

# Residual resampling in SMC

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The offspring counts are sampled according to:

$$v_t^{(i)} = \lfloor Nw_t^{(i)} \rfloor + X_i$$

$$X_i \sim \text{Multinomial}(N - k, (\bar{w}_t^{(1)}, \dots, \bar{w}_t^{(N)}))$$

where  $k := \sum_{i=1}^N \lfloor Nw_t^{(i)} \rfloor$  is the number of offspring assigned deterministically, and  $\bar{w}_t^{(i)} := \frac{Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor}{N - k}$  are the residual weights. Let us also define the residuals  $r_i := Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor$ . So  $\sum_{i=1}^N r_i = N - k$ .

The coalescence rate is defined as

$$c_N(t) := \frac{1}{(N)_2} \sum_{i=1}^N (v_t^{(i)})_2.$$

We will use  $c_N^m(t)$  and  $c_N^r(t)$  to denote the coalescence rates with multinomial and residual resampling respectively. The expectation then comes out as

$$\begin{aligned} \mathbb{E}[(v_t^{(i)})_2 | \mathcal{F}_{t-1}] &= \mathbb{E}[(v_t^{(i)})^2 | \mathcal{F}_{t-1}] - \mathbb{E}[v_t^{(i)} | \mathcal{F}_{t-1}] \\ &= \mathbb{E}[\lfloor Nw_t^{(i)} \rfloor^2 | \mathcal{F}_{t-1}] + 2\mathbb{E}[\lfloor Nw_t^{(i)} \rfloor r_i | \mathcal{F}_{t-1}] + \mathbb{E}\left[r_i \left(1 - \frac{r_i}{N - k} + r_i\right) | \mathcal{F}_{t-1}\right] - \mathbb{E}[Nw_t^{(i)} | \mathcal{F}_{t-1}] \\ &= \mathbb{E}[\lfloor Nw_t^{(i)} \rfloor^2 | \mathcal{F}_{t-1}] - \mathbb{E}[\lfloor Nw_t^{(i)} \rfloor | \mathcal{F}_{t-1}] + 2\mathbb{E}[\lfloor Nw_t^{(i)} \rfloor r_i | \mathcal{F}_{t-1}] + \mathbb{E}\left[r_i^2 \left(1 - \frac{1}{N - k}\right) | \mathcal{F}_{t-1}\right] \\ &= \mathbb{E}[(Nw_t^{(i)})^2 | \mathcal{F}_{t-1}] - \mathbb{E}[\lfloor Nw_t^{(i)} \rfloor | \mathcal{F}_{t-1}] - \mathbb{E}\left[\frac{r_i^2}{N - k} | \mathcal{F}_{t-1}\right] \end{aligned}$$

so we get

$$\begin{aligned} \mathbb{E}[c_N^r(t) | \mathcal{F}_{t-1}] &= \frac{1}{(N)_2} \sum_{i=1}^N \mathbb{E}[(v_t^{(i)})_2 | \mathcal{F}_{t-1}] \\ &= \frac{N}{N - 1} \sum_{i=1}^N \mathbb{E}[(w_t^{(i)})^2 | \mathcal{F}_{t-1}] - \frac{1}{(N)_2} \sum_{i=1}^N \mathbb{E}\left[\frac{r_i^2}{N - k} | \mathcal{F}_{t-1}\right] - \frac{1}{(N)_2} \mathbb{E}[k | \mathcal{F}_{t-1}] \\ &= \mathbb{E}[c_N^m(t) | \mathcal{F}_{t-1}] \left(1 + \frac{1}{N - 1}\right) - \frac{1}{(N)_2} \mathbb{E}\left[\frac{\sum_{i=1}^N (Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor)^2}{\sum_{j=1}^N (Nw_t^{(j)} - \lfloor Nw_t^{(j)} \rfloor)} \middle| \mathcal{F}_{t-1}\right] \\ &\quad - \frac{1}{(N)_2} \mathbb{E}\left[\sum_{i=1}^N \lfloor Nw_t^{(i)} \rfloor \middle| \mathcal{F}_{t-1}\right] \end{aligned} \tag{1}$$

## Sanity check:

When the weights are all equal,  $w_t^{(i)} \equiv 1/N$ , we should have  $\mathbb{E}[c_N^r(t) | \mathcal{F}_{t-1}] = 0$  since each particle will have exactly one offspring so it is impossible for any lineages to coalesce. In this case we have  $\mathbb{E}[c_N^m(t) | \mathcal{F}_{t-1}] = \sum_{i=1}^N \mathbb{E}[(w_t^{(i)})^2 | \mathcal{F}_{t-1}] = 1/N$  for multinomial resampling. We also have that  $Nw_t^{(i)} \equiv \lfloor Nw_t^{(i)} \rfloor \equiv 1$  and hence  $r_i = 0$  and  $k = N$ . Thus the RHS comes out as

$$\frac{1}{N} \frac{N}{N - 1} - 0 - \frac{1}{(N)_2} N = \frac{1}{N - 1} - \frac{1}{N - 1} = 0$$

as expected.

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We can write it in a different form by combining the second and third terms of (1):

$$\begin{aligned}
& -\frac{1}{(N)_2} \mathbb{E} \left[ \frac{\sum_{i=1}^N (Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor)^2}{\sum_{j=1}^N (Nw_t^{(j)} - \lfloor Nw_t^{(j)} \rfloor)} \middle| \mathcal{F}_{t-1} \right] - \frac{1}{(N)_2} \mathbb{E} \left[ \sum_{i=1}^N \lfloor Nw_t^{(i)} \rfloor \middle| \mathcal{F}_{t-1} \right] \\
& =: -\frac{1}{(N)_2} \mathbb{E}[A | \mathcal{F}_{t-1}]
\end{aligned}$$

Then

$$\begin{aligned}
A &= \frac{\sum_{i=1}^N (Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor)^2}{\sum_{j=1}^N (Nw_t^{(j)} - \lfloor Nw_t^{(j)} \rfloor)} + \sum_{i=1}^N \lfloor Nw_t^{(i)} \rfloor \\
&= \frac{\sum_{i=1}^N (Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor)^2 + \sum_{i=1}^N \sum_{k=1}^N \lfloor Nw_t^{(k)} \rfloor (Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor)}{\sum_{j=1}^N (Nw_t^{(j)} - \lfloor Nw_t^{(j)} \rfloor)} \\
&=: \frac{A'}{\sum_{j=1}^N (Nw_t^{(j)} - \lfloor Nw_t^{(j)} \rfloor)}
\end{aligned}$$

Then

$$\begin{aligned}
A' &= \sum_{i=1}^N (Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor)^2 + \sum_{i=1}^N \sum_{k=1}^N \lfloor Nw_t^{(k)} \rfloor (Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor) \\
&= \sum_{i=1}^N \left\{ \left( Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor \right)^2 + \lfloor Nw_t^{(i)} \rfloor \left( Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor \right) + \sum_{k \neq i} \lfloor Nw_t^{(k)} \rfloor \left( Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor \right) \right\} \\
&= \sum_{i=1}^N \left\{ (Nw_t^{(i)})^2 - Nw_t^{(i)} \lfloor Nw_t^{(i)} \rfloor + \sum_{k \neq i} \lfloor Nw_t^{(k)} \rfloor \left( Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor \right) \right\} \\
&= \sum_{i=1}^N \left\{ \left( Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor \right) \left( Nw_t^{(i)} + \sum_{k \neq i} \lfloor Nw_t^{(k)} \rfloor \right) \right\}
\end{aligned}$$

So we have

$$\begin{aligned}
\mathbb{E}[c_N^r(t) | \mathcal{F}_{t-1}] &= \mathbb{E}[c_N^m(t) | \mathcal{F}_{t-1}] \left( 1 + \frac{1}{N-1} \right) \\
&\quad - \frac{1}{(N)_2} \mathbb{E} \left[ \frac{\sum_{i=1}^N \left( Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor \right) \left( Nw_t^{(i)} + \sum_{k \neq i} \lfloor Nw_t^{(k)} \rfloor \right)}{\sum_{j=1}^N \left( Nw_t^{(j)} - \lfloor Nw_t^{(j)} \rfloor \right)} \middle| \mathcal{F}_{t-1} \right]
\end{aligned}$$