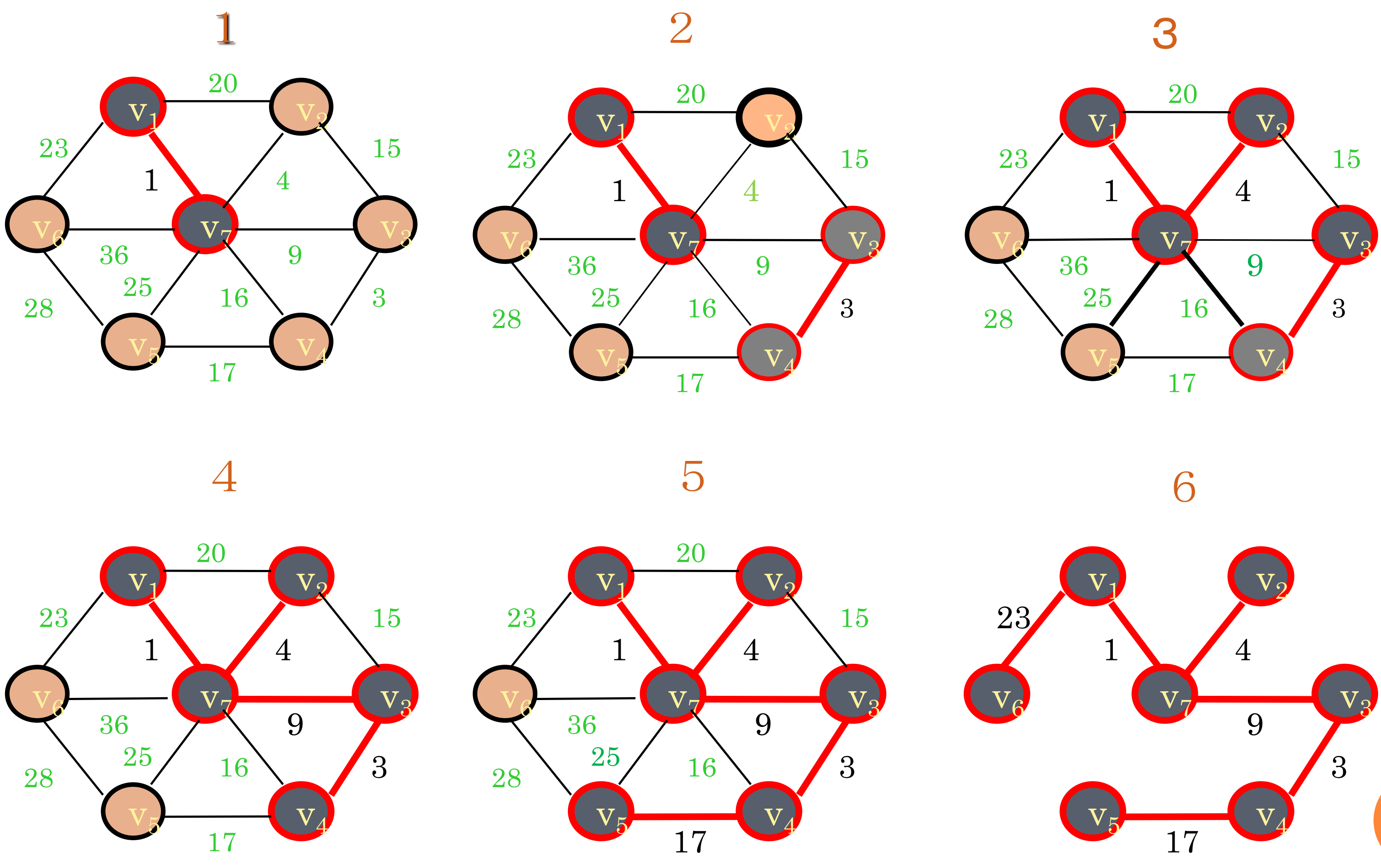
KRUSKAL'S ALGORITHM

1) Pick the cheapest edge available and add it to the **MST**

$$\boldsymbol{e_0} = \min_{\mathbf{v,u}} \boldsymbol{w(v,u)}, \qquad \boldsymbol{A}=\{\boldsymbol{e_0}\}$$

2) Choose next cheapest edge $\boldsymbol{e}=(\boldsymbol{v},\boldsymbol{w})$
3) If adding $\boldsymbol{e}$ to the $\boldsymbol{A}$ makes a cycle, do not add it.
4) Repeat the above 2) and 3) until all edges are chosen.

KRUSKAL'S ALGORITHM



KRUSKAL'S ALGORITHM

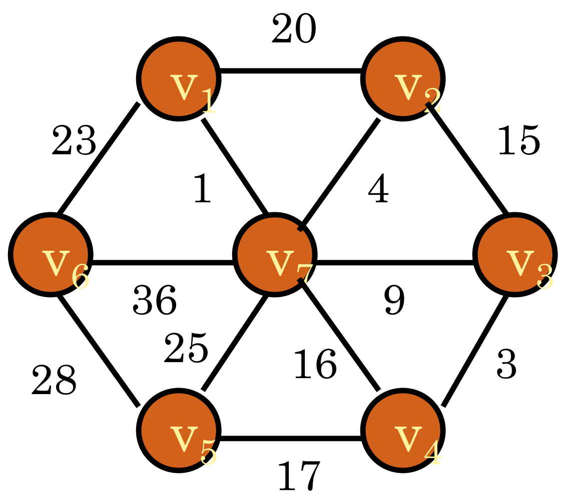
- Implementation:

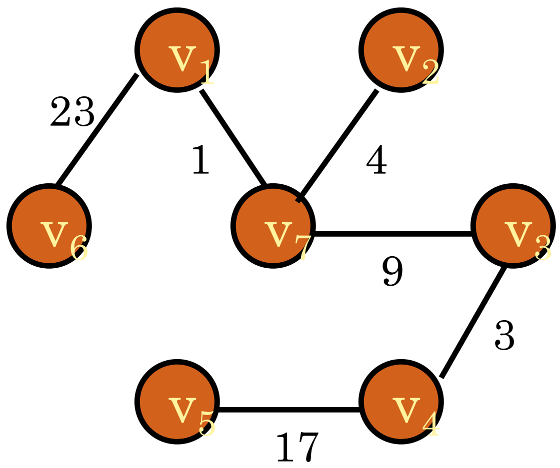
```
def  MST-KRUSKAL (G, w)
//  Graph G with set of nodes G.V, weight matrix w.
//  MST is the edges set A={}.
     A = ∅
     for each v ∈ G.V:
        MAKE-SET (v)
     Sort edges of G.E into non-decreasing order by weight w
     for each edge (u, v) ∈ G.E, taken in non-decreasing order of w:
        if FIND-SET (u) ≠ FIND-SET (v):
           A = A ∪ {(u, v)}
           UNION (u, v)
     return A
```

KRUSKAL'S ALGORITHM

- Implementation notes.
  - **UNION-FIND** data structure:
    - Given a node ***u*** the operation **FIND-SET (u)** will return the name of the set containing ***u***.
    - To test if two nodes ***u*** and ***v*** are in the same set, we simply check if **FIND-SET(u) = FIND-SET(v)**
    - The operation **UNION (u, v)** will take two sets containing ***u*** and ***v*** respectively and will merge them into a single set.
    - To make a set from one or several nodes, we use the **MAKE-SET** () operation.

KRUSKAL'S ALGORITHM

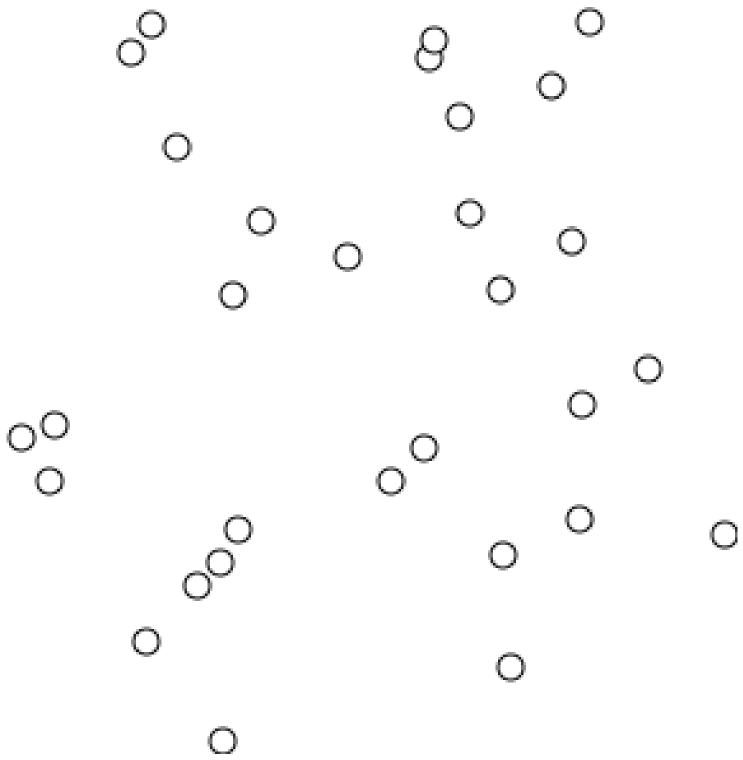




Edge	Action	Sets
		$\{v_1\},\{v_2\},\{v_3\},\{v_4\},\{v_5\},\{v_6\},\{v_7\}$
(v_1, v_7)	Add	$\{v_1,v_7\},\{v_2\},\{v_3\},\{v_4\},\{v_5\},\{v_6\}$
(v_3, v_4)	Add	$\{v_1,v_7\},\{v_2\},\{v_3,v_4\},\{v_5\},\{v_6\}$
(v_2, v_7)	Add	$\{v_1,v_2,v_7\},\{v_3,v_4\},\{v_5\},\{v_6\}$
(v_3, v_7)	Add	$\{v_1,v_2,v_3,v_4,v_7\},\{v_5\},\{v_6\}$
(v_2, v_3)	Reject	
(v_4, v_7)	Reject	
(v_4, v_5)	Add	$\{v_1,v_2,v_3,v_4,v_5,v_7\},\{v_6\}$
(v_1, v_2)	Reject	
(v_1, v_6)	Add	$\{v_1,v_2,v_3,v_4,v_5,v_6,v_7\}$

KRUSKAL'S ALGORITHM

- Animated example based on Euclidean distance:



KRUSKAL'S ALGORITHM

- Complexity.
  - Initializing set $\boldsymbol{A}$ takes $\boldsymbol{O}(1)$.
  - Making $|\mathbf{V}|$ sets takes $\boldsymbol{O}(\boldsymbol{V})$ time.
  - Time to sort the edges by weight is $\boldsymbol{O}(\boldsymbol{E}log\boldsymbol{E})$.
  - There are $|\mathbf{E}|$ FIND-SET and UNION operations taking $\boldsymbol{O}(\boldsymbol{E})$ time.
  - Since the graph is connected, $|\boldsymbol{E}| \geq |\boldsymbol{V}|$-1 and $|\boldsymbol{E}| < |\boldsymbol{V}|^2$, $log\ |\boldsymbol{V}|^2 = 2log|\boldsymbol{V}|$ which is $\boldsymbol{O}(log\boldsymbol{V})$.
  - Total running time is $\boldsymbol{O}(\boldsymbol{E}log\boldsymbol{V})$.

That's All for Today!