Weak convergence proof (in progress)

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Theorem 1. *Let $\nu_t^{(1:N)}$ denote the offspring numbers in an interacting particle system satisfying the standing assumption and such that, for any N sufficiently large, for all finite t, $\mathbb{P}\{\tau_N(t) = \infty\} = 0$. Suppose that there exists a deterministic sequence $(b_N)_{N\geq 1}$ such that $\lim_{N\to\infty} b_N = 0$ and*

$$\frac{1}{(N)_3}\sum_{i=1}^{N}\mathbb{E}_t\{(\nu_t^{(i)})_3\} \leq b_N \frac{1}{(N)_2}\sum_{i=1}^{N}\mathbb{E}_t\{(\nu_t^{(i)})_2\} \tag{1}$$

for all N, uniformly in $t \geq 1$. Then the rescaled genealogical process $(G^{(n,N)}_{\tau_N(t)})_{t\geq 0}$ converges weakly to Kingman's n-coalescent as $N \to \infty$.

Proof. Define $p_t := \max_{\xi\in E}\{1 - p_{\xi\xi}(t)\} = 1 - p_{\Delta\Delta}(t)$, where Δ denotes the trivial partition of $\{1, \dots, n\}$ into singletons. For a proof that the maximum is attained at $\xi = \Delta$, see Lemma 1. Following Möhle (1999), we now construct the two-dimensional Markov process $(Z_t, S_t)_{t\in\mathbb{N}}$ with transition probabilities

$$\mathbb{P}(Z_t = j, S_t = \eta \mid Z_{t-1} = i, S_{t-1} = \xi) = \begin{cases} 1 - p_t & \text{if } j = i \text{ and } \xi = \eta \\ p_{\xi\xi}(t) + p_t - 1 & \text{if } j = i+1 \text{ and } \xi = \eta \\ p_{\xi\eta}(t) & \text{if } j = i+1 \text{ and } \xi \neq \eta \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The construction is such that the marginal (S_t) has the same distribution as the genealogical process of interest, and (Z_t) has jumps at all the times (S_t) does plus some extra jumps. (The definition of p_t ensures that the probability in the second case is non-negative, attaining the value zero when $\xi = \Delta$.)

Denote by $0 = T_0^{(N)} < T_1^{(N)} < \dots$ the jump times of the rescaled process $(Z_{\tau_N(t)})_{t\geq 0}$, and $\omega_i^{(N)} := T_i^{(N)} - T_{i-1}^{(N)}$ the corresponding holding times ($i \in \mathbb{N}$).

... $\square$

Lemma 1. $\max_{\xi\in E}(1 - p_{\xi\xi}(t)) = 1 - p_{\Delta\Delta}(t)$.

Proof. Consider any $\xi \in E$ consisting of k blocks ($1 \leq k \leq n-1$), and any $\xi' \in E$ consisting of $k+1$ blocks. From the definition of $p_{\xi\eta}(t)$ (Koskela et al., 2018, Equation (1)),

$$p_{\xi\xi}(t) = \frac{1}{(N)_k}\sum_{\substack{i_1,\dots,i_k \\ \text{all distinct}}} \nu_t^{(i_1)} \cdots \nu_t^{(i_k)}. \tag{3}$$

Similarly,

$$\begin{aligned} p_{\xi'\xi'}(t) &= \frac{1}{(N)_{k+1}}\sum_{\substack{i_1,\dots,i_k,i_{k+1} \\ \text{all distinct}}} \nu_t^{(i_1)} \cdots \nu_t^{(i_k)} \nu_t^{(i_{k+1})} \\ &= \frac{1}{(N)_k(N-k)}\sum_{\substack{i_1,\dots,i_k \\ \text{all distinct}}} \left\{ \nu_t^{(i_1)} \cdots \nu_t^{(i_k)} \sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^{N} \nu_t^{(i_{k+1})} \right\}. \end{aligned} \tag{4}$$

Discarding the zero summands,

$$p_{\xi'\xi'}(t) = \frac{1}{(N)_k(N-k)} \sum_{\substack{i_1,\dots,i_k \\ \text{all distinct:} \\ \nu_t^{(i_1)},\dots,\nu_t^{(i_k)}>0}} \left\{ \nu_t^{(i_1)} \cdots \nu_t^{(i_k)} \sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^{N} \nu_t^{(i_{k+1})} \right\}. \tag{5}$$

The inner sum is

$$\sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^{N} \nu_t^{(i_{k+1})} = \left\{ \sum_{i=1}^{N} \nu_t^{(i)} - \sum_{i\in\{i_1,\dots,i_k\}} \nu_t^{(i)} \right\} \leq N-k \tag{6}$$

since $\nu_t^{(i_1)},\dots,\nu_t^{(i_k)}$ are all at least 1. Hence

$$p_{\xi'\xi'}(t) \leq \frac{N-k}{(N)_k(N-k)} \sum_{\substack{i_1,\dots,i_k \\ \text{all distinct:} \\ \nu_t^{(i_1)},\dots,\nu_t^{(i_k)}>0}} \nu_t^{(i_1)} \cdots \nu_t^{(i_k)} = p_{\xi\xi}(t). \tag{7}$$

Thus $p_{\xi\xi}(t)$ is decreasing in the number of blocks of ξ, and is therefore minimised by taking $\xi = \Delta$, which achieves the maximum n blocks. This choice in turn maximises $1 - p_{\xi\xi}(t)$, as required. □

Lemma 2. *For any* $0 < t < \infty$,

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1-p_r) \right] = e^{-\alpha t} \tag{8}$$

where $\alpha := n(n-1)/2$.

Proof. The strategy is to find upper and lower bounds on $\mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right]$, both of which converge to $e^{-\alpha t}$.

Lower Bound

From Brown et al. (2020, Equation (14)), taking $\xi = \Delta$, we have

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \alpha(1 + O(N^{-1}))\left[c_N(t) + B_n D_N(t)\right] \tag{9}$$

where $B_n > 0$ and the $O(N^{-1})$ term does not depend on t. In particular,

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \frac{N^{n-2}}{(N-2)_{n-2}} \alpha c_N(t) - \frac{N^{n-3}}{(N-3)_{n-3}} B_n D_N(t). \tag{10}$$

Since $D_N(t) \leq c_N(t)$, a sufficient condition for the bound to be positive is

$$E_t := \left\{ c_N(t) < \frac{(N-3)_{n-3}}{N^{n-3}} \left(\alpha\left(1 + \frac{2}{N-2}\right) + B_n \right)^{-1} \right\}. \tag{11}$$

Hence, by a multinomial expansion,

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1-p_r) &\geq \prod_{r=1}^{\tau_N(t)} \left\{1 - \alpha(1+O(N^{-1}))\left[c_N(r) + B_n D_N(r)\right]\right\} \times \prod_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r} \\
&= \left(1 + \sum_{k=1}^{\tau_N(t)} \sum_{\substack{r_1<\cdots<r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^{k} \left\{-\alpha(1+O(N^{-1}))\left[c_N(r_j) + B_n D_N(r_j)\right]\right\} \right) \times \prod_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r} \\
&= \prod_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r} + \sum_{k=1}^{\tau_N(t)} \left\{-\alpha(1+O(N^{-1}))\right\}^k \left(\prod_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r} \right) \sum_{\substack{r_1<\cdots<r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^{k} \left\{ c_N(r_j) + B_n D_N(r_j) \right\}.
\end{aligned} \tag{12}$$

Taking expectations,

$$
\begin{aligned}
\mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right] \geq{}& \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}\mathbb{1}_{E_r}\right] \\
&+\mathbb{E}\left[\sum_{k=1}^{\infty}\left\{-\alpha(1+O(N^{-1}))\right\}^k \mathbb{1}_{\{k\leq\tau_N(t)\}}\mathbb{1}_{\bigcap E_r}\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)}\prod_{j=1}^{k}\{c_N(r_j)+B_nD_N(r_j)\}\right] \\
={}& \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)}E_r\right] \\
&+\sum_{k=1}^{\infty}\left\{-\alpha(1+O(N^{-1}))\right\}^k \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)}\prod_{j=1}^{k}\{c_N(r_j)+B_nD_N(r_j)\}\middle| k\leq\tau_N(t),\bigcap_{r=1}^{\tau_N(t)}E_r\right] \\
&\times\mathbb{P}\left[k\leq\tau_N(t),\bigcap_{r=1}^{\tau_N(t)}E_r\right]. \qquad (13)
\end{aligned}
$$

We want to show that the conditional expectation on the right converges to $t^k/k!$, for reasons that will become clear. The strategy is to upper and lower bound this expectation by quantities that converge to $t^k/k!$.

First the lower bound. Assume that $k\leq\tau_N(t)$, ensuring that the sum is non-empty. From Koskela et al. (2018, Equation (8)),

$$
\begin{aligned}
\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)}\prod_{j=1}^{k}\{c_N(r_j)+B_nD_N(r_j)\} &\geq \sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)}\prod_{j=1}^{k}c_N(r_j) \\
&\geq \frac{1}{k!}\left(\sum_{s=1}^{\tau_N(t)}c_N(s)\right)^k-\frac{1}{k!}\binom{k}{2}\left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right)\left(\sum_{s=1}^{\tau_N(t)}c_N(s)\right)^{k-2} \\
&\geq \frac{1}{k!}t^k-\frac{1}{k!}\binom{k}{2}(t+1)^{k-2}\left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right) \qquad (14)
\end{aligned}
$$

by the definition of τ_N. Then, since the conditioning can only decrease the values of $c_N(s)$,

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)}\prod_{j=1}^{k}\{c_N(r_j)+B_nD_N(r_j)\}\middle| k\leq\tau_N(t),\bigcap_{r=1}^{\tau_N(t)}E_r\right] \\
&\quad\geq \frac{1}{k!}t^k-\frac{1}{k!}\binom{k}{2}(t+1)^{k-2}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)}c_N(s)^2\middle| k\leq\tau_N(t),\bigcap_{r=1}^{\tau_N(t)}E_r\right] \\
&\quad= \frac{1}{k!}t^k-\frac{1}{k!}\binom{k}{2}(t+1)^{k-2}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)}c_N(s)^2\mathbb{1}_{\{k\leq\tau_N(t)\}}\mathbb{1}_{\{\bigcap_{r=1}^{\tau_N(t)}E_r\}}\right]\left(\mathbb{P}\left[k\leq\tau_N(t),\bigcap_{r=1}^{\tau_N(t)}E_r\right]\right)^{-1} \\
&\quad\geq \frac{1}{k!}t^k-\frac{1}{k!}\binom{k}{2}(t+1)^{k-2}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right]\left(\mathbb{P}\left[k\leq\tau_N(t),\bigcap_{r=1}^{\tau_N(t)}E_r\right]\right)^{-1}\longrightarrow\frac{1}{k!}t^k \qquad (15)
\end{aligned}
$$

as $N\to\infty$ using Brown et al. (2020, Equation (5)) and Lemma 3.

Now for the upper bound. We start with a multinomial expansion and some manipulations of the sums:

$$
\begin{aligned}
\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)} \prod_{j=1}^{k} \{c_N(r_j)+B_n D_N(r_j)\} &= \frac{1}{k!} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \prod_{j=1}^{k} \{c_N(r_j)+B_n D_N(r_j)\} \\
&= \frac{1}{k!} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \sum_{\mathcal{I}\subseteq\{1,\dots,k\}} (B_n)^{k-|\mathcal{I}|} \left\{\prod_{i\in\mathcal{I}} c_N(r_i)\right\} \left\{\prod_{j\notin\mathcal{I}} D_N(r_j)\right\} \\
&= \frac{1}{k!} \sum_{\mathcal{I}\subseteq\{1,\dots,k\}} (B_n)^{k-|\mathcal{I}|} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \left\{\prod_{i\in\mathcal{I}} c_N(r_i)\right\} \left\{\prod_{j\notin\mathcal{I}} D_N(r_j)\right\} \\
&= \frac{1}{k!} \sum_{I=0}^{k} \binom{k}{I} (B_n)^{k-I} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \left\{\prod_{i=1}^{I} c_N(r_i)\right\} \left\{\prod_{j=I+1}^{k} D_N(r_j)\right\} \\
&= \frac{1}{k!} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \left\{\prod_{i=1}^{k} c_N(r_i)\right\} \\
&\quad + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (B_n)^{k-I} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \left\{\prod_{i=1}^{I} c_N(r_i)\right\} \left\{\prod_{j=I+1}^{k} D_N(r_j)\right\}. \qquad (16)
\end{aligned}
$$

Then, using that $D_N(s) \leq c_N(s)$ for all s (Koskela et al., 2018, p.9), along with the definition of τ_N,

$$
\begin{aligned}
&\frac{1}{k!} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \left\{\prod_{i=1}^{k} c_N(r_i)\right\} + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (B_n)^{k-I} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \left\{\prod_{i=1}^{I} c_N(r_i)\right\} \left\{\prod_{j=I+1}^{k} D_N(r_j)\right\} \\
&\leq \frac{1}{k!} \left(\sum_{r=1}^{\tau_N(t)} c_N(r)\right)^{k} + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (B_n)^{k-I} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \left\{\prod_{i=1}^{k-1} c_N(r_i)\right\} D_N(r_k) \\
&\leq \frac{1}{k!} \{t + c_N(\tau_N(t))\}^k + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (B_n)^{k-I} \left\{ \sum_{\substack{r_1\neq\cdots\neq r_{k-1}\\ \text{all distinct}}}^{\tau_N(t)} \prod_{i=1}^{k-1} c_N(r_i)\right\} \left(\sum_{r_k=1}^{\tau_N(t)} D_N(r_k)\right) \\
&\leq \frac{1}{k!} \{t + c_N(\tau_N(t))\}^k + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (B_n)^{k-I} \left(\sum_{r=1}^{\tau_N(t)} c_N(r)\right)^{k-1} \left(\sum_{r=1}^{\tau_N(t)} D_N(r)\right) \\
&\leq \frac{1}{k!} \{t + c_N(\tau_N(t))\}^k + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (B_n)^{k-I} (t+1)^{k-1} \left(\sum_{r=1}^{\tau_N(t)} D_N(r)\right). \qquad (17)
\end{aligned}
$$

Taking expectations,

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)}\prod_{j=1}^{k}\{c_N(r_j)+B_nD_N(r_j)\}\middle|k\le\tau_N(t),\bigcap_{r=1}^{\tau_N(t)}E_r\right]\\
&\quad\le\frac{1}{k!}\mathbb{E}\left[\{t+c_N(\tau_N(t))\}^k\middle|k\le\tau_N(t),\bigcap_{r=1}^{\tau_N(t)}E_r\right]\\
&\qquad+\frac{1}{k!}\sum_{I=0}^{k-1}\binom{k}{I}(B_n)^{k-I}(t+1)^{k-1}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)}D_N(r)\middle|k\le\tau_N(t),\bigcap_{r=1}^{\tau_N(t)}E_r\right]\\
&\quad=\frac{1}{k!}\mathbb{E}\left[\{t+c_N(\tau_N(t))\}^k\,\mathbb{1}_{\{k\le\tau_N(t)\}}\mathbb{1}_{\{\bigcap E_r\}}\right]\left(\mathbb{P}\left[k\le\tau_N(t),\bigcap_{r=1}^{\tau_N(t)}E_r\right]\right)^{-1}\\
&\qquad+\frac{1}{k!}\sum_{I=0}^{k-1}\binom{k}{I}(B_n)^{k-I}(t+1)^{k-1}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)}D_N(r)\mathbb{1}_{\{k\le\tau_N(t)\}}\mathbb{1}_{\{\bigcap E_r\}}\right]\left(\mathbb{P}\left[k\le\tau_N(t),\bigcap_{r=1}^{\tau_N(t)}E_r\right]\right)^{-1}\\
&\quad\le\left(\frac{1}{k!}\mathbb{E}\left[\{t+c_N(\tau_N(t))\}^k\right]+\frac{1}{k!}\sum_{I=0}^{k-1}\binom{k}{I}(B_n)^{k-I}(t+1)^{k-1}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)}D_N(r)\right]\right)\left(\mathbb{P}\left[k\le\tau_N(t),\bigcap_{r=1}^{\tau_N(t)}E_r\right]\right)^{-1}\\
&\quad\longrightarrow\frac{1}{k!}t^k.
\end{aligned}
\tag{18}
$$

The limit follows from Lemma 3 and Brown et al. (2020, Equations (3),(4)) along with the fact that, since $c_N(s)\in[0,1]$ for all s, $\mathbb{E}[c_N(s)^k]\le\mathbb{E}[c_N(s)]$ for all $k\ge1$, and the expansion

$$
\mathbb{E}\left[\{t+c_N(\tau_N(t))\}^k\right]=\sum_{i=0}^{k}\binom{k}{i}t^i\,\mathbb{E}\left[c_N(\tau_N(t))^{k-i}\right]\longrightarrow t^k. \tag{19}
$$

Combining the upper and lower limits, we conclude that

$$
\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)}\prod_{j=1}^{k}\{c_N(r_j)+B_nD_N(r_j)\}\middle|k\le\tau_N(t),\bigcap_{r=1}^{\tau_N(t)}E_r\right]=\frac{1}{k!}t^k \tag{20}
$$

and thus

$$\lim_{N\to\infty}\mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}\mathbb{1}_{E_r}+\sum_{k=1}^{\tau_N(t)}\left\{-\alpha(1+O(N^{-1}))\right\}^k\left(\prod_{r=1}^{\tau_N(t)}\mathbb{1}_{E_r}\right)\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)}\prod_{j=1}^{k}\{c_N(r_j)+B_nD_N(r_j)\}\right]$$

$$\begin{aligned}
&=\lim_{N\to\infty}\mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)}E_r\right]\\
&\quad+\lim_{N\to\infty}\sum_{k=1}^{\infty}\left\{-\alpha(1+O(N^{-1}))\right\}^k\mathbb{E}\left[\mathbb{1}_{\{k\le\tau_N(t)\}}\left(\prod_{r=1}^{\tau_N(t)}\mathbb{1}_{E_r}\right)\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)}\prod_{j=1}^{k}\{c_N(r_j)+B_nD_N(r_j)\}\right]\\
&=\lim_{N\to\infty}\mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)}E_r\right]\\
&\quad+\sum_{k=1}^{\infty}(-\alpha)^k\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)}\prod_{j=1}^{k}\{c_N(r_j)+B_nD_N(r_j)\}\middle|k\le\tau_N(t),\bigcap_{r=1}^{\tau_N(t)}E_r\right]\times\lim_{N\to\infty}\mathbb{P}\left[k\le\tau_N(t),\bigcap_{r=1}^{\tau_N(t)}E_r\right]\\
&=1+\sum_{k=1}^{\infty}(-\alpha)^k\frac{t^k}{k!}\times 1=e^{-\alpha t}
\end{aligned}\tag{21}$$

as $N\to\infty$, where the last line follows from (20) and Lemma 3.

Upper Bound

From Koskela et al. (2018, Lemma 1 Case 1), taking $\xi=\Delta$, we have

$$1-p_t=p_{\Delta\Delta}(t)\le 1-\alpha(1+O(N^{-1}))\left[c_N(t)-B'_nD_N(t)\right].\tag{22}$$

A multinomial expansion as before yields

$$\prod_{r=1}^{\tau_N(t)}(1-p_r)\le 1+\sum_{k=1}^{\tau_N(t)}\left\{-\alpha(1+O(N^{-1}))\right\}^k\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)}\prod_{j=1}^{k}\{c_N(r_j)-B'_nD_N(r_j)\}.\tag{23}$$

Analogously to (16), assuming $k\le\tau_N(t)$ we can write

$$\begin{aligned}
\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)}\prod_{j=1}^{k}\{c_N(r_j)-B'_nD_N(r_j)\}&=\frac{1}{k!}\sum_{\substack{r_1\neq\cdots\neq r_k\\\text{all distinct}}}^{\tau_N(t)}\left\{\prod_{i=1}^{k}c_N(r_i)\right\}\\
&\quad+\frac{1}{k!}\sum_{I=0}^{k-1}\binom{k}{I}(-B'_n)^{k-I}\sum_{\substack{r_1\neq\cdots\neq r_k\\\text{all distinct}}}\left\{\prod_{i=1}^{I}c_N(r_i)\right\}\left\{\prod_{j=I+1}^{k}D_N(r_j)\right\}.
\end{aligned}\tag{24}$$

We start by dealing with the second term:

$$\begin{aligned}
&\frac{1}{k!}\sum_{I=0}^{k-1}\binom{k}{I}(-B'_n)^{k-I}\sum_{\substack{r_1\neq\cdots\neq r_k\\\text{all distinct}}}\left\{\prod_{i=1}^{I}c_N(r_i)\right\}\left\{\prod_{j=I+1}^{k}D_N(r_j)\right\}\\
&\qquad=\frac{1}{k!}\sum_{\substack{I=0:\\k-I\text{ even}}}^{k-1}\binom{k}{I}(B'_n)^{k-I}\sum_{\substack{r_1\neq\cdots\neq r_k\\\text{all distinct}}}\left\{\prod_{i=1}^{I}c_N(r_i)\right\}\left\{\prod_{j=I+1}^{k}D_N(r_j)\right\}\\
&\qquad\quad-\frac{1}{k!}\sum_{\substack{I=0:\\k-I\text{ odd}}}^{k-1}\binom{k}{I}(B'_n)^{k-I}\sum_{\substack{r_1\neq\cdots\neq r_k\\\text{all distinct}}}\left\{\prod_{i=1}^{I}c_N(r_i)\right\}\left\{\prod_{j=I+1}^{k}D_N(r_j)\right\}.
\end{aligned}\tag{25}$$

This is lower bounded by

$$0-\frac{1}{k!}\sum_{\substack{I=0\\k-I\text{ odd}}}^{k-1}\binom{k}{I}(B_n')^{k-I}(t+1)^{k-1}\left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right) \tag{26}$$

using that $c_N(r), D_N(r)\geq 0$ for all r to bound the even terms below, and arguments from (17) to bound the odd terms above. The same arguments lead to the upper bound

$$\frac{1}{k!}\sum_{\substack{I=0\\k-I\text{ even}}}^{k-1}\binom{k}{I}(B_n')^{k-I}(t+1)^{k-1}\left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)-0. \tag{27}$$

By Brown et al. (2020, Equation (4)), both bounds have vanishing expectation as $N\to\infty$. We are left with the first term in (24), which is upper bounded by

$$\frac{1}{k!}\sum_{\substack{r_1\neq\cdots\neq r_k\\\text{all distinct}}}^{\tau_N(t)}\left\{\prod_{i=1}^{k}c_N(r_i)\right\}\leq\frac{1}{k!}\left(\sum_{s=1}^{\tau_N(t)}c_N(s)\right)^k\leq\frac{1}{k!}\{t+c_N(\tau_N(t))\}^k \tag{28}$$

the expectation of which converges to $t^k/k!$ as in (19). We use Koskela et al. (2018, Equation (8)) to construct a lower bound:

$$\begin{aligned}\frac{1}{k!}\sum_{\substack{r_1\neq\cdots\neq r_k\\\text{all distinct}}}^{\tau_N(t)}\left\{\prod_{i=1}^{k}c_N(r_i)\right\}&\geq\frac{1}{k!}\left(\sum_{s=1}^{\tau_N(t)}c_N(s)\right)^k-\frac{1}{k!}\binom{k}{2}\left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right)\left(\sum_{s=1}^{\tau_N(t)}c_N(s)\right)^{k-2}\\&\geq\frac{1}{k!}t^k-\frac{1}{k!}\binom{k}{2}(t+1)^{k-2}\left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right)\end{aligned} \tag{29}$$

The expectation of this bound also converges to $t^k/k!$, using Brown et al. (2020, Equation (5)). We can now conclude that

$$\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)}\prod_{j=1}^{k}\{c_N(r_j)-B_n'D_N(r_j)\}\right]=\frac{1}{k!}t^k \tag{30}$$

and thus, by calculations analogous to (21),

$$\lim_{N\to\infty}\mathbb{E}\left[1+\sum_{k=1}^{\tau_N(t)}\left\{-\alpha(1+O(N^{-1}))\right\}^k\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)}\prod_{j=1}^{k}\{c_N(r_j)-B_n'D_N(r_j)\}\right]=1+\sum_{k=1}^{\infty}(-\alpha)^k\frac{1}{k!}t^k=e^{-\alpha t} \tag{31}$$

as $N\to\infty$.

We now have upper and lower bounds on $\lim_{N\to\infty}\mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right]$, both of which are equal to $e^{-\alpha t}$, and the result follows. ☐

Lemma 3. *For any $n\leq N\in\mathbb{N}$, for all $t\in\mathbb{N}$, define*

$$E_t:=\left\{c_N(t)<\frac{(N-3)_{n-3}}{N^{n-3}}\left(\alpha\left(1+\frac{2}{N-2}\right)+B_n\right)^{-1}\right\} \tag{32}$$

where α and B_n are positive constants as in (9). Then, for all $t>0$,

$$\lim_{N\to\infty}\mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)}E_r\right]=1. \tag{33}$$

Proof.

$$\mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} E_r\right] = 1 - \mathbb{P}\left[\bigcup_{r=1}^{\tau_N(t)} E_r^c\right] = 1 - \mathbb{E}\left[\mathbb{1}_{\bigcup E_r^c}\right] \geq 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r^c}\right]$$
$$= 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[\mathbb{1}_{E_r^c} \mid \mathcal{F}_{r-1}]\right] = 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{P}[E_r^c \mid \mathcal{F}_{r-1}]\right] \tag{34}$$

where the inequality holds by considering the two possible values of $\mathbb{1}_{\bigcup E_r^c}$, and the second line follows from Koskela et al. (2018, Lemma 2) when the function $c_N(r)$ is replaced by $\mathbb{1}_{E_r^c}$. Using the generalised Markov inequality,

$$\mathbb{P}[E_r^c \mid \mathcal{F}_{r-1}] = \mathbb{P}\left[c_N(r) \geq \frac{(N-3)_{n-3}}{N^{n-3}}\left(\alpha\left(1+\frac{2}{N-2}\right)+B_n\right)^{-1} \middle| \mathcal{F}_{r-1}\right]$$
$$\leq \mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}] \frac{N^{2(n-3)}}{(N-3)_{n-3}^2}\left(\alpha\left(1+\frac{2}{N-2}\right)+B_n\right)^2. \tag{35}$$

Now, using Koskela et al. (2018, Lemma 2) again, but with $c_N(r)$ replaced by $c_N(r)^2$,

$$\mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} E_r\right] \geq 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}] \frac{N^{2(n-3)}}{(N-3)_{n-3}^2}\left(\alpha\left(1+\frac{2}{N-2}\right)+B_n\right)^2\right]$$
$$= 1 - \frac{N^{2(n-3)}}{(N-3)_{n-3}^2}\left(\alpha\left(1+\frac{2}{N-2}\right)+B_n\right)^2 \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}]\right]$$
$$= 1 - \frac{N^{2(n-3)}}{(N-3)_{n-3}^2}\left(\alpha\left(1+\frac{2}{N-2}\right)+B_n\right)^2 \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} c_N(r)^2\right]$$
$$\xrightarrow{N\to\infty} 1 - (\alpha + B_n)^2 \times 0 = 1. \tag{36}$$

□

Lemma 4. *For all $k \geq 1$, for all $t > 0$,*

$$\lim_{N\to\infty} \mathbb{P}\left[k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r\right] = 1. \tag{37}$$

Proof. We will construct a constant C_1 such that

$$\mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) < C_1\}\right] \leq \mathbb{P}\left[k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r\right] \tag{38}$$

and

$$\lim_{N\to\infty} \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) < C_1\}\right] = 1. \tag{39}$$

Any $C_1 \leq \frac{(N-3)_{n-3}}{N^{n-3}}\left(\alpha\left(1+\frac{2}{N-2}\right)+B_n\right)^{-1}$ will be sufficient for $\bigcap_{r=1}^{\tau_N(t)} E_r$. Furthermore, we can write

$$\{\tau_N(t) \geq k\} = \left\{\min\left\{s \geq 1 : \sum_{r=1}^{s} c_N(r) \geq t\right\} \geq k\right\} = \left\{\sum_{r=1}^{k-1} c_N(r) < t\right\} \supseteq \bigcap_{r=1}^{k-1}\left\{c_N(r) < \frac{t}{k}\right\}. \tag{40}$$

A suitable choice to satisfy (38) would thus be

$$C_1 = \min\left\{\frac{(N-3)_{n-3}}{N^{n-3}}\left(\alpha\left(1+\frac{2}{N-2}\right)+B_n\right)^{-1}, \frac{t}{k}\right\}. \tag{41}$$

Noting that t is fixed, it can be shown that (39) is satisfied by modifying the constant in the proof of Lemma 3. □

References

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