

# Genealogies of Sequential Monte Carlo Algorithms

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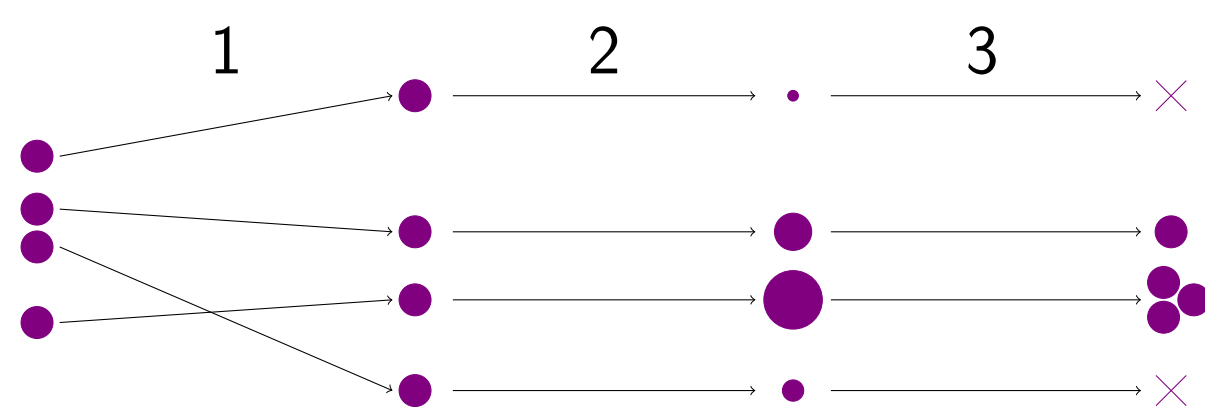
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## Sequential Monte Carlo

SMC<sup>1</sup> is an effective Monte Carlo method for models with strong correlation between parameters. For example, inferring a sequence of underlying parameters from an autocorrelated time series.

We approximate posterior distributions by samples of “particles”. The algorithm starts with  $N$  equally-weighted particles and repeats the steps:



1. **Propagate:** update particle positions for the next parameter
2. **Weight:** weight particles according to agreement with the observation
3. **Resample:** duplicate high-weight particles and kill off low-weight particles to obtain a new sample of  $N$  particles.

The figure shows a particle approximation for a tractable model. The exact posterior mode and 95% HPD set are obtained with the Kalman filter.

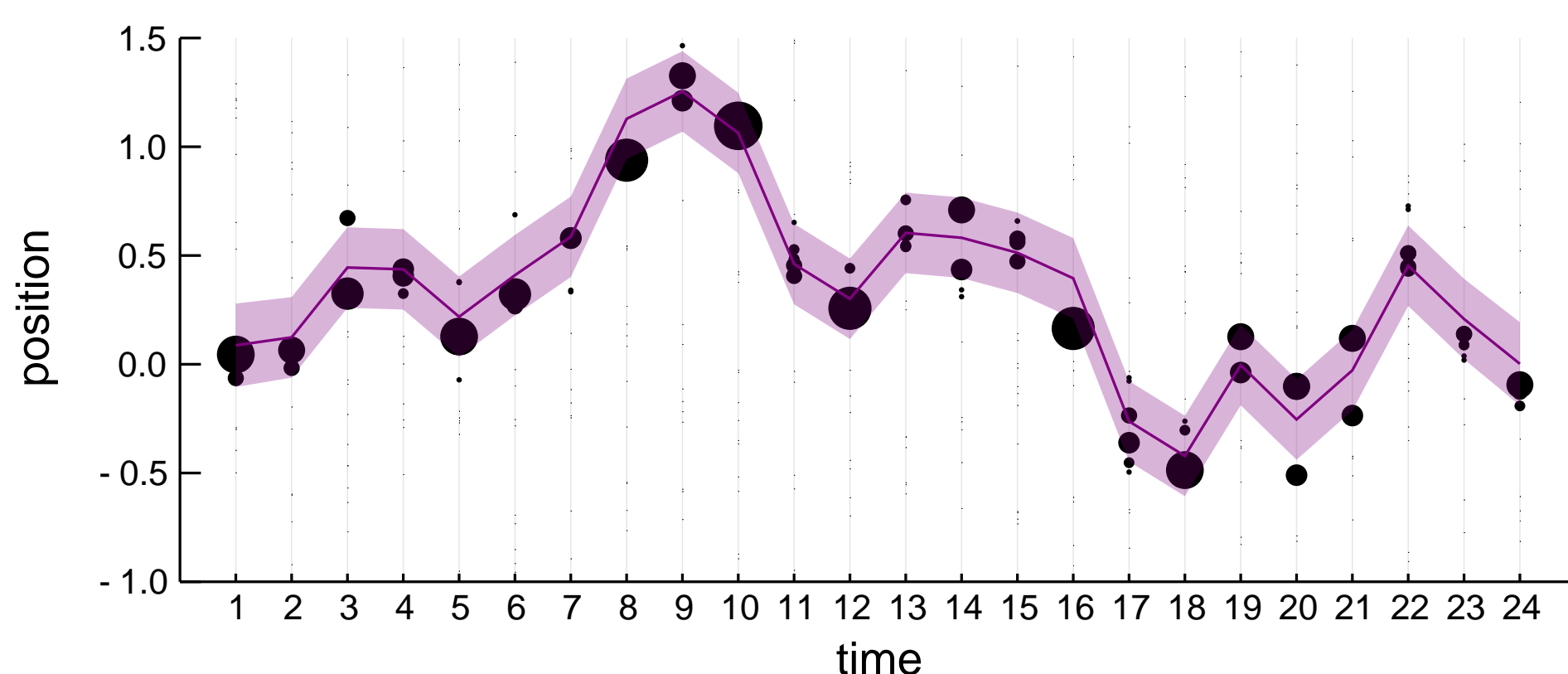


Figure: Exact posterior (purple) and weighted SMC particles before resampling (black) for a linear Gaussian HMM.

## Ancestral Degeneracy

For some applications, we want to approximate the *joint* distribution over all parameters. We have  $N$  samples from this distribution: the “trajectories” of each of the  $N$  particles.

However, due to resampling these trajectories overlap, coalescing backwards in time. At some point all the trajectories coalesce, and we approximate that parameter’s distribution with just one sample!

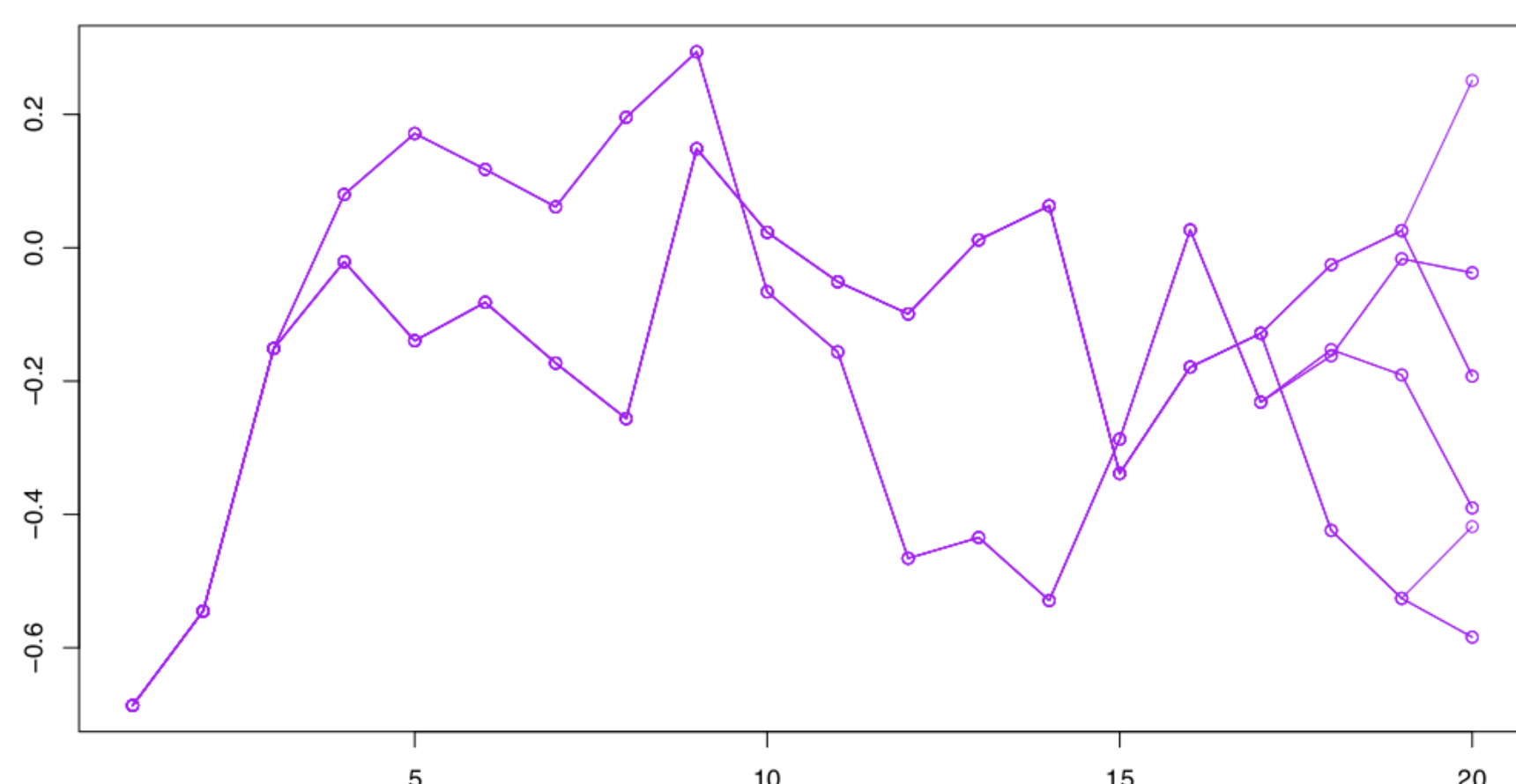


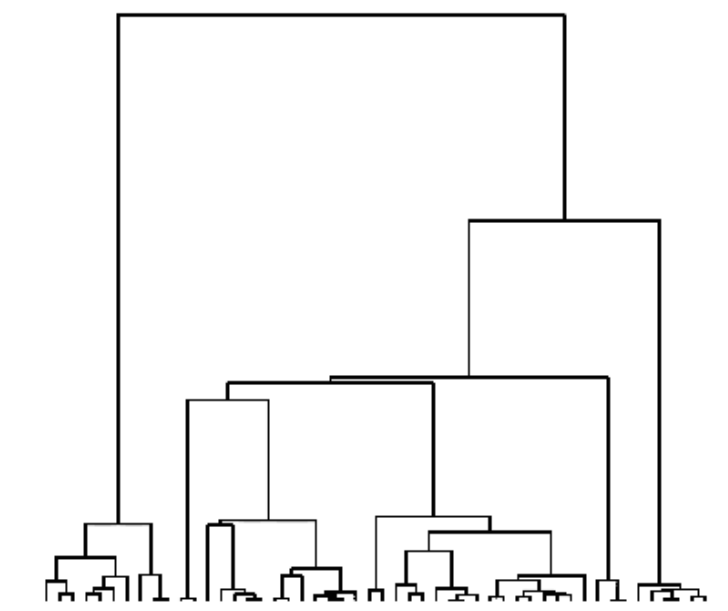
Figure: Trajectories from a sample of  $N=6$  particles. At most points there are just one or two distinct samples.

This phenomenon is known as *ancestral degeneracy*<sup>2</sup>, since the joint structure of the trajectories can be viewed as a *genealogy* of particles. It is one of the main issues in SMC, so we would like to analyse it.

## The Kingman Coalescent<sup>3</sup>

Kingman’s  $n$ -coalescent is the coalescent process on a sample of  $n$  individuals from a population of size  $N$  in which each pair of lineages merges with unit rate.

The Kingman coalescent arises as an extension of the  $n$ -coalescent in the  $N \rightarrow \infty$  limit. It is the limiting genealogy for a large class of population models.



A realisation of the 50-coalescent.

Source: Wikimedia Commons

## SMC Genealogies

The genealogy depends on the number of offspring of each particle upon resampling at each time  $t$ , denoted  $(\nu_t^{(1)}, \dots, \nu_t^{(N)})$ . A key quantity in the asymptotic analysis is the coalescence rate

$$c_N(t) := \frac{1}{(N)_2} \sum_{i=1}^N (\nu_t^{(i)})_2.$$

To obtain convergence to the Kingman coalescent, we rescale time by the inverse coalescence rate to achieve the required unit rate.

It has been shown that, under such a rescaling and some standard assumptions, genealogies induced by multinomial resampling converge (in the sense of finite-dimensional distributions) to the Kingman coalescent<sup>4</sup>. Simulations show the same structure even for finite  $N$ .

We have since extended this result to cover conditional SMC<sup>5</sup> with multinomial resampling. In this algorithm, a certain “immortal trajectory” is conditioned to survive all the resampling steps. This still admits the Kingman coalescent as its limiting genealogy, but simulations show different behaviour for  $N < \infty$ .

CSMC treeheight,  $N=8192$

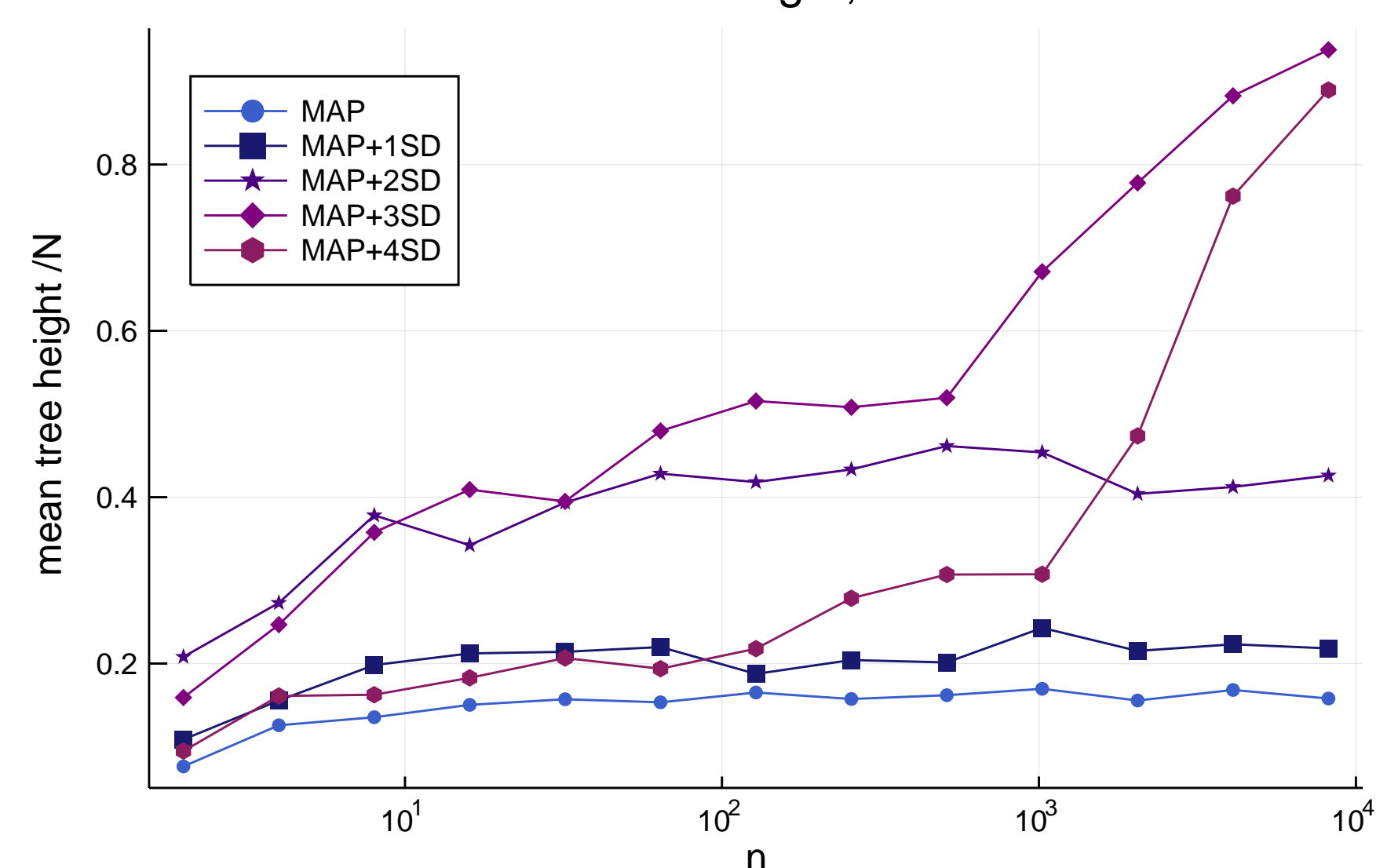


Figure: Average height of subtrees with  $n$  randomly chosen nodes, for different choices of immortal trajectory.

It appears that the process undergoes a phase transition when the immortal trajectory is particularly “unlikely” and  $n$  approaches  $N$ .

1. Gordon, N. J., Salmond, D. J., & Smith, A. F. (1993). Novel approach to nonlinear/non-Gaussian Bayesian state estimation. *IEE:F* 140(3), 107-113. IET Digital Library.
2. Doucet, A., & Johansen, A. M. (2011). A tutorial on particle filtering and smoothing: Fifteen years later. *Handbook of nonlinear filtering*, 656-704. OUP.
3. Kingman, J.F.C. (1982). The coalescent. *Stochastic processes and their applications*, 13(3), 235-248.
4. Koskela, J. et al. (2018). Asymptotic genealogies of interacting particle systems with an application to sequential Monte Carlo. arXiv:1804.01811.
5. Andrieu, C., Doucet, A., & Holenstein, R. (2010). Particle Markov chain Monte Carlo methods. *RSS:B*, 72(3), 269-342.