Conditional SMC — with updated assumptions

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July 31, 2019

Theorem 1. *Under the time scaling of Koskela et al. (2018, Theorem 1) and the conditions of Koskela et al. (2018, Lemma 3), genealogies of SMC algorithms with multinomial resampling converge to Kingman's n-coalescent in the sense of finite-dimensional distributions as $N \to \infty$.*

Proof. The standing assumption holds by exchangeability of the Multinomial distribution. We also need to show that there exists a deterministic sequence $(b_N)_{N\in\mathbb{N}}$ such that $\lim_{N\to\infty} b_N = 0$ and

$$\frac{\frac{1}{(N)_3}\sum_{i=1}^N \mathbb{E}[(\nu_t^{(i)})_3 | \mathcal{F}_{t-1}}{\frac{1}{(N)_2}\sum_{i=1}^N \mathbb{E}[(\nu_t^{(i)})_2 | \mathcal{F}_{t-1}]} \le b_N \tag{1}$$

for all $N \in \mathbb{N}$. For the denominator, we apply Koskela et al. (2018, Lemma 3) directly to obtain, for some constants $a \ge \varepsilon \ge 0$,

$$\sum_{i=1}^N \mathbb{E}[(\nu_t^{(i)})_2 | \mathcal{F}_{t-1}] \ge \frac{(N)_2 \varepsilon^4}{N a^4}$$

For the numerator, we use that $\nu_t^{(i)} \longrightarrow (\nu_t^{(i)})_3$ is $\{i\}$-increasing, along with the argument from the proof of Koskela et al. (2018, Lemma 3), to obtain

$$\sum_{i=1}^N \mathbb{E}[(\nu_t^{(i)})_3 | \mathcal{F}_{t-1}] \le \sum_{i=1}^N \mathbb{E}[(\tilde{a}_t^{(i)})_3]$$

where

$$\tilde{a}_t^{(j)} \sim^{iid} \text{Categorical}\left(\left(\frac{a}{\varepsilon}\right)^{\mathbb{I}_{\{i=1\}} - \mathbb{I}_{\{i\neq 1\}}}, \left(\frac{a}{\varepsilon}\right)^{\mathbb{I}_{\{i=2\}} - \mathbb{I}_{\{i\neq 2\}}}, \dots, \left(\frac{a}{\varepsilon}\right)^{\mathbb{I}_{\{i=N\}} - \mathbb{I}_{\{i\neq N\}}} \right) \tag{2}$$

We can calculate the expectation for $\tilde{a}_t^{(1:N)}$, so we obtain

$$\begin{aligned}
\sum_{i=1}^N \mathbb{E}[(\nu_t^{(i)})_3 | \mathcal{F}_{t-1}] &\le (N)_3 \left(\frac{1}{(N-1)\varepsilon/a + a/\varepsilon} \right)^3 \left[(N-1)\left(\frac{\varepsilon}{a}\right)^3 + \left(\frac{a}{\varepsilon}\right)^3 \right] \\
&= \frac{(N)_3}{((N-1)\varepsilon/a + a/\varepsilon)^3} \left(a^6 + (N-1)\varepsilon^6 \right) \\
&\le \frac{(N)_3}{N^3 \varepsilon^6} (N a^6) = \frac{(N)_3}{N^2} \frac{a^6}{\varepsilon^6}
\end{aligned}$$

where the last inequality follows because $a \ge \varepsilon$. Putting these together, we can bound the ratio above by

$$b_N := \frac{(N)_3 \frac{(N)_3}{N^2} \frac{a^6}{\varepsilon^6}}{(N)_2 \frac{(N)_2 \varepsilon^4}{N a^4}} = \frac{1}{N} \frac{a^{10}}{\varepsilon^{10}} \quad \xrightarrow{N\to\infty} 0 \tag{3}$$

We conclude the result by applying Theorem ? [the KJJS Thm1 with updated assns]. ☐

Theorem 2. *Under the time scaling of Koskela et al. (2018, Theorem 1) and the conditions of Koskela et al. (2018, Lemma 3), genealogies of conditional SMC algorithms with multinomial resampling converge to Kingman's n-coalescent in the sense of finite-dimensional distributions as $N \to \infty$.*

Proof. Denote the vector of particle weights $w_t^{(1:N)} = (w_t^{(1)}, w_t^{(2)}, \dots, w_t^{(N)})$. Let $\tilde{\nu}_t^{(1:N)} = (\tilde{\nu}_t^{(1)}, \tilde{\nu}_t^{(2)}, \dots, \tilde{\nu}_t^{(N)})$ denote the associated offspring counts under conditional SMC. Assuming WLOG that the immortal particle is particle 1, the offspring counts are distributed according to

$$\tilde{\nu}_t^{(1:N)} \mid w_t^{(1:N)} \overset{d}{=} (1, 0, 0, \dots, 0) + \text{Multinomial}(N-1, w_t^{(1:N)})$$

The standing assumption holds by exchangeability of the offspring assignments in Algorithm **??**. As before, we need to show that

$$\frac{\frac{1}{(N)_3}\sum_{i=1}^N \mathbb{E}[(\tilde{\nu}_t^{(i)})_3 | \mathcal{F}_{t-1}}{\frac{1}{(N)_2}\sum_{i=1}^N \mathbb{E}[(\tilde{\nu}_t^{(i)})_2 | \mathcal{F}_{t-1}]} \le b_N \tag{4}$$

for some deterministic sequence $b_N \lim_{N\to\infty} 0$. For the denominator, we find

$$\begin{aligned}
\sum_{i=1}^N \mathbb{E}[(\tilde{\nu}_t^{(i)})_2 | \mathcal{F}_{t-1} &= \mathbb{E}\left[(\tilde{\nu}_t^{(1)})_2 | \mathcal{F}_{t-1}\right] + \sum_{i=2}^N \mathbb{E}\left[(\tilde{\nu}_t^{(i)})_2 | \mathcal{F}_{t-1}\right] \\
&= (N-1)_2 \mathbb{E}[(w_t^{(1)})^2 | \mathcal{F}_{t-1}] + 2(N-1)\mathbb{E}[w_t^{(1)} | \mathcal{F}_{t-1}] + \sum_{i=2}^N (N-1)_2 \mathbb{E}[(w_t^{(i)})^2 | \mathcal{F}_{t-1}] \\
&= (N-1)_2 \sum_{i=1}^N \mathbb{E}[(w_t^{(i)})^2 | \mathcal{F}_{t-1}] + 2(N-1)\mathbb{E}[w_t^{(1)} | \mathcal{F}_{t-1}]
\end{aligned}$$

using that $(X+1)_2 = (X)_2 + 2(X)_1$ and the factorial moments of the Multinomial distribution (Mosimann, 1962). For the numerator, we have

$$\begin{aligned}
\sum_{i=1}^N \mathbb{E}[(\tilde{\nu}_t^{(i)})_3 | \mathcal{F}_{t-1} &= \mathbb{E}\left[(\tilde{\nu}_t^{(1)})_3 | \mathcal{F}_{t-1}\right] + \sum_{i=2}^N \mathbb{E}\left[(\tilde{\nu}_t^{(i)})_3 | \mathcal{F}_{t-1}\right] \\
&= (N-1)_3 \mathbb{E}[(w_t^{(1)})^3 | \mathcal{F}_{t-1}] + 3(N-1)_2\mathbb{E}[(w_t^{(1)})^2 | \mathcal{F}_{t-1}] + \sum_{i=2}^N (N-1)_3 \mathbb{E}[(w_t^{(i)})^3 | \mathcal{F}_{t-1}] \\
&= (N-1)_3 \sum_{i=1}^N \mathbb{E}[(w_t^{(i)})^3 | \mathcal{F}_{t-1}] + 3(N-1)_2\mathbb{E}[(w_t^{(1)})^2 | \mathcal{F}_{t-1}]
\end{aligned}$$

using similarly that $(X+1)_3 = (X)_3 + 3(X)_2$. Combining these expressions, the ratio in (4) becomes

$$\begin{aligned}
\frac{\frac{1}{(N)_3}\sum_{i=1}^N \mathbb{E}[(\tilde{\nu}_t^{(i)})_3 | \mathcal{F}_{t-1}}{\frac{1}{(N)_2}\sum_{i=1}^N \mathbb{E}[(\tilde{\nu}_t^{(i)})_2 | \mathcal{F}_{t-1}]} &= \frac{1}{N-2} \frac{(N-1)_3 \sum_{i=1}^N \mathbb{E}[(w_t^{(i)})^3 | \mathcal{F}_{t-1}] + 3(N-1)_2\mathbb{E}[(w_t^{(1)})^2 | \mathcal{F}_{t-1}]}{(N-1)_2 \sum_{i=1}^N \mathbb{E}[(w_t^{(i)})^2 | \mathcal{F}_{t-1}] + 2(N-1)\mathbb{E}[w_t^{(1)} | \mathcal{F}_{t-1}]} \\
&\le \frac{1}{N-2} \frac{(N-1)_3 \sum_{i=1}^N \mathbb{E}[(w_t^{(i)})^3 | \mathcal{F}_{t-1}]}{(N-1)_2 \sum_{i=1}^N \mathbb{E}[(w_t^{(i)})^2 | \mathcal{F}_{t-1}]} + \frac{1}{N-2} \frac{3(N-1)_2\mathbb{E}[(w_t^{(1)})^2 | \mathcal{F}_{t-1}]}{(N-1)_2 \sum_{i=1}^N \mathbb{E}[(w_t^{(i)})^2 | \mathcal{F}_{t-1}]} \\
&\le \frac{1}{N-2} \frac{(N-1)_3 \sum_{i=1}^N \mathbb{E}[(w_t^{(i)})^3 | \mathcal{F}_{t-1}]}{(N-1)_2 \sum_{i=1}^N \mathbb{E}[(w_t^{(i)})^2 | \mathcal{F}_{t-1}]} + \frac{1}{N-2} \frac{3(N-1)_2\mathbb{E}[(w_t^{(1)})^2 | \mathcal{F}_{t-1}]}{(N-1)_2 \mathbb{E}[(w_t^{(1)})^2 | \mathcal{F}_{t-1}]} \\
&< \frac{1}{N-2} \frac{\sum_{i=1}^N \mathbb{E}[(\nu_t^{(i)})_3 | \mathcal{F}_{t-1}]}{\sum_{i=1}^N \mathbb{E}[(\nu_t^{(i)})_2 | \mathcal{F}_{t-1}]} + \frac{3}{N-2}
\end{aligned}$$

where the last inequality comes from $\frac{(N-1)_3}{(N-1)_2} = N-3 < N-2 = \frac{(N)_3}{(N)_2}$. Then, using Theorem 1, we can bound this by

$$b_N := \frac{1}{N}\frac{a^6}{\varepsilon^6} + \frac{3}{N-2} \quad \overset{N\to\infty}{\longrightarrow} 0$$

as required, where a and ε are constants defined in the conditions of Koskela et al. (2018, Lemma 3). We conclude the result by applying Theorem ? [the KJJS Thm1 with updated assns]. $\square$

References

Koskela, J., Jenkins, P. A., Johansen, A. M. and Spanò, D. (2018), 'Asymptotic genealogies of interacting particle systems with an application to sequential Monte Carlo', *arXiv preprint arXiv:1804.01811* .

Mosimann, J. E. (1962), 'On the compound multinomial distribution, the multivariate β-distribution, and correlations among proportions', *Biometrika* **49**(1/2), 65–82.