

Residual resampling in SMC

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The offspring counts are sampled according to:

$$v_t^{(i)} = \lfloor Nw_t^{(i)} \rfloor + X_i$$

$$X_i \sim \text{Multinomial}(N - k, (\bar{w}_t^{(1)}, \dots, \bar{w}_t^{(N)}))$$

where $k := \sum_{i=1}^N \lfloor Nw_t^{(i)} \rfloor$ is the number of offspring assigned deterministically, and $\bar{w}_t^{(i)} := \frac{Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor}{N - k}$ are the residual weights. Let us also define the residuals $r_i := Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor$.

1 Coalescence rate

We shall calculate the expected coalescence rate in the case of residual resampling. The coalescence rate is defined in Koskela et al. (2018) as

$$c_N(t) := \frac{1}{(N)_2} \sum_{i=1}^N \mathbb{E}[(v_t^{(i)})_2].$$

The inner expectation comes out as

$$\begin{aligned} \mathbb{E}[(v_t^{(i)})_2] &= \mathbb{E}[(v_t^{(i)})^2] - \mathbb{E}[v_t^{(i)}] \\ &= \lfloor Nw_t^{(i)} \rfloor^2 + 2\lfloor Nw_t^{(i)} \rfloor r_i + r_i \left(1 - \frac{r_i}{N - k} + r_i\right) - Nw_t^{(i)} \\ &= \lfloor Nw_t^{(i)} \rfloor^2 - \lfloor Nw_t^{(i)} \rfloor + 2\lfloor Nw_t^{(i)} \rfloor r_i + r_i^2 \left(1 - \frac{1}{N - k}\right) \\ &= (Nw_t^{(i)})^2 - \lfloor Nw_t^{(i)} \rfloor - \frac{r_i^2}{N - k} \end{aligned}$$

so we get

$$\begin{aligned} \mathbb{E}[c_N^r(t) | \mathcal{F}_{t-1}] &= \frac{1}{(N)_2} \mathbb{E} \left[\sum_{i=1}^N \mathbb{E}[(v_t^{(i)})_2] | \mathcal{F}_{t-1} \right] \\ &= \frac{N}{N-1} \sum_{i=1}^N \mathbb{E}[(w_t^{(i)})^2 | \mathcal{F}_{t-1}] - \frac{1}{(N)_2} \sum_{i=1}^N \mathbb{E} \left[\frac{r_i^2}{N - k} | \mathcal{F}_{t-1} \right] - \frac{1}{(N)_2} \mathbb{E}[k | \mathcal{F}_{t-1}] \\ &= \mathbb{E}[c_N^m(t) | \mathcal{F}_{t-1}] \left(1 + \frac{1}{N-1}\right) - \frac{1}{(N)_2} \mathbb{E} \left[\frac{\sum_{i=1}^N (Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor)^2}{\sum_{j=1}^N (Nw_t^{(j)} - \lfloor Nw_t^{(j)} \rfloor)} | \mathcal{F}_{t-1} \right] \\ &\quad - \frac{1}{(N)_2} \mathbb{E} \left[\sum_{i=1}^N \lfloor Nw_t^{(i)} \rfloor | \mathcal{F}_{t-1} \right] \end{aligned}$$

Sanity check:

When the weights are all equal, $w_t^{(i)} \equiv 1/N$, we should have $\mathbb{E}[c_N^r(t) | \mathcal{F}_{t-1}] = 0$ since each particle will have exactly one offspring so it is impossible for any lineages to coalesce. In this case we have $\mathbb{E}[c_N^m(t) | \mathcal{F}_{t-1}] =$

$\sum_{i=1}^N \mathbb{E}[(w_t^{(i)})^2 | \mathcal{F}_{t-1}] = 1/N$ for multinomial resampling. We also have that $Nw_t^{(i)} \equiv \lfloor Nw_t^{(i)} \rfloor \equiv 1$ and hence $r_i = 0$ and $k = N$. Thus the RHS comes out as

$$\frac{1}{N} \frac{N}{N-1} - 0 - \frac{1}{(N)_2} N = \frac{1}{N-1} - \frac{1}{N-1} = 0$$

as expected.

2 Squared coalescence rate and other awful expressions

First let's write down various moments that will be needed (where $i \neq j$).

$$\begin{aligned} \mathbb{E}[(v_t^{(i)})^2 | \mathcal{F}_{t-1}] &= \lfloor Nw_t^{(i)} \rfloor^2 + 2\lfloor Nw_t^{(i)} \rfloor Nw_t^{(i)} + (N)_2 (w_t^{(i)})^2 \\ \mathbb{E}[(v_t^{(i)})^3 | \mathcal{F}_{t-1}] &= \lfloor Nw_t^{(i)} \rfloor^3 + 3\lfloor Nw_t^{(i)} \rfloor^2 Nw_t^{(i)} + 3\lfloor Nw_t^{(i)} \rfloor (N)_2 (w_t^{(i)})^2 + (N)_3 (w_t^{(i)})^3 \\ \mathbb{E}[(v_t^{(i)})^4 | \mathcal{F}_{t-1}] &= \lfloor Nw_t^{(i)} \rfloor^4 + 4\lfloor Nw_t^{(i)} \rfloor^3 Nw_t^{(i)} + 6\lfloor Nw_t^{(i)} \rfloor^2 (N)_2 (w_t^{(i)})^2 + 4\lfloor Nw_t^{(i)} \rfloor (N)_3 (w_t^{(i)})^3 + (N)_4 (w_t^{(i)})^4 \\ \mathbb{E}[v_t^{(i)} v_t^{(j)} | \mathcal{F}_{t-1}] &= \lfloor Nw_t^{(i)} \rfloor \lfloor Nw_t^{(j)} \rfloor + \lfloor Nw_t^{(i)} \rfloor Nw_t^{(j)} + \lfloor Nw_t^{(j)} \rfloor Nw_t^{(i)} + (N)_2 w_t^{(i)} w_t^{(j)} \\ \mathbb{E}[(v_t^{(i)})^2 v_t^{(j)} | \mathcal{F}_{t-1}] &= \lfloor Nw_t^{(i)} \rfloor^2 \lfloor Nw_t^{(j)} \rfloor + \lfloor Nw_t^{(i)} \rfloor^2 Nw_t^{(j)} + 2\lfloor Nw_t^{(i)} \rfloor \lfloor Nw_t^{(j)} \rfloor Nw_t^{(i)} + 2\lfloor Nw_t^{(i)} \rfloor (N)_2 w_t^{(i)} w_t^{(j)} \\ &\quad + \lfloor Nw_t^{(j)} \rfloor (N)_2 (w_t^{(i)})^2 + (N)_3 (w_t^{(i)})^2 w_t^{(j)} \\ \mathbb{E}[(v_t^{(i)})^2 (v_t^{(j)})^2 | \mathcal{F}_{t-1}] &= \lfloor Nw_t^{(i)} \rfloor^2 \lfloor Nw_t^{(j)} \rfloor^2 + 2\lfloor Nw_t^{(i)} \rfloor^2 \lfloor Nw_t^{(j)} \rfloor Nw_t^{(j)} + 2\lfloor Nw_t^{(i)} \rfloor \lfloor Nw_t^{(j)} \rfloor^2 Nw_t^{(i)} \\ &\quad + \lfloor Nw_t^{(i)} \rfloor^2 (N)_2 (w_t^{(j)})^2 + 4\lfloor Nw_t^{(i)} \rfloor \lfloor Nw_t^{(j)} \rfloor (N)_2 w_t^{(i)} w_t^{(j)} + \lfloor Nw_t^{(j)} \rfloor^2 (N)_2 (w_t^{(i)})^2 \\ &\quad + 2\lfloor Nw_t^{(i)} \rfloor (N)_3 w_t^{(i)} (w_t^{(j)})^2 + 2\lfloor Nw_t^{(j)} \rfloor (N)_3 (w_t^{(i)})^2 w_t^{(j)} + (N)_4 (w_t^{(i)})^2 (w_t^{(j)})^2 \end{aligned}$$

For the squared coalescence rate, expanding the falling factorials appropriately, we get

$$\begin{aligned} \mathbb{E}[(c_N^r(t))^2 | \mathcal{F}_{t-1}] &= \frac{1}{(N)_2^2} \left(\sum_{i=1}^N \mathbb{E}[(v_t^{(i)})_2^2 | \mathcal{F}_{t-1}] + \sum_{i=1}^N \sum_{j \neq i} \mathbb{E}[(v_t^{(i)})_2 (v_t^{(j)})_2 | \mathcal{F}_{t-1}] \right) \\ &= \frac{1}{(N)_2^2} \sum_{i=1}^N \left(\mathbb{E}[(v_t^{(i)})^4 | \mathcal{F}_{t-1}] - 2\mathbb{E}[(v_t^{(i)})^3 | \mathcal{F}_{t-1}] + \mathbb{E}[(v_t^{(i)})^2 | \mathcal{F}_{t-1}] \right) \\ &\quad + \frac{1}{(N)_2^2} \sum_{i=1}^N \sum_{j \neq i} \left(\mathbb{E}[(v_t^{(i)})^2 (v_t^{(j)})^2 | \mathcal{F}_{t-1}] - \mathbb{E}[(v_t^{(i)})^2 v_t^{(j)} | \mathcal{F}_{t-1}] - \mathbb{E}[v_t^{(i)} (v_t^{(j)})^2 | \mathcal{F}_{t-1}] + \mathbb{E}[v_t^{(i)} v_t^{(j)} | \mathcal{F}_{t-1}] \right) \end{aligned}$$

Then we can try plugging in the expressions derived above and find that none of the terms cancel. Maybe if we're clever we can factorise it or something.

3 Mega-merger rate

Now the rate of super0binary mergers...

$$\begin{aligned} \mathbb{E}[D_N(t) | \mathcal{F}_{t-1}] &= \frac{1}{N(N)_2} \sum_{i=1}^N \left(\mathbb{E}[(v_t^{(i)})^3 | \mathcal{F}_{t-1}] - \mathbb{E}[(v_t^{(i)})^2 | \mathcal{F}_{t-1}] \right) \\ &\quad + \frac{1}{N(N)_2} \sum_{i=1}^N \sum_{j \neq i} \left(\mathbb{E}[(v_t^{(i)})^2 (v_t^{(j)})^2 | \mathcal{F}_{t-1}] - \mathbb{E}[v_t^{(i)} (v_t^{(j)})^2 | \mathcal{F}_{t-1}] \right) \end{aligned}$$

References

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