

Extension to conditional SMC

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We now consider extending the results of Koskela et al. (2018) to the case of conditional SMC. In particular, the SMC updates will be conditioned on a particular trajectory surviving. We concentrate on the exchangeable model, so we may take WLOG that the “immortal line” is the trajectory containing individual 1 from each generation. We first assume the simplest case, with multinomial resampling; analogous to the standard SMC case where

$$v_t^{(i)} \stackrel{d}{=} \text{Bin}(N, w_t^{(i)}), \quad i = 1, \dots, N$$

yielding the coalescence rate

$$c_N(t) := \frac{1}{(N)_2} \sum_{i=1}^N \mathbb{E} \left[(v_t^{(i)})_2 \right] = \sum_{i=1}^N \mathbb{E} \left[(w_t^{(i)})^2 \right]. \quad (1)$$

But now, since the first line is immortal, in each time step the first individual must have at least one offspring. The remaining $N - 1$ offspring are assigned multinomially to the N possible parents as usual, giving the offspring numbers:

$$\begin{aligned} \tilde{v}_t^{(1)} &\stackrel{d}{=} 1 + \text{Bin}(N - 1, w_t^{(1)}) \\ \tilde{v}_t^{(i)} &\stackrel{d}{=} \text{Bin}(N - 1, w_t^{(i)}), \quad i = 2, \dots, N. \end{aligned}$$

We therefore have the following moments (using tower property):

$$\begin{aligned} \mathbb{E}[\tilde{v}_t^{(i)}] &= (N - 1)\mathbb{E}[w_t^{(i)}] \\ \mathbb{E}[(\tilde{v}_t^{(i)})^2] &= (N - 1)(N - 2)\mathbb{E}[(w_t^{(i)})^2] + (N - 1)\mathbb{E}[w_t^{(i)}] & i = 2, \dots, N \\ \mathbb{E}[\tilde{v}_t^{(1)}] &= (N - 1)\mathbb{E}[w_t^{(1)}] + 1 \\ \mathbb{E}[(\tilde{v}_t^{(1)})^2] &= (N - 1)(N - 2)\mathbb{E}[(w_t^{(1)})^2] + 3(N - 1)\mathbb{E}[w_t^{(1)}] + 1 \end{aligned}$$

and we can derive the altered coalescence rate:

$$\begin{aligned} \tilde{c}_N(t) &= \frac{1}{(N)_2} \sum_{i=1}^N \mathbb{E} \left[(\tilde{v}_t^{(i)})_2 \right] \\ &= \frac{1}{(N)_2} \mathbb{E} \left[(\tilde{v}_t^{(1)})^2 - \tilde{v}_t^{(1)} \right] + \frac{1}{(N)_2} \sum_{i=2}^N \mathbb{E} \left[(\tilde{v}_t^{(i)})^2 - \tilde{v}_t^{(i)} \right] \\ &= \frac{1}{(N)_2} \left[(N - 1)(N - 2)\mathbb{E}[(w_t^{(1)})^2] + 2(N - 1)\mathbb{E}[w_t^{(1)}] \right] + \frac{1}{(N)_2} \sum_{i=2}^N (N - 1)(N - 2)\mathbb{E}[(w_t^{(i)})^2] \\ &= \frac{1}{(N)_2} \sum_{i=1}^N (N - 1)(N - 2)\mathbb{E}[(w_t^{(i)})^2] + \frac{1}{(N)_2} 2(N - 1)\mathbb{E}[w_t^{(1)}] \\ &= \frac{N - 2}{N} c_N(t) + \frac{2}{N} \mathbb{E}[w_t^{(1)}] \end{aligned} \quad (2)$$

Under the conditions of Koskela et al. (2018, Corollary 2), we have that $\mathbb{E}[w_t^{(1)}] = O(N^{-1})$, and hence

$$\tilde{c}_N(t) - c_N(t) = O(N^{-2}).$$

Koskela et al. (2018) gives the following bounds on $c_N(t)$:

$$\frac{C_*}{N-1} \leq c_N(t) \leq \frac{C}{N-1}$$

Then, since $\tilde{c}_N(t)$ differs from $c_N(t)$ by $O(N^{-2})$, for sufficiently large N there exist constants \tilde{C}, \tilde{C}_* such that

$$\frac{\tilde{C}_*}{N-1} \leq \tilde{c}_N(t) \leq \frac{\tilde{C}}{N-1}$$

and we can thus derive bounds analogous to Koskela et al. (2018, (5)-(6)):

$$\frac{N-1}{\tilde{C}_*}t \leq \tilde{\tau}_N(t) \leq \frac{N-1}{\tilde{C}}t \tag{3}$$

$$\frac{N-1}{\tilde{C}_*}(s-t) \leq \tilde{\tau}_N(s) - \tilde{\tau}_N(t) \leq \frac{N-1}{\tilde{C}}(s-t) \tag{4}$$

References

Koskela, J., Jenkins, P. A., Johansen, A. M. and Spano, D. (2018), ‘Asymptotic genealogies of interacting particle systems with an application to sequential monte carlo’, *arXiv preprint arXiv:1804.01811* .