Weak convergence proof v.2 (neater) (in progress)

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Lemma 1.

$$t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right) \binom{l}{2} (t+1)^{l-2} \leq \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \leq t^l + c_N(\tau_N(t))(t+1)^l. \tag{1}$$

Proof. As pointed out in Koskela et al. (2018, Equation (8)),

$$\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \geq \left(\sum_{s=0}^{\tau_N(t)} c_N(s)\right)^l - \binom{l}{2} \left(\sum_{s=0}^{\tau_N(t)} c_N(s)^2\right) \left(\sum_{s=0}^{\tau_N(t)} c_N(s)\right)^{l-2}. \tag{2}$$

By definition of τ_N,

$$t \leq \sum_{s=0}^{\tau_N(t)} c_N(s) \leq t+1. \tag{3}$$

Substituting these bounds into the RHS of (2) yields the lower bound.

It is a true fact that

$$\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \leq \left(\sum_{s=0}^{\tau_N(t)} c_N(s)\right)^l, \tag{4}$$

as can be seen by considering the multinomial expansion of the RHS. This is further bounded by

$$\left(\sum_{s=0}^{\tau_N(t)} c_N(s)\right)^l \leq \left(\sum_{s=0}^{\tau_N(t)-1} c_N(s) + c_N(\tau_N(t))\right)^l \leq [t + c_N(\tau_N(t))]^l, \tag{5}$$

again using the definition of τ_N. A binomial expansion yields

$$[t + c_N(\tau_N(t))]^l = t^l + \sum_{i=0}^{l-1} \binom{l}{i} t^i c_N(\tau_N(t))^{l-i} = t^l + c_N(\tau_N(t)) \sum_{i=0}^{l-1} \binom{l}{i} t^i c_N(\tau_N(t))^{l-1-i}, \tag{6}$$

then since $c_N(s) \leq 1$ for all s,

$$\sum_{i=0}^{l-1} \binom{l}{i} t^i c_N(\tau_N(t))^{l-1-i} \leq \sum_{i=0}^{l-1} \binom{l}{i} t^i \leq (t+1)^l. \tag{7}$$

Putting this together yields the upper bound. □

Lemma 2. *Let B be a positive constant which may depend on n.*

$$\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) + B D_N(s_j)] \leq \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) + \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right) (t+1)^{l-1}(1+B)^l. \tag{8}$$

Proof. We start with a binomial expansion:

$$\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) + BD_N(s_j)] = \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \sum_{\mathcal{I}\subseteq[l]} B^{l-|\mathcal{I}|} \left(\prod_{i\in\mathcal{I}} c_N(s_i)\right) \left(\prod_{j\notin\mathcal{I}} D_N(s_j)\right)$$
$$= \sum_{\mathcal{I}\subseteq[l]} B^{l-|\mathcal{I}|} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i\in\mathcal{I}} c_N(s_i)\right) \left(\prod_{j\notin\mathcal{I}} D_N(s_j)\right) \tag{9}$$

where $[l] := \{1, \dots, l\}$. Since the sum is over all permutations of $r_1, \dots, r_l$, we may arbitrarily choose an ordering for $\{1, \dots, l\}$ such that $\mathcal{I} = \{1, \dots, |\mathcal{I}|\}$:

$$\sum_{\mathcal{I}\subseteq[l]} B^{l-|\mathcal{I}|} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i\in\mathcal{I}} c_N(s_i)\right) \left(\prod_{j\notin\mathcal{I}} D_N(s_j)\right) = \sum_{I=0}^{l} \binom{l}{I} B^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right) \left(\prod_{j=I+1}^{l} D_N(s_j)\right). \tag{10}$$

Separating the term $I = l$,

$$\sum_{I=0}^{l} \binom{l}{I} B^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right) \left(\prod_{j=I+1}^{l} D_N(s_j)\right) = \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j)$$
$$+ \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right) \left(\prod_{j=I+1}^{l} D_N(s_j)\right). \tag{11}$$

In the second line, there is always at least one D_N term, and $c_N(s) \leq D_N(s)$ for all s, so we can write

$$\sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right) \left(\prod_{j=I+1}^{l} D_N(s_j)\right) \leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{l-1} c_N(s_i)\right) D_N(s_l)$$
$$\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \left(\sum_{s_1\neq\cdots\neq s_{l-1}}^{\tau_N(t)} \prod_{i=1}^{l-1} c_N(s_i)\right) \sum_{s_l=1}^{\tau_N(t)} D_N(s_l)$$
$$\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} (t+1)^{l-1} \sum_{s=1}^{\tau_N(t)} D_N(s) \tag{12}$$

using (4) and (3). Finally, by the Binomial Theorem,

$$\sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} (t+1)^{l-1} \sum_{s=1}^{\tau_N(t)} D_N(s) \leq \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right) (t+1)^{l-1} (1+B)^l, \tag{13}$$

which, together with (11), concludes the proof. $\square$

Lemma 3. *Let B be a positive constant which may depend on n.*

$$\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - BD_N(s_j)] \geq \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) - \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right) (t+1)^{l-1} (1+B)^l. \tag{14}$$

Proof. A binomial expansion and subsequent manipulation as in (9)–(11) gives

$$\begin{aligned}
\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - BD_N(s_j)] &= \sum_{\mathcal{I}\subseteq[l]} (-B)^{l-|\mathcal{I}|} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i\in\mathcal{I}} c_N(s_i)\right)\left(\prod_{j\notin\mathcal{I}} D_N(s_j)\right) \\
&= \sum_{I=0}^{l} \binom{l}{I} (-B)^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right)\left(\prod_{j=I+1}^{l} D_N(s_j)\right) \\
&= \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) + \sum_{I=0}^{l-1} \binom{l}{I} (-B)^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right)\left(\prod_{j=I+1}^{l} D_N(s_j)\right) \\
&\geq \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) - \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right)\left(\prod_{j=I+1}^{l} D_N(s_j)\right)
\end{aligned} \tag{15}$$

where the last inequality just multiplies some positive terms by -1. Then (12)–(13) can be applied directly (noting that an upper bound on negative terms gives a lower bound overall):

$$-\sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right)\left(\prod_{j=I+1}^{l} D_N(s_j)\right) \geq \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)(t+1)^{l-1}(1+B)^l \tag{16}$$

which concludes the proof. ☐

Lemma 4 (Basis step)**.** *For any* $0 < t < \infty$*,*

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1-p_r)\right] = e^{-\alpha_n t} \tag{17}$$

where $\alpha_n := n(n-1)/2$.

Proof. We start by showing that $\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right] \leq e^{-\alpha_n t}$.
From Koskela et al. (2018, Lemma 1 Case 1), taking $\xi = \Delta$, we have

$$1 - p_t = p_{\Delta\Delta}(t) \leq 1 - \alpha_n(1 + O(N^{-1}))\,[c_N(t) - B'_n D_N(t)] \tag{18}$$

where the $O(N^{-1})$ term does not depend on t. Applying a multinomial expansion and then separating the positive and negative terms,

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1-p_r) &\leq 1 + \sum_{l=1}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1}))\frac{1}{l!} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - B'_n D_N(s_j)] \\
&= 1 + \sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1}))\frac{1}{l!} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - B'_n D_N(s_j)] \\
&\quad - \sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1}))\frac{1}{l!} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - B'_n D_N(s_j)]\,.
\end{aligned} \tag{19}$$

This is further bounded by applying Lemma 3 and then both bounds of Lemma 1:

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)}(1-p_r) &\leq 1 + \sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1}))\frac{1}{l!} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \\
&\quad - \sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1}))\frac{1}{l!} \left\{ \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) - \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)(t+1)^{l-1}(1+B_n')^l \right\} \\
&\leq 1 + \sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1}))\frac{1}{l!}\left\{t^l + c_N(\tau_N(t))(t+1)^l\right\} \\
&\quad - \sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1}))\frac{1}{l!} \left\{ t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right)\binom{l}{2}(t+1)^{l-2} - \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)(t+1)^{l-1}(1+B_n')^l \right\}.
\end{aligned} \tag{20}$$

A bit of tidying up and we have

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)}(1-p_r) &\leq \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1}))\frac{1}{l!}t^l + c_N(\tau_N(t)) \sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1}))\frac{1}{l!}(t+1)^l \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right) \sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1}))\frac{1}{l!}\binom{l}{2}(t+1)^{l-2} \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right) \sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1}))\frac{1}{l!}(t+1)^{l-1}(1+B_n')^l \\
&\leq \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1}))\frac{1}{l!}t^l + c_N(\tau_N(t)) \exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right) \frac{1}{2}\alpha_n^2 \exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right) \exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')].
\end{aligned} \tag{21}$$

Now, taking the expectation and limit, and applying Brown et al. (2021, Equations (3.3)–(3.5)) and Lemma ?? (Lemma 2 in the messy weakconv note; not written up here yet),

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right] &\leq \sum_{l=0}^{\infty}(-\alpha_n)^l \frac{1}{l!}t^l \lim_{N\to\infty}\mathbb{P}\left[\tau_N(t)\geq l\right] + \lim_{N\to\infty}\mathbb{E}\left[c_N(\tau_N(t))\right]\exp[\alpha_n(t+1)] \\
&\quad + \lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right]\frac{1}{2}\alpha_n^2\exp[\alpha_n(t+1)] \\
&\quad + \lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} D_N(s)\right]\exp[\alpha_n(t+1)(1+B_n')] \\
&= \sum_{l=0}^{\infty}(-\alpha_n)^l\frac{1}{l!}t^l = e^{-\alpha_n t}.
\end{aligned} \tag{22}$$

It remains to show that $\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right] \geq e^{-\alpha_n t}$.
From Brown et al. (2021, Equation (3.14)), taking $\xi = \Delta$, we have

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \alpha_n(1 + O(N^{-1}))\left[c_N(t) + B_n D_N(t)\right] \tag{23}$$

where $B_n > 0$ and the $O(N^{-1})$ term does not depend on t. In particular,

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \frac{N^{n-2}}{(N-2)_{n-2}}\alpha_n c_N(t) - \frac{N^{n-3}}{(N-3)_{n-3}} B_n D_N(t). \tag{24}$$

Since $D_N(s) \leq c_N(s)$ for all s (Koskela et al., 2018, p.9), a sufficient condition for this bound to be non-negative is

$$E_r := \left\{ c_N(r) \leq \frac{(N-3)_{n-3}}{N^{n-3}} \left(\alpha_n \left(1 + \frac{2}{N-2}\right) + B_n \right)^{-1} \right\}, \tag{25}$$

and we define $E := \bigcap_{r=1}^{\tau_N(t)} E_r$. We now apply a multinomial expansion to the product, and split into positive and negative terms:

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)}(1-p_r) &\geq \left\{ 1 + \sum_{l=1}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} \left[c_N(s_j) + B_n D_N(s_j)\right] \right\} \mathbb{1}_E \\
&= \left\{ 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} \left[c_N(s_j) + B_n D_N(s_j)\right] \right. \\
&\qquad \left. - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} \left[c_N(s_j) + B_n D_N(s_j)\right] \right\} \mathbb{1}_E
\end{aligned} \tag{26}$$

This is further bounded by applying Lemma 2 and both bounds in Lemma 1:

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)}(1-p_r) &\geq \left\{ 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \right. \\
&\qquad \left. - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left[\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1 + B_n)^l \right] \right\} \mathbb{1}_E \\
&\geq \left\{ 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left[t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \binom{l}{2} (t+1)^{l-2} \right] \right. \\
&\qquad \left. - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left[t^l + c_N(\tau_N(t))(t+1)^l + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1 + B_n)^l \right] \right\} \mathbb{1}_E.
\end{aligned} \tag{27}$$

Tidying things up,

$$
\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1-p_r) \geq& \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1}))\frac{1}{l!}t^l \mathbb{1}_E - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right) \sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1}))\frac{1}{l!}\binom{l}{2}(t+1)^{l-2} \\
&- c_N(\tau_N(t)) \sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1}))\frac{1}{l!}(t+1)^l \\
&- \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right) \sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1}))\frac{1}{l!}(t+1)^{l-1}(1+B_n)^l \\
\geq& \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1}))\frac{1}{l!}t^l \mathbb{1}_E - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right) \frac{1}{2}\alpha_n^2 \exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
&- c_N(\tau_N(t)) \exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
&- \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right) \exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n)]. \qquad (28)
\end{aligned}
$$

Now, taking the expectation and limit, and applying Brown et al. (2021, Equations (3.3)–(3.5)) and Lemma ?? (Lemma 2 in the messy weakconv note; not written up here yet),

$$
\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1-p_r)\right] \geq& \sum_{l=0}^{\infty} (-\alpha_n)^l (1+O(N^{-1}))\frac{1}{l!}t^l \lim_{N\to\infty} \mathbb{P}\left[\{\tau_N(t)\geq l\}\cap E\right] \\
&- \lim_{N\to\infty} \mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right] \frac{1}{2}\alpha_n^2 \exp[\alpha_n(t+1)] \\
&- \lim_{N\to\infty} \mathbb{E}\left[c_N(\tau_N(t))\right] \exp[\alpha_n(t+1)] \\
&- \lim_{N\to\infty} \mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} D_N(s)\right] \exp[\alpha_n(t+1)(1+B_n)] \\
=& \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l = e^{-\alpha_n t}. \qquad (29)
\end{aligned}
$$

Combining the upper and lower bounds in (22) and (29) respectively concludes the proof. □

References

Brown, S., Jenkins, P. A., Johansen, A. M. and Koskela, J. (2021), 'Simple conditions for convergence of sequential Monte Carlo genealogies with applications', *Electronic Journal of Probability* **26**(1), 1–22.

Koskela, J., Jenkins, P. A., Johansen, A. M. and Spanò, D. (2018), Asymptotic genealogies of interacting particle systems with an application to sequential Monte Carlo, Mathematics e-print 1804.01811, ArXiv.