Extension to conditional SMC

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May 9, 2018

We now consider extending the results of **?** to the case of conditional SMC. In particular, the SMC updates will be conditioned on a particular trajectory surviving. We concentrate on the exchangeable model, so we may take WLOG that the "immortal line" is the trajectory containing individual 1 from each generation. We first assume the simplest case, with multinomial resampling; analogous to the standard SMC case where

$$v_t^{(i)} \overset{d}{=} \text{Bin}(N, w_t^{(i)}), \qquad i = 1, \dots, N$$

yielding the coalescence rate

$$c_N(t) := \frac{1}{(N)_2} \sum_{i=1}^N \mathbb{E}\left[(v_t^{(i)})_2 \right] = \sum_{i=1}^N (w_t^{(i)})^2. \tag{1}$$

But now, since the first line is immortal, in each time step the first individual must have at least one offspring. The remaining $N-1$ offspring are assigned multinomially to the N possible parents as usual, giving the offspring numbers:

$$\begin{aligned} \tilde{v}_t^{(1)} &\overset{d}{=} 1 + \text{Bin}(N-1, w_t^{(1)}) \\ \tilde{v}_t^{(i)} &\overset{d}{=} \text{Bin}(N-1, w_t^{(i)}), \qquad i = 2, \dots, N. \end{aligned}$$

We therefore have the following moments:

$$\begin{aligned} &\mathbb{E}(\tilde{v}_t^{(i)}) = (N-1) w_t^{(i)} & &\text{Var}(\tilde{v}_t^{(i)}) = (N-1) w_t^{(i)} (1 - w_t^{(i)}) & &i = 2, \dots, N \\ &\mathbb{E}(\tilde{v}_t^{(1)}) = (N-1) w_t^{(1)} + 1 & &\text{Var}(\tilde{v}_t^{(1)}) = (N-1) w_t^{(1)} (1 - w_t^{(1)}) & & \end{aligned}$$

and we can derive the altered coalescence rate:

$$\begin{aligned} \tilde{c}_N(t) &= \frac{1}{(N)_2} \sum_{i=1}^N \mathbb{E}\left[(\tilde{v}_t^{(i)})_2 \right] \\ &= \frac{1}{(N)_2} \mathbb{E}\left[(\tilde{v}_t^{(1)})^2 - \tilde{v}_t^{(1)} \right] + \frac{1}{(N)_2} \sum_{i=2}^N \mathbb{E}\left[(\tilde{v}_t^{(i)})^2 - \tilde{v}_t^{(i)} \right] \\ &= \frac{1}{(N)_2} \sum_{i=2}^N (N-1)(N-2)(w_t^{(i)})^2 + (N^2 - 3N + 2)(w_t^{(1)})^2 + 2(N-1) w_t^{(1)} \\ &= \frac{1}{(N)_2} \sum_{i=1}^N (N-1)(N-2)(w_t^{(i)})^2 + 2(N-1) w_t^{(1)} \\ &= \frac{N-2}{N} c_N(t) + \frac{2}{N} w_t^{(1)} \end{aligned} \tag{2}$$

Since $w_t^{(1)} \leq 1$ for all t, as $N \to \infty$ we have

$$\tilde{c}_N(t) - c_N(t) = O(N^{-1})$$