Weak convergence proof v.2 (neater) (in progress)

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Bounds on sum-products

Lemma 1.

$$t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right) \binom{l}{2} (t+1)^{l-2} \leq \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \leq t^l + c_N(\tau_N(t))(t+1)^l. \tag{1}$$

Proof. As pointed out in Koskela et al. (2018, Equation (8)),

$$\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \geq \left(\sum_{s=0}^{\tau_N(t)} c_N(s)\right)^l - \binom{l}{2} \left(\sum_{s=0}^{\tau_N(t)} c_N(s)^2\right) \left(\sum_{s=0}^{\tau_N(t)} c_N(s)\right)^{l-2}. \tag{2}$$

By definition of τ_N,

$$t \leq \sum_{s=0}^{\tau_N(t)} c_N(s) \leq t+1. \tag{3}$$

Substituting these bounds into the RHS of (2) yields the lower bound.

It is a true fact that

$$\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \leq \left(\sum_{s=0}^{\tau_N(t)} c_N(s)\right)^l, \tag{4}$$

as can be seen by considering the multinomial expansion of the RHS. This is further bounded by

$$\left(\sum_{s=0}^{\tau_N(t)} c_N(s)\right)^l \leq \left(\sum_{s=0}^{\tau_N(t)-1} c_N(s) + c_N(\tau_N(t))\right)^l \leq \left[t + c_N(\tau_N(t))\right]^l, \tag{5}$$

again using the definition of τ_N. A binomial expansion yields

$$\left[t + c_N(\tau_N(t))\right]^l = t^l + \sum_{i=0}^{l-1} \binom{l}{i} t^i c_N(\tau_N(t))^{l-i} = t^l + c_N(\tau_N(t)) \sum_{i=0}^{l-1} \binom{l}{i} t^i c_N(\tau_N(t))^{l-1-i}, \tag{6}$$

then since $c_N(s) \leq 1$ for all s,

$$\sum_{i=0}^{l-1} \binom{l}{i} t^i c_N(\tau_N(t))^{l-1-i} \leq \sum_{i=0}^{l-1} \binom{l}{i} t^i \leq (t+1)^l. \tag{7}$$

Putting this together yields the upper bound. ∎

Lemma 2. *Let B be a positive constant which may depend on n.*

$$\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) + BD_N(s_j)] \leq \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1}(1+B)^l. \tag{8}$$

Proof. We start with a binomial expansion:

$$\begin{aligned} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) + BD_N(s_j)] &= \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \sum_{\mathcal{I}\subseteq[l]} B^{l-|\mathcal{I}|} \left(\prod_{i\in\mathcal{I}} c_N(s_i) \right) \left(\prod_{j\notin\mathcal{I}} D_N(s_j) \right) \\ &= \sum_{\mathcal{I}\subseteq[l]} B^{l-|\mathcal{I}|} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i\in\mathcal{I}} c_N(s_i) \right) \left(\prod_{j\notin\mathcal{I}} D_N(s_j) \right) \end{aligned} \tag{9}$$

where $[l] := \{1, \ldots, l\}$. Since the sum is over all permutations of $r_1, \ldots, r_l$, we may arbitrarily choose an ordering for $\{1, \ldots, l\}$ such that $\mathcal{I} = \{1, \ldots, |\mathcal{I}|\}$:

$$\sum_{\mathcal{I}\subseteq[l]} B^{l-|\mathcal{I}|} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i\in\mathcal{I}} c_N(s_i) \right) \left(\prod_{j\notin\mathcal{I}} D_N(s_j) \right) = \sum_{I=0}^{l} \binom{l}{I} B^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right). \tag{10}$$

Separating the term $I = l$,

$$\begin{aligned} \sum_{I=0}^{l} \binom{l}{I} B^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right) &= \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \\ &\quad + \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right). \end{aligned} \tag{11}$$

In the second line, there is always at least one D_N term, and $c_N(s) \leq D_N(s)$ for all s, so we can write

$$\begin{aligned} \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right) &\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{l-1} c_N(s_i) \right) D_N(s_l) \\ &\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \left(\sum_{s_1\neq\cdots\neq s_{l-1}}^{\tau_N(t)} \prod_{i=1}^{l-1} c_N(s_i) \right) \sum_{s_l=1}^{\tau_N(t)} D_N(s_l) \\ &\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} (t+1)^{l-1} \sum_{s=1}^{\tau_N(t)} D_N(s) \end{aligned} \tag{12}$$

using (4) and (3). Finally, by the Binomial Theorem,

$$\sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} (t+1)^{l-1} \sum_{s=1}^{\tau_N(t)} D_N(s) \leq \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1}(1+B)^l, \tag{13}$$

which, together with (11), concludes the proof. ■

Lemma 3. *Let B be a positive constant which may depend on n.*

$$\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - BD_N(s_j)] \geq \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1}(1+B)^l. \tag{14}$$

Proof. A binomial expansion and subsequent manipulation as in (9)–(11) gives

$$\begin{aligned}
\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - BD_N(s_j)] &= \sum_{\mathcal{I}\subseteq[l]} (-B)^{l-|\mathcal{I}|} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i\in\mathcal{I}} c_N(s_i)\right)\left(\prod_{j\notin\mathcal{I}} D_N(s_j)\right) \\
&= \sum_{I=0}^{l} \binom{l}{I} (-B)^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right)\left(\prod_{j=I+1}^{l} D_N(s_j)\right) \\
&= \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) + \sum_{I=0}^{l-1} \binom{l}{I} (-B)^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right)\left(\prod_{j=I+1}^{l} D_N(s_j)\right) \\
&\geq \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) - \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right)\left(\prod_{j=I+1}^{l} D_N(s_j)\right)
\end{aligned} \tag{15}$$

where the last inequality just multiplies some positive terms by -1. Then (12)–(13) can be applied directly (noting that an upper bound on negative terms gives a lower bound overall):

$$-\sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right)\left(\prod_{j=I+1}^{l} D_N(s_j)\right) \geq \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)(t+1)^{l-1}(1+B)^l \tag{16}$$

which concludes the proof. ∎

Main components of weak convergence

Lemma 4 (Basis step)**.** *For any* $0 < t < \infty$,

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1-p_r)\right] = e^{-\alpha_n t} \tag{17}$$

where $\alpha_n := n(n-1)/2$.

Proof. We start by showing that $\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right] \leq e^{-\alpha_n t}$.
From Koskela et al. (2018, Lemma 1 Case 1), taking $\xi = \Delta$, we have

$$1 - p_t = p_{\Delta\Delta}(t) \leq 1 - \alpha_n(1+O(N^{-1}))\,[c_N(t) - B_n' D_N(t)] \tag{18}$$

where the $O(N^{-1})$ term does not depend on t. Applying a multinomial expansion and then separating the positive and negative terms,

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1-p_r) &\leq 1 + \sum_{l=1}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1}))\frac{1}{l!} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - B_n' D_N(s_j)] \\
&= 1 + \sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1}))\frac{1}{l!} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - B_n' D_N(s_j)] \\
&\quad - \sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1}))\frac{1}{l!} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - B_n' D_N(s_j)]\,.
\end{aligned} \tag{19}$$

This is further bounded by applying Lemma 3 and then both bounds of Lemma 1:

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1-p_r) &\leq 1 + \sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \\
&\quad - \sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \left\{ \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1+B_n')^l \right\} \\
&\leq 1 + \sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \left\{ t^l + c_N(\tau_N(t))(t+1)^l \right\} \\
&\quad - \sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \left\{ t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \binom{l}{2} (t+1)^{l-2} - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1+B_n')^l \right\}.
\end{aligned} \tag{20}$$

A bit of tidying up and we have

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1-p_r) &\leq \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l + c_N(\tau_N(t)) \sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} (t+1)^l \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \binom{l}{2} (t+1)^{l-2} \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} (t+1)^{l-1} (1+B_n')^l \\
&\leq \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l + c_N(\tau_N(t)) \exp[\alpha_n (1+O(N^{-1}))(t+1)] \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n (1+O(N^{-1}))(t+1)] \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n (1+O(N^{-1}))(t+1)(1+B_n')].
\end{aligned} \tag{21}$$

Now, taking the expectation and limit, and applying Brown et al. (2021, Equations (3.3)–(3.5)) and Lemma ?? (Lemma 2 in the messy weakconv note; not written up here yet),

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1-p_r) \right] &\leq \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l \lim_{N\to\infty} \mathbb{P}\left[\tau_N(t) \geq l \right] + \lim_{N\to\infty} \mathbb{E}\left[c_N(\tau_N(t)) \right] \exp[\alpha_n (t+1)] \\
&\quad + \lim_{N\to\infty} \mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right] \frac{1}{2} \alpha_n^2 \exp[\alpha_n (t+1)] \\
&\quad + \lim_{N\to\infty} \mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} D_N(s) \right] \exp[\alpha_n (t+1)(1+B_n')] \\
&= \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l = e^{-\alpha_n t}.
\end{aligned} \tag{22}$$

It remains to show that $\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right] \geq e^{-\alpha_n t}$.
From Brown et al. (2021, Equation (3.14)), taking $\xi = \Delta$, we have

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \alpha_n(1 + O(N^{-1}))\left[c_N(t) + B_n D_N(t)\right] \tag{23}$$

where $B_n > 0$ and the $O(N^{-1})$ term does not depend on t. In particular,

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \frac{N^{n-2}}{(N-2)_{n-2}}\alpha_n c_N(t) - \frac{N^{n-3}}{(N-3)_{n-3}} B_n D_N(t). \tag{24}$$

Since $D_N(s) \leq c_N(s)$ for all s (Koskela et al., 2018, p.9), a sufficient condition for this bound to be non-negative is

$$E_r := \left\{ c_N(r) \leq \frac{(N-3)_{n-3}}{N^{n-3}} \left(\alpha_n \left(1 + \frac{2}{N-2}\right) + B_n \right)^{-1} \right\}, \tag{25}$$

and we define $E := \bigcap_{r=1}^{\tau_N(t)} E_r$. We now apply a multinomial expansion to the product, and split into positive and negative terms:

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)}(1-p_r) &\geq \left\{ 1 + \sum_{l=1}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} \left[c_N(s_j) + B_n D_N(s_j)\right] \right\} \mathbb{1}_E \\
&= \left\{ 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} \left[c_N(s_j) + B_n D_N(s_j)\right] \right. \\
&\qquad \left. - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} \left[c_N(s_j) + B_n D_N(s_j)\right] \right\} \mathbb{1}_E
\end{aligned} \tag{26}$$

This is further bounded by applying Lemma 2 and both bounds in Lemma 1:

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)}(1-p_r) &\geq \left\{ 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \right. \\
&\qquad \left. - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left[\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1 + B_n)^l \right] \right\} \mathbb{1}_E \\
&\geq \left\{ 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left[t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \binom{l}{2} (t+1)^{l-2} \right] \right. \\
&\qquad \left. - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left[t^l + c_N(\tau_N(t))(t+1)^l + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1 + B_n)^l \right] \right\} \mathbb{1}_E.
\end{aligned} \tag{27}$$

Tidying things up,

$$
\begin{aligned}
\prod_{r=1}^{\tau_N(t)}(1-p_r) \geq \sum_{l=0}^{\tau_N(t)}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_E - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right)\sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)} \alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\binom{l}{2}(t+1)^{l-2} \\
- c_N(\tau_N(t))\sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)} \alpha_n^l(1+O(N^{-1}))\frac{1}{l!}(t+1)^l \\
- \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)\sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)} \alpha_n^l(1+O(N^{-1}))\frac{1}{l!}(t+1)^{l-1}(1+B_n)^l \\
\geq \sum_{l=0}^{\tau_N(t)}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_E - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right)\frac{1}{2}\alpha_n^2\exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
- c_N(\tau_N(t))\exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
- \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n)]. \qquad (28)
\end{aligned}
$$

Now, taking the expectation and limit, and applying Brown et al. (2021, Equations (3.3)–(3.5)) and Lemma ?? (Lemma 2 in the messy weakconv note; not written up here yet),

$$
\begin{aligned}
\lim_{N\to\infty}\mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right] &\geq \sum_{l=0}^{\infty}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\lim_{N\to\infty}\mathbb{P}\left[\{\tau_N(t)\geq l\}\cap E\right] \\
&\quad - \lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right]\frac{1}{2}\alpha_n^2\exp[\alpha_n(t+1)] \\
&\quad - \lim_{N\to\infty}\mathbb{E}\left[c_N(\tau_N(t))\right]\exp[\alpha_n(t+1)] \\
&\quad - \lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} D_N(s)\right]\exp[\alpha_n(t+1)(1+B_n)] \\
&= \sum_{l=0}^{\infty}(-\alpha_n)^l\frac{1}{l!}t^l = e^{-\alpha_n t}. \qquad (29)
\end{aligned}
$$

Combining the upper and lower bounds in (22) and (29) respectively concludes the proof. ■

I have proofs for the next three lemmata, I'm just working on a presentation that might be intelligible to someone other than myself.

Lemma 5 (Induction step upper bound)**.** *Fix $k\in\mathbb{N}$, $i_0:=0$, $i_k:=k$. For any sequence of times $0=t_0\leq t_1\leq\cdots\leq t_k\leq t$,*

$$
\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\left(\prod_{i=1}^{k}p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)}(1-p_r)\right)\right] \leq \alpha_n^k\sum_{l=0}^{\infty}\frac{1}{l!}(-\alpha_n)^l t^l\sum_{\substack{i_1\leq\cdots\leq i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\geq j\forall j}}\prod_{j=1}^{k}\frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}. \qquad (30)
$$

Lemma 6 (Induction step lower bound). *Fix $k \in \mathbb{N}$, $i_0 := 0$, $i_k := k$. For any sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_k \leq t$,*

$$\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i \leq \tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i} \right) \left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r) \right) \right] \geq \alpha_n^k \sum_{l=0}^{\infty} \frac{1}{l!}(-\alpha_n)^l t^l \sum_{\substack{i_1\leq\cdots\leq i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\geq j\forall j}} \prod_{j=1}^{k} \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}. \tag{31}$$

Lemma 7. *Fix $l, k \in \mathbb{N}$, $i_0 := 0$, $i_k := k$. Let E be any event independent of $r_1, \dots, r_k$ such that $\lim_{N\to\infty} \mathbb{P}[E] = 1$. Then for any sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_k \leq t$,*

$$\lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_E \sum_{\substack{r_1<\cdots<r_k:\\ r_i \leq \tau_N(t_i)\forall i}} \prod_{i=1}^{k} c_N(r_i) \right] = \sum_{\substack{i_1\leq\cdots\leq i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\geq j\forall j}} \prod_{j=1}^{k} \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}. \tag{32}$$

Indicators

Lemma 8. *Let A, B be events. If $\lim \mathbb{P}[A] = 1$ and $\lim \mathbb{P}[B] = 1$ then $\lim \mathbb{P}[A \cap B] = 1$.*

The above might be so obvious as to go unstated, but it is very important because it means we don't have to deal with intersections of dependent events! Here is a little proof just to be sure:

Proof.

$$\begin{aligned}
&\lim \mathbb{P}[A] = 1 \textbf{ and } \lim \mathbb{P}[B] = 1\\
\Leftrightarrow &\lim \mathbb{P}[A^c] = 0 \textbf{ and } \lim \mathbb{P}[B^c] = 0\\
\Rightarrow &\lim \{\mathbb{P}[A^c] + \mathbb{P}[B^c]\} = 0\\
\Rightarrow &\lim \mathbb{P}[A^c \cup B^c] = 0\\
\Leftrightarrow &\lim \mathbb{P}[A \cap B] = 1.
\end{aligned} \tag{33}$$

The only part of this argument that I find potentially controversial is going from the third to the fourth line, which is an application of the sandwich theorem (since $0 \leq \mathbb{P}[A^c \cup B^c] \leq \mathbb{P}[A^c] + \mathbb{P}[B^c]$). ∎

Lemma 9. *Let K be a constant which may depend on n, N but not on r, such that $K^{-2} = O(1)$ as $N \to \infty$. Define the events $E_r := \{c_N(r) < K\}$ and denote $E := \bigcap_{r=1}^{\tau_N(t)} E_r$. Then $\lim_{N\to\infty} \mathbb{P}[E] = 1$.*

Proof.

$$\begin{aligned}
\mathbb{P}[E] = 1 - \mathbb{P}[E^c] = 1 - \mathbb{P}\left[\bigcup_{r=1}^{\tau_N(t)} E_r^c \right] &= 1 - \mathbb{E}\left[\mathbb{1}_{\bigcup E_r^c} \right] \geq 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r^c} \right]\\
= 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}\left[\mathbb{1}_{E_r^c} \mid \mathcal{F}_{r-1} \right] \right] &= 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{P}\left[E_r^c \mid \mathcal{F}_{r-1} \right] \right]
\end{aligned} \tag{34}$$

where for the second line we apply Lemma 13 with $f(r) = \mathbb{1}_{E_r^c}$. To see that this choice of f satisfies the conditions of Lemma 13, note that

$$\sum_{r=1}^{\tau_N(s)} \mathbb{1}_{\{c_N(r)\geq K\}} \leq \sum_{r=1}^{\tau_N(s)} \frac{c_N(r)}{K} \leq \frac{s+1}{K} < \infty. \tag{35}$$

By the generalised Markov inequality,

$$\mathbb{P}[E_r^c \mid \mathcal{F}_{r-1}] = \mathbb{P}[c_N(r) \geq K \mid \mathcal{F}_{r-1}] \leq \mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}] K^{-2}. \tag{36}$$

Substituting this into (34) and applying Lemma 13 again, this time with $f(r) = c_N(r)^2$,

$$\mathbb{P}[E] \geq 1 - K^{-2}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}]\right] = 1 - K^{-2}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} c_N(r)^2\right]. \tag{37}$$

Applying Brown et al. (2021, Equation (3.5)), the limit is

$$\lim_{N\to\infty} \mathbb{P}[E] = 1 - O(1) \times 0 = 1 \tag{38}$$

as required. ∎

Lemma 10. *Fix $t > 0$. For any $l \in \mathbb{R}$, $\lim_{N\to\infty} \mathbb{P}[\tau_N(t) \geq l] = 1$.*

Proof.

$$\{\tau_N(t) \geq l\} = \left\{\min\left\{s \geq 1 : \sum_{r=1}^{s} c_N(r) \geq t\right\} \geq l\right\} = \left\{\sum_{r=1}^{l-1} c_N(r) < t\right\} \supseteq \bigcap_{r=1}^{l-1} \left\{c_N(r) < \frac{t}{l}\right\} \supseteq \bigcap_{r=1}^{\tau_N(t)} \left\{c_N(r) < \frac{t}{l}\right\}. \tag{39}$$

Hence

$$\lim_{N\to\infty} \mathbb{P}[\tau_N(t) \geq l] \geq \lim_{N\to\infty} \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \left\{c_N(r) < \frac{t}{l}\right\}\right] \tag{40}$$

and the result follows by applying Lemma 9 with $K = t/l$. ∎

Lemma 11. *Define the event*

$$E^\star := \bigcap_{j=1}^{k} \left\{\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \leq \frac{(t_j - t_{j-1})^{i_j - i_{j-1} - k + 1}}{(i_j - i_{j-1})!}\right\}. \tag{41}$$

Then $\lim_{N\to\infty} \mathbb{P}[E^\star] = 1$.

Proof.

$$\begin{aligned} E^\star &\supseteq \left\{\sum_{j=1}^{k} \sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \leq \sum_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1} - k + 1}}{(i_j - i_{j-1})!}\right\} \\ &= \left\{\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \leq \sum_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1} - k + 1}}{(i_j - i_{j-1})!}\right\} \\ &\supseteq \left\{\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \leq \frac{1}{k!} \sum_{j=1}^{k} (t_j - t_{j-1})^{i_j - i_{j-1} - k + 1}\right\}. \end{aligned} \tag{42}$$

To simplify the RHS further, consider the possible values of $(i_j - i_{j-1} - k + 1) \in \{-k+1, \ldots, 1\}$: This simplification isn't necessary for the result, but it makes the expressions less cumbersome later on.
Case $(i_j - i_{j-1} - k + 1) < 0$:

$$\sum_{j=1}^{k} (t_j - t_{j-1})^{i_j - i_{j-1} - k + 1} \geq \sum_{j=1}^{k} t^{i_j - i_{j-1} - k + 1} \geq \sum_{j=1}^{k} t^{-k+1} = kt^{-k+1}. \tag{43}$$

Case $(i_j - i_{j-1} - k + 1) = 0$:

$$\sum_{j=1}^{k} (t_j - t_{j-1})^{i_j - i_{j-1} - k + 1} = \sum_{j=1}^{k} 1 = k. \tag{44}$$

Case $(i_j - i_{j-1} - k + 1) = 1$:

$$\sum_{j=1}^{k} (t_j - t_{j-1})^{i_j - i_{j-1} - k + 1} = \sum_{j=1}^{k} (t_j - t_{j-1}) = t_k - t_0 = t. \tag{45}$$

Altogether

$$\sum_{j=1}^{k} (t_j - t_{j-1})^{i_j - i_{j-1} - k + 1} \geq \min\{kt^{-k+1}, k, t\} = \min\{kt^{-k+1}, t\} = t \min\{kt^{-k}, 1\}, \tag{46}$$

so

$$E^\star \supseteq \left\{ \sum_{s=1}^{\tau_N(t)} c_N(s)^2 < \frac{t}{k!} \min\{kt^{-k}, 1\} \right\}. \tag{47}$$

Using Markov's inequality,

$$\begin{aligned}
\mathbb{P}[E^\star] &\geq \mathbb{P}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 < \frac{t}{k!} \min\{kt^{-k}, 1\} \right] = 1 - \mathbb{P}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \geq \frac{t}{k!} \min\{kt^{-k}, 1\} \right] \\
&\geq 1 - \frac{k!}{t} \max\{1, k^{-1}t^k\} \mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right], \qquad (48)
\end{aligned}$$

and by Brown et al. (2021, Equation (3.5))

$$\lim_{N\to\infty} \mathbb{P}[E^\star] = 1 - O(1) \times 0 = 1 \tag{49}$$

as required. ∎

Lemma 12. *Let K be a constant not depending on N, r, but which may depend on n.*

$$\lim_{N\to\infty} \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) \geq K D_N(r)\} \right] = 1. \tag{50}$$

Proof will be courtesy of Jere's note with a few edits and probably more explicit workings.

Proof.

$$\begin{aligned}
\mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) \geq K D_N(r)\} \right] &\geq \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) > K D_N(r)\} \right] \\
&= 1 - \mathbb{P}\left[\bigcup_{r=1}^{\tau_N(t)} \{c_N(r) \leq K D_N(r)\} \right] \\
&= 1 - \mathbb{E}\left[\mathbb{1}_{\bigcup \{c_N(r) \leq K D_N(r)\}} \right] \\
&\geq 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{1}_{\{c_N(r) \leq K D_N(r)\}} \right] \\
&= 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{P}[c_N(r) \leq K D_N(r) \mid \mathcal{F}_{r-1}] \right] \qquad (51)
\end{aligned}$$

where the final inequality is an application of Lemma 13 with $f(r) = \mathbb{1}_{\{c_N(r) \leq K D_N(r)\}}$.

Fix $0 < \varepsilon < (n-3)(n-4)$ and let $N > \max\{\varepsilon^{-1}, (\binom{n-2}{2} - 2\varepsilon)^{-1}\}$. For each r, i define the event $A_i(r) := \{\nu_r^{(i)} \leq N\varepsilon\}$. Conditional on $\mathcal{F}_{r-1}$, we have

$$D_N(r) = \frac{1}{N(N)_2} \sum_{i=1}^{N} (\nu_r^{(j)})_2 \left[\nu_r^{(i)} + \frac{1}{N} \sum_{j \neq i} (\nu_r^{(i)})^2 \right] \mathbb{1}_{A_i(r)^c} + \frac{1}{N(N)_2} \sum_{i=1}^{N} (\nu_r^{(i)})_2 \left[\nu_r^{(i)} + \frac{1}{N} \sum_{j \neq i} (\nu_r^{(j)})^2 \right] \mathbb{1}_{A_i(r)}. \tag{52}$$

For the first term,

$$\frac{1}{N(N)_2} \sum_{i=1}^{N} (\nu_r^{(i)})_2 \left[\nu_r^{(i)} + \frac{1}{N} \sum_{j \neq i} (\nu_r^{(j)})^2 \right] \mathbb{1}_{A_i(r)^c} \leq \sum_{i=1}^{N} \mathbb{1}_{A_i(r)^c}. \tag{53}$$

For the second term,

$$\begin{aligned}
\frac{1}{N(N)_2} \sum_{i=1}^{N} (\nu_r^{(i)})_2 \left[\nu_r^{(i)} + \frac{1}{N} \sum_{j \neq i} (\nu_r^{(j)})^2 \right] \mathbb{1}_{A_i(r)} &\leq \frac{1}{N(N)_2} \sum_{i=1}^{N} (\nu_r^{(i)})_2 \nu_r^{(i)} \mathbb{1}_{A_i(r)} + \frac{1}{N^2(N)_2} \sum_{i=1}^{N} (\nu_r^{(i)})_2 \sum_{j=1}^{N} (\nu_r^{(j)})^2 \mathbb{1}_{A_i(r)} \\
&\leq \frac{1}{N} c_N(r) N\varepsilon + \frac{1}{N^2(N)_2} \sum_{i=1}^{N} (\nu_r^{(i)})_2 \sum_{j=1}^{N} (\nu_r^{(j)})_2 \mathbb{1}_{A_i(r)} \\
&\quad + \frac{1}{N^2(N)_2} \sum_{i=1}^{N} (\nu_r^{(i)})_2 \sum_{j=1}^{N} (\nu_r^{(j)}) \mathbb{1}_{A_i(r)} \\
&\leq \varepsilon c_N(r) + \frac{1}{N^2} \sum_{i=1}^{N} \nu_r^{(i)} N\varepsilon c_N(r) + \frac{1}{N^2} c_N(r) N \\
&= c_N(r) \left(2\varepsilon + \frac{1}{N} \right).
\end{aligned} \tag{54}$$

Hence, conditional on $\mathcal{F}_{r-1}$,

$$\begin{aligned}
\{c_N(r) \geq K D_N(r)\} &\supseteq \left\{ c_N(r) \leq K c_N(r)(2\varepsilon + N^{-1}) + K \sum_{i=1}^{N} \mathbb{1}_{A_i(r)^c} \right\} \\
&= \left\{ K^{-1} - 2\varepsilon - \frac{1}{N} \leq \sum_{i=1}^{N} \mathbb{1}_{A_i(r)^c} c_N(r)^{-1} \right\}
\end{aligned} \tag{55}$$

where the ratio $\mathbb{1}_{A_i(r)^c} / c_N(r)$ is well-defined because

$$A_i(r)^c \Rightarrow c_N(r) := \frac{1}{(N)_2} \sum_{j=1}^{N} (\nu_r^{(j)})_2 \geq \frac{1}{(N)_2} (\nu_r^{(j)})_2 \geq \frac{\varepsilon(N\varepsilon - 1)}{N-1} \geq \varepsilon \left(\varepsilon - \frac{1}{N} \right) > 0. \tag{56}$$

... The rest of the proof follows the rest of Jere's note... ■

Other useful results

The following Lemma is taken from Koskela et al. (2018, Lemma 2), where the function is set to $f(t) = c_N(t)$, but the authors remark that the result holds for other choices of function.

Lemma 13. *Let $(\mathcal{F}_t)$ be the backwards-in-time filtration generated by the offspring counts $\nu_t^{(1:N)}$ at each generation t, and let $f(t)$ be any deterministic function of $\nu_t^{(1:N)}$ that is non-negative and bounded. In particular, for all t there exists $B < \infty$ such that $0 \leq f(t) \leq B$. Then*

$$\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} f(r) \right] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}] \right]. \tag{57}$$

Proof. Define

$$M_s := \sum_{r=1}^{s} \left\{ f(r) - \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}] \right\}. \tag{58}$$

It is easy to establish that (M_s) is a martingale with respect to $(\mathcal{F}_s)$, and $M_0 = 0$. Now fix $K \geq 1$ and note that $\tau_N(t) \wedge K$ is a bounded $\mathcal{F}_t$-stopping time. Hence we can apply the optional stopping theorem:

$$\mathbb{E}[M_{\tau_N(t)\wedge K}] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} \{f(r) - \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\}\right] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} f(r)\right] - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right] = 0. \tag{59}$$

Since this holds for all $K \geq 1$,

$$\lim_{K\to\infty} \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} f(r)\right] = \lim_{K\to\infty} \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right]. \tag{60}$$

The monotone convergence theorem allows the limit to pass inside the expectation on each side (since increasing K can only increase each sum, by possibly adding non-negative terms). Hence

$$\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} f(r)\right] = \mathbb{E}\left[\lim_{K\to\infty} \sum_{r=1}^{\tau_N(t)\wedge K} f(r)\right] = \mathbb{E}\left[\lim_{K\to\infty} \sum_{r=1}^{\tau_N(t)\wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right] \tag{61}$$

which concludes the proof. ∎

References

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