Corollaries 1–3 proofs with details...

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Definition 1. A function f is said to be i-increasing if it is an increasing function in $v_t^{(i)} = |\{j : a_t^{(j)} = i\}|$.

Multinomial resampling

Corollary 1. *Under the time scaling (**??**), supposing there exist constants* $0 < \varepsilon \leq 1 \leq a < \infty$ *such that*

$$\frac{1}{a} \leq g_t(x, x') \leq a \tag{1}$$

$$\varepsilon h(x') \leq q_t(x, x') \leq \frac{1}{\varepsilon} h(x'), \tag{2}$$

genealogies of SMC algorithms with multinomial resampling converge to Kingman's n*-coalescent in the sense of finite-dimensional distributions as* $N \to \infty$.

Lemma 1. *Let* $a_t^{(i)}$ *be the parental indices from a SMC algorithm with multinomial resampling. For any function* f *that is* i*-increasing,*

$$\mathbb{E}[f(\mathbf{a}_t) \mid \mathcal{H}_t] \leq \mathbb{E}[f(\mathbf{A}_1)]$$
$$\mathbb{E}[f(\mathbf{a}_t) \mid \mathcal{H}_t] \geq \mathbb{E}[f(\mathbf{A}_2)]$$

where the elements of $\mathbf{A}_1, \mathbf{A}_2$ *are all mutually independent and independent of* $\mathcal{F}_\infty$*, and distributed according to*

$$A_1^{(j)} \sim \text{Categorical}\left(\left(\frac{a}{\varepsilon}\right)^{\mathbb{1}_{\{i=1\}} - \mathbb{1}_{\{i \neq 1\}}}, \dots, \left(\frac{a}{\varepsilon}\right)^{\mathbb{1}_{\{i=N\}} - \mathbb{1}_{\{i \neq N\}}} \right)$$
$$A_2^{(j)} \sim \text{Categorical}\left(\left(\frac{\varepsilon}{a}\right)^{\mathbb{1}_{\{i=1\}} - \mathbb{1}_{\{i \neq 1\}}}, \dots, \left(\frac{\varepsilon}{a}\right)^{\mathbb{1}_{\{i=N\}} - \mathbb{1}_{\{i \neq N\}}} \right),$$

where the arguments of Categorical and Multinomial distributions are given up to a normalising constant here and throughout this document.

Proof. The result follows using the bounds given in equations (1), (2) with a balls-in-bins coupling, and cancelling h from the top and bottom. $\square$

Define the corresponding "family sizes" $V_1^{(i)} := |\{j : A_1^{(j)} = i\}|$ and $V_2^{(i)} := |\{j : A_2^{(j)} = i\}|$ for $i = 1, \dots, N$. The distributions of $\mathbf{A}_1, \mathbf{A}_2$ imply the following:

$$\mathbf{V}_1 \sim \text{Multinomial}\left(N, \left(\frac{a}{\varepsilon}\right)^{\mathbb{1}_{\{i=1\}} - \mathbb{1}_{\{i \neq 1\}}}, \dots, \left(\frac{a}{\varepsilon}\right)^{\mathbb{1}_{\{i=N\}} - \mathbb{1}_{\{i \neq N\}}} \right)$$
$$\mathbf{V}_2 \sim \text{Multinomial}\left(N, \left(\frac{\varepsilon}{a}\right)^{\mathbb{1}_{\{i=1\}} - \mathbb{1}_{\{i \neq 1\}}}, \dots, \left(\frac{\varepsilon}{a}\right)^{\mathbb{1}_{\{i=N\}} - \mathbb{1}_{\{i \neq N\}}} \right),$$

Notice that the function $f_i(\mathbf{a}_t) := (v_t^{(i)})_2$ is i-increasing for each $i = 1, \dots, N$. Applying Lemma 1 and the Multinomial moments formula (Mosimann, 1962), we obtain the following lower bound:

$$\mathbb{E}_t[f_i(\mathbf{a}_t)] \geq \mathbb{E}[f_i(\mathbf{A}_2)] = \mathbb{E}[(V_2^{(i)})_2]$$
$$= \frac{(N)_2 (\varepsilon/a)^2}{[(\varepsilon/a) + (N-1)(a/\varepsilon)]^2} \geq \frac{(N)_2 (\varepsilon/a)^2}{N^2 (a/\varepsilon)^2} = \frac{(N)_2}{N^2} \frac{\varepsilon^4}{a^4}.$$

So we can lower bound the denominator by

$$\mathbb{E}_t[c_N(t)] = \frac{1}{(N)_2}\sum_{i=1}^{N}\mathbb{E}_t[(v_t^{(i)})_2] \geq \frac{N}{(N)_2}\frac{(N)_2}{N^2}\frac{\varepsilon^4}{a^4} = \frac{\varepsilon^4}{Na^4}.$$

To upper bound the numerator, consider the function $f_i(\mathbf{a}_t) := (v_t^{(i)})_3$, which is i-increasing for each $i = 1, \dots, N$. Again using Lemma 1 and (Mosimann, 1962), we obtain the following lower bound:

$$\begin{aligned}\mathbb{E}_t[f_i(\mathbf{a}_t)] &\leq \mathbb{E}[f_i(\mathbf{A}_1)] = \mathbb{E}[(V_1^{(i)})_3] \\ &= \frac{(N)_3(a/\varepsilon)^3}{[(a/\varepsilon)+(N-1)(\varepsilon/a)]^3} \geq \frac{(N)_3(a/\varepsilon)^3}{N^3(\varepsilon/a)^3} = \frac{(N)_3}{N^3}\frac{a^6}{\varepsilon^6}.\end{aligned}$$

and the numerator is therefore bounded above by

$$\frac{1}{(N)_3}\sum_{i=1}^{N}\mathbb{E}_t[(v_t^{(i)})_3] \leq \frac{N}{(N)_3}\frac{(N)_3}{N^3}\frac{\varepsilon^6}{a^6} = \frac{\varepsilon^6}{N^2a^6}.$$

The ratio is therefore bounded above by

$$\frac{\frac{1}{(N)_3}\sum_{i=1}^{N}\mathbb{E}_t[(v_t^{(i)})_3]}{\frac{1}{(N)_2}\sum_{i=1}^{N}\mathbb{E}_t[(v_t^{(i)})_2]} \leq \frac{N}{(N)_3}\frac{(N)_3}{N^3}\frac{\varepsilon^6}{a^6} = \frac{\varepsilon^6}{N^2a^6}\frac{Na^4}{\varepsilon^4} = \frac{a^{10}}{N\varepsilon^{10}} =: b_N \lim_{N\to\infty} 0.$$

We can thus conclude the proof of Corollary 1 by applying Theorem 1.

Conditional SMC with multinomial resampling

References

Mosimann, J. E. (1962), 'On the compound multinomial distribution, the multivariate β-distribution, and correlations among proportions', *Biometrika* **49**(1/2), 65–82.