prior: assume $y_1, y_2 \cdots y_T$ are in a line.

$y_1 \ y_2 \ \cdots \ y_T$

So. AR(n) $\begin{cases} y_T = y_{T-1} \\ y_T = a_0 + a_1 y_{T-1} + \cdots + a_n y_{T-n} \end{cases}$

$$a_0 = N(0, \delta_0^2) \qquad a_i = (\tfrac{1}{n}, \delta_i^2) \quad i = 1, 2, \cdots, n$$

likelihood:

$$[y_T, \cdots, y_1; \theta] = N(\mu_y, \delta_y^2) \prod_{t=2}^{T} N(a_0 + a_1 y_{t-1}, \delta_e^2)$$

posterior $\propto$ likelihood $\times$ prior

$$\propto N(\mu_y, \delta_y^2) \prod_{t=2}^{T} N(a_0 + a_1 y_{t-1}, \delta_e^2) \ N(0, \delta_0^2) \prod_{i=1}^{n} N(\tfrac{1}{n}, \delta_i^2)$$

$$\text{posterior} = \frac{1}{\sqrt{2\pi}\delta_y} \exp\left(-\frac{(y_t - \mu_y)^2}{2\delta_y^2}\right) \cdot \prod_{t=2}^{T} \frac{1}{\sqrt{2\pi}\delta_e} \exp\left(-\frac{(y_t - a_0 - a_1 y_{t-1})^2}{2\delta_e^2}\right)$$

$$\cdot \frac{1}{\sqrt{2\pi}\delta_0} \exp\left(-\frac{a_0^2}{2\delta_0^2}\right) \cdot \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\delta_i} \exp\left(-\frac{(a_i - \frac{1}{n})^2}{2\delta_i^2}\right)$$

$$\ln(\text{posterior}) = -\frac{1}{2}\ln(2\pi\delta_y^2) - \frac{(y_t - \mu_y)^2}{2\delta_y^2} - \sum_{t=2}^{T} \frac{1}{2}\ln(2\pi\delta_e^2) - \sum_{t=2}^{T} \frac{(y_t - a_0 - a_1 y_{t-1})^2}{2\delta_e^2}$$

$$- \frac{1}{2}\ln(2\pi\delta_0^2) - \frac{a_0^2}{2\delta_0^2} - \sum_{i=1}^{n} \frac{1}{2}\ln(2\pi\delta_i^2) - \sum_{i=1}^{n} \frac{(a_i - \frac{1}{n})^2}{2\delta_i^2}$$

$$\frac{\partial \ln}{\partial a_0} = \sum_{t=2}^{T} \frac{y_t - a_0 - a_1 y_{t-1}}{\delta_e^2} - \frac{a_0}{2\delta_0^2} = 0$$

$$\frac{\partial \ln}{\partial a_i} = \sum_{t=2}^{T} \frac{(y_t - a_0 - a_i y_{t-1}) y_{t-1}}{\delta_e^2} - \sum_{i=1}^{n} \frac{a_i - \frac{1}{n}}{\delta_i^2} = 0$$

$$\frac{\partial \ln}{\partial \delta_e^2} = -\frac{T-1}{2\delta_e^2} + \frac{\sum (y_t - a_0 - a_1 y_{t-1})^2}{2\delta_e^4} = 0 \qquad \delta_e^2 = \sum_{t=2}^{T} (y_t - a_0 - a_1 y_{t-1})^2 / (T-1)$$

2 & 3

1) Data Observation

Raw data can seen in fig.1.

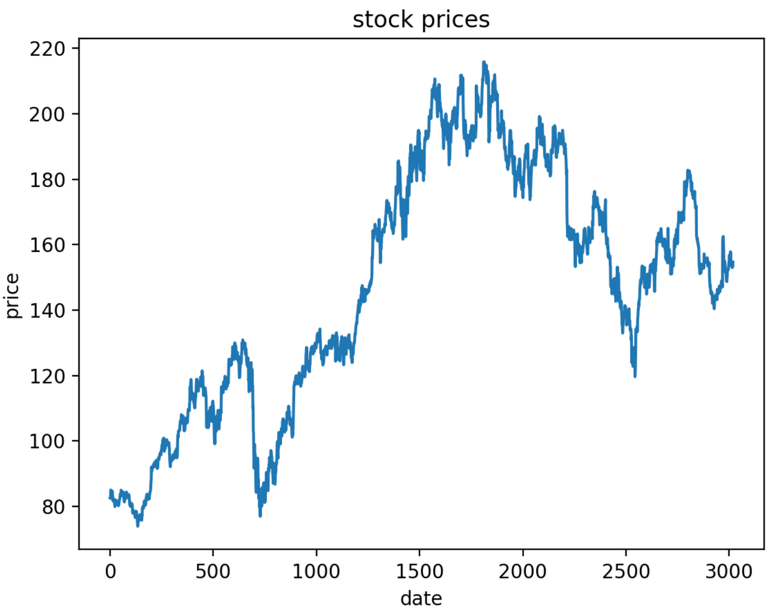
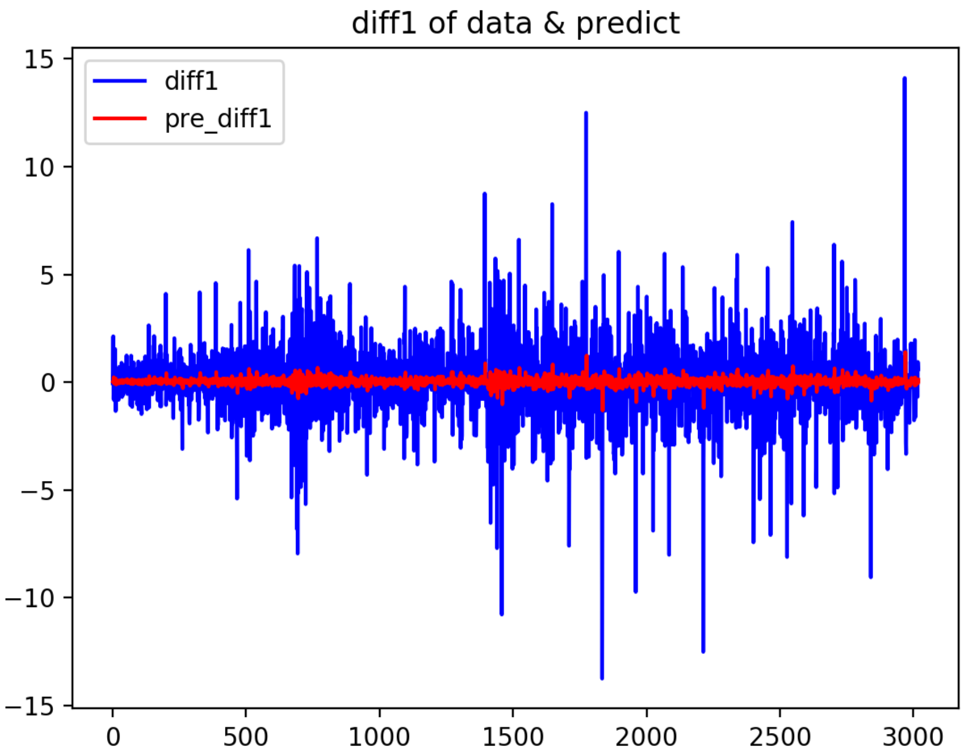


fig.1 Raw Data

2) Difference



It is obvious that the raw data is not a stationary TS.
So we make a first order difference, seen in fig.2.

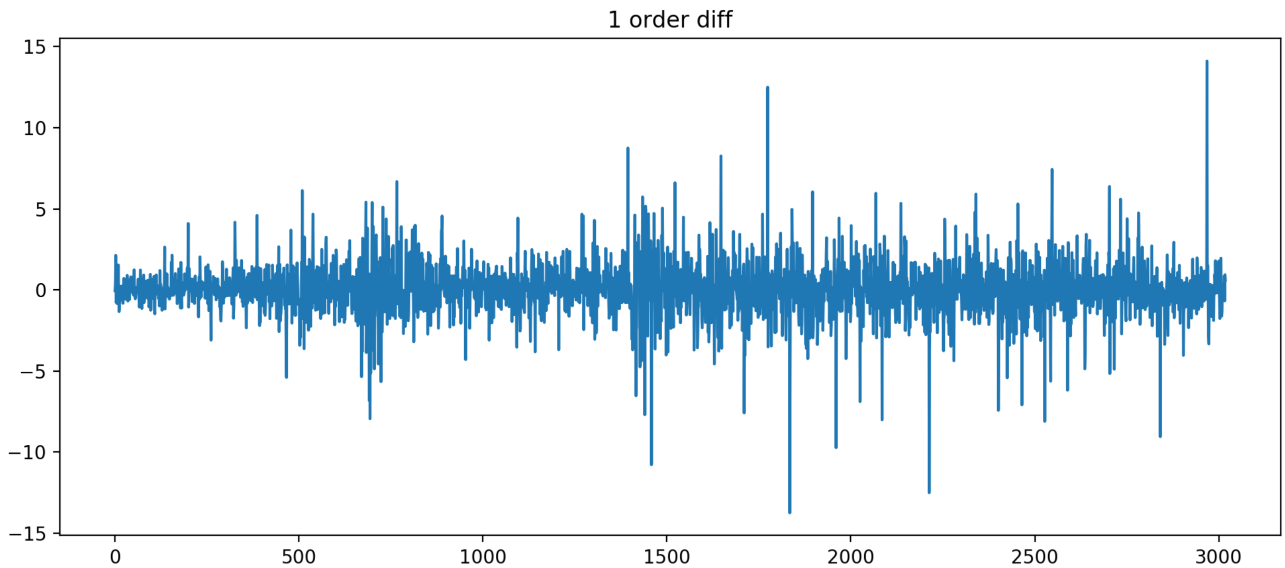


fig.2 first order difference

It can be seen that these values fluctuate around zero and there is no obvious trend. So think of it as a stationary sequence.

3) Estimation of AR(1)

fig.3 first order difference of raw data, and the data calculated by AR(1) model

Difference of raw data is stationary, we use statsmodels library to figure out the parameters, a0=0.023923, a1=097104, source code see enclosure. fig.3 is show the first order difference of raw data (blue), and the data calculated by AR(1) model(red).

4) Best order AR model
We use the absolution of difference between origin data and data from AR model as fitting error, the fitting error of different order can seen in fig.4 . As we can see, 8 order AR model has the minimum fitting error, so the best AR model order is 8.

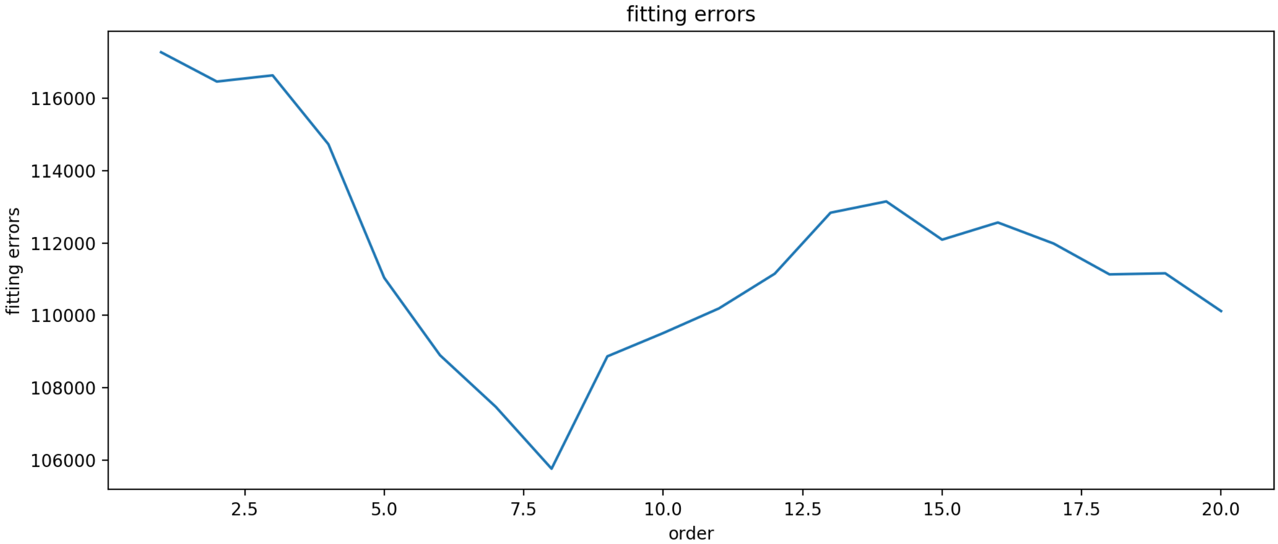


fig.4 fitting error of different AR model order

5) Plot
According to 4, the best AR model order is 8, so we figure out 8 order AR model, parameters are: a0=0.023826, a1=0.095265, a2=-0.012151, a3=0.002853, a4=-0.019528, a5=-0.042574, a6=-0.023492, a7=-0.014651, a8=-0.019640. The data, the data synthesised by using the 8 order AR model can see in fig.5. the error histogram is shown in fig.6 .

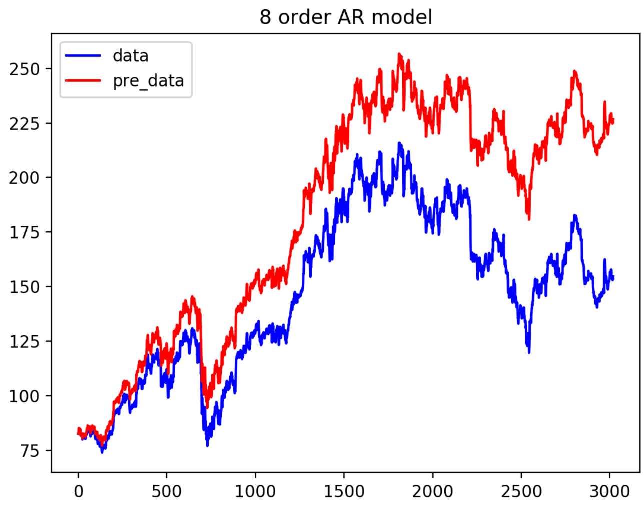


fig.5 best order AR model and raw data

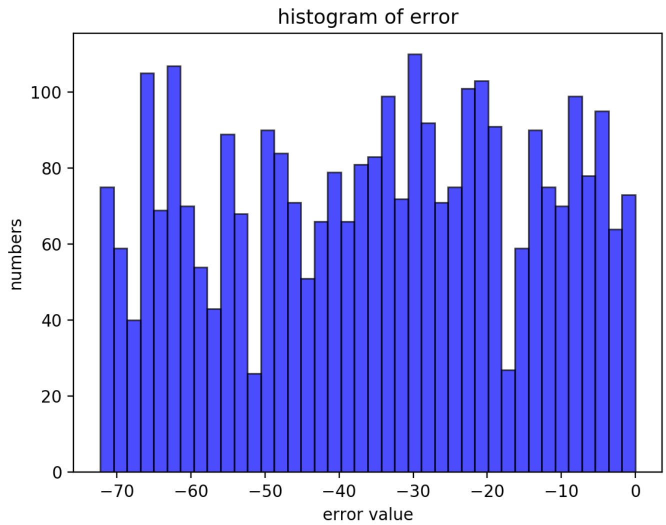


fig.6 histogram of error