Optimal Control Applied to Pesticide Treatment of a Predator-Prey Population

Milestone #3
Math 325

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May 3, 2024

1 Motivation

Control of pest populations is integral to the farming industry. For example, sugarcane harvests across the United States are impacted by aphid infestations, which were responsible for 79 million dollars in losses and pesticide-related costs to farmers between 2014 and 2016 [2]. The price of these creatures is felt by farmers, distributors, and consumers, creating high demand for pest control solutions. Insects such as lady beetles and parasitic wasps are natural predators of aphids, but are highly sensitive to insecticides [3]. These predators do not harm crops and assist pest control efforts by feeding on aphid populations. Researchers may then be interested in calculating the optimal amount of insecticide to spray onto a sugarcane field to minimize aphid populations while remaining below the amount that could harm predator populations. This flavor of problem is a perfect candidate for an optimal control approach.

This example lays the framework for our more generalized predator-prey model. In this report, we model a predator-prey population where the prey is a pest, treated with a pesticide that harms both populations. We begin by developing the differential equation model, casting the problem as an optimal control, finding necessary conditions, and numerically simulating results in MATLAB.

2 Model Development

2.1 Assumptions

The basis of our model is the standard Lotka-Volterra model for predator-prey dynamics, given by

$$\begin{aligned}N_1' &= (\alpha_1 - \beta_1 N_2(t))N_1(t), N_1(0) = N_{10},\\ N_2' &= (\beta_2 N_1(t) - \alpha_2)N_2(t), N_2(0) = N_{20}.\end{aligned}$$

The prey population is given by N_1 while the predator population is given by N_2. To further generalize the problem, t is scaled to an arbitrary time unit, meaning that parameter choices must agree with whichever unit is needed for the problem. Constants $\alpha_1, \beta_1, \alpha_2,$ and β_2 are positive weights that determine how much changes in one population affect the other; for our purposes, we let $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1$, which simplifies our problem to

$$\begin{aligned}N_1' &= (1 - N_2(t))N_1(t), N_1(0) = N_{10},\\ N_2' &= (N_1(t) - 1)N_2(t), N_2(0) = N_{20}.\end{aligned}$$

In this form, the inverse relationship between the two populations is clear: when $N_2 = 0$, the prey population grows exponentially, whereas when $N_1 = 0$, the predator population decays exponentially. Increases in the prey population increase the predator population, which in turn decreases the prey population, and the cycle continues; this logically follows from what we observe in nature, where a basic predator-prey system is periodic until an equilibrium is reached.

We incorporate the effect of the pesticide with the control $u(t)$, which represents the rate of application at time t. Though an ideal pesticide will only harm pests, leave no residue, and work in a density-dependent manner, this is not typically the case in applied situations; for simplicity, we assume the latter two to be true in our model. If we let $d_1, d_2 > 0$ be parameters representing the effect of pesticide density on N_1 and N_2 respectively, then the total effect of the pesticide on N_1 depends on d_1, the population size N_1, and the rate of pesticide application $u(t)$, without loss of generality to the predator population. Subtracting the product $d_1 N_1 u$ from the differential equation reflects the negative impact of pesticide use. We also require that $N_{10}, N_{20} \geq 0$, since population size is non-negative. The system then becomes

$$\begin{aligned} N_1' &= (1 - N_2(t))N_1(t) - d_1 N_1(t)u(t), N_1(0) = N_{10} \geq 0, \\ N_2' &= (N_1(t) - 1)N_2(t) - d_2 N_2(t)u(t), N_2(0) = N_{20} \geq 0. \end{aligned}$$

With the differential equation system well-defined, we continue with an optimal control approach.

2.2 Optimal Control

While the best method of controlling the pest population would be to apply high amounts of pesticide, this harms the predator population and potentially the environment. Pest control can also become costly, introducing an economic restriction. Consequently, we assume that environmental or economic regulations specify a maximum application level $M > 0$ at any given time t as well as a fixed total application amount $B > 0$ over the entire time interval $[0, T]$. In our model, this translates on a bounded control $0 \leq u(t) \leq M$ and an isoperimetric constraint $\int_0^T u(t) = B$.

We define the objective functional $J(u) = u^2$. This ensures that lower rates of pesticide application $u(t)$ more effectively minimize $J(u)$, with the added benefit of preventing linearity in the control, a mathematical nightmare. We incorporate a weight parameter A to represent the importance we assign to minimizing the pest population as opposed to reducing the control. As we seek to minimize the prey population at a particular final time $t = T$, we add the payoff term $N_1(T)$ to our objective functional. We now cast the formal optimal control problem:

$$\begin{aligned} &\min_u N_1(T) + \frac{A}{2}\int_0^T u(t)^2 dt \\ \text{subject to } N_1' =& (1 - N_2(t))N_1(t) - d_1 N_1(t)u(t), N_1(0) = N_{10} \geq 0, \\ N_2' =& (N_1(t) - 1)N_2(t) - d_2 N_2(t)u(t), N_2(0) = N_{20} \geq 0. \\ &0 \leq u(t) \leq M, \int_0^T u(t)dt = B. \end{aligned}$$

Notable in this system is the interdependence of parameters B, M, and T, since $B = \int_0^T u(t)dt \leq \int_0^T M dt = MT$. Therefore in later simulations, we require that parameter selection satisfies $B \leq MT$ for the system to be well-defined. Additionally, this two-state system includes an isoperimetric constraint, which will influence our solution in the following section.

3 Model Solution

We incorporate the aforementioned isoperimetric constraint $\int_0^T u(t)^2 = B$ by defining a new state variable $z(t) = \int_0^T u(t)$; in the form of a differential equation, this becomes, $z'(t) = u(t)$. The integral bounds provide initial and final conditions $z(0) = 0$ and $z(T) = B$. Our new optimal control problem may be cast as

$$
\begin{aligned}
&\min_u N_1(T) + \frac{A}{2}\int_0^T u(t)^2 dt \\
\text{subject to } &N_1' = (1 - N_2(t))N_1(t) - d_1 N_1(t)u(t), N_1(0) = N_{10} \geq 0, \\
&N_2' = (N_1(t) - 1)N_2(t) - d_2 N_2(t)u(t), N_2(0) = N_{20} \geq 0. \\
&z' = u, z(0) = 0, z(T) = B \\
&0 \leq u(t) \leq M.
\end{aligned}
$$

With this three-state, bounded control system, we can apply our usual solving techniques. We continue by defining the Hamiltonian below; for brevity, we denote functions of time $u(t), N_1(t), N_2(t)$, etc. as u, N_1, N_2.

$$
H = \frac{A}{2}u^2 + \lambda_1[(1 - N_2)N_1 - d_1 N_1 u] + \lambda_2[(N_1 - 1)N_2 - d_2 N_2 u] + \lambda_3 u. \tag{1}
$$

From Equation 1 we can derive the adjoint equations, transversality conditions, and optimality conditions by Pontryagin's Maximum Principle. The adjoints λ_1, λ_2 and λ_3, defined by

$$
\lambda_1' = -\frac{\partial H}{\partial N_1}, \qquad \lambda_2' = -\frac{\partial H}{\partial N_2}, \qquad \lambda_3' = -\frac{\partial H}{\partial z},
$$

append the state equations to the objective functional. Differentiation yields

$$
\begin{aligned}
\lambda_1' &= \lambda_1(N_2 - 1 + d_1 u) - \lambda_2 N_2, \\
\lambda_2' &= \lambda_2(1 - N_1 + d_2 u) + \lambda_1 N_1, \\
\lambda_3' &= 0.
\end{aligned}
$$

These adjoints, along with the state equations, create a system of differential equations whose solution is the optimized states. To provide us with a particular solution, we derive transversality conditions for each adjoint equation. For $\lambda_1(T)$, we define a function $\phi(N_1(t))$ to represent the payoff term for N_1; as $\phi(N_1(t)) = N_1(t)$, the transversality condition is $\phi'(N_1(t)) = \lambda_1(T) = 1$. See [1] for more details about this technique. As N_2 has no payoff terms (or sign-on bonuses), we default to $\lambda_2(T) = 0$. Finally, since $z'(t)$ has two fixed points, we do not have a transversality condition for λ_3 so that the system is not overly defined.

For our optimality condition, we calculate $\frac{\partial H}{\partial u} = Au - \lambda_1 d_1 N_1 - \lambda_2 d_2 N_2 + \lambda_3$. Since we will be using MATLAB as a numerical solver, it suffices to solve for u^* from $\frac{\partial H}{\partial u} = 0$, yielding

$$
u^* = \frac{1}{A}(\lambda_1 d_1 N_1 + \lambda_2 d_2 N_2 - \lambda_3)
$$

The bounds on $u(t)$ and the continuity of the control function allow us to define

$$
u^* = \min(M, \max(0, \frac{1}{A}(\lambda_1 d_1 N_1 + \lambda_2 d_2 N_2 - \lambda_3)))
$$

The complexity of this system prevents us from finding an analytical solution. Additionally, a traditional forward-backward sweep using Runge-Kutta 4 in MATLAB is insufficient for a state equation with two fixed points such as our $z'(t)$. Therefore, we adopt a more refined approach that Lenhart calls the adjusted forward-backward sweep.

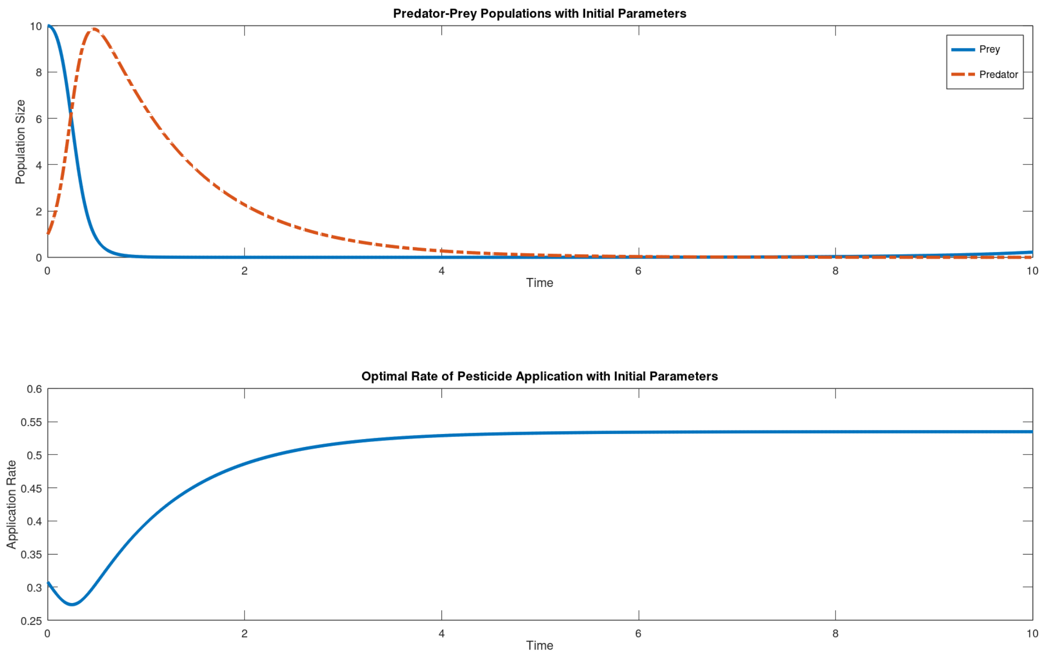

Figure 1: Plot of optimal pest-predator populations (top) with optimal pesticide application rate (bottom) for parameters in Equation 2.

4 Experimentation

4.1 Baseline Simulations

We begin with a simulation of the system using the following parameters:

$$d_1 = 0.1, \qquad d_2 = 0.1, \qquad N_{10} = 10, \qquad N_{20} = 1, \qquad M = 1, \qquad A = 1, \qquad B = 5, \qquad T = 10. \tag{2}$$

We see a sharp decrease in the pest population early on in the simulation followed by a spike in the predator population. This is expected since the pests serve as their food source. The control $u(t)$ begins relatively low, but increases on the interval $0.5 \leq t \leq 2$ before reaching near-constant application for the remainder of the time period. The period of increase causes a sudden elimination of the pest population, but this has the unintended effect of decimating the predator population as well. Notice that $d_1 = d_2$, meaning that the pesticide kills both populations with the same efficacy. It would be impractical to select a pesticide that equally harms the intended population and a harmless one. Since this regimen is clearly too strong, we consider decreasing B, the total application of pesticide, as well. With a few further adjustments to satisfy the relationship between B, M, and T, we run another simulation with the following parameters:

$$d_1 = 0.1, \qquad d_2 = 0.01, \qquad N_{10} = 5, \qquad N_{20} = 2, \qquad M = 1, \qquad A = 1, \qquad B = 1, \qquad T = 5. \tag{3}$$

As shown in Figure 2, by decreasing d_2 in proportion to d_1, the predator population is somewhat more steady, though still eradicated by the end of the time interval. The optimal pesticide regimen begins near the minimum value, increasing over time until reaching a near-constant value. Note that for our first simulation, $u(t) = 0.53$ on $[2, 10]$ while with these adjusted parameters $u(t) = 0.2$ on $[2, 5]$. This decrease is due to the reduction in B, the total amount of pesticide that can be used. Even by using only 20% of the previous amount of pesticide with only 10% of the harm to predators compared to the first simulation, we still see the concerning high mortality of the predator population. The following sections test the effects of varying individual parameters from this basis system, with the overarching goal of reducing the pest population while minimizing harm to the predator population.

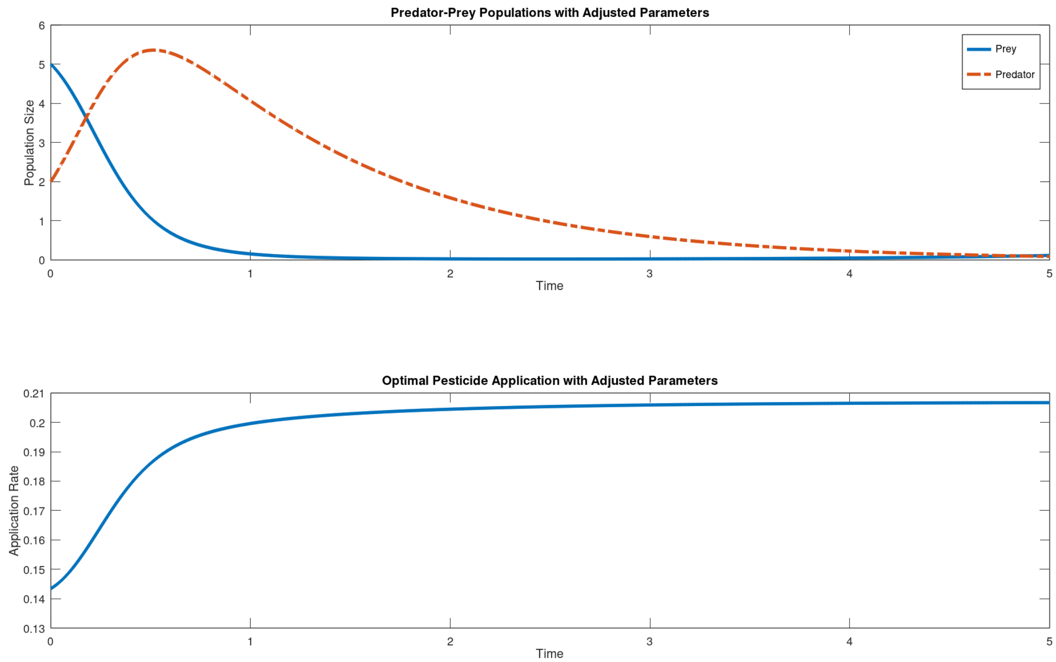

Figure 2: Plot of optimal pest-predator populations and optimal pesticide application rate with parameters in Equation 3.

4.2 Varying System Parameters

To address this issue, we can increase the weight parameter A from $A = 1$ to $A = 10$. This will emphasize minimizing pesticide cost over the time interval instead of minimizing the pest population. We simulate this in Figure 3. Since the total amount of pesticide B is fixed, variations in A only change the distribution of the application rate. Although the control becomes near-constant on the entire time interval, this does little to change the optimal states. This is not reassuring.

We now assess the impact of the density parameters d_1 and d_2. We postulate that an increase in d_1 would result in a higher pesticide efficacy on the intended population; however, we see in Figure 4 that this is not the case. Instead, this causes a lower initial application of pesticide which increases near the end of the time interval, resulting in states identical to Figures 2 and 3. Next we consider the effects of d_2. We simulate pesticides with varying effects on the predator population with $d_2 = 0.01$ and $d_2 = 0.03$ in Figure 5. This change causes similar effects as increasing d_1, having no effect on the predator and prey populations.

We continue by testing the impact of M and B. In Figure 6, we see that the optimal control is identical for $M = 1$ and $M = 1.5$, as are the optimal populations. A 50% decrease in B causes a proportional decrease in the optimal pesticide application rate over the entire time interval, maintaining the same distribution as when $B = 1$. Surprisingly, halving the pesticide amount has no effect on the optimal populations.

4.3 Varying Initial Conditions

We next consider the impact of changes in N_{10} and N_{20} to the model. Since these directly affect the prey N_1 and predator N_2 populations, we do see differences at $t = 0$. However, as shown in Figures 8 and 9, the general shape of population curves is the same with identical results near the end of the time interval. Interestingly, the initial pesticide application is higher for a higher value of N_{10}.

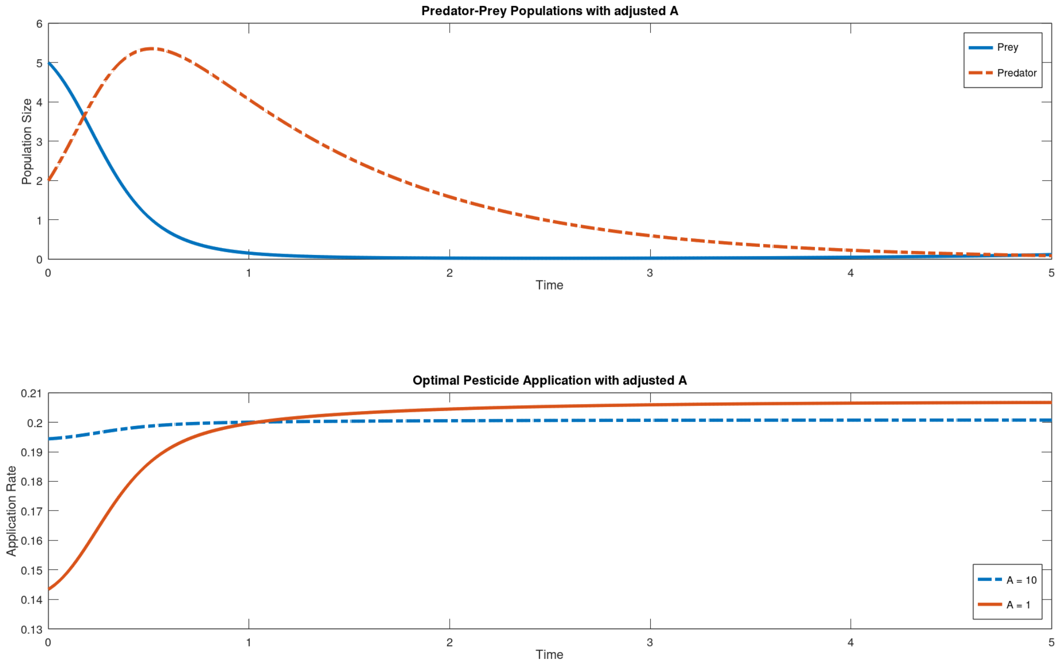

Figure 3: Plot of optimal pest-predator populations and optimal pesticide application rate with parameters in Equation 3 and $A = 10$. This increase in A assigns more importance to minimizing the objective functional u^2 compared to the payoff term $N_1(T)$.

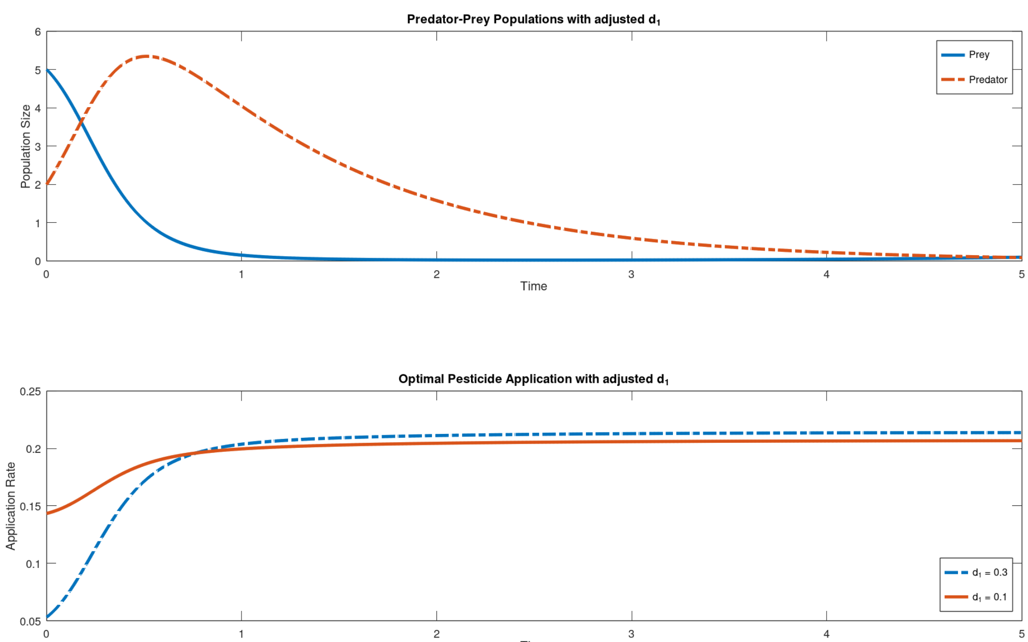

Figure 4: Plot of optimal states and control for adjusted parameters in Equation 3 with $d_1 = 0.3$. Tripling the efficacy of the pesticide on the pest population only impacts the optimal application rate.

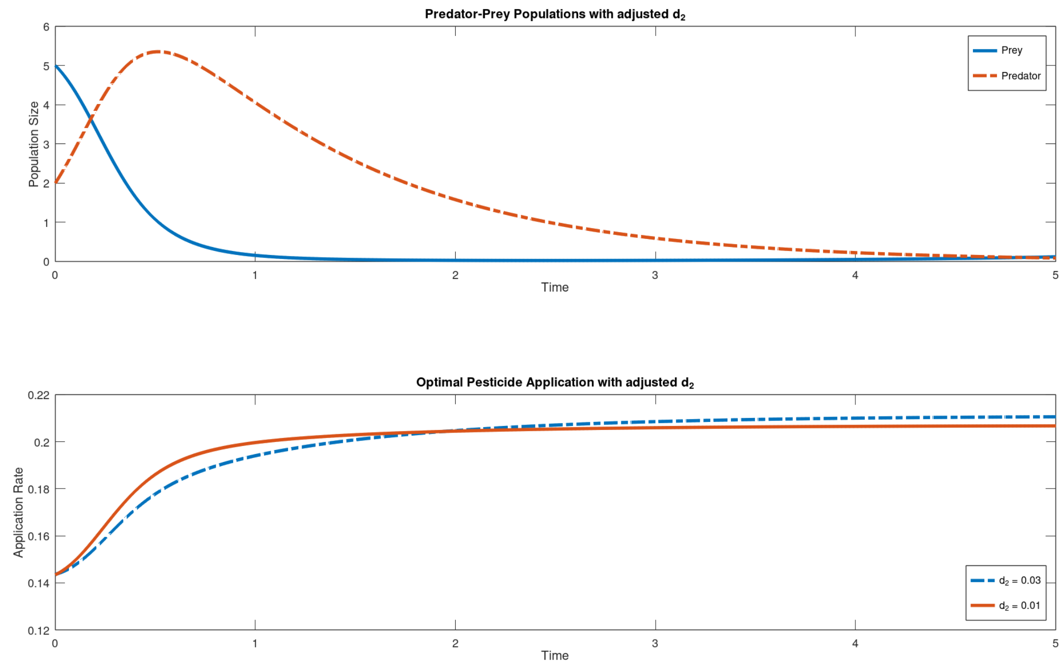

Figure 5: Plot of optimal states and control for adjusted parameters in Equation 3 with $d_2 = 0.03$.

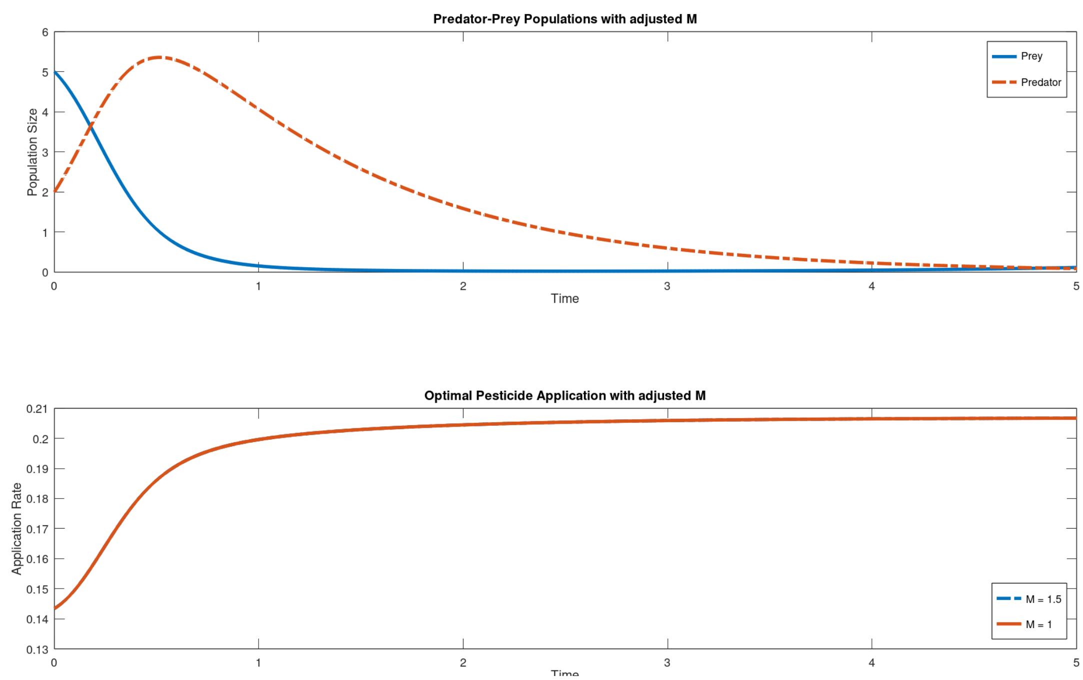

Figure 6: Plot of optimal states and control for adjusted parameters in Equation 3 with $M = 1.5$.

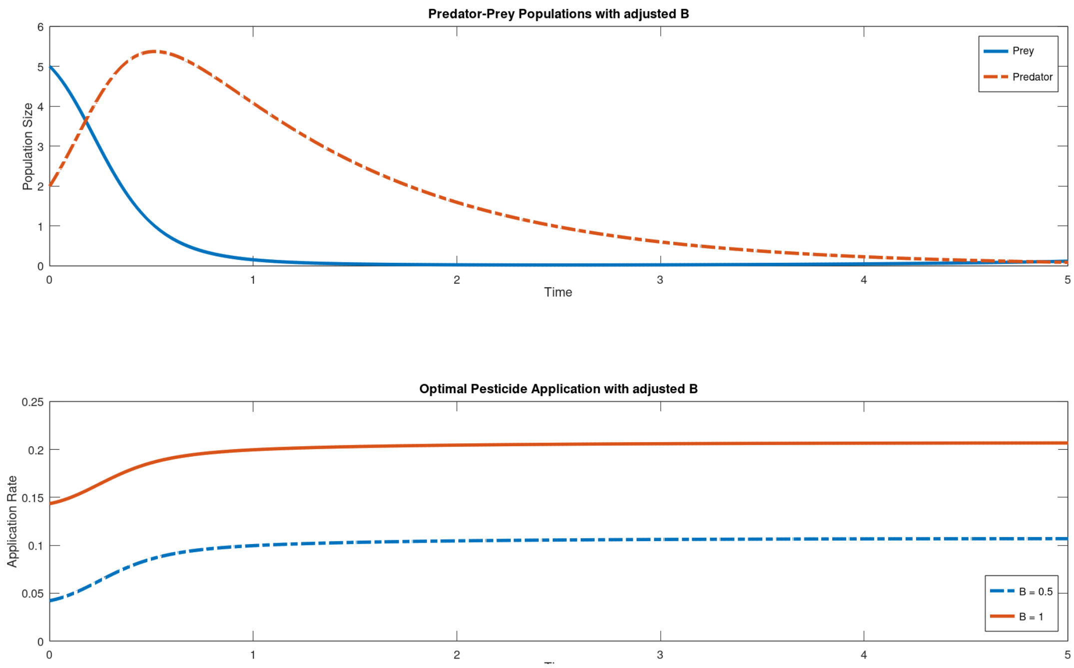

Figure 7: Plot of optimal states and control for adjusted parameters in Equation 3 with $B = 0.5$.

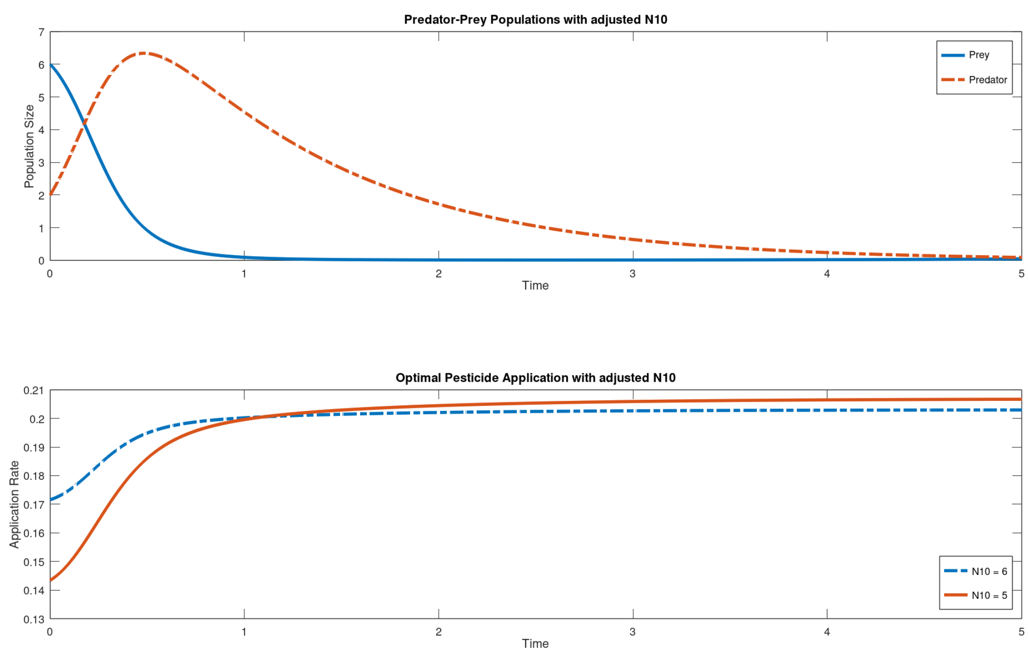

Figure 8: Plot of optimal states and control for adjusted parameters in Equation 3 with $N_{10} = 6$.

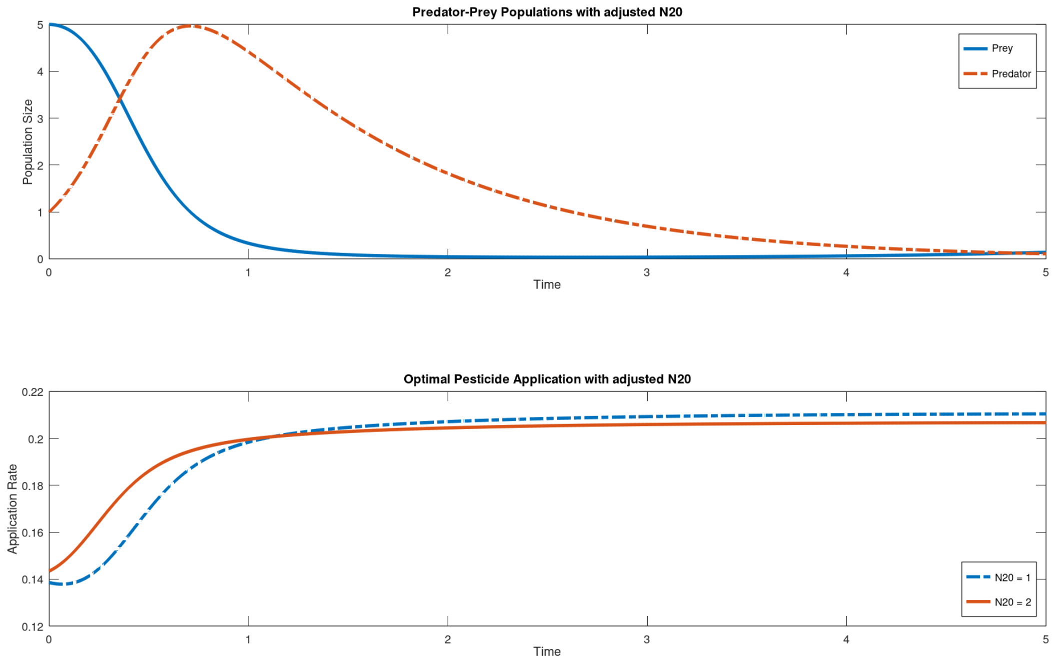

Figure 9: Plot of optimal states and control for adjusted parameters in Equation 3 with $N_{20} = 2$.

4.4 Varying Time Horizon T

In Figure 10 we test the impact of a longer time horizon. On the interval $0 \leq t \leq 5$, the populations are identical to the previous cases, despite a radically different pesticide treatment regimen. The optimal application begins closer to the maximum dose and decreases closer to the minimum, held constant for the majority of the time interval afterwards. However, we see a steady growth in the prey population near $t = 7$ that is sharply cut off by a spike in predators and a small pesticide injection.

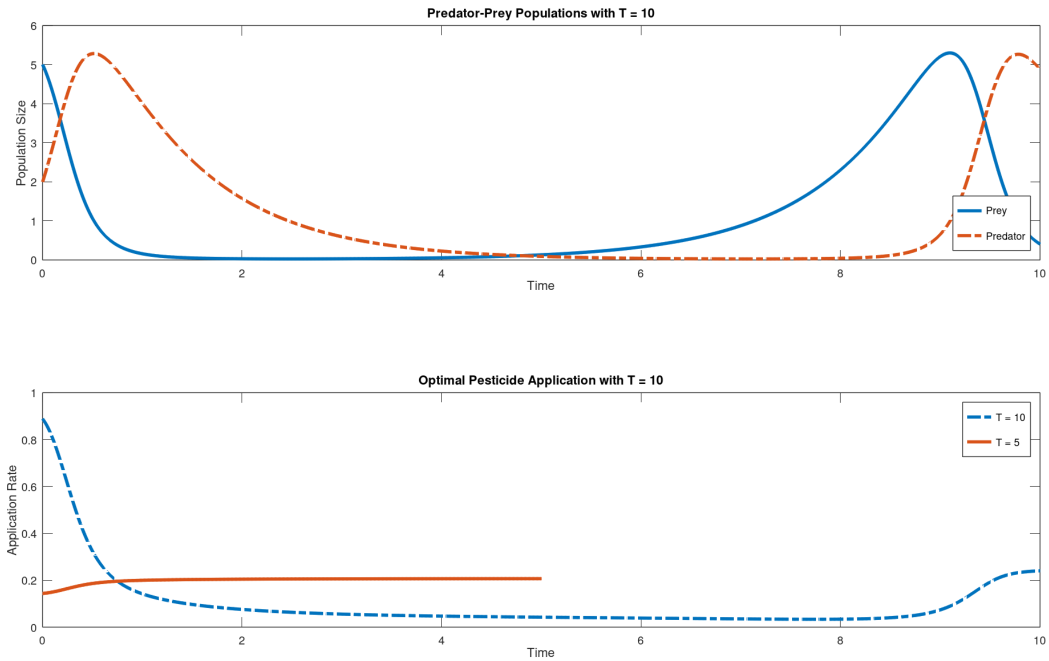

Figure 10: T

5 Discussion

5.1 General Results

If we measure success of the model by its performance in minimizing the final pest population and the total amount of pesticide used, the model was highly successful. For simulations 1 - 9, the pest population was eliminated at $t = T$ and the total amount of pesticide remained within bounds $0 \leq u(t) \leq M$ while conforming to the isoperimetric constraint $\int_0^T u(t)dt = B$. However, we faced the unintended consequence of eliminating the predator population as well. It makes sense that our model would have this behavior, as our objective functional did not take N_2 into account. The predator population was incidental to minimizing pests and the cost of pesticide application, resulting in its decimation. Since the differential equation for N_2 exponentially decays when the prey population is zero, it makes sense that without a stable pest population as a food source, the predator population would tend toward extinction. The added harmful effect of the pesticide on the predators further decreases their population size.

The shape of the optimal pesticide application curve was somewhat uniform in simulations 1-9; beginning with a minimum dose, the application rate $u(t)$ slowly increased until becoming near-constant for the remainder of the time interval. This tells farmers that it is best practice to slowly introduce pesticide use into crop maintenance and hold rates steady over time. The initial and final application rates will vary based on the pesticide strength on both populations, importance of minimizing cost, or amount that can be applied, both overall and instantaneously.

5.2 Trivial Parameters

The adjusted parameters from Equation 3 represent an ideal scenario to apply pesticide: relatively low pest population, higher predator population, and high efficacy on pests with little effect on predators. Yet we still do not achieve an optimal result, since the predator population is decimated. From this simulation, we varied parameters individually to determine whether any changes could improve the optimal states. Parameters that we found to have little effect on the optimal populations were A, d_1, d_2, M, and B. It is somewhat shocking that the efficacy of the pesticide on both populations did not radically change $N_1(t)$ or

$N_2(t)$, as this implies that the choice of pesticide is not fully relevant to the optimal populations. Even in Figure 1, where the pesticide killed both populations with the same density, the population curves took on a similar shape. This is an alarming result of the model. Additionally, the lack of efficacy associated with maximum application rate and total pesticide amount is concerning. It is not only irrelevant what our choice of pesticide is, but also how much we apply, since simply adjusting the application regimen can yield the same results. Optimal control systems can account for changes in parameter values by adjusting the control and states, but this system leans heavily to adjusting the former to maintain ideal population levels. This tells us that any variation in pesticide efficacy or amount can be accounted for by changing the application regimen, which is good news for farmers who may be restricted in their pesticide options.

5.3 Impactful Parameters

Initial conditions of both populations, though having a noticeable effect on optimal states, maintained the same general shape as previous iterations. Notably, higher initial pest populations require a larger initial application of pesticide to eliminate the population by the end of the time interval. Farmers facing more advanced infestations will have to use more pesticide initially, leaving less available for the remainder of the time interval. We conclude that it is better to begin treatment at earlier stages of infestation to achieve optimal results.

Only modifications in the time horizon T showed significant changes in both overall population behavior and optimal application regimen. The population curves are near-identical for $T = 5$ and $T = 10$ on the interval $[0, 5]$, with the $T = 10$ case exhibiting wildly different behavior on $[5, 10]$. Since we adjusted T without changing the total amount of pesticide B, the optimal application rate for most of the interval was significantly lower than the $T = 5$ case. This resulted in a pest population boom near $t = 9$, which quickly decreased as some pesticide was applied and the predator population fed on the pests. This predator boom seems implausible since the population was nearly zero, and presumably there is a minimum number of living individuals required for population growth. Including an Allee effect in N_2' would solve this problem without further complicating our numerical solution.

This case has the lowest prey population at $t = T$ that both conforms to constraints on the control and maintains the predator population, providing us with the best overall results. However, the peak of the pest population around $t = 9$ is concerning for real-life scenarios, since a high concentration of pests can lead to crop damage for farmers. Additionally, we are unsure of population dynamics after the treatment regimen has concluded; while we can assume that the predator population will decay once the pests have died out, there is no indication of whether we should expect the prey population to remain low. A return to initial population levels would mean that the pesticide treatment was ineffective; considering the cost of the pesticide to the farmer, the optimal treatment could result in overall losses. Ensuring that T is well-chosen is essential to avoid this issue. For example, letting T be the length of the harvest season for a particular crop would ensure that the pest population is minimized during the time they can do the most damage. Another suggestion is letting T be the time period of pest reproduction, as often larvae cause the most crop damage as they develop.

5.4 Concluding Remarks & Further Research

Though the model is highly effective at minimizing the prey population and cost of pesticide while conforming to environmental/economic constraints on the control, it fails to minimize harm to the predator population. A possible remedy for this could include adjusting the differential equation N_2'. The predator population experiences exponential decay without a steady pest population to feed on, but this is unrealistic in nature since most predators can feed on different species depending on availability. Returning to our introductory example, the lady beetle feeds on mites and insect eggs while parasitoid wasps can drink flower nectar [4]. If an alternate food source is available, the predator population need not be eliminated when the pest population is. This many explain why we saw similar states after adjusting d_2: it is likely that the predator population was under more stress from a lack of a food source than direct side effects of pesticides. The negative effects on the predator population are particularly alarming from an environmental standpoint, stressing the importance of a modification. We also suggest the incorporation of an Allee effect for N_1' and N_2' to increase the accuracy of the model to real-life scenarios.

The robustness of the model with parameter variation is another avenue for further research. Our results imply that the amount and strength of pesticide used is not as important as the time that it is applied, as we can achieve the same optimal populations by varying the pesticide application rate. While this solution is more cost effective since it allows farmers to dilute or adjust the amount of pesticide they purchase, it does require more precision in application, recalling that for many of our simulations, application rate must be held steady with up to two decimal points of precision. The realism of this is to be questioned further. This is a great first step to modeling a pest-predator scenario and finding optimal cost-effective treatment for farmers.

References

[1] Lenhart, S., and Workman, J. T. (2007). Optimal control applied to biological models. Chapman and Hall/CRC.

[2] https://texasfarmbureau.org/report-sugarcane-aphids-cost-texas-millionshttps://texasfarmbureau.org/report-sugarcane-aphids-cost-texas-millions/

[3] https://extension.entm.purdue.edu/publications/E-96/E-96.html

[4] https://extension.umd.edu/resource/parasitoid-wasps/