

Weak convergence proof v.2

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Bounds on sum-products

Lemma 1. Fix $t > 0$, $l \in \mathbb{N}$.

$$t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \binom{l}{2} (t+1)^{l-2} \leq \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l c_N(s_j) \leq t^l + c_N(\tau_N(t))(t+1)^l. \quad (1)$$

Proof. As pointed out in Koskela et al. (2018, Equation (8)),

$$\sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l c_N(s_j) \geq \left(\sum_{s=0}^{\tau_N(t)} c_N(s) \right)^l - \binom{l}{2} \left(\sum_{s=0}^{\tau_N(t)} c_N(s)^2 \right) \left(\sum_{s=0}^{\tau_N(t)} c_N(s) \right)^{l-2}. \quad (2)$$

By definition of τ_N ,

$$t \leq \sum_{s=0}^{\tau_N(t)} c_N(s) \leq t+1. \quad (3)$$

Substituting these bounds into the RHS of (2) yields the lower bound.

It is a true fact that

$$\sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l c_N(s_j) \leq \left(\sum_{s=0}^{\tau_N(t)} c_N(s) \right)^l, \quad (4)$$

as can be seen by considering the multinomial expansion of the RHS. This is further bounded by

$$\left(\sum_{s=0}^{\tau_N(t)} c_N(s) \right)^l \leq \left(\sum_{s=0}^{\tau_N(t)-1} c_N(s) + c_N(\tau_N(t)) \right)^l \leq [t + c_N(\tau_N(t))]^l, \quad (5)$$

again using the definition of τ_N . A binomial expansion yields

$$[t + c_N(\tau_N(t))]^l = t^l + \sum_{i=0}^{l-1} \binom{l}{i} t^i c_N(\tau_N(t))^{l-i} = t^l + c_N(\tau_N(t)) \sum_{i=0}^{l-1} \binom{l}{i} t^i c_N(\tau_N(t))^{l-1-i}, \quad (6)$$

then since $c_N(s) \leq 1$ for all s ,

$$\sum_{i=0}^{l-1} \binom{l}{i} t^i c_N(\tau_N(t))^{l-1-i} \leq \sum_{i=0}^{l-1} \binom{l}{i} t^i \leq (t+1)^l. \quad (7)$$

Putting this together yields the upper bound. ■

Lemma 2. Fix $t > 0$, $l \in \mathbb{N}$. Let B be a positive constant which may depend on n .

$$\sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l [c_N(s_j) + BD_N(s_j)] \leq \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l c_N(s_j) + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1+B)^l. \quad (8)$$

Proof. We start with a binomial expansion:

$$\begin{aligned} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l [c_N(s_j) + BD_N(s_j)] &= \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \sum_{\mathcal{I} \subseteq [l]} B^{l-|\mathcal{I}|} \left(\prod_{i \in \mathcal{I}} c_N(s_i) \right) \left(\prod_{j \notin \mathcal{I}} D_N(s_j) \right) \\ &= \sum_{\mathcal{I} \subseteq [l]} B^{l-|\mathcal{I}|} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \left(\prod_{i \in \mathcal{I}} c_N(s_i) \right) \left(\prod_{j \notin \mathcal{I}} D_N(s_j) \right) \end{aligned} \quad (9)$$

where $[l] := \{1, \dots, l\}$. Since the sum is over all permutations of s_1, \dots, s_l , we may arbitrarily choose an ordering for $\{1, \dots, l\}$ such that $\mathcal{I} = \{1, \dots, |\mathcal{I}|\}$:

$$\sum_{\mathcal{I} \subseteq [l]} B^{l-|\mathcal{I}|} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \left(\prod_{i \in \mathcal{I}} c_N(s_i) \right) \left(\prod_{j \notin \mathcal{I}} D_N(s_j) \right) = \sum_{I=0}^l \binom{l}{I} B^{l-I} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^I c_N(s_i) \right) \left(\prod_{j=I+1}^l D_N(s_j) \right). \quad (10)$$

Separating the term $I = l$,

$$\begin{aligned} \sum_{I=0}^l \binom{l}{I} B^{l-I} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^I c_N(s_i) \right) \left(\prod_{j=I+1}^l D_N(s_j) \right) &= \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l c_N(s_j) \\ &\quad + \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^I c_N(s_i) \right) \left(\prod_{j=I+1}^l D_N(s_j) \right). \end{aligned} \quad (11)$$

In the second line, there is always at least one D_N term, and $c_N(s) \geq D_N(s)$ for all s (Koskela et al., 2018, p.9), so we can write

$$\begin{aligned} \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^I c_N(s_i) \right) \left(\prod_{j=I+1}^l D_N(s_j) \right) &\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{l-1} c_N(s_i) \right) D_N(s_l) \\ &\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \left(\sum_{s_1 \neq \dots \neq s_{l-1}}^{\tau_N(t)} \prod_{i=1}^{l-1} c_N(s_i) \right) \sum_{s_l=1}^{\tau_N(t)} D_N(s_l) \\ &\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} (t+1)^{l-1} \sum_{s=1}^{\tau_N(t)} D_N(s) \end{aligned} \quad (12)$$

using (4) and (3). Finally, by the Binomial Theorem,

$$\sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} (t+1)^{l-1} \sum_{s=1}^{\tau_N(t)} D_N(s) \leq \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1+B)^l, \quad (13)$$

which, together with (11), concludes the proof. ■

Lemma 3. Fix $t > 0$, $l \in \mathbb{N}$. Let B be a positive constant which may depend on n .

$$\sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l [c_N(s_j) - BD_N(s_j)] \geq \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l c_N(s_j) - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1+B)^l. \quad (14)$$

Proof. A binomial expansion and subsequent manipulation as in (9)–(11) gives

$$\begin{aligned}
\sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l [c_N(s_j) - BD_N(s_j)] &= \sum_{\mathcal{I} \subseteq [l]} (-B)^{l-|\mathcal{I}|} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \left(\prod_{i \in \mathcal{I}} c_N(s_i) \right) \left(\prod_{j \notin \mathcal{I}} D_N(s_j) \right) \\
&= \sum_{I=0}^l \binom{l}{I} (-B)^{l-I} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^I c_N(s_i) \right) \left(\prod_{j=I+1}^l D_N(s_j) \right) \\
&= \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l c_N(s_j) + \sum_{I=0}^{l-1} \binom{l}{I} (-B)^{l-I} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^I c_N(s_i) \right) \left(\prod_{j=I+1}^l D_N(s_j) \right) \\
&\geq \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l c_N(s_j) - \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^I c_N(s_i) \right) \left(\prod_{j=I+1}^l D_N(s_j) \right)
\end{aligned} \tag{15}$$

where the last inequality just multiplies some positive terms by -1 . Then (12)–(13) can be applied directly (noting that an upper bound on negative terms gives a lower bound overall):

$$-\sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^I c_N(s_i) \right) \left(\prod_{j=I+1}^l D_N(s_j) \right) \geq - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1+B)^l \tag{16}$$

which concludes the proof. ■

Main components of weak convergence

Lemma 4 (Basis step). *For any $0 < t < \infty$,*

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\prod_{r=1}^{\tau_N(t)} (1 - p_r) \right] = e^{-\alpha_n t} \tag{17}$$

where $\alpha_n := n(n-1)/2$.

Proof. We start by showing that $\lim_{N \rightarrow \infty} \mathbb{E} \left[\prod_{r=1}^{\tau_N(t)} (1 - p_r) \right] \leq e^{-\alpha_n t}$.

From Koskela et al. (2018, Lemma 1 Case 1), taking $\xi = \Delta$, we have for each r

$$1 - p_r = p_{\Delta\Delta}(r) \leq 1 - \alpha_n(1 + O(N^{-1})) [c_N(r) - B'_n D_N(r)] \tag{18}$$

where the $O(N^{-1})$ term does not depend on r . When N is large enough, a sufficient condition to ensure the bound in (18) is non-negative is the event

$$E_r := \{c_N(r) \leq \alpha_n^{-1}\}. \tag{19}$$

We will also need to control the sign of $c_N(r) - B'_n D_N(r)$, for which we define the event

$$E'_r := \{c_N(r) \geq B'_n D_N(r)\}, \tag{20}$$

and we define $E := \bigcap_{r=1}^{\tau_N(t)} E_r$ and $E' := \bigcap_{r=1}^{\tau_N(t)} E'_r$. Then

$$1 - p_r = p_{\Delta\Delta}(r) \leq 1 - \alpha_n(1 + O(N^{-1})) [c_N(r) - B'_n D_N(r)] \mathbb{1}_{E \cap E'}. \tag{21}$$

Applying a multinomial expansion and then separating the positive and negative terms,

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1 - p_r) &\leq 1 + \sum_{l=1}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \dots \neq s_l} \prod_{j=1}^l [c_N(s_j) - B'_n D_N(s_j)] \mathbb{1}_{E \cap E'} \\
&= 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \dots \neq s_l} \prod_{j=1}^l [c_N(s_j) - B'_n D_N(s_j)] \mathbb{1}_{E \cap E'} \\
&\quad - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \dots \neq s_l} \prod_{j=1}^l [c_N(s_j) - B'_n D_N(s_j)] \mathbb{1}_{E \cap E'}. \tag{22}
\end{aligned}$$

This is further bounded by applying Lemma 3 and then both bounds of Lemma 1:

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1 - p_r) &\leq 1 + \left\{ \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \dots \neq s_l} \prod_{j=1}^l c_N(s_j) \right. \\
&\quad \left. - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left[\sum_{s_1 \neq \dots \neq s_l} \prod_{j=1}^l c_N(s_j) - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1 + B'_n)^l \right] \right\} \mathbb{1}_{E \cap E'} \\
&\leq 1 + \left\{ \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \{t^l + c_N(\tau_N(t))(t+1)^l\} \right. \\
&\quad \left. - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left[t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \binom{l}{2} (t+1)^{l-2} - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1 + B'_n)^l \right] \right\} \mathbb{1}_{E \cap E'}. \tag{23}
\end{aligned}$$

Collecting some terms,

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1 - p_r) &\leq 1 + \sum_{l=1}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} \mathbb{1}_{E \cap E'} + c_N(\tau_N(t)) \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} (t+1)^l \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \binom{l}{2} (t+1)^{l-2} \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} (t+1)^{l-1} (1 + B'_n)^l \\
&\leq 1 + \sum_{l=1}^{\infty} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_{\{\tau_N(t) \geq l\}} \mathbb{1}_{E \cap E'} + c_N(\tau_N(t)) \exp[\alpha_n (1 + O(N^{-1}))(t+1)] \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n (1 + O(N^{-1}))(t+1)] \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n (1 + O(N^{-1}))(t+1)(1 + B'_n)]. \tag{24}
\end{aligned}$$

Now, taking the expectation and limit, then applying Brown et al. (2021, Equations (3.3)–(3.5)), and Lemmata 9,

10 and 12 to deal with the indicators,

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathbb{E} \left[\prod_{r=1}^{\tau_N(t)} (1 - p_r) \right] &\leq 1 + \sum_{l=1}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l \lim_{N \rightarrow \infty} \mathbb{P} [\{\tau_N(t) \geq l\} \cap E \cap E'] + \lim_{N \rightarrow \infty} \mathbb{E} [c_N(\tau_N(t))] \exp[\alpha_n(t+1)] \\
&\quad + \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right] \frac{1}{2} \alpha_n^2 \exp[\alpha_n(t+1)] \\
&\quad + \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{s=1}^{\tau_N(t)} D_N(s) \right] \exp[\alpha_n(t+1)(1+B'_n)] \\
&= 1 + \sum_{l=1}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l = e^{-\alpha_n t}.
\end{aligned} \tag{25}$$

It remains to show the corresponding lower bound $\lim_{N \rightarrow \infty} \mathbb{E} \left[\prod_{r=1}^{\tau_N(t)} (1 - p_r) \right] \geq e^{-\alpha_n t}$. From Brown et al. (2021, Equation (3.14)), taking $\xi = \Delta$, we have

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \alpha_n(1 + O(N^{-1})) [c_N(t) + B_n D_N(t)] \tag{26}$$

where $B_n > 0$ and the $O(N^{-1})$ term does not depend on t . In particular,

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \frac{N^{n-2}}{(N-2)_{n-2}} \alpha_n [c_N(t) + B_n D_N(t)]. \tag{27}$$

Since $D_N(s) \leq c_N(s)$ for all s (Koskela et al., 2018, p.9), a sufficient condition for this bound to be non-negative is

$$E_r := \left\{ c_N(r) \leq \frac{(N-2)_{n-2}}{N^{n-2}} \alpha_n^{-1} (1 + B_n)^{-1} \right\}, \tag{28}$$

and we again define $E := \bigcap_{r=1}^{\tau_N(t)} E_r$. We now apply a multinomial expansion to the product, and split into positive and negative terms:

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1 - p_r) &\geq \left\{ 1 + \sum_{l=1}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \dots \neq s_l} \prod_{j=1}^l [c_N(s_j) + B_n D_N(s_j)] \right\} \mathbb{1}_E \\
&= \left\{ 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \dots \neq s_l} \prod_{j=1}^l [c_N(s_j) + B_n D_N(s_j)] \right. \\
&\quad \left. - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \dots \neq s_l} \prod_{j=1}^l [c_N(s_j) + B_n D_N(s_j)] \right\} \mathbb{1}_E
\end{aligned} \tag{29}$$

This is further bounded by applying Lemma 2 and both bounds in Lemma 1:

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1 - p_r) &\geq \left\{ 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \dots \neq s_l} \prod_{j=1}^l c_N(s_j) \right. \\
&\quad \left. - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left[\sum_{s_1 \neq \dots \neq s_l} \prod_{j=1}^l c_N(s_j) + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1 + B_n)^l \right] \right\} \mathbb{1}_E \\
&\geq \left\{ 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left[t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \binom{l}{2} (t+1)^{l-2} \right] \right. \\
&\quad \left. - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left[t^l + c_N(\tau_N(t)) (t+1)^l + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1 + B_n)^l \right] \right\} \mathbb{1}_E.
\end{aligned} \tag{30}$$

Collecting terms,

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1 - p_r) &\geq \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_E - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \binom{l}{2} (t+1)^{l-2} \\
&\quad - c_N(\tau_N(t)) \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} (t+1)^l \\
&\quad - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} (t+1)^{l-1} (1 + B_n)^l \\
&\geq \sum_{l=0}^{\infty} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_E \mathbb{1}_{\{\tau_N(t) \geq l\}} - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n (1 + O(N^{-1})) (t+1)] \\
&\quad - c_N(\tau_N(t)) \exp[\alpha_n (1 + O(N^{-1})) (t+1)] \\
&\quad - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n (1 + O(N^{-1})) (t+1) (1 + B_n)]. \tag{31}
\end{aligned}$$

Now, taking the expectation and limit, and applying Brown et al. (2021, Equations (3.3)–(3.5)) to show that all but the first sum vanish, and Lemmata 10 and 9 to show that $\lim_{N \rightarrow \infty} \mathbb{P}[\{\tau_N(t) \geq l\} \cap E] = 1$,

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathbb{E} \left[\prod_{r=1}^{\tau_N(t)} (1 - p_r) \right] &\geq \sum_{l=0}^{\infty} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \lim_{N \rightarrow \infty} \mathbb{P}[\{\tau_N(t) \geq l\} \cap E] \\
&\quad - \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right] \frac{1}{2} \alpha_n^2 \exp[\alpha_n (t+1)] \\
&\quad - \lim_{N \rightarrow \infty} \mathbb{E} [c_N(\tau_N(t))] \exp[\alpha_n (t+1)] \\
&\quad - \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{s=1}^{\tau_N(t)} D_N(s) \right] \exp[\alpha_n (t+1) (1 + B_n)] \\
&= \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l = e^{-\alpha_n t}. \tag{32}
\end{aligned}$$

Combining the upper and lower bounds in (25) and (32) respectively concludes the proof. ■

Lemma 5 (Induction step upper bound). *Fix $k \in \mathbb{N}$, $i_0 := 0$, $i_k := k$. For any sequence of times $0 = t_0 \leq t_1 \leq \dots \leq t_k \leq t$,*

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^k p_{r_i} \right) \left(\prod_{\substack{r=1 \\ r \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) \right) \right] \leq \alpha_n^k \sum_{l=0}^{\infty} \frac{1}{l!} (-\alpha_n)^l t^l \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}. \tag{33}$$

Proof. We use the bound on $(1 - p_r)$ from (18) and apply a multinomial expansion, defining as in (19) and (20)

respectively the events E and E' which ensure the bounds are non-negative:

$$\begin{aligned}
\prod_{\substack{r=1 \\ r \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) &\leq \prod_{\substack{r=1 \\ r \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} \{1 - \alpha_n(1 + O(N^{-1}))[c_N(r) - B'_n D_N(r)] \mathbb{1}_{E \cap E'}\} \\
&= 1 + \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{\substack{s_1 \neq \dots \neq s_l \\ s_i \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} \prod_{j=1}^l [c_N(s_j) - B'_n D_N(s_j)] \mathbb{1}_{E \cap E'} \\
&= 1 + \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l [c_N(s_j) - B'_n D_N(s_j)] \mathbb{1}_{E \cap E'} \\
&\quad - \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{\substack{s_1 \neq \dots \neq s_l \\ \exists i, i': s_i = r_{i'}}}^{\tau_N(t)} \prod_{j=1}^l [c_N(s_j) - B'_n D_N(s_j)] \mathbb{1}_{E \cap E'}. \tag{34}
\end{aligned}$$

The penultimate line above is exactly the expansion we had in the basis step (22), except for the limit on l , and as such following the same arguments gives a bound analogous to that in (24):

$$\begin{aligned}
1 + \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l [c_N(s_j) - B'_n D_N(s_j)] \mathbb{1}_{E \cap E'} \\
\leq 1 + \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_{E \cap E'} + c_N(\tau_N(t)) \exp[\alpha_n(1 + O(N^{-1}))(t + 1)] \\
\quad + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n(1 + O(N^{-1}))(t + 1)] \\
\quad + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n(1 + O(N^{-1}))(t + 1)(1 + B'_n)]. \tag{35}
\end{aligned}$$

For the last line of (34),

$$\begin{aligned}
& - \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{\substack{s_1 \neq \dots \neq s_l \\ \exists i, i': s_i = r_{i'}}} \prod_{j=1}^l \{c_N(s_j) - B'_n D_N(s_j)\} \mathbb{1}_{E \cap E'} \\
& \leq \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{\substack{s_1 \neq \dots \neq s_l \\ \exists i, i': s_i = r_{i'}}} \prod_{j=1}^l \{c_N(s_j) + B'_n D_N(s_j)\} \\
& \leq \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{\substack{s_1 \neq \dots \neq s_l \\ \exists i, i': s_i = r_{i'}}} (1 + B'_n)^l \prod_{j=1}^l c_N(s_j) \\
& \leq \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1 + O(N^{-1})) \frac{1}{(l-1)!} \sum_{s_1 \in \{r_1, \dots, r_k\}} \sum_{s_2 \neq \dots \neq s_l}^{\tau_N(t)} (1 + B'_n)^l \prod_{j=1}^l c_N(s_j) \\
& = \sum_{s \in \{r_1, \dots, r_k\}} c_N(s) \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1 + O(N^{-1})) \frac{1}{(l-1)!} (1 + B'_n)^l \sum_{s_1 \neq \dots \neq s_{l-1}}^{\tau_N(t)} \prod_{j=1}^{l-1} c_N(s_j) \\
& \leq \sum_{j=1}^k c_N(r_j) \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1 + O(N^{-1})) \frac{1}{(l-1)!} (1 + B'_n)^l (t+1)^{l-1} \\
& \leq \left(\sum_{j=1}^k c_N(r_j) \right) \alpha_n (1 + B'_n) \exp[\alpha_n (1 + O(N^{-1})) (1 + B'_n) (t+1)]. \tag{36}
\end{aligned}$$

Putting these together, we have

$$\begin{aligned}
& \prod_{\substack{r=1 \\ r \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) \leq 1 + \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} \mathbb{1}_{E \cap E'} + c_N(\tau_N(t)) \exp[\alpha_n (1 + O(N^{-1})) (t+1)] \\
& \quad + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n (1 + O(N^{-1})) (t+1)] \\
& \quad + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n (1 + O(N^{-1})) (t+1) (1 + B'_n)] \\
& \quad + \left(\sum_{j=1}^k c_N(r_j) \right) \alpha_n (1 + B'_n) \exp[\alpha_n (1 + O(N^{-1})) (1 + B'_n) (t+1)]. \tag{37}
\end{aligned}$$

Meanwhile, using the bound on p_r from (26) then applying a modification of Lemma 2,

$$\begin{aligned}
& \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k p_{r_i} \leq \alpha_n^k (1 + O(N^{-1})) \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k [c_N(r_i) + B_n D_N(r_i)] \\
& \leq \alpha_n^k (1 + O(N^{-1})) \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \alpha_n^k (1 + O(N^{-1})) (t+1)^{k-1} (1 + B_n)^k. \tag{38}
\end{aligned}$$

A more liberal (but simpler) bound can be arrived at thus:

$$\begin{aligned}
\prod_{i=1}^k p_{r_i} &\leq \alpha_n^k (1 + O(N^{-1})) \prod_{i=1}^k [c_N(r_i) + B_n D_N(r_i)] \\
&\leq \alpha_n^k (1 + O(N^{-1})) \prod_{i=1}^k c_N(r_i) (1 + B_n) \\
&\leq \alpha_n^k (1 + O(N^{-1})) (1 + B_n)^k \prod_{i=1}^k c_N(r_i)
\end{aligned} \tag{39}$$

which also leads to the deterministic bound

$$\begin{aligned}
\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k p_{r_i} &\leq \alpha_n^k (1 + O(N^{-1})) (1 + B_n)^k \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) \\
&\leq \alpha_n^k (1 + O(N^{-1})) (1 + B_n)^k \frac{1}{k!} \sum_{r_1 \neq \dots \neq r_k}^{\tau_N(t)} \prod_{i=1}^k c_N(r_i) \\
&\leq \alpha_n^k (1 + O(N^{-1})) (1 + B_n)^k \frac{1}{k!} (t + 1)^k.
\end{aligned} \tag{40}$$

Combining (37) with the other product, the expression inside the expectation in (33) is bounded above by

$$\begin{aligned}
&\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^k p_{r_i} \right) \left(\prod_{\substack{r=1 \\ r \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) \right) \leq \left\{ 1 + \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_{E \cap E'} \right\} \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k p_{r_i} \\
&\quad + \left\{ c_N(\tau_N(t)) \exp[\alpha_n (1 + O(N^{-1}))(t + 1)] + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n (1 + O(N^{-1}))(t + 1)] \right. \\
&\quad \left. + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n (1 + O(N^{-1}))(t + 1)(1 + B'_n)] \right\} \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k p_{r_i} \\
&\quad + \exp[\alpha_n (1 + O(N^{-1}))(1 + B'_n)(t + 1)] \alpha_n (1 + B'_n) \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \sum_{j=1}^k c_N(r_j) \prod_{i=1}^k p_{r_i}.
\end{aligned} \tag{41}$$

Applying the various bounds (38)–(40), we have

$$\begin{aligned}
& \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^k p_{r_i} \right) \left(\prod_{\substack{r=1 \\ r \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) \right) \\
& \leq \alpha_n^k (1 + O(N^{-1})) \left\{ 1 + \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_{E \cap E'} \right\} \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) \\
& \quad + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \alpha_n^k (1 + O(N^{-1})) (t+1)^{k-1} (1 + B_n)^k \sum_{l=0}^{\tau_N(t)} (\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \\
& \quad + \left\{ c_N(\tau_N(t)) \exp[\alpha_n (1 + O(N^{-1})) (t+1)] + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n (1 + O(N^{-1})) (t+1)] \right. \\
& \quad \left. + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n (1 + O(N^{-1})) (t+1) (1 + B'_n)] \right\} \alpha_n^k (1 + O(N^{-1})) (1 + B_n)^k \frac{1}{k!} (t+1)^k \\
& \quad + \exp[\alpha_n (1 + B'_n) (t+1)] \alpha_n (1 + B'_n) \alpha_n^k (1 + O(N^{-1})) (1 + B_n)^k \\
& \quad \times \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \sum_{j=1}^k c_N(r_j) \prod_{i=1}^k c_N(r_i). \tag{42}
\end{aligned}$$

Upon taking the expectation and limit, we have

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^k p_{r_i} \right) \left(\prod_{\substack{r=1 \\ r \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) \right) \right] \leq \alpha_n^k \lim_{N \rightarrow \infty} \mathbb{E} \left[\left(1 + \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l \frac{1}{l!} t^l \mathbb{1}_{E \cap E'} \right) \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) \right] \\
& \quad + \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{s=1}^{\tau_N(t)} D_N(s) \right] \alpha_n^k (t+1)^{k-1} (1 + B_n)^k \exp[\alpha_n t] \\
& \quad + \left\{ \lim_{N \rightarrow \infty} \mathbb{E} [c_N(\tau_N(t))] \exp[\alpha_n (t+1)] + \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right] \frac{1}{2} \alpha_n^2 \exp[\alpha_n (t+1)] \right. \\
& \quad \left. + \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{s=1}^{\tau_N(t)} D_N(s) \right] \exp[\alpha_n (t+1) (1 + B'_n)] \right\} \alpha_n^k (1 + B_n)^k \frac{1}{k!} (t+1)^k \\
& \quad + \exp[\alpha_n (1 + B'_n) (t+1)] \alpha_n^{k+1} (1 + B'_n) (1 + B_n)^k \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \sum_{j=1}^k c_N(r_j) \prod_{i=1}^k c_N(r_i) \right]. \tag{43}
\end{aligned}$$

The middle terms vanish due to Brown et al. (2021, Equations (3.3)–(3.5)) and the expression becomes

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^k p_{r_i} \right) \left(\prod_{\substack{r=1 \\ r \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) \right) \right] &\leq \alpha_n^k \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) \right] \\
&+ \alpha_n^k \sum_{l=1}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l \lim_{N \rightarrow \infty} \mathbb{E} \left[\mathbb{1}_{\{\tau_N(t) \geq k+l\}} \mathbb{1}_{E \cap E'} \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) \right] \\
&+ \exp[\alpha_n(1 + B'_n)(t + 1)] \alpha_n^{k+1} (1 + B'_n)(1 + B_n)^k \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \sum_{j=1}^k c_N(r_j) \prod_{i=1}^k c_N(r_i) \right]. \quad (44)
\end{aligned}$$

To simplify the last line,

$$\begin{aligned}
\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \sum_{j=1}^k c_N(r_j) \prod_{i=1}^k c_N(r_i) &\leq \frac{1}{k!} \sum_{r_1 \neq \dots \neq r_k}^{\tau_N(t)} \sum_{j=1}^k c_N(r_j) \prod_{i=1}^k c_N(r_i) \\
&= \frac{1}{k!} \sum_{r_1 \neq \dots \neq r_k}^{\tau_N(t)} \sum_{j=1}^k c_N(r_j)^2 \prod_{i \neq j} c_N(r_i) \\
&\leq \frac{1}{k!} \sum_{j=1}^k \sum_{s=1}^{\tau_N(t)} c_N(s)^2 \sum_{r_1 \neq \dots \neq r_{k-1}}^{\tau_N(t)} \prod_{i=1}^{k-1} c_N(r_i) \\
&\leq \frac{1}{(k-1)!} \sum_{s=1}^{\tau_N(t)} c_N(s)^2 (t+1)^{k-1} \quad (45)
\end{aligned}$$

hence

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \sum_{s \in \{r_1, \dots, r_k\}} c_N(s) \prod_{i=1}^k c_N(r_i) \right] \leq \frac{1}{(k-1)!} (t+1)^{k-1} \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right] = 0 \quad (46)$$

by Brown et al. (2021, Equation (3.5)). By Lemmata 10, 9 and 12, $\lim_{N \rightarrow \infty} \mathbb{P}[\{\tau_N(t) \geq k+l\} \cap E \cap E'] = 1$, so we can apply Lemma 7 to the remaining expectations in (44), yielding

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^k p_{r_i} \right) \left(\prod_{\substack{r=1 \\ r \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) \right) \right] &\leq \alpha_n^k \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \\
&= \alpha_n^k e^{-\alpha_n t} \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \quad (47)
\end{aligned}$$

as required. ■

Lemma 6 (Induction step lower bound). *Fix $k \in \mathbb{N}$, $i_0 := 0$, $i_k := k$. For any sequence of times $0 = t_0 \leq t_1 \leq \dots \leq t_k \leq t$,*

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^k p_{r_i} \right) \left(\prod_{\substack{r=1 \\ r \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) \right) \right] \geq \alpha_n^k \sum_{l=0}^{\infty} \frac{1}{l!} (-\alpha_n)^l t^l \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}. \quad (48)$$

Proof. Firstly,

$$\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^k p_{r_i} \right) \left(\prod_{\substack{r=1 \\ r \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) \right) \geq \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^k p_{r_i} \right) \left(\prod_{r=1}^{\tau_N(t)} (1 - p_r) \right). \quad (49)$$

Now the second product does not depend on r_1, \dots, r_k , and we can use the lower bound from (31):

$$\begin{aligned} \prod_{r=1}^{\tau_N(t)} (1 - p_r) &\geq \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_E - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n(1 + O(N^{-1}))(t+1)] \\ &\quad - c_N(\tau_N(t)) \exp[\alpha_n(1 + O(N^{-1}))(t+1)] \\ &\quad - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n(1 + O(N^{-1}))(t+1)(1 + B_n)] \end{aligned} \quad (50)$$

where E is defined as in (28). We will also need an upper bound on this product, which is formed from (24) with a further deterministic bound:

$$\begin{aligned} \prod_{r=1}^{\tau_N(t)} (1 - p_r) &\leq \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l + c_N(\tau_N(t)) \exp[\alpha_n(1 + O(N^{-1}))(t+1)] \\ &\quad + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n(1 + O(N^{-1}))(t+1)] \\ &\quad + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n(1 + O(N^{-1}))(t+1)(1 + B'_n)] \\ &\leq \exp[\alpha_n(1 + O(N^{-1}))t] + \exp[\alpha_n(1 + O(N^{-1}))(t+1)] \\ &\quad + \frac{1}{2} \alpha_n^2 (t+1) \exp[\alpha_n(1 + O(N^{-1}))(t+1)] + (t+1) \exp[\alpha_n(1 + O(N^{-1}))(t+1)(1 + B'_n)] \\ &\leq \left(2 + \frac{\alpha_n^2(t+1)}{2} \right) \exp[\alpha_n(1 + O(N^{-1}))(t+1)] + (t+1) \exp[\alpha_n(1 + O(N^{-1}))(t+1)(1 + B'_n)]. \end{aligned} \quad (51)$$

Now let us consider the remaining sum-product on the RHS of (49). We use the same bound on p_r as in (18):

$$p_r = 1 - p_{\Delta\Delta}(r) \geq \alpha_n(1 + O(N^{-1})) [c_N(r) - B'_n D_N(r)] \quad (52)$$

where the $O(N^{-1})$ term does not depend on r . When N is large enough for the factor of $(1 + O(N^{-1}))$ to be non-negative, a sufficient condition to ensure the bound in (52) is non-negative is the event E' that was defined in (20). Then

$$\prod_{i=1}^k p_{r_i} \geq \alpha_n^k (1 + O(N^{-1})) \prod_{i=1}^k [c_N(r_i) - B'_n D_N(r_i)] \mathbb{1}_{E'}. \quad (53)$$

Applying a modification of Lemma 3,

$$\begin{aligned}
\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^k p_{r_i} \right) &\geq \alpha_n^k (1 + O(N^{-1})) \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k [c_N(r_i) - B'_n D_N(r_i)] \mathbb{1}_{E'} \\
&\geq \alpha_n^k (1 + O(N^{-1})) \left\{ \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) \mathbb{1}_{E'} - \frac{1}{k!} \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{k-1} (1 + B'_n)^k \right\}.
\end{aligned} \tag{54}$$

The above expression is already split into positive and negative terms; a lower bound on (49) can be formed by multiplying the positive terms by the lower bound (50) and the negative terms by the upper bound (51). Thus

$$\begin{aligned}
\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^k p_{r_i} \right) \left(\prod_{\substack{r=1 \\ r \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) \right) &\geq \alpha_n^k (1 + O(N^{-1})) \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) \mathbb{1}_{E'} \left\{ \right. \\
&\quad \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_E \\
&\quad - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n (1 + O(N^{-1})) (t+1)] \\
&\quad - c_N(\tau_N(t)) \exp[\alpha_n (1 + O(N^{-1})) (t+1)] \\
&\quad \left. - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n (1 + O(N^{-1})) (t+1) (1 + B_n)] \right\} \\
&\quad - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \alpha_n^k (1 + O(N^{-1})) \frac{1}{k!} (t+1)^{k-1} (1 + B'_n)^k \left\{ \right. \\
&\quad \left(2 + \frac{\alpha_n^2 (t+1)}{2} \right) \exp[\alpha_n (1 + O(N^{-1})) (t+1)] \\
&\quad \left. + (t+1) \exp[\alpha_n (1 + O(N^{-1})) (t+1) (1 + B'_n)] \right\}.
\end{aligned} \tag{55}$$

Due to Brown et al. (2021, Equations (3.3)–(3.5)), all but the first two lines in the above have vanishing expectation, leaving

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^k p_{r_i} \right) \left(\prod_{\substack{r=1 \\ r \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) \right) \right] \\
\geq \lim_{N \rightarrow \infty} \mathbb{E} \left[\alpha_n^k (1 + O(N^{-1})) \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) \mathbb{1}_{E'} \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_E \right] \\
= \alpha_n^k \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l \lim_{N \rightarrow \infty} \mathbb{E} \left[\mathbb{1}_{\{\tau_N(t) \geq l\}} \mathbb{1}_{E \cap E'} \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) \right]
\end{aligned} \tag{56}$$

Lemmata 9 and 12 establish that $\lim_{N \rightarrow \infty} \mathbb{P}[E \cap E'] = 1$ and Lemma 10 deals with the other indicator. We can

therefore apply Lemma 7 to conclude that

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^k p_{r_i} \right) \left(\prod_{\substack{r=1 \\ r \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) \right) \right] &\geq \alpha_n^k \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \\
&= \alpha_n^k e^{-\alpha_n t} \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \quad (57)
\end{aligned}$$

as required. ■

Lemma 7. Fix $k \in \mathbb{N}$, $i_0 := 0$, $i_k := k$. Let E be any event independent of r_1, \dots, r_k such that $\lim_{N \rightarrow \infty} \mathbb{P}[E] = 1$. Then for any sequence of times $0 = t_0 \leq t_1 \leq \dots \leq t_k \leq t$,

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\mathbb{1}_E \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) \right] = \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}. \quad (58)$$

Proof. As pointed out by Möhle (1999, p. 460), the sum-product on the left hand side can be expanded as

$$\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) = \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \sum_{\substack{r_{i_{j-1}+1} < \dots < r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i). \quad (59)$$

Moreover,

$$\sum_{\substack{r_{i_{j-1}+1} < \dots < r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) = \frac{1}{(i_j - i_{j-1})!} \sum_{\substack{r_{i_{j-1}+1} \neq \dots \neq r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i). \quad (60)$$

By a modification of the upper bound in Lemma 1,

$$\begin{aligned}
\sum_{\substack{r_{i_{j-1}+1} \neq \dots \neq r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) &\leq (t_j - t_{j-1})^{i_j - i_{j-1}} + c_N(\tau_N(t_j))(t_j - t_{j-1} + 1)^{i_j - i_{j-1}} \\
&\leq (t_j - t_{j-1})^{i_j - i_{j-1}} + c_N(\tau_N(t_j))(t_j - t_{j-1} + 1)^k. \quad (61)
\end{aligned}$$

Now, taking the product on the outside,

$$\begin{aligned}
\prod_{j=1}^k \sum_{\substack{r_{i_{j-1}+1} < \dots < r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) &\leq \prod_{j=1}^k \left\{ \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} + c_N(\tau_N(t_j)) \frac{(1 + t_j - t_{j-1})^k}{(i_j - i_{j-1})!} \right\} \\
&\leq \prod_{j=1}^k \left\{ \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} + c_N(\tau_N(t_j)) (1 + t_j - t_{j-1})^k \right\} \\
&= \sum_{\mathcal{I} \subseteq [k]} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} c_N(\tau_N(t_j)) (1 + t_j - t_{j-1})^k \right) \\
&= \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \\
&\quad + \sum_{\mathcal{I} \subset [k]} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} c_N(\tau_N(t_j)) (1 + t_j - t_{j-1})^k \right) \\
&\leq \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \\
&\quad + \sum_{\mathcal{I} \subset [k]} c_N(\tau_N(t_{j^*})) \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) (1 + t_j - t_{j-1})^{k^2} \tag{62}
\end{aligned}$$

where, say, $j^* := \min\{j \notin \mathcal{I}\}$. Now we are in a position to evaluate the limit in (58):

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathbb{E} \left[\mathbb{1}_E \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) \right] &\leq \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) \right] \\
&\leq \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \\
&\quad + \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \sum_{\mathcal{I} \subset [k]} \lim_{N \rightarrow \infty} \mathbb{E} [c_N(\tau_N(t_{j^*}))] \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) (1 + t_j - t_{j-1})^{k^2} \\
&= \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \tag{63}
\end{aligned}$$

using Brown et al. (2021, Equation (3.3)).

For the corresponding lower bound, by a modification of the lower bound in Lemma 1,

$$\begin{aligned}
\sum_{\substack{r_{i_{j-1}+1} \neq \dots \neq r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) &\geq (t_j - t_{j-1})^{i_j - i_{j-1}} - \binom{i_j - i_{j-1}}{2} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{i_j - i_{j-1} - 2} \\
&\geq (t_j - t_{j-1})^{i_j - i_{j-1}} - (i_j - i_{j-1})! \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{k-1}. \tag{64}
\end{aligned}$$

Define the event

$$E_j^\star = \left\{ \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) \leq \frac{(t_j - t_{j-1})^{i_j - i_{j-1} - k + 1}}{(i_j - i_{j-1})!} \right\}, \quad (65)$$

(If $t_j = t_{j-1}$ then E_j^\star has probability 1 even without the limit; otherwise this is in the right form for Lemma 11.) which is sufficient to ensure the j^{th} term in the following product is non-negative, and define $E^\star := \bigcap_{j=1}^k E_j^\star$. Now, taking a product over j ,

$$\begin{aligned} \prod_{j=1}^k \sum_{\substack{r_{i_{j-1}+1} < \dots < r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) &\geq \prod_{j=1}^k \left\{ \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} - \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{k-1} \right\} \mathbb{1}_{E^\star} \\ &= \sum_{\mathcal{I} \subseteq [k]} (-1)^{k-|\mathcal{I}|} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{k-1} \right) \mathbb{1}_{E^\star} \\ &= \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E^\star} \\ &\quad + \sum_{\mathcal{I} \subset [k]} (-1)^{k-|\mathcal{I}|} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{k-1} \right) \mathbb{1}_{E^\star} \\ &\geq \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E^\star} \\ &\quad - \sum_{\mathcal{I} \subset [k]} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{k-1} \right) \\ &\geq \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E^\star} \\ &\quad - \sum_{\mathcal{I} \subset [k]} \left(\sum_{s=\tau_N(t_{j^*-1})+1}^{\tau_N(t_{j^*})} c_N(s)^2 \right) \left(\prod_{j \in \mathcal{I}} (t_j - t_{j-1})^k \right) \left(\prod_{j \notin \mathcal{I}} (t_j - t_{j-1} + 1)^k \right) \\ &\geq \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E^\star} - \sum_{\mathcal{I} \subset [k]} \left(\sum_{s=\tau_N(t_{j^*-1})+1}^{\tau_N(t_{j^*})} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{k^2}, \end{aligned} \quad (66)$$

where again we have arbitrarily set $j^* := \min\{j \notin \mathcal{I}\}$. We can now evaluate the limit:

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathbb{E} \left[\mathbb{1}_E \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) \right] &\geq \lim_{N \rightarrow \infty} \mathbb{E} \left[\mathbb{1}_{E \cap E^*} \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right] \\
&\quad - \lim_{N \rightarrow \infty} \mathbb{E} \left[\mathbb{1}_E \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \sum_{\mathcal{I} \subset [k]} \left(\sum_{s=\tau_N(t_{j^*-1})+1}^{\tau_N(t_{j^*})} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{k^2} \right] \\
&\geq \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \lim_{N \rightarrow \infty} \mathbb{E}[\mathbb{1}_{E \cap E^*}] \\
&\quad - \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \sum_{\mathcal{I} \subset [k]} \left(\sum_{s=\tau_N(t_{j^*-1})+1}^{\tau_N(t_{j^*})} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{k^2} \right] \\
&= \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \lim_{N \rightarrow \infty} \mathbb{P}[E \cap E^*] \\
&\quad - \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \sum_{\mathcal{I} \subset [k]} \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{s=\tau_N(t_{j^*-1})+1}^{\tau_N(t_{j^*})} c_N(s)^2 \right] (t_j - t_{j-1} + 1)^{k^2} \\
&= \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \tag{67}
\end{aligned}$$

where for the last equality we use Brown et al. (2021, Equation (3.5)) to show that the second sum vanishes and Lemma 11 to show that $\lim_{N \rightarrow \infty} \mathbb{P}[E \cap E^*] = 1$. We have shown that the upper and lower bounds coincide, so the result follows. \blacksquare

Indicators

Lemma 8. *Let A, B be events. If $\lim \mathbb{P}[A] = 1$ and $\lim \mathbb{P}[B] = 1$ then $\lim \mathbb{P}[A \cap B] = 1$.*

The above might be so obvious as to go unstated, but it is very important because it means we don't have to deal with intersections of dependent events! Here is a little proof just to be sure:

Proof.

$$\begin{aligned}
&\lim \mathbb{P}[A] = 1 \text{ and } \lim \mathbb{P}[B] = 1 \\
&\Leftrightarrow \lim \mathbb{P}[A^c] = 0 \text{ and } \lim \mathbb{P}[B^c] = 0 \\
&\Rightarrow \lim \{\mathbb{P}[A^c] + \mathbb{P}[B^c]\} = 0 \\
&\Rightarrow \lim \mathbb{P}[A^c \cup B^c] = 0 \\
&\Leftrightarrow \lim \mathbb{P}[A \cap B] = 1. \tag{68}
\end{aligned}$$

The only part of this argument that I find potentially controversial is going from the third to the fourth line, which is an application of the sandwich theorem (since $0 \leq \mathbb{P}[A^c \cup B^c] \leq \mathbb{P}[A^c] + \mathbb{P}[B^c]$). ■

Lemma 9. *Let $K > 0$ be a constant which may depend on n, N but not on r , such that $K^{-2} = O(1)$ as $N \rightarrow \infty$. Define the events $E_r := \{c_N(r) < K\}$ and denote $E := \bigcap_{r=1}^{\tau_N(t)} E_r$. Then $\lim_{N \rightarrow \infty} \mathbb{P}[E] = 1$.*

Proof.

$$\begin{aligned} \mathbb{P}[E] &= 1 - \mathbb{P}[E^c] = 1 - \mathbb{P}\left[\bigcup_{r=1}^{\tau_N(t)} E_r^c\right] = 1 - \mathbb{E}\left[\mathbb{1}_{\bigcup_{r=1}^{\tau_N(t)} E_r^c}\right] \geq 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r^c}\right] \\ &= 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[\mathbb{1}_{E_r^c} \mid \mathcal{F}_{r-1}]\right] = 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{P}[E_r^c \mid \mathcal{F}_{r-1}]\right] \end{aligned} \quad (69)$$

where for the second line we apply Lemma 13 with $f(r) = \mathbb{1}_{E_r^c}$. By the generalised Markov inequality,

$$\mathbb{P}[E_r^c \mid \mathcal{F}_{r-1}] = \mathbb{P}[c_N(r) \geq K \mid \mathcal{F}_{r-1}] \leq K^{-2} \mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}]. \quad (70)$$

Substituting this into (69) and applying Lemma 13 again, this time with $f(r) = c_N(r)^2$,

$$\mathbb{P}[E] \geq 1 - K^{-2} \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}]\right] = 1 - K^{-2} \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} c_N(r)^2\right]. \quad (71)$$

Applying Brown et al. (2021, Equation (3.5)), the limit is

$$\lim_{N \rightarrow \infty} \mathbb{P}[E] = 1 - O(1) \times 0 = 1 \quad (72)$$

as required. ■

Lemma 10. *Fix $t > 0$. For any $l \in \mathbb{N}$, $\lim_{N \rightarrow \infty} \mathbb{P}[\tau_N(t) \geq l] = 1$.*

Proof. We can replace the event $\{\tau_N(t) \geq l\}$ with an event of the form of E in Lemma 9:

$$\{\tau_N(t) \geq l\} = \left\{ \min \left\{ s \geq 1 : \sum_{r=1}^s c_N(r) \geq t \right\} \geq l \right\} = \left\{ \sum_{r=1}^{l-1} c_N(r) < t \right\} \supseteq \bigcap_{r=1}^{l-1} \left\{ c_N(r) < \frac{t}{l} \right\} \supseteq \bigcap_{r=1}^{\tau_N(t)} \left\{ c_N(r) < \frac{t}{l} \right\}. \quad (73)$$

Hence

$$\lim_{N \rightarrow \infty} \mathbb{P}[\tau_N(t) \geq l] \geq \lim_{N \rightarrow \infty} \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \left\{ c_N(r) < \frac{t}{l} \right\}\right] = 1 \quad (74)$$

by applying Lemma 9 with $K = t/l$. ■

Lemma 11. *Fix $k \in \mathbb{N}$, a sequence of times $0 = t_0 \leq t_1 \leq \dots \leq t_k \leq t$, and let K_1, \dots, K_k be strictly positive constants such that for each j , $K_j^{-1} = O(1)$ as $N \rightarrow \infty$. Define the event*

$$E^* := \bigcap_{j=1}^k \left\{ \sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \leq K_j \right\}. \quad (75)$$

Then $\lim_{N \rightarrow \infty} \mathbb{P}[E^] = 1$.*

Proof.

$$\mathbb{P}[E^*] = 1 - \mathbb{P}[(E^*)^c] = 1 - \mathbb{P}\left[\bigcup_{j=1}^k \left\{ \sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 > K_j \right\}\right] \geq 1 - \sum_{j=1}^k \mathbb{P}\left[\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \geq K_j\right]. \quad (76)$$

Applying Markov's inequality,

$$\mathbb{P}[E^*] \geq 1 - \sum_{j=1}^k K_j^{-1} \mathbb{E}\left[\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2\right] \xrightarrow{N \rightarrow \infty} 1 - \sum_{j=1}^k O(1) \times 0 = 1 \quad (77)$$

by Brown et al. (2021, Equation (3.5)). ■

Lemma 12. Fix $t > 0$. Let K be a constant not depending on N, r , but which may depend on n .

$$\lim_{N \rightarrow \infty} \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) \geq K D_N(r)\}\right] = 1. \quad (78)$$

Proof.

$$\begin{aligned} \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) \geq K D_N(r)\}\right] &\geq \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) > K D_N(r)\}\right] \\ &= 1 - \mathbb{P}\left[\bigcup_{r=1}^{\tau_N(t)} \{c_N(r) \leq K D_N(r)\}\right] \\ &= 1 - \mathbb{E}\left[\mathbb{1}_{\bigcup_{r=1}^{\tau_N(t)} \{c_N(r) \leq K D_N(r)\}}\right] \\ &\geq 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{1}_{\{c_N(r) \leq K D_N(r)\}}\right] \\ &= 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{P}[c_N(r) \leq K D_N(r) \mid \mathcal{F}_{r-1}]\right] \end{aligned} \quad (79)$$

where the final inequality is an application of Lemma 13 with $f(r) = \mathbb{1}_{\{c_N(r) \leq K D_N(r)\}}$.

Fix $0 < \varepsilon < K^{-1}/2$ and assume $N > \max\{\varepsilon^{-1}, (K^{-1} - 2\varepsilon)^{-1}\}$. For each r, i define the event $A_i(r) := \{\nu_r^{(i)} \leq N\varepsilon\}$. Conditional on \mathcal{F}_{r-1} , we have

$$D_N(r) = \frac{1}{N(N)_2} \sum_{i=1}^N (\nu_r^{(j)})_2 \left[\nu_r^{(i)} + \frac{1}{N} \sum_{j \neq i} (\nu_r^{(j)})^2 \right] \mathbb{1}_{A_i^c(r)} + \frac{1}{N(N)_2} \sum_{i=1}^N (\nu_r^{(i)})_2 \left[\nu_r^{(i)} + \frac{1}{N} \sum_{j \neq i} (\nu_r^{(j)})^2 \right] \mathbb{1}_{A_i(r)}. \quad (80)$$

For the first term,

$$\frac{1}{N(N)_2} \sum_{i=1}^N (\nu_r^{(i)})_2 \left[\nu_r^{(i)} + \frac{1}{N} \sum_{j \neq i} (\nu_r^{(j)})^2 \right] \mathbb{1}_{A_i^c(r)} \leq \sum_{i=1}^N \mathbb{1}_{A_i^c(r)}. \quad (81)$$

For the second term,

$$\begin{aligned}
\frac{1}{N(N)_2} \sum_{i=1}^N (\nu_r^{(i)})_2 \left[\nu_r^{(i)} + \frac{1}{N} \sum_{j \neq i} (\nu_r^{(j)})^2 \right] \mathbb{1}_{A_i(r)} &\leq \frac{1}{N(N)_2} \sum_{i=1}^N (\nu_r^{(i)})_2 \nu_r^{(i)} \mathbb{1}_{A_i(r)} + \frac{1}{N^2(N)_2} \sum_{i=1}^N (\nu_r^{(i)})_2 \sum_{j=1}^N (\nu_r^{(j)})^2 \mathbb{1}_{A_i(r)} \\
&\leq \frac{1}{N} c_N(r) N \varepsilon + \frac{1}{N^2(N)_2} \sum_{i=1}^N (\nu_r^{(i)})_2 \sum_{j=1}^N (\nu_r^{(j)})_2 \mathbb{1}_{A_i(r)} \\
&\quad + \frac{1}{N^2(N)_2} \sum_{i=1}^N (\nu_r^{(i)})_2 \sum_{j=1}^N (\nu_r^{(j)}) \mathbb{1}_{A_i(r)} \\
&\leq \varepsilon c_N(r) + \frac{1}{N^2} \sum_{i=1}^N \nu_r^{(i)} N \varepsilon c_N(r) + \frac{1}{N^2} c_N(r) N \\
&= c_N(r) \left(2\varepsilon + \frac{1}{N} \right). \tag{82}
\end{aligned}$$

Altogether we have

$$D_N(r) \leq c_N(r) \left(2\varepsilon + \frac{1}{N} \right) + \sum_{i=1}^N \mathbb{1}_{A_i^c(r)}. \tag{83}$$

Hence, still conditional on \mathcal{F}_{r-1} ,

$$\begin{aligned}
\{c_N(r) \leq K D_N(r)\} &\subseteq \left\{ c_N(r) \leq K c_N(r) (2\varepsilon + N^{-1}) + K \sum_{i=1}^N \mathbb{1}_{A_i^c(r)} \right\} \\
&= \left\{ K^{-1} - 2\varepsilon - \frac{1}{N} \leq \sum_{i=1}^N \frac{\mathbb{1}_{A_i^c(r)}}{c_N(r)} \right\} \tag{84}
\end{aligned}$$

where the ratio $\mathbb{1}_{A_i^c(r)}/c_N(r)$ is well-defined because

$$A_i^c(r) \Rightarrow c_N(r) := \frac{1}{(N)_2} \sum_{j=1}^N (\nu_r^{(j)})_2 \geq \frac{1}{(N)_2} (\nu_r^{(i)})_2 \geq \frac{\varepsilon(N\varepsilon - 1)}{N - 1} \geq \varepsilon \left(\varepsilon - \frac{1}{N} \right) > 0. \tag{85}$$

Hence by Markov's inequality (the conditions on ε, N ensuring the constant is always strictly positive),

$$\begin{aligned}
\mathbb{P}[c_N(r) \leq K D_N(r) \mid \mathcal{F}_{r-1}] &\leq \mathbb{P} \left[\sum_{i=1}^N \mathbb{1}_{A_i^c(r)} \geq \left(K^{-1} - 2\varepsilon - \frac{1}{N} \right) \varepsilon \left(\varepsilon - \frac{1}{N} \right) \middle| \mathcal{F}_{r-1} \right] \\
&\leq \frac{1}{(K^{-1} - 2\varepsilon - \frac{1}{N}) \varepsilon (\varepsilon - \frac{1}{N})} \mathbb{E} \left[\sum_{i=1}^N \mathbb{1}_{A_i^c(r)} \middle| \mathcal{F}_{r-1} \right] \\
&\leq \frac{1}{(K^{-1} - 2\varepsilon - \frac{1}{N}) \varepsilon (\varepsilon - \frac{1}{N})} \mathbb{E} \left[\sum_{i=1}^N \frac{(\nu_r^{(i)})_3}{(N\varepsilon)_3} \middle| \mathcal{F}_{r-1} \right] \\
&\leq \frac{1}{(K^{-1} - 2\varepsilon - \frac{1}{N}) \varepsilon (\varepsilon - \frac{1}{N})} \mathbb{E} \left[\frac{N(N)_2}{(N\varepsilon)_3} D_N(r) \middle| \mathcal{F}_{r-1} \right]. \tag{86}
\end{aligned}$$

Applying Lemma 13 once more, with $f(r) = D_N(r)$,

$$\begin{aligned}
\mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} \mathbb{P}[c_N(r) \leq K D_N(r) \mid \mathcal{F}_{r-1}] \right] &\leq \frac{1}{(K^{-1} - 2\varepsilon - \frac{1}{N}) \varepsilon (\varepsilon - \frac{1}{N})} \frac{N(N)_2}{(N\varepsilon)_3} \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[D_N(r) \mid \mathcal{F}_{r-1}] \right] \\
&= \frac{1}{(K^{-1} - 2\varepsilon - \frac{1}{N}) \varepsilon (\varepsilon - \frac{1}{N})} \frac{N(N)_2}{(N\varepsilon)_3} \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} D_N(r) \right] \\
&\xrightarrow{N \rightarrow \infty} \frac{1}{(K^{-1} - 2\varepsilon) \varepsilon^5} \times 0 = 0. \tag{87}
\end{aligned}$$

Substituting this back into (79) concludes the proof. ■

Other useful results

The following Lemma is taken from Koskela et al. (2018, Lemma 2), where the function is set to $f(r) = c_N(r)$, but the authors remark that the result holds for other choices of function.

Lemma 13. Fix $t > 0$. Let (\mathcal{F}_r) be the backwards-in-time filtration generated by the offspring counts $\nu_r^{(1:N)}$ at each generation r , and let $f(r)$ be any deterministic function of $\nu_r^{(1:N)}$ that is non-negative and bounded. In particular, for all r there exists $B < \infty$ such that $0 \leq f(r) \leq B$. Then

$$\mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} f(r) \right] = \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}] \right]. \quad (88)$$

Proof. Define

$$M_s := \sum_{r=1}^s \{f(r) - \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\}. \quad (89)$$

It is easy to establish that (M_s) is a martingale with respect to (\mathcal{F}_s) , and $M_0 = 0$. Now fix $K \geq 1$ and note that $\tau_N(t) \wedge K$ is a bounded \mathcal{F}_t -stopping time. Hence we can apply the optional stopping theorem:

$$\mathbb{E}[M_{\tau_N(t) \wedge K}] = \mathbb{E} \left[\sum_{r=1}^{\tau_N(t) \wedge K} \{f(r) - \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\} \right] = \mathbb{E} \left[\sum_{r=1}^{\tau_N(t) \wedge K} f(r) \right] - \mathbb{E} \left[\sum_{r=1}^{\tau_N(t) \wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}] \right] = 0. \quad (90)$$

Since this holds for all $K \geq 1$,

$$\lim_{K \rightarrow \infty} \mathbb{E} \left[\sum_{r=1}^{\tau_N(t) \wedge K} f(r) \right] = \lim_{K \rightarrow \infty} \mathbb{E} \left[\sum_{r=1}^{\tau_N(t) \wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}] \right]. \quad (91)$$

The monotone convergence theorem allows the limit to pass inside the expectation on each side (since increasing K can only increase each sum, by possibly adding non-negative terms). Hence

$$\mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} f(r) \right] = \mathbb{E} \left[\lim_{K \rightarrow \infty} \sum_{r=1}^{\tau_N(t) \wedge K} f(r) \right] = \mathbb{E} \left[\lim_{K \rightarrow \infty} \sum_{r=1}^{\tau_N(t) \wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}] \right] = \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}] \right] \quad (92)$$

which concludes the proof. ■

Dependency graph

References

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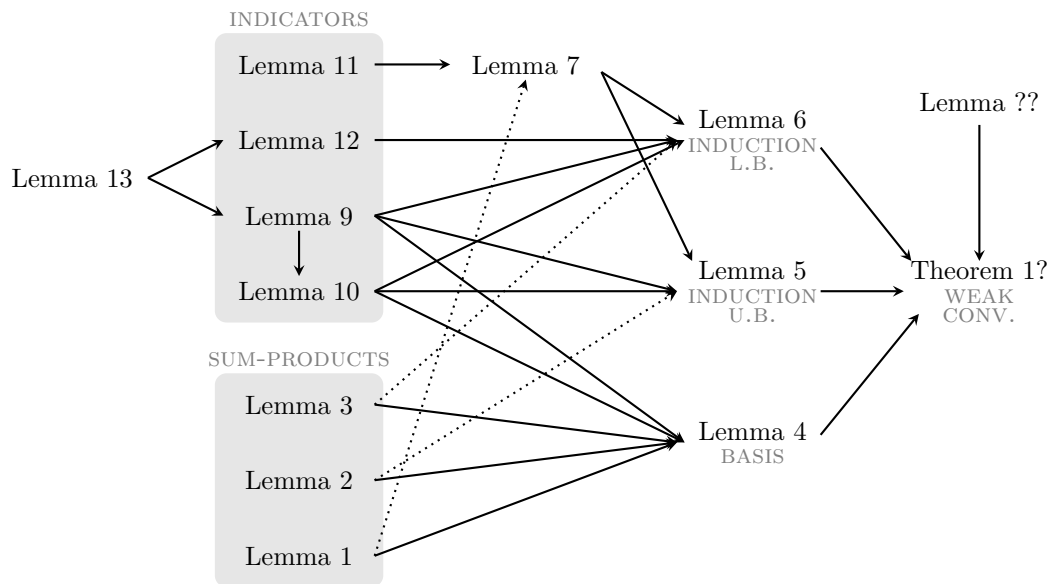


Figure 1: Graph showing dependencies between the lemmata used to prove weak convergence. Dotted arrows indicate dependence via a slight modification of the preceding lemma.