ALGORITHMS AND DATA STRUCTURES II

Lecture 8
Divide and Conquer Algorithm Design,
Matrix multiplication (MM),
Strassen Algorithm for MM.

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DIVIDE AND CONQUER

- Recursive algorithms solve a given problem by calling themselves recursively. They follow a **divide-and-conquer** approach:
  - break the problem into several sub-problems that are similar to the original problem but smaller in size,
  - solve the sub-problems recursively,
  - combine these solutions to create a solution to the original problem.

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DIVIDE AND CONQUER

- The divide-and-conquer paradigm has three steps at each level of the recursion:

1. **Divide** the problem into several sub-problems.
2. **Conquer** the sub-problems by solving them recursively. If the sub-problem sizes are small enough, then solve the sub-problem straightforwardly.
3. **Combine** the solutions to the sub-problems into the solution for the original problem.

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DIVIDE AND CONQUER

- The **merge sort** algorithm is an example of divide-and-conquer approach:

1. **Divide** : Divide an n element sequence to be sorted into two subsequences of $n/2$ elements each.
2. **Conquer** : Sort the two subsequences recursively using merge sort.
3. **Combine** : Merge the two sorted subsequences to get the answer.

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MERGE SORT

The **DIVIDE** step

5 2 4 6 | 1 3 2 6

5 2 | 4 6

1 3 | 2 6

5 | 2

4 | 6

1 | 3

2 | 6

5
2
4
6
1
3
2
6

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MERGE SORT

5
2
4
6
1
3
2
6

The **Conquer** step:
sort the two subsequences recursively using merge sort.

Recursion goes on until our subsequences come down to length one. Then they are sorted and we have nothing to do.

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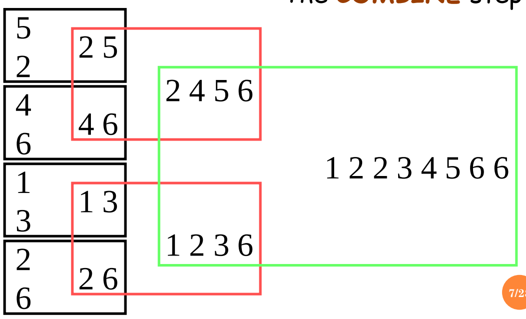
MERGE SORT

The **COMBINE** step



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MERGE SORT

- In Lecture 1, we analyzed the merge sort algorithm and found that the time complexity is:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 2T(n/2) + \Theta(n) & \text{if } n > 1 \end{cases}$$

which we said that can be solved and gives:

$$T(n) = O(n \log n)$$

- At the end of this lecture we will prove it using the so called Master theorem.

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MATRIX MULTIPLICATION

- Let A and B be two $n \times n$ matrices. The product of A and B is defined as $C = AB$, where for $1 \leq i, j \leq n$,

$$C[i, j] = \sum_{k=1}^{n} A[i,k] \times B[k, j]$$

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MATRIX MULTIPLICATION

- If n is a power of 2, we can partition each of A and B into four $(n/2) \times (n/2)$ matrices and express the product of A and B in terms of these $(n/2) \times (n/2)$ matrices as:

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \times \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

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MATRIX MULTIPLICATION

- If we treat A and B as 2×2 matrices, whose elements are $(n/2) \times (n/2)$ matrices, then the C can be expressed in terms of sums and products of these $(n/2) \times (n/2)$ matrices:

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21}$$
$$C_{12} = A_{11} \times B_{12} + A_{12} \times B_{22}$$
$$C_{21} = A_{21} \times B_{11} + A_{22} \times B_{21}$$
$$C_{22} = A_{21} \times B_{12} + A_{22} \times B_{22}$$

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MATRIX MULTIPLICATION

- Recursive algorithm for matrix multiplication:

```
def MAT-MULT (A, B):
    n = A.rows
    C = new (n×n) matrix
    if n == 1: C11=A11·B11
    else:  // partition A, B and C
        C11=MAT-MULT (A11, B11) + MAT-MULT (A12, B21)
        C12=MAT-MULT (A11, B12) + MAT-MULT (A12, B22)
        C21=MAT-MULT (A21, B11) + MAT-MULT (A22, B21)
        C22=MAT-MULT (A21, B12) + MAT-MULT (A22, B22)
    return C
```

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MATRIX MULTIPLICATION

- Analysis of the recursive algorithm for matrix multiplication.
  - If $n = 1$, we do only one scalar multiplication -> $T(1) = \Theta(1)$
  - For $n > 1$, each recursive call multiplies two $n/2 \times n/2$ matrices contributing $T(n/2)$ time. There are 8 such recursive calls -> $8T(n/2)$.
  - Four matrix additions take $\Theta(4n^2/4) = \Theta(n^2)$

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 8T(n/2) + \Theta(n^2) & \text{if } n > 1 \end{cases}$$

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STRASSEN ALGORITHM

- In 1969, Strassen proposed an algorithm which is faster than the recursive matrix multiplication. It has four steps:
  - **Step 1**. Divide input matrices A and B into $n/2 \times n/2$ matrices.
  - **Step 2**. Create 10 matrices $S_1, S_2, \ldots S_{10}$, each of which is sum or difference of the matrices created at Step 1.
  - **Step 3**. Using matrices from Step 1 and Step 2 compute 7 matrices $P_1, P_2, \ldots P_7$.
  - **Step 4**. Compute $C_{11}, C_{12}, C_{21}, C_{22}$ by adding and subtracting various combinations of P_i matrices.

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STRASSEN ALGORITHM

- Strassen algorithm matrix computation:

Step 1: $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$

Step 2:

$$\begin{aligned} S_1 &= B_{12} - B_{22} & S_2 &= A_{11} + A_{12} \\ S_3 &= A_{21} + A_{22} & S_4 &= B_{21} - B_{11} \\ S_5 &= A_{11} + A_{22} & S_6 &= B_{11} + B_{22} \\ S_7 &= A_{12} - A_{22} & S_8 &= B_{21} + B_{22} \\ S_9 &= A_{11} - A_{21} & S_{10} &= B_{11} + B_{12} \end{aligned}$$

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STRASSEN ALGORITHM

- Strassen algorithm matrix computation:

Step 3:

$$\begin{aligned} &P_1 = A_{11}S_1 \quad P_2 = S_2B_{22} \quad P_3 = S_3B_{11} \\ &P_4 = A_{22}S_4 \quad P_5 = S_5S_6 \quad P_6 = S_7S_8 \quad P_7 = S_9S_{10} \end{aligned}$$

Step 4:

$$\begin{aligned} C_{11} &= P_5 + P_4 - P_2 + P_6 \\ C_{12} &= P_1 + P_2 \\ C_{21} &= P_3 + P_4 \\ C_{22} &= P_1 + P_5 - P_3 - P_7 \end{aligned} \qquad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

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STRASSEN ALGORITHM

- Strassen algorithm matrix computation - check:

$$\begin{aligned} C_{12} &= P_1 + P_2 = A_{11}S_1 + S_2B_{22} \\ &= A_{11}B_{12} - A_{11}B_{22} + A_{11}B_{22} + A_{12}B_{22} \\ &= A_{11}B_{12} + A_{12}B_{22} \end{aligned}$$

$$\begin{aligned} C_{21} &= P_3 + P_4 = S_3B_{11} + A_{22}S_4 \\ &= A_{21}B_{11} + A_{22}B_{11} + A_{22}B_{21} - A_{22}B_{11} \\ &= A_{21}B_{11} + A_{22}B_{21} \end{aligned}$$

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STRASSEN ALGORITHM

- Strassen algorithm analysis.
  - If $n = 1$, we do only one scalar multiplication -> $T(1) = \Theta(1)$
  - For $n > 1$, **at step 2** each recursive call multiplies two $n/2 \times n/2$ matrices contributing $T(n/2)$ time. There are 7 such recursive calls -> $7T(n/2)$. The number of additions is 18.
  - Matrix additions take $\Theta(18n^2/4) = \Theta(n^2)$

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 7T(n/2) + \Theta(n^2) & \text{if } n > 1 \end{cases}$$

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SOLVING RECURRENCES

- A **recurrence** is an equation that describes a function in terms of its value in smaller inputs.
- There are three main methods for solving recurrences:
  - In the substitution method, we guess a bound and then use induction to prove it.
  - The recursion tree method converts the recurrence into a tree and uses bounding summations.
  - The master method provides bounds for recurrences of the form:

$$T(n) = aT(n/b) + f(n)$$

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THE MASTER METHOD

- The **master method** depends on the following theorem:

THEOREM* (Master theorem)

Let $a \geq 1$, $b>1$ and $c>1$ be constants, and let $T(n)$ be defined on the nonnegative integers by the recurrence:

$$T(n) = \begin{cases} b & \text{if } n = 1 \\ aT(n/c) + bn & \text{if } n > 1 \end{cases}$$

* Simplified version

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THE MASTER METHOD

- (theorem continuation)

Then, if n is a power of c, $T(n)$ has the following asymptotic bounds :

$$T(n) = \begin{cases} O(n), & \text{if } a < c, \\ O(n \log n), & \text{if } a = c, \\ O(n^{\log_c a}), & \text{if } a > c, \end{cases}$$

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THE MASTER METHOD

- Lets find the solution for our algorithms:
  - MERGE SORT.

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 2T(n/2) + \Theta(n) & \text{if } n > 1 \end{cases}$$

we have $a = c$, therefore (according to the second row)

$$T(n) = O(n \log n)$$

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THE MASTER METHOD

- Lets find the solution for our algorithms:
  - RECURSIVE MATRIX MULTIPLICATION.

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 8T(n/2) + \Theta(n^2) & \text{if } n > 1 \end{cases}$$

we have $a=8 > c=2$, therefore (according to the third row)

$$T(n) = O(n^{\log_c a}) = O(n^{\log_2 8}) = O(n^3)$$

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THE MASTER METHOD

- Lets find the solution for our algorithms:
  - STRASSEN ALGORITHM.

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 7T(n/2) + \Theta(n^2) & \text{if } n > 1 \end{cases}$$

we have $a=7 > c=2$, therefore (according to the third row)

$$T(n) = O(n^{\log_c a}) = O(n^{\log_2 7}) = O(n^{2.81})$$

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VINOGRAD ALGORITHM

- The **Vinograd algorithm** is a variant of the Strassen algorithm which requires (the same) *7* multiplications, but only *15* additions/subtractions.
- Vinograd algorithm complexity is the same, but the reduced number of additions/subtractions has **practical** significance.

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DISCUSSION

- There are two key issues when efficiently applying Strassen algorithm to arbitrary matrices.
  - First the constraint that the matrix size be a power of 2 must be handled.
    - One solution – zero padding.
  - The second key issue for efficiency of Strassen algorithm is controlling the depth of recursion.
    - For small n, Strassen algorithm is actually slower!

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DISCUSSION

- Matrix multiplication is a fundamental operation and is critical when attempting to speed up scientific computations.
- The performance of matrix multiplication is dependent on two elements:
  - ✓ the **operation count** and
  - ✓ the **memory reference count**.
- Minimizing both of these factors will produce an optimal algorithm.

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THAT'S ALL FOR TODAY!

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