Weak convergence proof (in progress)

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Theorem 1. *Let $\nu_t^{(1:N)}$ denote the offspring numbers in an interacting particle system satisfying the standing assumption and such that, for any N sufficiently large, for all finite t, $\mathbb{P}\{\tau_N(t)=\infty\}=0$. Suppose that there exists a deterministic sequence $(b_N)_{N\geq 1}$ such that $\lim_{N\to\infty} b_N = 0$ and*

$$\frac{1}{(N)_3}\sum_{i=1}^{N}\mathbb{E}_t\{(\nu_t^{(i)})_3\} \leq b_N \frac{1}{(N)_2}\sum_{i=1}^{N}\mathbb{E}_t\{(\nu_t^{(i)})_2\} \tag{1}$$

for all N, uniformly in $t\geq 1$. Then the rescaled genealogical process $(G_{\tau_N(t)}^{(n,N)})_{t\geq 0}$ converges weakly to Kingman's n-coalescent as $N\to\infty$.

Proof. Define $p_t := \max_{\xi\in E}\{1-p_{\xi\xi}(t)\} = 1-p_{\Delta\Delta}(t)$, where Δ denotes the trivial partition of $\{1,\dots,n\}$ into singletons. For a proof that the maximum is attained at $\xi=\Delta$, see Lemma 1. Following Möhle (1999), we now construct the two-dimensional Markov process $(Z_t,S_t)_{t\in\mathbb{N}}$ with transition probabilities

$$\mathbb{P}(Z_t=j, S_t=\eta \mid Z_{t-1}=i, S_{t-1}=\xi) = \begin{cases} 1-p_t & \text{if } j=i \text{ and } \xi=\eta \\ p_{\xi\xi}(t)+p_t-1 & \text{if } j=i+1 \text{ and } \xi=\eta \\ p_{\xi\eta}(t) & \text{if } j=i+1 \text{ and } \xi\neq\eta \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The construction is such that the marginal (S_t) has the same distribution as the genealogical process of interest, and (Z_t) has jumps at all the times (S_t) does plus some extra jumps. (The definition of p_t ensures that the probability in the second case is non-negative, attaining the value zero when $\xi=\Delta$.)

Denote by $0=T_0^{(N)}<T_1^{(N)}<\dots$ the jump times of the rescaled process $(Z_{\tau_N(t)})_{t\geq 0}$, and $\omega_i^{(N)} := T_i^{(N)}-T_{i-1}^{(N)}$ the corresponding holding times ($i\in\mathbb{N}$).

... □

Useful results

Lemma 1. $\max_{\xi\in E}(1-p_{\xi\xi}(t)) = 1-p_{\Delta\Delta}(t)$.

Proof. Consider any $\xi\in E$ consisting of k blocks ($1\leq k\leq n-1$), and any $\xi'\in E$ consisting of $k+1$ blocks. From the definition of $p_{\xi\eta}(t)$ (Koskela et al., 2018, Equation (1)),

$$p_{\xi\xi}(t) = \frac{1}{(N)_k}\sum_{\substack{i_1,\dots,i_k \\ \text{all distinct}}} \nu_t^{(i_1)}\cdots\nu_t^{(i_k)}. \tag{3}$$

Similarly,

$$\begin{aligned} p_{\xi'\xi'}(t) &= \frac{1}{(N)_{k+1}}\sum_{\substack{i_1,\dots,i_k,i_{k+1} \\ \text{all distinct}}} \nu_t^{(i_1)}\cdots\nu_t^{(i_k)}\nu_t^{(i_{k+1})} \\ &= \frac{1}{(N)_k(N-k)}\sum_{\substack{i_1,\dots,i_k \\ \text{all distinct}}} \left\{ \nu_t^{(i_1)}\cdots\nu_t^{(i_k)} \sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^{N} \nu_t^{(i_{k+1})} \right\}. \end{aligned} \tag{4}$$

Discarding the zero summands,

$$p_{\xi'\xi'}(t) = \frac{1}{(N)_k (N-k)} \sum_{\substack{i_1,\dots,i_k \\ \text{all distinct:} \\ \nu_t^{(i_1)},\dots,\nu_t^{(i_k)}>0}} \left\{ \nu_t^{(i_1)} \cdots \nu_t^{(i_k)} \sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^{N} \nu_t^{(i_{k+1})} \right\}. \tag{5}$$

The inner sum is

$$\sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^{N} \nu_t^{(i_{k+1})} = \left\{ \sum_{i=1}^{N} \nu_t^{(i)} - \sum_{i\in\{i_1,\dots,i_k\}} \nu_t^{(i)} \right\} \leq N-k \tag{6}$$

since $\nu_t^{(i_1)}, \dots, \nu_t^{(i_k)}$ are all at least 1. Hence

$$p_{\xi'\xi'}(t) \leq \frac{N-k}{(N)_k (N-k)} \sum_{\substack{i_1,\dots,i_k \\ \text{all distinct:} \\ \nu_t^{(i_1)},\dots,\nu_t^{(i_k)}>0}} \nu_t^{(i_1)} \cdots \nu_t^{(i_k)} = p_{\xi\xi}(t). \tag{7}$$

Thus $p_{\xi\xi}(t)$ is decreasing in the number of blocks of ξ, and is therefore minimised by taking $\xi = \Delta$, which achieves the maximum n blocks. This choice in turn maximises $1 - p_{\xi\xi}(t)$, as required. □

Lemma 2. *For all $k \in \mathbb{N}$, for all $t > 0$,*

$$\lim_{N\to\infty} \mathbb{P}\left[k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] = 1 \tag{8}$$

and consequently

$$\lim_{N\to\infty} \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} E_r \right] = 1 \qquad \text{and} \qquad \lim_{N\to\infty} \mathbb{P}\left[k \leq \tau_N(t) \right] = 1. \tag{9}$$

Proof. We first construct a constant C_1 such that

$$\mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) < C_1\} \right] \leq \mathbb{P}\left[k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \tag{10}$$

and

$$\lim_{N\to\infty} \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) < C_1\} \right] = 1. \tag{11}$$

Any $C_1 \leq \frac{(N-3)_{n-3}}{N^{n-3}} \left(\alpha_n \left(1 + \frac{2}{N-2}\right) + B_n \right)^{-1}$ will suffice to ensure that $\{c_N(r) < C_1\} \subseteq E_r$ for all r. Furthermore, we can write

$$\{\tau_N(t) \geq k\} = \left\{ \min\left\{ s \geq 1 : \sum_{r=1}^{s} c_N(r) \geq t \right\} \geq k \right\} = \left\{ \sum_{r=1}^{k-1} c_N(r) < t \right\} \supseteq \bigcap_{r=1}^{k-1} \left\{ c_N(r) < \frac{t}{k} \right\}. \tag{12}$$

A suitable choice to satisfy (10) would thus be

$$C_1 = \min\left\{ \frac{(N-3)_{n-3}}{N^{n-3}} \left(\alpha_n \left(1 + \frac{2}{N-2}\right) + B_n \right)^{-1}, \frac{t}{k} \right\}. \tag{13}$$

Now we show that this choice of C_1 satisfies (11).

$$\begin{aligned} \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) < C_1\} \right] &= 1 - \mathbb{P}\left[\bigcup_{r=1}^{\tau_N(t)} \{c_N(r) \geq C_1\} \right] = 1 - \mathbb{E}\left[\mathbb{1}_{\bigcup\{c_N(r)\geq C_1\}} \right] \geq 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{1}_{\{c_N(r)\geq C_1\}} \right] \\ &= 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[\mathbb{1}_{\{c_N(r)\geq C_1\}} \mid \mathcal{F}_{r-1}] \right] = 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{P}[\{c_N(r) \geq C_1\} \mid \mathcal{F}_{r-1}] \right] \end{aligned} \tag{14}$$

where the inequality holds by considering the two possible values of $\mathbb{1}_{\bigcup\{c_N(r)\geq C_1\}}$, and the second line follows from Lemma 3 with $f(r) = \mathbb{1}_{\{c_N(r)\geq C_1\}}$. (To see that this choice of f satisfies the conditions of Lemma 3, note that $\sum_{r=1}^{\tau_N(s)} \mathbb{1}_{\{c_N(r)\geq C_1\}} \leq \sum_{r=1}^{\tau_N(s)} c_N(r)/C_1 \leq (s+1)/C_1$.) Using the generalised Markov inequality,

$$\mathbb{P}[\{c_N(r) \geq C_1\} \mid \mathcal{F}_{r-1}] \leq C_1^{-2}\, \mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}]. \tag{15}$$

Now, using Lemma 3 again, this time with $f(r) = c_N(r)^2$,

$$\begin{aligned}
\mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) < C_1\}\right] &\geq 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} C_1^{-2}\, \mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}]\right] \\
&= 1 - C_1^{-2}\, \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}]\right] \\
&= 1 - C_1^{-2}\, \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} c_N(r)^2\right] \\
&\overset{N\to\infty}{\longrightarrow} 1 - \min\{(\alpha_n + B_n)^2, k^2/t^2\} \times 0 = 1. \qquad (16)
\end{aligned}$$

Clearly $\mathbb{P}\left[k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r\right] \leq \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} E_r\right]$ and $\mathbb{P}\left[k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r\right] \leq \mathbb{P}\left[k \leq \tau_N(t)\right]$, so (9) follows as an immediate corollary. □

The following Lemma is taken from Koskela et al. (2018, Lemma 2), where the function is set to $f(t) = c_N(t)$, but the authors remark that the result holds for other choices of function.

Lemma 3. *Let $(\mathcal{F}_t)$ be the backwards-in-time filtration generated by the offspring counts $\nu_t^{(1:N)}$ at each generation t, and let $f(t)$ be any non-negative deterministic function of $\nu_t^{(1:N)}$ such that for any fixed s, $\sum_{r=1}^{\tau_N(s)} f(r) < \infty$. Then*

$$\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} f(r)\right] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right]. \tag{17}$$

Proof. Define

$$M_s := \sum_{r=1}^{s} \{f(r) - \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\}. \tag{18}$$

It is easy to establish that (M_s) is a martingale with respect to $(\mathcal{F}_s)$, and $M_0 = 0$. Now fix $K > 0$ and note that $\tau_N(t) \wedge K$ is a bounded $\mathcal{F}_t$-stopping time. Hence we can apply the optional stopping theorem:

$$\mathbb{E}[M_{\tau_N(t)\wedge K}] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} \{f(r) - \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\}\right] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} f(r)\right] - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right] = 0. \tag{19}$$

Taking $K \to \infty$ and applying the monotone convergence theorem concludes the proof. □

Basis step

Lemma 4. *For any $0 < t < \infty$,*

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1 - p_r)\right] = e^{-\alpha_n t} \tag{20}$$

where $\alpha_n := n(n-1)/2$.

Proof. The strategy is to find upper and lower bounds on $\mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1 - p_r)\right]$, both of which converge to $e^{-\alpha_n t}$.

Lower Bound

From Brown et al. (2020, Equation (14)), taking $\xi = \Delta$, we have

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \alpha_n(1 + O(N^{-1}))\,[c_N(t) + B_n D_N(t)] \tag{21}$$

where $B_n > 0$ and the $O(N^{-1})$ term does not depend on t. In particular,

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \frac{N^{n-2}}{(N-2)_{n-2}} \alpha_n c_N(t) - \frac{N^{n-3}}{(N-3)_{n-3}} B_n D_N(t). \tag{22}$$

Since $D_N(s) \leq c_N(s)$ for all s (Koskela et al., 2018, p.9), a sufficient condition for the bound to be positive is

$$E_t := \left\{ c_N(t) < \frac{(N-3)_{n-3}}{N^{n-3}} \left(\alpha_n \left(1 + \frac{2}{N-2} \right) + B_n \right)^{-1} \right\}. \tag{23}$$

Hence, by a multinomial expansion,

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1 - p_r) &\geq \prod_{r=1}^{\tau_N(t)} \left\{ 1 - \alpha_n (1 + O(N^{-1})) \left[c_N(r) + B_n D_N(r) \right] \right\} \times \prod_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r} \\
&= \left(1 + \sum_{k=1}^{\tau_N(t)} \sum_{\substack{r_1 < \cdots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^{k} \left\{ -\alpha_n (1 + O(N^{-1})) \left[c_N(r_j) + B_n D_N(r_j) \right] \right\} \right) \times \prod_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r} \\
&= \prod_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r} + \sum_{k=1}^{\tau_N(t)} \left\{ -\alpha_n (1 + O(N^{-1})) \right\}^k \left(\prod_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r} \right) \sum_{\substack{r_1 < \cdots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^{k} \left\{ c_N(r_j) + B_n D_N(r_j) \right\}.
\end{aligned} \tag{24}$$

Taking expectations,

$$\begin{aligned}
\mathbb{E} \left[\prod_{r=1}^{\tau_N(t)} (1 - p_r) \right] &\geq \mathbb{E} \left[\prod_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r} \right] \\
&\quad + \mathbb{E} \left[\sum_{k=1}^{\infty} \left\{ -\alpha_n (1 + O(N^{-1})) \right\}^k \mathbb{1}_{\{k \leq \tau_N(t)\}} \mathbb{1}_{\bigcap E_r} \sum_{\substack{r_1 < \cdots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^{k} \left\{ c_N(r_j) + B_n D_N(r_j) \right\} \right] \\
&= \mathbb{P} \left[\bigcap_{r=1}^{\tau_N(t)} E_r \right] \\
&\quad + \sum_{k=1}^{\infty} \left\{ -\alpha_n (1 + O(N^{-1})) \right\}^k \mathbb{E} \left[\sum_{\substack{r_1 < \cdots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^{k} \left\{ c_N(r_j) + B_n D_N(r_j) \right\} \middle| k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \\
&\quad \times \mathbb{P} \left[k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right].
\end{aligned} \tag{25}$$

Swapping the expectation with the infinite sum is justified by the dominated convergence theorem, the calculations for which are almost identical to the invocation of Tannery's theorem in equations (33)–(38).

We want to show that the conditional expectation on the right converges to $t^k/k!$, for reasons that will become clear. The strategy is to upper and lower bound this expectation by quantities that converge to $t^k/k!$.

First the lower bound. Fix $k \leq \tau_N(t)$, so that the sum is non-empty. From Koskela et al. (2018, Equation (8)),

$$\begin{aligned}
\sum_{\substack{r_1 < \cdots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^{k} \left\{ c_N(r_j) + B_n D_N(r_j) \right\} &\geq \sum_{\substack{r_1 < \cdots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^{k} c_N(r_j) \\
&\geq \frac{1}{k!} \left(\sum_{s=1}^{\tau_N(t)} c_N(s) \right)^k - \frac{1}{k!} \binom{k}{2} \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \left(\sum_{s=1}^{\tau_N(t)} c_N(s) \right)^{k-2} \\
&\geq \frac{1}{k!} t^k - \frac{1}{k!} \binom{k}{2} (t+1)^{k-2} \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right)
\end{aligned} \tag{26}$$

by the definition of τ_N. Then

$$\begin{aligned}
&\mathbb{E}\left[\sum_{\substack{r_1 < \cdots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^{k} \{c_N(r_j) + B_n D_N(r_j)\} \middle| k \le \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \\
&\qquad \ge \frac{1}{k!} t^k - \frac{1}{k!} \binom{k}{2} (t+1)^{k-2} \mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \middle| k \le \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \\
&\qquad = \frac{1}{k!} t^k - \frac{1}{k!} \binom{k}{2} (t+1)^{k-2} \mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \mathbb{1}_{\{k \le \tau_N(t)\}} \mathbb{1}_{\{\bigcap_{r=1}^{\tau_N(t)} E_r\}} \right] \left(\mathbb{P}\left[k \le \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \right)^{-1} \\
&\qquad \ge \frac{1}{k!} t^k - \frac{1}{k!} \binom{k}{2} (t+1)^{k-2} \mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right] \left(\mathbb{P}\left[k \le \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \right)^{-1} \longrightarrow \frac{1}{k!} t^k
\end{aligned} \tag{27}$$

as $N \to \infty$ using Brown et al. (2020, Equation (5)) and Lemma 2.

Now for the upper bound. We start with a multinomial expansion and some manipulations of the sums:

$$\begin{aligned}
\sum_{\substack{r_1 < \cdots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^{k} \{c_N(r_j) + B_n D_N(r_j)\} &= \frac{1}{k!} \sum_{\substack{r_1 \ne \cdots \ne r_k \\ \text{all distinct}}}^{\tau_N(t)} \prod_{j=1}^{k} \{c_N(r_j) + B_n D_N(r_j)\} \\
&= \frac{1}{k!} \sum_{\substack{r_1 \ne \cdots \ne r_k \\ \text{all distinct}}}^{\tau_N(t)} \sum_{\mathcal{I} \subseteq \{1,\ldots,k\}} (B_n)^{k-|\mathcal{I}|} \left\{ \prod_{i \in \mathcal{I}} c_N(r_i) \right\} \left\{ \prod_{j \notin \mathcal{I}} D_N(r_j) \right\} \\
&= \frac{1}{k!} \sum_{\mathcal{I} \subseteq \{1,\ldots,k\}} (B_n)^{k-|\mathcal{I}|} \sum_{\substack{r_1 \ne \cdots \ne r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i \in \mathcal{I}} c_N(r_i) \right\} \left\{ \prod_{j \notin \mathcal{I}} D_N(r_j) \right\} \\
&= \frac{1}{k!} \sum_{I=0}^{k} \binom{k}{I} (B_n)^{k-I} \sum_{\substack{r_1 \ne \cdots \ne r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^{I} c_N(r_i) \right\} \left\{ \prod_{j=I+1}^{k} D_N(r_j) \right\} \\
&= \frac{1}{k!} \sum_{\substack{r_1 \ne \cdots \ne r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^{k} c_N(r_i) \right\} \\
&\qquad + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (B_n)^{k-I} \sum_{\substack{r_1 \ne \cdots \ne r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^{I} c_N(r_i) \right\} \left\{ \prod_{j=I+1}^{k} D_N(r_j) \right\}.
\end{aligned} \tag{28}$$

Then, using that $D_N(s) \leq c_N(s)$ for all s, along with the definition of τ_N,

$$\begin{aligned}
&\frac{1}{k!} \sum_{\substack{r_1 \neq \cdots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^{k} c_N(r_i) \right\} + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (B_n)^{k-I} \sum_{\substack{r_1 \neq \cdots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^{I} c_N(r_i) \right\} \left\{ \prod_{j=I+1}^{k} D_N(r_j) \right\} \\
&\qquad \leq \frac{1}{k!} \left(\sum_{r=1}^{\tau_N(t)} c_N(r) \right)^k + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (B_n)^{k-I} \sum_{\substack{r_1 \neq \cdots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^{k-1} c_N(r_i) \right\} D_N(r_k) \\
&\qquad \leq \frac{1}{k!} \left\{ t + c_N(\tau_N(t)) \right\}^k + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (B_n)^{k-I} \left\{ \sum_{\substack{r_1 \neq \cdots \neq r_{k-1} \\ \text{all distinct}}}^{\tau_N(t)} \prod_{i=1}^{k-1} c_N(r_i) \right\} \left(\sum_{r_k=1}^{\tau_N(t)} D_N(r_k) \right) \\
&\qquad \leq \frac{1}{k!} \left\{ t + c_N(\tau_N(t)) \right\}^k + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (B_n)^{k-I} \left(\sum_{r=1}^{\tau_N(t)} c_N(r) \right)^{k-1} \left(\sum_{r=1}^{\tau_N(t)} D_N(r) \right) \\
&\qquad \leq \frac{1}{k!} \left\{ t + c_N(\tau_N(t)) \right\}^k + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (B_n)^{k-I} (t+1)^{k-1} \left(\sum_{r=1}^{\tau_N(t)} D_N(r) \right).
\end{aligned} \tag{29}$$

Taking expectations,

$$\begin{aligned}
&\mathbb{E} \left[\sum_{\substack{r_1 < \cdots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^{k} \{ c_N(r_j) + B_n D_N(r_j) \} \middle| k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \\
&\quad \leq \frac{1}{k!} \mathbb{E} \left[\{ t + c_N(\tau_N(t)) \}^k \middle| k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \\
&\qquad + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (B_n)^{k-I} (t+1)^{k-1} \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} D_N(r) \middle| k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \\
&\quad = \frac{1}{k!} \mathbb{E} \left[\{ t + c_N(\tau_N(t)) \}^k \, \mathbb{1}_{\{k \leq \tau_N(t)\}} \mathbb{1}_{\{\bigcap E_r\}} \right] \left(\mathbb{P} \left[k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \right)^{-1} \\
&\qquad + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (B_n)^{k-I} (t+1)^{k-1} \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} D_N(r) \mathbb{1}_{\{k \leq \tau_N(t)\}} \mathbb{1}_{\{\bigcap E_r\}} \right] \left(\mathbb{P} \left[k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \right)^{-1} \\
&\quad \leq \left(\frac{1}{k!} \mathbb{E} \left[\{ t + c_N(\tau_N(t)) \}^k \right] + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (B_n)^{k-I} (t+1)^{k-1} \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} D_N(r) \right] \right) \left(\mathbb{P} \left[k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \right)^{-1} \\
&\quad \longrightarrow \frac{1}{k!} t^k.
\end{aligned} \tag{30}$$

The limit follows from Lemma 2 and Brown et al. (2020, Equations (3),(4)) along with the fact that, since $c_N(s) \in [0, 1]$ for all s, $\mathbb{E}[c_N(s)^k] \leq \mathbb{E}[c_N(s)]$ for all $k \geq 1$, and the expansion

$$\mathbb{E} \left[\{ t + c_N(\tau_N(t)) \}^k \right] = \sum_{i=0}^{k} \binom{k}{i} t^i \, \mathbb{E} \left[c_N(\tau_N(t))^{k-i} \right] \longrightarrow t^k \tag{31}$$

which holds for any fixed k.

Combining the upper and lower limits, we conclude that, for any fixed $k \leq \tau_N(t)$,

$$\lim_{N \to \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \cdots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^{k} \{ c_N(r_j) + B_n D_N(r_j) \} \middle| k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] = \frac{1}{k!} t^k. \tag{32}$$

Now, starting with (25), we have

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1-p_r)\right] \geq \lim_{N\to\infty} \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} E_r\right]$$
$$+ \lim_{N\to\infty} \sum_{k=1}^{\infty} \left\{-\alpha_n(1+O(N^{-1}))\right\}^k \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)} \prod_{j=1}^{k} \{c_N(r_j) + B_n D_N(r_j)\} \middle| k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r\right] \mathbb{P}\left[k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r\right] \tag{33}$$

Since $\tau_N(t) \to \infty$, some care must be taken when exchanging the limit with the sum. We will use Tannery's theorem (a special case of dominated convergence) to show that this is okay. Let

$$a_k(N) := \{-\alpha_n(1+O(N^{-1}))\}^k \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)} \prod_{j=1}^{k} \{c_N(r_j) + B_n D_N(r_j)\} \middle| k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r\right] \mathbb{P}\left[k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r\right]. \tag{34}$$

We know from (32) and Lemma 2 that

$$\lim_{N\to\infty} a_k(N) = (-\alpha_n)^k \frac{1}{k!} t^k. \tag{35}$$

Furthermore, using (30),

$$\begin{aligned}
|a_k(N)| &\leq \{\alpha_n(1+O(N^{-1}))\}^k \left(\frac{1}{k!}(t+1)^k + \frac{1}{k!}\sum_{I=0}^{k-1}\binom{k}{I}(B_n)^{k-I}(t+1)^{k-1}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} c_N(r)\right]\right) \times 1 \\
&\leq \{\alpha_n(1+O(N^{-1}))\}^k \left(\frac{1}{k!}(t+1)^k + \frac{1}{k!}\sum_{I=0}^{k-1}\binom{k}{I}(B_n)^{k-I}(t+1)^k\right) \\
&= \{\alpha_n(1+O(N^{-1}))\}^k \frac{1}{k!}(t+1)^k \left(1 + \sum_{I=0}^{k-1}\binom{k}{I}(B_n)^{k-I}\right) \\
&= \{\alpha_n(1+O(N^{-1}))\}^k \frac{1}{k!}(t+1)^k (1+B_n)^k =: M_k.
\end{aligned} \tag{36}$$

Now

$$\sum_{k=0}^{\infty} M_k = \exp\left\{\alpha_n(1+O(N^{-1}))(t+1)(1+B_n)\right\} < \infty \tag{37}$$

so we can apply Tannery's theorem:

$$\lim_{N\to\infty} \sum_{k=1}^{\infty} a_k(N) = \sum_{k=1}^{\infty} \lim_{N\to\infty} a_k(N) = e^{-\alpha_n t} - 1 \tag{38}$$

and finally, applying Lemma 2,

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1-p_r)\right] \geq 1 + e^{-\alpha_n t} - 1 = e^{-\alpha_n t}. \tag{39}$$

Upper Bound

From Koskela et al. (2018, Lemma 1 Case 1), taking $\xi = \Delta$, we have

$$1 - p_t = p_{\Delta\Delta}(t) \leq 1 - \alpha_n(1+O(N^{-1}))\left[c_N(t) - B'_n D_N(t)\right] \tag{40}$$

where the $O(N^{-1})$ term does not depend on t. A multinomial expansion as in the lower bound yields

$$\prod_{r=1}^{\tau_N(t)} (1-p_r) \leq 1 + \sum_{k=1}^{\tau_N(t)} \left\{-\alpha_n(1+O(N^{-1}))\right\}^k \sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)} \prod_{j=1}^{k} \{c_N(r_j) - B'_n D_N(r_j)\}. \tag{41}$$

Analogously to (28), for fixed $k \le \tau_N(t)$ we can write

$$\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)} \prod_{j=1}^{k} \{c_N(r_j) - B_n' D_N(r_j)\} = \frac{1}{k!} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^{k} c_N(r_i) \right\} \tag{42}$$

$$+ \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (-B_n')^{k-I} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}} \left\{ \prod_{i=1}^{I} c_N(r_i) \right\} \left\{ \prod_{j=I+1}^{k} D_N(r_j) \right\}.$$

We start by dealing with the second term:

$$\frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (-B_n')^{k-I} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}} \left\{ \prod_{i=1}^{I} c_N(r_i) \right\} \left\{ \prod_{j=I+1}^{k} D_N(r_j) \right\}$$

$$= \frac{1}{k!} \sum_{\substack{I=0:\\ k-I \text{ even}}}^{k-1} \binom{k}{I} (B_n')^{k-I} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}} \left\{ \prod_{i=1}^{I} c_N(r_i) \right\} \left\{ \prod_{j=I+1}^{k} D_N(r_j) \right\}$$

$$- \frac{1}{k!} \sum_{\substack{I=0:\\ k-I \text{ odd}}}^{k-1} \binom{k}{I} (B_n')^{k-I} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}} \left\{ \prod_{i=1}^{I} c_N(r_i) \right\} \left\{ \prod_{j=I+1}^{k} D_N(r_j) \right\}. \tag{43}$$

This is lower bounded by

$$0 - \frac{1}{k!} \sum_{\substack{I=0\\ k-I \text{ odd}}}^{k-1} \binom{k}{I} (B_n')^{k-I} (t+1)^{k-1} \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \tag{44}$$

using that $c_N(r), D_N(r) \ge 0$ for all r to bound the even terms below, and arguments from (29) to bound the odd terms above. The same arguments lead to the upper bound

$$\frac{1}{k!} \sum_{\substack{I=0\\ k-I \text{ even}}}^{k-1} \binom{k}{I} (B_n')^{k-I} (t+1)^{k-1} \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) - 0. \tag{45}$$

By Brown et al. (2020, Equation (4)), both bounds have vanishing expectation as $N \to \infty$. We are left with the first term in (42), which is upper bounded by

$$\frac{1}{k!} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^{k} c_N(r_i) \right\} \le \frac{1}{k!} \left(\sum_{s=1}^{\tau_N(t)} c_N(s) \right)^k \le \frac{1}{k!} \{t + c_N(\tau_N(t))\}^k \tag{46}$$

the expectation of which (conditional on $k \le \tau_N(t)$; otherwise the sum is empty and has expectation zero) converges to $t^k/k!$ as in (31). We use Koskela et al. (2018, Equation (8)) to construct a lower bound:

$$\frac{1}{k!} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^{k} c_N(r_i) \right\} \ge \frac{1}{k!} \left(\sum_{s=1}^{\tau_N(t)} c_N(s) \right)^k - \frac{1}{k!} \binom{k}{2} \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \left(\sum_{s=1}^{\tau_N(t)} c_N(s) \right)^{k-2}$$

$$\ge \frac{1}{k!} t^k - \frac{1}{k!} \binom{k}{2} (t+1)^{k-2} \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \tag{47}$$

The expectation of this bound (conditional on $k \le \tau_N(t)$) also converges to $t^k/k!$, using Brown et al. (2020, Equation (5)). We can now conclude that

$$\lim_{N\to\infty} \mathbb{E} \left[\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)} \prod_{j=1}^{k} \{c_N(r_j) - B_n' D_N(r_j)\} \,\middle|\, k \le \tau_N(t) \right] = \frac{1}{k!} t^k \tag{48}$$

and thus, applying dominated convergence and Tannery's theorem as in the lower bound, along with Lemma 2,

$$
\begin{aligned}
&\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1-p_r)\right] \\
&\quad \le \lim_{N\to\infty} \mathbb{E}\left[1 + \sum_{k=1}^{\tau_N(t)} \left\{-\alpha_n(1+O(N^{-1}))\right\}^k \sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)} \prod_{j=1}^{k} \{c_N(r_j) - B_n' D_N(r_j)\}\right] \\
&\quad = 1 + \sum_{k=1}^{\infty} \lim_{N\to\infty} \{-\alpha_n(1+O(N^{-1}))\}^k \lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)} \prod_{j=1}^{k} \{c_N(r_j) - B_n' D_N(r_j)\} \middle| k \le \tau_N(t)\right] \lim_{N\to\infty} \mathbb{P}\left[k \le \tau_N(t)\right] \\
&\quad = 1 + \sum_{k=1}^{\infty} (-\alpha_n)^k \frac{1}{k!} t^k = e^{-\alpha_n t}. \qquad (49)
\end{aligned}
$$

We now have upper and lower bounds on $\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right]$, both of which are equal to $e^{-\alpha_n t}$, and the result follows. $\square$

Induction step: upper bound

Lemma 5. *Fix $l, k \in \mathbb{N}$, $i_0 = 0$, $i_k = k$. For any sequence of times $0 = t_0 \le t_1 \le \cdots \le t_k \le t$, independently of l,*

$$
\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i \le \tau_N(t_i)\forall i}} \prod_{i=1}^{k} c_N(r_i) \middle| l+k \le \tau_N(t)\right] \le \sum_{\substack{i_1 \le \cdots \le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j \ge j \forall j}} \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}. \qquad (50)
$$

Proof. As pointed out in Möhle (1999, p. 460), the sum-product on the left hand side can be expanded as

$$
\sum_{\substack{r_1<\cdots<r_k:\\ r_i \le \tau_N(t_i)\forall i}} \prod_{i=1}^{k} c_N(r_i) = \sum_{\substack{i_1 \le \cdots \le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j \ge j \forall j}} \prod_{j=1}^{k} \sum_{\substack{r_{i_{j-1}+1}<\cdots<r_{i_j}\\ =\tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i). \qquad (51)
$$

Following Koskela et al. (2018, (8)–(9)), the inner sum-product in (51) may be written

$$
\begin{aligned}
\sum_{\substack{r_{i_{j-1}+1}<\cdots<r_{i_j}\\ =\tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) &= \frac{1}{(i_j - i_{j-1})!} \sum_{\substack{r_{i_{j-1}+1}\neq\cdots\neq r_{i_j}\\ =\tau_N(t_{j-1})+1\\ \text{all distinct}}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) \\
&\le \frac{1}{(i_j - i_{j-1})!} \left[t_j - t_{j-1} + c_N(\tau_N(t_j))\right]^{i_j - i_{j-1}} \\
&= \frac{1}{(i_j - i_{j-1})!} \sum_{l=0}^{i_j - i_{j-1}} \binom{i_j - i_{j-1}}{l} (t_j - t_{j-1})^l \left[c_N(\tau_N(t_j))\right]^{i_j - i_{j-1} - l} \\
&= \frac{1}{(i_j - i_{j-1})!} (t_j - t_{j-1})^{i_j - i_{j-1}} \\
&\qquad + \frac{1}{(i_j - i_{j-1})!} c_N(\tau_N(t_j)) \sum_{l=0}^{i_j - i_{j-1} - 1} \binom{i_j - i_{j-1}}{l} (t_j - t_{j-1})^l \left[c_N(\tau_N(t_j))\right]^{i_j - i_{j-1} - 1 - l} \\
&\le \frac{1}{(i_j - i_{j-1})!} (t_j - t_{j-1})^{i_j - i_{j-1}} + c_N(\tau_N(t_j))(1 + t_j - t_{j-1})^k, \qquad (52)
\end{aligned}
$$

using in the last line that $c_N \le 1$ and $0 \le i_j - i_{j-1} \le k$. Now taking a product on the outside,

$$
\begin{aligned}
\prod_{j=1}^{k} \sum_{\substack{r_{i_{j-1}+1}<\cdots<r_{i_j} \\ =\tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) &= \prod_{j=1}^{k} \left\{ \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} + c_N(\tau_N(t_j))(1 + t_j - t_{j-1})^k \right\} \\
&= \sum_{\mathcal{I} \subseteq [k]} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} c_N(\tau_N(t_j))(1 + t_j - t_{j-1})^k \right) \\
&= \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \\
&\quad + \sum_{\mathcal{I} \subset [k]} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} c_N(\tau_N(t_j))(1 + t_j - t_{j-1})^k \right) \\
&\le \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \\
&\quad + \sum_{\mathcal{I} \subset [k]} c_N(\tau_N(t_{j^*})) \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{\substack{j \notin \mathcal{I} \\ j \neq j^*}} (1 + t_j - t_{j-1})^k \right)
\end{aligned} \tag{53}
$$

where, say, $j^* := \min\{j \notin \mathcal{I}\}$. Now we are in a position to evaluate the limit in Lemma 5:

$$
\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k: \\ r_i \le \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i) \middle| l + k \le \tau_N(t) \right] &\le \sum_{\substack{i_1 \le \cdots \le i_{k-1} \\ \in \{0,\dots,k\}: \\ i_j \ge j \forall j}} \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \\
&\quad + \sum_{\substack{i_1 \le \cdots \le i_{k-1} \\ \in \{0,\dots,k\}: \\ i_j \ge j \forall j}} \sum_{\mathcal{I} \subset [k]} \lim_{N\to\infty} \mathbb{E}\left[c_N(\tau_N(t_{j^*})) \mid l + k \le \tau_N(t) \right] \\
&\qquad \times \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \prod_{\substack{j \notin \mathcal{I} \\ j \neq j^*}} (1 + t_j - t_{j-1})^k \\
&= \sum_{\substack{i_1 \le \cdots \le i_{k-1} \\ \in \{0,\dots,k\}: \\ i_j \ge j \forall j}} \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}
\end{aligned} \tag{54}
$$

using Brown et al. (2020, Equation (3)). □

Lemma 6.

$$
\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k: \\ r_i \le \tau_N(t_i) \forall i}} \left(\prod_{i=1}^{k} p_{r_i} \right) \left(\prod_{\substack{r=1 \\ \notin \{r_1,\dots,r_k\}}}^{\tau_N(t)} (1 - p_r) \right) \right] \le \alpha_n^k \sum_{l=0}^{\infty} \frac{1}{l!} (-\alpha_n)^l t^l \lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k: \\ r_i \le \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i) \middle| l + k \le \tau_N(t) \right] \tag{55}
$$

Proof. I will expand each of the two products on the left hand side to get a number of terms that constitute an upper bound. In particular, I will write

$$
\prod_{i=1}^{k} p_{r_i} \le \mathcal{A}_1 + \mathcal{A}_2, \qquad \prod_{\substack{r=1 \\ \notin \{r_1,\dots,r_k\}}}^{\tau_N(t)} (1 - p_r) \le \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 + \mathcal{B}_4, \tag{56}
$$

where the $\mathcal{A}_i$ and $\mathcal{B}_i$ are positive terms to be defined later (actually, $\mathcal{B}_1$ may be negative, but that's okay). In this way we are left with only summation inside the expectation, so we can proceed using linearity:

$$\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\left(\prod_{i=1}^k p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)}(1-p_r)\right)\right]\le\mathbb{E}\left[\sum\mathcal{A}_1\mathcal{B}_1\right]+\mathbb{E}\left[\sum\mathcal{A}_1\mathcal{B}_2\right]+\cdots+\mathbb{E}\left[\sum\mathcal{A}_2\mathcal{B}_4\right]. \tag{57}$$

The expansions will be such that the only cross-term contributing to the limit on the LHS of (55) is $\mathcal{A}_1\mathcal{B}_1$. I will show that the other cross-terms vanish and that the $\mathcal{A}_1\mathcal{B}_1$ term is bounded as in (55).

First, using (21),

$$\begin{aligned}
\prod_{i=1}^k p_{r_i}&\le\prod_{i=1}^k\alpha_n(1+O(N^{-1}))\{c_N(r_i)+B_nD_N(r_i)\}\\
&=\{\alpha_n(1+O(N^{-1}))\}^k\sum_{\mathcal{I}\subseteq[k]}\left(\prod_{i\in\mathcal{I}}c_N(r_i)\right)\left(\prod_{j\notin\mathcal{I}}B_nD_N(r_j)\right)\\
&=\{\alpha_n(1+O(N^{-1}))\}^k\prod_{i=1}^k c_N(r_i)+\{\alpha_n(1+O(N^{-1}))\}^k\sum_{\mathcal{I}\subset[k]}B_n^{k-|\mathcal{I}|}\left(\prod_{i\in\mathcal{I}}c_N(r_i)\right)\left(\prod_{j\notin\mathcal{I}}B_nD_N(r_j)\right)\\
&=\mathcal{A}_1+\mathcal{A}_2
\end{aligned}\tag{58}$$

where we have defined

$$\begin{aligned}
\mathcal{A}_1&:=\{\alpha_n(1+O(N^{-1}))\}^k\prod_{i=1}^k c_N(r_i),\\
\mathcal{A}_2&:=\{\alpha_n(1+O(N^{-1}))\}^k\sum_{\mathcal{I}\subset[k]}B_n^{k-|\mathcal{I}|}\left(\prod_{i\in\mathcal{I}}c_N(r_i)\right)\left(\prod_{j\notin\mathcal{I}}B_nD_N(r_j)\right).
\end{aligned}\tag{59}$$

For the cross-terms involving $\mathcal{A}_2$ we will use the following bound, which follows from (28) and (29):

$$\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\mathcal{A}_2\le\left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)\{\alpha_n(1+O(N^{-1}))\}^k(t+1)^{k-1}\frac{1}{k!}(1+B_n)^k. \tag{60}$$

We also have the following deterministic bound using the argument of (46), which will be used for the cross-terms with $\mathcal{B}_2$, $\mathcal{B}_3$ and $\mathcal{B}_4$:

$$\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\mathcal{A}_1\le\{\alpha_n(1+O(N^{-1}))\}^k\sum_{\substack{r_1<\cdots<r_k\\ =1}}^{\tau_N(t)}\prod_{i=1}^k c_N(r_i)\le\{\alpha_n(1+O(N^{-1}))\}^k\frac{1}{k!}(t+1)^k. \tag{61}$$

Now for those $\mathcal{B}$ terms... First, expanding the relevant product in (55),

$$\begin{aligned}
\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)}(1-p_r)\le&\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)}\big(1-\alpha_n(1+O(N^{-1}))\{c_N(r)-B_n'D_N(r)\}\big)\\
&=1+\sum_{l=1}^{\tau_N(t)-k}\{-\alpha_n(1+O(N^{-1}))\}^l\sum_{\substack{s_1<\cdots<s_l\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)}\prod_{j=1}^l\{c_N(s_j)-B_n'D_N(s_j)\}\\
&=1+\sum_{l=1}^{\tau_N(t)-k}\{-\alpha_n(1+O(N^{-1}))\}^l\sum_{\substack{s_1<\cdots<s_l\\ =1}}^{\tau_N(t)}\prod_{j=1}^l\{c_N(s_j)-B_n'D_N(s_j)\}\\
&\quad-\sum_{l=1}^{\tau_N(t)-k}\{-\alpha_n(1+O(N^{-1}))\}^l\sum_{\substack{s_1<\cdots<s_l\\ \in\{r_1,\dots,r_k\}}}\prod_{j=1}^l\{c_N(s_j)-B_n'D_N(s_j)\}.
\end{aligned}\tag{62}$$

Now we further expand the RHS of (62), starting with the first line:

$$
\begin{aligned}
1+&\sum_{l=1}^{\tau_N(t)-k}\{-\alpha_n(1+O(N^{-1}))\}^l\sum_{\substack{s_1<\dots<s_l\\=1}}^{\tau_N(t)}\prod_{j=1}^{l}\{c_N(s_j)-B_n' D_N(s_j)\}\\
&=1+\sum_{l=1}^{\tau_N(t)-k}\frac{1}{l!}\{\alpha_n(1+O(N^{-1}))\}^l\sum_{\substack{s_1\neq\dots\neq s_l\\=1}}^{\tau_N(t)}\left\{\sum_{\substack{I=0\\ \text{even}}}^{l}\binom{l}{I}(B_n')^{l-I}\left(\prod_{i=1}^{I}c_N(s_i)\right)\left(\prod_{j=I+1}^{l}D_N(s_j)\right)\right.\\
&\qquad\left.-\sum_{\substack{I=0\\ \text{odd}}}^{l}\binom{l}{I}(B_n')^{l-I}\left(\prod_{i=1}^{I}c_N(s_i)\right)\left(\prod_{j=I+1}^{l}D_N(s_j)\right)\right\}\\
&\leq 1+\sum_{l=1}^{\tau_N(t)-k}\frac{1}{l!}\{\alpha_n(1+O(N^{-1}))\}^l\sum_{\substack{s_1\neq\dots\neq s_l\\=1}}^{\tau_N(t)}\prod_{i=1}^{l}c_N(s_i)\\
&\qquad+\sum_{l=1}^{\tau_N(t)-k}\frac{1}{l!}\{\alpha_n(1+O(N^{-1}))\}^l\sum_{\substack{s_1\neq\dots\neq s_l\\=1}}^{\tau_N(t)}\sum_{\substack{I=0\\ \text{even}}}^{l-1}\binom{l}{I}(B_n')^{l-I}\left(\prod_{i=1}^{I}c_N(s_i)\right)\left(\prod_{j=I+1}^{l}D_N(s_j)\right)\\
&\leq \sum_{l=0}^{\tau_N(t)-k}\frac{1}{l!}\{-\alpha_n(1+O(N^{-1}))\}^l\{t+c_N(\tau_N(t))\}^l\\
&\qquad+\sum_{l=1}^{\tau_N(t)-k}\frac{1}{l!}\{\alpha_n(1+O(N^{-1}))\}^l\sum_{\substack{I=0\\ \text{even}}}^{l-1}\binom{l}{I}(B_n')^{l-I}(t+1)^{l-1}\left(\sum_{r=1}^{\tau_N(t)}D_N(r)\right)\\
&\leq \sum_{l=0}^{\tau_N(t)-k}\frac{1}{l!}\{-\alpha_n(1+O(N^{-1}))\}^l t^l+\sum_{l=0}^{\tau_N(t)}\frac{1}{l!}\{\alpha_n(1+O(N^{-1}))\}^l c_N(\tau_N(t))\sum_{i=0}^{l-1}\binom{l}{i}t^i\left\{c_N(\tau_N(t))\right\}^{l-1-i}\\
&\qquad+\left(\sum_{r=1}^{\tau_N(t)}D_N(r)\right)\sum_{l=1}^{\tau_N(t)-k}\frac{1}{l!}\{\alpha_n(1+O(N^{-1}))\}^l(t+1)^{l-1}(1+B_n')^l\\
&\leq \sum_{l=0}^{\tau_N(t)-k}\frac{1}{l!}\{-\alpha_n(1+O(N^{-1}))\}^l t^l+c_N(\tau_N(t))\exp\left[\alpha_n(1+O(N^{-1}))(t+1)\right]\\
&\qquad+\left(\sum_{r=1}^{\tau_N(t)}D_N(r)\right)\exp\left[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')\right].
\end{aligned}
\tag{63}
$$

Now the second line:

$$
\begin{aligned}
&-\sum_{l=1}^{\tau_N(t)-k}\{-\alpha_n(1+O(N^{-1}))\}^l \sum_{\substack{s_1<\cdots<s_l\\ \in\{r_1,\ldots,r_k\}}}\prod_{j=1}^{l}\{c_N(s_j)-B_n' D_N(s_j)\}\\
&\qquad = -\sum_{l=1}^{k\wedge(\tau_N(t)-k)}\frac{1}{l!}\{\alpha_n(1+O(N^{-1}))\}^l \sum_{\substack{s_1\neq\cdots\neq s_l\\ \in\{r_1,\ldots,r_k\}}}\left\{\sum_{\substack{I=0\\ \text{even}}}^{l}\binom{l}{I}(B_n')^{l-I}\left(\prod_{i=1}^{I}c_N(s_i)\right)\left(\prod_{j=I+1}^{l}D_N(s_j)\right)\right.\\
&\qquad\qquad\left.-\sum_{\substack{I=0\\ \text{odd}}}^{l}\binom{l}{I}(B_n')^{l-I}\left(\prod_{i=1}^{I}c_N(s_i)\right)\left(\prod_{j=I+1}^{l}D_N(s_j)\right)\right\}\\
&\qquad \le \sum_{l=1}^{k}\frac{1}{l!}\{\alpha_n(1+O(N^{-1}))\}^l\sum_{\substack{I=0\\ \text{odd}}}^{l}\binom{l}{I}(B_n')^{l-I}\sum_{\substack{s_1\neq\cdots\neq s_l\\ \in\{r_1,\ldots,r_k\}}}\left(\prod_{i=1}^{I}c_N(s_i)\right)\left(\prod_{j=I+1}^{l}D_N(s_j)\right)\\
&\qquad \le \sum_{l=1}^{k}\frac{1}{l!}\{\alpha_n(1+O(N^{-1}))\}^l\sum_{\substack{I=0\\ \text{odd}}}^{l}\binom{l}{I}(B_n')^{l-I}\sum_{s\in\{r_1,\ldots,r_k\}}c_N(s)\sum_{\substack{s_1\neq\cdots\neq s_{l-1}\\ \in\{r_1,\ldots,r_k\}\setminus\{s\}}}\prod_{i=1}^{l-1}c_N(s_i)\\
&\qquad \le \sum_{l=1}^{k}\frac{1}{l!}\{\alpha_n(1+O(N^{-1}))\}^l(1+B_n')^l\left(\sum_{s\in\{r_1,\ldots,r_k\}}c_N(s)\right)\frac{(k-1)!}{(k-l)!}\\
&\qquad \le \left(\sum_{s\in\{r_1,\ldots,r_k\}}c_N(s)\right)\left[1+\alpha_n(1+O(N^{-1}))(1+B_n')\right]^k. \qquad (64)
\end{aligned}
$$

Putting these together, we have

$$
\begin{aligned}
\prod_{\substack{r=1\\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)}(1-p_r) &\le \sum_{l=0}^{\tau_N(t)-k}\frac{1}{l!}\{-\alpha_n(1+O(N^{-1}))\}^l t^l + c_N(\tau_N(t))\exp\left[\alpha_n(1+O(N^{-1}))(t+1)\right]\\
&\quad+\left(\sum_{r=1}^{\tau_N(t)}D_N(r)\right)\exp\left[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')\right]\\
&\quad+\left(\sum_{s\in\{r_1,\ldots,r_k\}}c_N(s)\right)\left[1+\alpha_n(1+O(N^{-1}))(1+B_n')\right]^k\\
&= \mathcal{B}_1+\mathcal{B}_2+\mathcal{B}_3+\mathcal{B}_4 \qquad (65)
\end{aligned}
$$

where we define

$$
\begin{aligned}
\mathcal{B}_1 &:= \sum_{l=0}^{\tau_N(t)-k}\frac{1}{l!}\{-\alpha_n(1+O(N^{-1}))\}^l t^l,\\
\mathcal{B}_2 &:= c_N(\tau_N(t))\exp\left[\alpha_n(1+O(N^{-1}))(t+1)\right],\\
\mathcal{B}_3 &:= \left(\sum_{r=1}^{\tau_N(t)}D_N(r)\right)\exp\left[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')\right],\\
\mathcal{B}_4 &:= \left(\sum_{s\in\{r_1,\ldots,r_k\}}c_N(s)\right)\left[1+\alpha_n(1+O(N^{-1}))(1+B_n')\right]^k. \qquad (66)
\end{aligned}
$$

We also have the deterministic bounds, independent of $r_1, \dots, r_k$:

$$\begin{aligned}
\mathcal{B}_1 &\le \exp\left[\alpha_n(1+O(N^{-1}))t\right], \\
\mathcal{B}_2 &\le \exp\left[\alpha_n(1+O(N^{-1}))(t+1)\right], \\
\mathcal{B}_3 &\le (t+1)\exp\left[\alpha_n(1+O(N^{-1}))(t+1)(1+B'_n)\right], \\
\mathcal{B}_4 &\le k\left[1+\alpha_n(1+O(N^{-1}))(1+B'_n)\right]^k.
\end{aligned} \tag{67}$$

Now we are in a position to bound the cross-terms.

Using the bounds in (67), which are all non-negative, deterministic and do not depend on $r_1, \dots, r_k$, we can bound

$$\mathbb{E}\left[\sum \mathcal{A}_2\mathcal{B}_1\right] + \mathbb{E}\left[\sum \mathcal{A}_2\mathcal{B}_2\right] + \mathbb{E}\left[\sum \mathcal{A}_2\mathcal{B}_3\right] + \mathbb{E}\left[\sum \mathcal{A}_2\mathcal{B}_4\right] \tag{68}$$

by the sum of the bounds in (67), multiplied by the expectation of the bound from (60), which is

$$\mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \mathcal{A}_2\right] \le \mathbb{E}\left[\left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)\{\alpha_n(1+O(N^{-1}))\}^k(t+1)^{k-1}\frac{1}{k!}(1+B_n)^k\right] \longrightarrow 0 \tag{69}$$

as $N\to\infty$ by Brown et al. (2020, Equation (4)). So all the terms involving $\mathcal{A}_2$ vanish in the limit. Now the remaining terms are those involving $\mathcal{A}_1$. We see from their definitions in (66) that $\mathcal{B}_2$ and $\mathcal{B}_3$ do not depend on $r_1, \dots, r_k$. Along with the deterministic bound (61), this means we can write

$$\begin{aligned}
\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \mathcal{A}_1(\mathcal{B}_2+\mathcal{B}_3)\right] &\le \lim_{N\to\infty}\mathbb{E}\left[\mathcal{B}_2+\mathcal{B}_3\right]\times\alpha_n^k\frac{1}{k!}(t+1)^k \\
&\le \left\{\lim_{N\to\infty}\mathbb{E}\left[c_N(\tau_N(t))\right]\exp[\alpha_n(t+1)] + \lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} D_N(s)\right]\exp[\alpha_n(t+1)(1+B'_n)]\right\}\times\alpha_n^k\frac{1}{k!}(t+1)^k \\
&= 0
\end{aligned} \tag{70}$$

by Brown et al. (2020, Equations (3)–(4)). The other nuisance term $\mathcal{A}_1\mathcal{B}_4$ requires more delicate handling, since $\mathcal{B}_4$ depends on $r_1, \dots, r_k$. Using (66) and (59),

$$\sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \mathcal{A}_1\mathcal{B}_4 \le \{\alpha_n(1+O(N^{-1}))\}^k\left[1+\alpha_n(1+O(N^{-1}))(1+B'_n)\right]^k \sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\{c_N(r_1)+\dots+c_N(r_k)\}\prod_{i=1}^k c_N(r_i). \tag{71}$$

Expanding the sum on the RHS,

$$\begin{aligned}
\sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\{c_N(r_1)+\dots+c_N(r_k)\}\prod_{i=1}^k c_N(r_i) &= \sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\sum_{j=1}^k c_N(r_j)^2\prod_{i\ne j}c_N(r_i) \\
&\le \sum_{j=1}^k\sum_{\substack{r_1\ne\dots\ne r_k:\\ =1}}^{\tau_N(t)} c_N(r_j)^2\prod_{i\ne j}c_N(r_i) \\
&= k\sum_{s=1}^{\tau_N(t)}c_N(s)^2\sum_{\substack{r_1\ne\dots\ne r_{k-1}:\\ =1}}^{\tau_N(t)}\prod_{i=1}^{k-1}c_N(r_i) \\
&\le k\left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right)(t+1)^{k-1}
\end{aligned} \tag{72}$$

where the last inequality follows from (46). Hence

$$\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \mathcal{A}_1\mathcal{B}_4\right] \le \alpha_n^k \left[1+\alpha_n(1+B'_n)\right]^k k(t+1)^{k-1} \lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right] = 0 \tag{73}$$

by Brown et al. (2020, Equation (4)).

We have shown that all the cross-terms vanish in the limit, except for $\mathcal{A}_1\mathcal{B}_1$, which will give us the RHS of (55):

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \mathcal{A}_1\mathcal{B}_1\right] &= \alpha_n^k \lim_{N\to\infty}\mathbb{E}\left[\sum_{l=0}^{\tau_N(t)-k} \frac{1}{l!}\{-\alpha_n(1+O(N^{-1}))\}^l t^l \sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^k c_N(r_i)\right] \\
&= \alpha_n^k \lim_{N\to\infty}\mathbb{E}\left[\sum_{l=0}^{\infty} \frac{1}{l!}\{-\alpha_n(1+O(N^{-1}))\}^l t^l \mathbb{1}_{l+k\le\tau_N(t)} \sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^k c_N(r_i)\right] \\
&= \alpha_n^k \sum_{l=0}^{\infty}\frac{1}{l!}(-\alpha_n)^l t^l \lim_{N\to\infty}\mathbb{E}\left[\mathbb{1}_{l+k\le\tau_N(t)} \sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^k c_N(r_i)\right] \\
&= \alpha_n^k \sum_{l=0}^{\infty}\frac{1}{l!}(-\alpha_n)^l t^l \lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^k c_N(r_i) \,\middle|\, l+k\le\tau_N(t)\right] \lim_{N\to\infty}\mathbb{P}\left[l+k\le\tau_N(t)\right] \\
&= \alpha_n^k \sum_{l=0}^{\infty}\frac{1}{l!}(-\alpha_n)^l t^l \lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^k c_N(r_i) \,\middle|\, l+k\le\tau_N(t)\right]
\end{aligned} \tag{74}$$

by Lemma 2. Altogether we have shown that

$$\begin{aligned}
\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^k p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\right] &\le \lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} (\mathcal{A}_1+\mathcal{A}_2)(\mathcal{B}_1+\mathcal{B}_2+\mathcal{B}_3+\mathcal{B}_4)\right] \\
&\le \alpha_n^k \sum_{l=0}^{\infty}\frac{1}{l!}(-\alpha_n)^l t^l \lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^k c_N(r_i) \,\middle|\, l+k\le\tau_N(t)\right]
\end{aligned} \tag{75}$$

which concludes the proof. $\square$

Induction step: lower bound

References

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