Fix $\varepsilon > 0$. Let N be large enough that $\varepsilon > 1/N$, and let $A_i(r) := \{\nu_r^{(i)} \leq N\varepsilon\}$. Following [Möhle and Sagitov, 2003, Proof of Lemma 5.5],

$$\begin{aligned}
&\frac{1}{N(N)_2} \sum_{i=1}^{N} (\nu_r^{(i)})_2 \left[\nu_r^{(i)} + \frac{1}{N} \sum_{j \neq i} (\nu_r^{(j)})^2 \right] \mathbb{1}_{A_i(r)} \\
&\leq \frac{1}{N(N)_2} \sum_{i=1}^{N} (\nu_r^{(i)})_2 \left[N\varepsilon + \frac{1}{N} \sum_{j=1}^{N} (\nu_r^{(j)})_2 + \frac{1}{N} \sum_{j=1}^{N} \nu_r^{(j)} \right] \mathbb{1}_{A_i(r)} \\
&\leq \left[\varepsilon c_N(r) + \frac{1}{N} c_N(r) + \frac{N\varepsilon}{N^2 (N)_2} \sum_{i=1}^{N} \nu_r^{(i)} \sum_{j=1}^{N} (\nu_r^{(j)})_2 \right] \mathbb{1}_{A_i(r)} = \left(2\varepsilon + \frac{1}{N} \right) c_N(r),
\end{aligned}$$

and

$$\frac{1}{N(N)_2} \sum_{i=1}^{N} (\nu_r^{(i)})_2 \left[\nu_r^{(i)} + \frac{1}{N} \sum_{j \neq i} (\nu_r^{(j)})^2 \right] \mathbb{1}_{A_i(r)^c} \leq \sum_{i=1}^{N} \mathbb{1}_{A_i(r)^c}.$$

Thus

$$\mathbb{E}\left[\sum_{r=\tau_N(t_0)+1}^{\tau_N(t_k)} \mathbb{1}_{\{c_N(r) \leq \binom{n-2}{2} D_N(r)\}} \right] \leq \mathbb{E}\left[\sum_{r=\tau_N(t_0)+1}^{\tau_N(t_k)} \mathbb{1}_{\left\{ \left(\binom{n-2}{2}^{-1} - 2\varepsilon - 1/N \right) \leq \sum_{i=1}^{N} \mathbb{1}_{A_i(r)^c} / c_N(r) \right\}} \right]$$

where the ratio $\mathbb{1}_{A_i(r)^c} / c_N(r)$ is well defined because

$$A_i(r)^c \Rightarrow c_N(r) = \frac{1}{(N)_2} \sum_{i=1}^{N} (\nu_t^{(i)})_2 \geq \frac{\varepsilon(N\varepsilon - 1)}{N-1} = \varepsilon\Big(\varepsilon - \frac{1}{N}\Big) > 0.$$

Hence also

$$\mathbb{E}\left[\sum_{r=\tau_N(t_0)+1}^{\tau_N(t_k)} \mathbb{1}_{\{c_N(r) \leq \binom{n-2}{2} D_N(r)\}} \right] \leq \mathbb{E}\left[\sum_{r=\tau_N(t_0)+1}^{\tau_N(t_k)} \mathbb{1}_{\left\{ \left(\binom{n-2}{2}^{-1} - 2\varepsilon - 1/N \right) \varepsilon(\varepsilon - 1/N) \leq \sum_{i=1}^{N} \mathbb{1}_{A_i(r)^c} \right\}} \right],$$

whereupon [Koskela et al., 2018, Lemma 2] and the conditional Markov inequality yield

$$\begin{aligned}
&\mathbb{E}\left[\sum_{r=\tau_N(t_0)+1}^{\tau_N(t_k)} \mathbb{1}_{\{c_N(r) \leq \binom{n-2}{2} D_N(r)\}} \right] \\
&\leq \frac{1}{\left(\binom{n-2}{2}^{-1} - 2\varepsilon - 1/N \right) \varepsilon(\varepsilon - 1/N)} \mathbb{E}\left[\sum_{r=\tau_N(t_0)+1}^{\tau_N(t_k)} \sum_{i=1}^{N} \mathbb{1}_{A_i(r)^c} \right].
\end{aligned}$$

A further two invocations of [Koskela et al., 2018, Lemma 2] with the conditional Markov inequality in between result in

$$\begin{aligned}
&\mathbb{E}\left[\sum_{r=\tau_N(t_0)+1}^{\tau_N(t_k)} \mathbb{1}_{\{c_N(r) \leq \binom{n-2}{2} D_N(r)\}} \right] \\
&\leq \frac{1}{\left(\binom{n-2}{2}^{-1} - 2\varepsilon - 1/N \right) \varepsilon(\varepsilon - 1/N)} \mathbb{E}\left[\sum_{r=\tau_N(t_0)+1}^{\tau_N(t_k)} \sum_{i=1}^{N} \frac{(\nu_r^{(i)})_3}{(N\varepsilon)_3} \right] \\
&\leq \frac{N(N)_2}{\left(\binom{n-2}{2}^{-1} - 2\varepsilon - 1/N \right) \varepsilon(\varepsilon - 1/N)(N\varepsilon)_3} \mathbb{E}\left[\sum_{r=\tau_N(t_0)+1}^{\tau_N(t_k)} D_N(r) \right] \\
&\to \frac{1}{\left(\binom{n-2}{2}^{-1} - 2\varepsilon \right) \varepsilon^5} \times 0
\end{aligned}$$

by [Koskela et al., 2018, Eq. (3)] as $N \to \infty$, as required.

References

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