# Stratified resampling ++

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THIS DOCUMENT IS OBSOLETE. For the most up-to-date version of these calculations, see thesis Background & Applications chapters. —SB

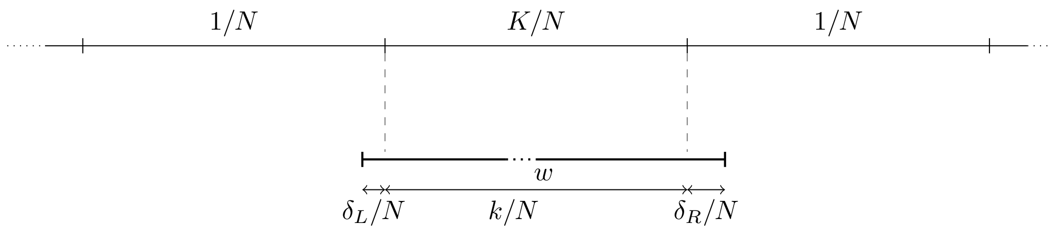
(A1) The conditional distribution of parental indices $a_t^{(1:N)}$ given offspring counts $\nu_t^{(1:N)}$ is uniform over all assignments such that $|\{j : a_t^{(j)} = i\}| = \nu_t^{(i)}$ for all $i$.

There are complex dependencies between the offspring counts, but we can still find some constraints on the distribution of each count conditional on the corresponding weight. Write the $i^{th}$ weight in the form $w_t^{(i)} = (K + \delta)/N$, where $\delta \in [0, 1)$ and $K \in \{0, \dots, N - 1\}$. The distribution of $\nu_t^{(i)}$ depends not only on $w_t^{(i)}$ but also on where the $i^{th}$ weight interval falls with respect to the length-$(1/N)$ intervals for inversion sampling. There are two cases to consider, which are illustrated in Figure 1. Note that Case (b) cannot happen if $K = 0$.

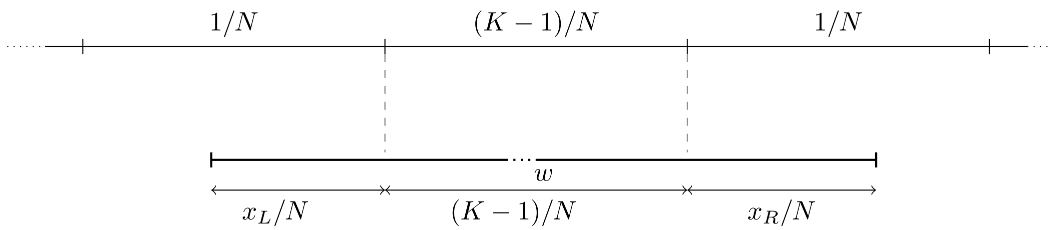
In any case $\nu_t^{(i)} \in \{k - 1, k, k + 1, k + 2\}$ almost surely. To define a probability distribution over these four values, we introduce the notation $p_j := \mathbb{P}[\nu_t^{(i)} = \lfloor N w_t^{(i)} \rfloor + j \mid w_t^{(i)}]$, for $j = -1, 0, 1, 2$. Since the sample within each interval of length $1/N$ is uniform over that interval, we find the probabilities given in Table 1, in terms of $\delta$ and the other quantities $\delta_L, \delta_R \in [0, \delta]$ and $x_L, x_R \in [\delta, 1]$ defined in Figure 1. The probabilities do not depend on $k$, but of course the corresponding values of $\nu_t^{(i)}$ do. By definition $\delta_L + \delta_R = \delta$ and $x_L + x_R = 1 + \delta$.

| | Case (a) | Case (b) | L.B. | U.B. |
|---|---|---|---|---|
| $p_{-1}$ | $0$ | $x_L x_R - \delta$ | $0$ | $1/4$ |
| $p_0$ | $1 - \delta + \delta_L \delta_R$ | $1 + \delta - 2x_L x_R$ | $(1 - \delta)^2/2$ | $1 - 3\delta/4$ |
| $p_1$ | $\delta - 2\delta_L \delta_R$ | $x_L x_R$ | $\delta/2$ | $(1 + \delta)/2$ |
| $p_2$ | $\delta_L \delta_R$ | $0$ | $0$ | $1/4$ |

Table 1: Marginal probability distribution of $\nu_t^{(i)}$ conditional on $w_t^{(i)}$, in terms of $\delta$ and the quantities defined in Figure 1, along with upper and lower bounds on these in terms of $\delta$ only, which hold in both cases i.e. whenever $w_t^{(i)} = (K + \delta)/N$.




(a) The parent under consideration is automatically assigned $K$ offspring, plus up to two more. ($\delta_L + \delta_R = \delta$.)




(b) This case can only occur when $K \geq 1$. The parent under consideration is automatically assigned $K-1$ offspring, plus up to two more. ($x_L + x_R = 1 + \delta$.)

Figure 1: Cases for stratified resampling with a fixed weight $w = (K+\delta)/N$

**Theorem 1.** *Let $\nu_t^{(1:N)}$ denote the offspring numbers in an interacting particle system satisfying (A1) such that, for any $N$ sufficiently large, $\mathbb{P}[\tau_N(t) = \infty] = 0$ for all finite $t$. Suppose that there exists a deterministic sequence $(b_N)_{N \geq 1}$ such that $\lim_{N\to\infty} b_N = 0$ and*

$$\frac{1}{(N)_3} \sum_{i=1}^{N} \mathbb{E}_t[(\nu_t^{(i)})_3] \leq b_N \frac{1}{(N)_2} \sum_{i=1}^{N} \mathbb{E}_t[(\nu_t^{(i)})_2] \tag{1}$$

*for all $N$, uniformly in $t \geq 1$. Fix $n \leq N$ and consider a randomly chosen sample of $n$ terminal particles. Then the resulting rescaled genealogical process $(G^{(n,N)}_{\tau_N(t)})_{t\geq 0}$ converges in the sense of finite-dimensional distributions to Kingman's $n$-coalescent as $N \to \infty$.*

**Corollary 2.** *Consider an SMC algorithm using stratified resampling, such that (A1) is satisfied. Assume that there exists a constant $a \in [1, \infty)$ such that for all $x, x', t$,*

$$\frac{1}{a} \leq g_t(x, x') \leq a. \tag{2}$$

*Assume that $\mathbb{P}[\tau_N(t) = \infty] = 0$ for all finite $t$. Let $(G_t^{(n,N)})_{t\geq 0}$ denote the genealogy of a random sample of $n$ terminal particles from the output of the algorithm when the total number of particles used is $N$. Then, for any fixed $n$, the time-scaled genealogy $(G_{\tau_N(t)}^{(n,N)})_{t\geq 0}$ converges to Kingman's $n$-coalescent as $N \to \infty$, in the sense of finite-dimensional distributions.*

*Proof.* Recall that the sequence of $\sigma$-algebras

$$\mathcal{H}_t := \sigma(X_{t-1}^{(1:N)}, X_t^{(1:N)}, w_{t-1}^{(1:N)}, w_t^{(1:N)}) \tag{3}$$

are such that $\nu_t^{(1:N)}$ is conditionally independent of the filtration $\mathcal{F}_{t-1}$ given $\mathcal{H}_t$. With stratified resampling, conditional on the weights each offspring count almost surely takes one of four values: $\nu_t^{(i)} \in \{\lfloor N w_t^{(i)} \rfloor - 1, \lfloor N w_t^{(i)} \rfloor, \lfloor N w_t^{(i)} \rfloor + 1, \lfloor N w_t^{(i)} \rfloor + 2\}$. Denote $p_j^{(i)} := \mathbb{P}[\nu_t^{(i)} = \lfloor N w_t^{(i)} \rfloor + j \mid \mathcal{H}_t]$ for $j = -1, 0, 1, 2$. Now

$$\begin{aligned}\mathbb{E}[(\nu_t^{(i)})_2 \mid \mathcal{H}_t] = p_{-1}^{(i)}(\lfloor N w_t^{(i)} \rfloor - 1)_2 + p_0^{(i)}(\lfloor N w_t^{(i)} \rfloor)_2 + p_1^{(i)}(\lfloor N w_t^{(i)} \rfloor + 1)_2 \\ + p_2^{(i)}(\lfloor N w_t^{(i)} \rfloor + 2)_2\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}[(\nu_t^{(i)})_3 \mid \mathcal{H}_t] &= p_{-1}^{(i)}(\lfloor N w_t^{(i)} \rfloor - 1)_3 + p_0^{(i)}(\lfloor N w_t^{(i)} \rfloor)_3 + p_1^{(i)}(\lfloor N w_t^{(i)} \rfloor + 1)_3 \\ &\quad + p_2^{(i)}(\lfloor N w_t^{(i)} \rfloor + 2)_3 \\ &= p_{-1}^{(i)}(\lfloor N w_t^{(i)} \rfloor - 3)(\lfloor N w_t^{(i)} \rfloor - 1)_2 + p_0^{(i)}(\lfloor N w_t^{(i)} \rfloor - 2)(\lfloor N w_t^{(i)} \rfloor)_2 \\ &\quad + p_1^{(i)}(\lfloor N w_t^{(i)} \rfloor - 1)(\lfloor N w_t^{(i)} \rfloor + 1)_2 + p_2^{(i)}\lfloor N w_t^{(i)} \rfloor(\lfloor N w_t^{(i)} \rfloor + 2)_2 \\ &\leq \lfloor N w_t^{(i)} \rfloor \{p_0^{(i)}(\lfloor N w_t^{(i)} \rfloor)_2 + p_1^{(i)}(\lfloor N w_t^{(i)} \rfloor + 1)_2\} \\ &= \lfloor N w_t^{(i)} \rfloor \mathbb{E}[(\nu_t^{(i)})_2 \mid \mathcal{H}_t] \\ &\leq a^2 \mathbb{E}[(\nu_t^{(i)})_2 \mid \mathcal{H}_t]\end{aligned}$$

The last line uses the almost sure bound $w_t^{(i)} \leq a^2/N$ which follows from (2) along with the form of the weights in Algorithm **??**. Note that some terms in the above expressions may be equal to zero when $w_t^{(i)}$ is small enough, but the bound always holds nonetheless. Since the above holds for all $i$, applying the tower rule we have

$$\frac{1}{(N)_3} \sum_{i=1}^{N} \mathbb{E}_t[(\nu_t^{(i)})_3] \leq \frac{a^2}{N-2} \frac{1}{(N)_2} \sum_{i=1}^{N} \mathbb{E}_t[(\nu_t^{(i)})_2]$$

satisfying (1) with $b_N := a^2/(N-2) \to 0$. The result then follows by applying Theorem 1. ■

**Lemma 3.** *Consider an SMC algorithm using stratified resampling. Suppose that*

$$\varepsilon \leq q_t(x, x') \leq \varepsilon^{-1}$$

*uniformly in* $x, x'$ *for some* $\varepsilon \in (0, 1]$*, and that there exist* $\zeta > 0$ *and* $\delta \in (0, 1)$ *such that*

$$\mathbb{P}[\max_i w_t^{(i)} - \min_i w_t^{(i)} \geq 2\delta/N \mid \mathcal{F}_{t-1}] \geq \zeta$$

*for infinitely many* $t$*. Then, for all* $N > 1$*,* $\mathbb{P}[\tau_N(t) = \infty] = 0$ *for all finite* $t$*.*

*Proof.* It is sufficient by a Borel-Cantelli argument, which is written somewhere else —SB to prove that under the stated conditions

$$\sum_{r=0}^{\infty} \mathbb{P}[c_N(r) > 2/N^2 \mid \mathcal{F}_{r-1}] = \infty.$$

Firstly,

$$\begin{aligned} \mathbb{P}[c_N(t) \leq 2/N^2 \mid \mathcal{H}_t] &= \mathbb{P}[c_N(t) = 0 \mid \mathcal{H}_t] = \mathbb{P}[\nu_t^{(i)} = 1 \, \forall i \in \{1, \dots, N\} \mid \mathcal{H}_t] \\ &\leq \mathbb{P}[\nu_t^{(i^\star)} = 1 \mid \mathcal{H}_t], \end{aligned} \tag{4}$$

where $i^\star := \operatorname{argmax}_i \{w_t^{(i)}\}$ (but note that the inequality holds when $i^\star$ is taken to be any particular index). Define for any $k \in \mathbb{Z}$

$$p_k^{(i)} := \mathbb{P}\left[\nu_t^{(i)} = \lfloor N w_t^{(i)} \rfloor + k \,\middle|\, \mathcal{H}_t\right].$$

Since stratified resampling almost surely results in $\nu_t^{(i)} \in \{\lfloor N w_t^{(i)} \rfloor - 1, \lfloor N w_t^{(i)} \rfloor, \lfloor N w_t^{(i)} \rfloor + 1, \lfloor N w_t^{(i)} \rfloor + 2\}$ we have that $p_k^{(i)} \equiv 0$ for $k \notin \{-1, 0, 1, 2\}$, and

$$\sum_{k=-1}^{2} p_k^{(i)} = \sum_{k=-1}^{2} \mathbb{P}\left[\nu_t^{(i)} = \lfloor N w_t^{(i)} \rfloor + k \,\middle|\, w_t^{(1:N)}\right] = 1.$$

Up to a proportionality constant $C$,

$$\begin{aligned} p_k^{(i)} = C \, \mathbb{P}&\left[\nu_t^{(i)} = \lfloor N w_t^{(i)} \rfloor + k \,\middle|\, w_t^{(1:N)}\right] \\ &\times \sum_{\substack{a_{1:N} \in \{1, \dots, N\}^N : \\ |\{j : a_j = i\}| = \lfloor N w_t^{(i)} \rfloor + k}} \mathbb{P}\left[a_t^{(1:N)} = a_{1:N} \,\middle|\, \nu_t^{(i)}, w_t^{(1:N)}\right] \prod_{j=1}^{N} q_{t-1}(X_t^{(a_j)}, X_{t-1}^{(j)}) \end{aligned}$$

for each $k \in \{-1, 0, 1, 2\}$. We can bound each probability above and below using the almost sure bounds on $q_{t-1}$ stated in the Lemma:

$$C \, \mathbb{P}\left[\nu_t^{(i)} = \lfloor N w_t^{(i)} \rfloor + k \,\middle|\, w_t^{(1:N)}\right] \varepsilon^N \leq p_k^{(i)} \leq C \, \mathbb{P}\left[\nu_t^{(i)} = \lfloor N w_t^{(i)} \rfloor + k \,\middle|\, w_t^{(1:N)}\right] \varepsilon^{-N}$$

then eliminate the constant $C$ by normalising, to obtain lower bounds

$$\begin{aligned} p_k^{(i)} &\geq \frac{C\,\mathbb{P}[\nu_t^{(i)} = \lfloor N w_t^{(i)} \rfloor + k \mid w_t^{(1:N)}]\varepsilon^N}{\sum_{j=-1}^{2} C\,\mathbb{P}[\nu_t^{(i)} = \lfloor N w_t^{(i)} \rfloor + j \mid w_t^{(1:N)}]\varepsilon^{-N}} \\ &= \mathbb{P}[\nu_t^{(i)} = \lfloor N w_t^{(i)} \rfloor + k \mid w_t^{(1:N)}]\varepsilon^{2N}. \end{aligned} \tag{5}$$

Suppose that $\max_i w_t^{(i)} - \min_i w_t^{(i)} \geq 2\delta/N$. Then that at least one of $\{\max_i w_t^{(i)} \geq (1+\delta)/N\}$ and $\{\min_i w_t^{(i)} \leq (1-\delta)/N\}$ occurs. We will now examine each of these possibilities.

We can always write the maximum weight as $w_t^{(i^\star)} = \frac{1+\gamma}{N}$ for some $\gamma \geq 0$. Then, using (4),

$$\mathbb{P}[c_N(t) > 2/N^2 \mid \mathcal{H}_t] \geq 1 - \mathbb{P}[\nu_t^{(i^\star)} = 1 \mid \mathcal{H}_t] = \begin{cases} 0 & \text{if } \gamma = 0 \\ 1 - p_0^{(i^\star)} & \text{if } \gamma \in (0,1) \\ 1 - p_{-1}^{(i^\star)} & \text{if } \gamma \in [1,2) \\ 1 & \text{if } \gamma \geq 2. \end{cases}$$

If $\gamma \in (0,1)$ then

$$1 - p_0^{(i^\star)} \geq \frac{3\gamma\varepsilon^{2N}}{4}$$

using (5) and Table 1 ($p_0$, U.B.). Similarly, if $\gamma \in [1,2)$ then by Table 1 ($p_{-1}$, U.B.),

$$1 - p_{-1}^{(i^\star)} \geq \left(1 - \frac{1}{4}\right)\varepsilon^{2N} \geq \frac{3\varepsilon^{2N}}{4}.$$

So overall, under the constraint $\max_i w_t^{(i)} \geq (1+\delta)/N$, we have

$$\mathbb{P}[c_N(t) > 2/N^2 \mid \mathcal{H}_t] \geq \min_{\gamma\geq\delta} \left\{ \frac{3\gamma\varepsilon^{2N}}{4} \mathbb{1}_{\{\gamma\in[0,1)\}} + \frac{3\varepsilon^{2N}}{4} \mathbb{1}_{\{\gamma\in[1,2)\}} + \mathbb{1}_{\{\gamma\geq 2\}} \right\} = \frac{3\delta\varepsilon^{2N}}{4}.$$

Now for the minimum weight. Let $j^\star := \operatorname{argmin}_i\{w_t^{(i)}\}$ and write $w_t^{(j^\star)} = \frac{1-\gamma}{N}$, for some $\gamma \in [0,1]$. Then we have

$$\mathbb{P}[c_N(t) > 2/N^2 \mid \mathcal{H}_t] \geq 1 - \mathbb{P}[\nu_t^{(j^\star)} = 1 \mid \mathcal{H}_t] = \begin{cases} 1 - p_1^{(j^\star)} & \text{if } \gamma \in (0,1] \\ 0 & \text{if } \gamma = 0. \end{cases}$$

If $\gamma \in (0,1]$ then

$$1 - p_1^{(j^\star)} \geq \left(1 - \frac{1 + (1-\gamma)}{2}\right)\varepsilon^{2N} = \frac{\gamma\varepsilon^{2N}}{2},$$

again using Table 1 ($p_1$, U.B.). Therefore, under the constraint $\min_i w_t^{(i)} \leq (1-\delta)/N$, we have

$$\mathbb{P}[c_N(t) > 2/N^2 \mid \mathcal{H}_t] \geq \min_{\gamma\geq\delta} \left\{ \frac{\gamma\varepsilon^{2N}}{2} \right\} = \frac{\delta\varepsilon^{2N}}{2}.$$

Combining both cases, we find for arbitrary $r$

$$\mathbb{P}[c_N(r) > 2/N^2 \mid \mathcal{H}_r] \geq \frac{\delta\varepsilon^{2N}}{2} \mathbb{1}_{\{\max_i w_r^{(i)} - \min_i w_r^{(i)} \geq 2\delta/N\}}$$

so

$$\begin{aligned}\mathbb{P}[c_N(r) > 2/N^2 \mid \mathcal{F}_{r-1}] &\geq \frac{\delta \varepsilon^{2N}}{2} \mathbb{P}[\max_i w_r^{(i)} - \min_i w_r^{(i)} \geq 2\delta/N \mid \mathcal{F}_{r-1}] \\ &\geq \zeta \frac{\delta \varepsilon^{2N}}{2} > 0\end{aligned}$$

for infinitely many $r$. Hence

$$\sum_{r=0}^{\infty} \mathbb{P}[c_N(r) > 2/N^2 \mid \mathcal{F}_{r-1}] = \infty$$

as required. ■