Extension to conditional SMC

Suzie Brown

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We now consider extending the results of Koskela et al. (2018) to the case of conditional SMC. In particular, the SMC updates will be conditioned on a particular trajectory surviving. We concentrate on the exchangeable model, so we may take WLOG that the "immortal line" is the trajectory containing individual 1 from each generation. We first assume the simplest case, with multinomial resampling; analogous to the standard SMC case where

$$v_t^{(i)} \stackrel{d}{=} \mathrm{Bin}(N, w_t^{(i)}), \qquad i = 1, \dots, N$$

yielding the coalescence rate

$$c_N(t) := \frac{1}{(N)_2} \sum_{i=1}^{N} \mathbb{E}\left[(v_t^{(i)})_2\right] = \sum_{i=1}^{N} \mathbb{E}\left[(w_t^{(i)})^2\right]. \tag{1}$$

But now, since the first line is immortal, in each time step the first individual must have at least one offspring. The remaining $N-1$ offspring are assigned multinomially to the N possible parents as usual, giving the offspring numbers:

$$\begin{aligned}
\tilde{v}_t^{(1)} &\stackrel{d}{=} 1 + \mathrm{Bin}(N-1, w_t^{(1)}) \\
\tilde{v}_t^{(i)} &\stackrel{d}{=} \mathrm{Bin}(N-1, w_t^{(i)}), \qquad i = 2, \dots, N.
\end{aligned}$$

We therefore have the following moments (using tower property):

$$\begin{aligned}
&\mathbb{E}[\tilde{v}_t^{(i)}] = (N-1)\mathbb{E}[w_t^{(i)}] \\
&\mathbb{E}[(\tilde{v}_t^{(i)})^2] = (N-1)(N-2)\mathbb{E}[(w_t^{(i)})^2] + (N-1)\mathbb{E}[w_t^{(i)}] \qquad\qquad i = 2, \dots, N \\
&\mathbb{E}[\tilde{v}_t^{(1)}] = (N-1)\mathbb{E}[w_t^{(1)}] + 1 \\
&\mathbb{E}[(\tilde{v}_t^{(1)})^2] = (N-1)(N-2)\mathbb{E}[(w_t^{(i)})^2] + 3(N-1)\mathbb{E}[w_t^{(1)}] + 1
\end{aligned}$$

and we can derive the altered coalescence rate:

$$\begin{aligned}
\tilde{c}_N(t) &= \frac{1}{(N)_2} \sum_{i=1}^{N} \mathbb{E}\left[(\tilde{v}_t^{(i)})_2\right] \\
&= \frac{1}{(N)_2} \mathbb{E}\left[(\tilde{v}_t^{(1)})^2 - \tilde{v}_t^{(1)}\right] + \frac{1}{(N)_2} \sum_{i=2}^{N} \mathbb{E}\left[(\tilde{v}_t^{(i)})^2 - \tilde{v}_t^{(i)}\right] \\
&= \frac{1}{(N)_2} \left[(N-1)(N-2)\mathbb{E}[(w_t^{(1)})^2] + 2(N-1)\mathbb{E}[w_t^{(1)}]\right] + \frac{1}{(N)_2} \sum_{i=2}^{N} (N-1)(N-2)\mathbb{E}[(w_t^{(i)})^2] \\
&= \frac{1}{(N)_2} \sum_{i=1}^{N} (N-1)(N-2)\mathbb{E}[(w_t^{(i)})^2] + \frac{1}{(N)_2} 2(N-1)\mathbb{E}[w_t^{(1)}] \\
&= \frac{N-2}{N} c_N(t) + \frac{2}{N} \mathbb{E}[w_t^{(1)}]
\end{aligned} \tag{2}$$

Since $w_t^{(1)} \leq 1$ for all t, as $N \to \infty$ we have

$$\tilde{c}_N(t) - c_N(t) = O(N^{-1})$$

Under the conditions on the weights $w_t^{(i)}$ assumed in Koskela et al. (2018), we have that $\mathbb{E}[w_t^{(1)}] = O(N^{-1})$, and hence

$$\tilde{c}_N(t) = \frac{N-2}{N} c_N(t) + O(N^{-2}) = c_N(t) + O\left(\frac{c_N(t)}{N}\right)$$

Koskela et al. (2018) gives the following bounds on $c_N(t)$:

$$\frac{C_*}{N-1} \leq c_N(t) \leq \frac{C}{N-1}$$

Then, since $\tilde{c}_N(t)$ differs from $c_N(t)$ by $O(N^{-1})$, for sufficiently large N there exist constants $\tilde{C}, \tilde{C}_*$ such that

$$\frac{\tilde{C}_*}{N-1} \leq \tilde{c}_N(t) \leq \frac{\tilde{C}}{N-1}$$

and we can thus derive bounds analogous to Koskela et al. (2018, (5)-(6)):

$$\frac{N-1}{\tilde{C}_*} t \leq \tilde{\tau}_N(t) \leq \frac{N-1}{\tilde{C}} t \tag{3}$$

$$\frac{N-1}{\tilde{C}_*}(s-t) \leq \tilde{\tau}_N(s) - \tilde{\tau}_N(t) \leq \frac{N-1}{\tilde{C}}(s-t) \tag{4}$$

References

Koskela, J., Jenkins, P. A., Johansen, A. M. and Spano, D. (2018), 'Asymptotic genealogies of interacting particle systems with an application to sequential monte carlo', *arXiv preprint arXiv:1804.01811* .