Weak convergence proof: an aside

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October 2, 2020

Lemma 1. *For any $n \leq N \in \mathbb{N}$, for all $t > 0$, define*

$$E_t := \left\{ c_N(t) < \frac{(N-3)_{n-3}}{N^{n-3}} \left(\alpha \left(1 + \frac{2}{N-2} \right) + B_n \right)^{-1} \right\} \tag{1}$$

where α and B_n are positive constants as in equation whatever. Then, for all $t > 0$,

$$\lim_{N\to\infty} \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} E_r \right] = 1. \tag{2}$$

Proof.

$$\mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} E_r \right] = 1 - \mathbb{P}\left[\bigcup_{r=1}^{\tau_N(t)} E_r^c \right] \geq 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{P}[E_r^c] \right]. \tag{3}$$

Using the generalised Markov inequality,

$$\begin{aligned} \mathbb{P}[E_r^c] &= \mathbb{P}\left[c_N(t) \geq \frac{(N-3)_{n-3}}{N^{n-3}} \left(\alpha \left(1 + \frac{2}{N-2} \right) + B_n \right)^{-1} \right] \\ &\leq \mathbb{E}[c_N(r)^2] \frac{N^{2(n-3)}}{(N-3)_{n-3}^2} \left(\alpha \left(1 + \frac{2}{N-2} \right) + B_n \right)^2. \end{aligned} \tag{4}$$

Now

$$\begin{aligned} \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} E_r \right] &\geq 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[c_N(r)^2] \frac{N^{2(n-3)}}{(N-3)_{n-3}^2} \left(\alpha \left(1 + \frac{2}{N-2} \right) + B_n \right)^2 \right] \\ &= 1 - \frac{N^{2(n-3)}}{(N-3)_{n-3}^2} \left(\alpha \left(1 + \frac{2}{N-2} \right) + B_n \right)^2 \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} c_N(r)^2 \right] \\ &\xrightarrow{N\to\infty} 1 - (\alpha + B_n)^2 \times 0 = 1. \end{aligned} \tag{5}$$

□