

Weak convergence proof (in progress)

Suzie Brown

September 23, 2020

Theorem 1. Let $\nu_t^{(1:N)}$ denote the offspring numbers in an interacting particle system satisfying the standing assumption and such that, for any N sufficiently large, $\mathbb{P}\{\tau_N(t) = \infty\} = 0$ for all finite t . Suppose that there exists a deterministic sequence $(b_N)_{N \geq 1}$ such that $\lim_{N \rightarrow \infty} b_N = 0$ and

$$\frac{1}{(N)_3} \sum_{i=1}^N \mathbb{E}_t\{(\nu_t^{(i)})_3\} \leq b_N \frac{1}{(N)_2} \sum_{i=1}^N \mathbb{E}_t\{(\nu_t^{(i)})_2\} \quad (1)$$

for all N , uniformly in $t \geq 1$. Then the rescaled genealogical process $(G_{\tau_N(t)}^{(n,N)})_{t \geq 0}$ converges weakly to Kingman's n -coalescent as $N \rightarrow \infty$.

Proof. Define $p_t := \max_{\xi \in E} \{1 - p_{\xi\xi}(t)\} = 1 - p_{\Delta\Delta}(t)$, where Δ denotes the trivial partition of $\{1, \dots, n\}$ into singletons. For a proof that the maximum is attained at $\xi = \Delta$, see Lemma 1. Following Möhle (1999), we now construct the two-dimensional Markov process $(Z_t, S_t)_{t \in \mathbb{N}}$ with transition probabilities

$$\mathbb{P}(Z_t = j, S_t = \eta \mid Z_{t-1} = i, S_{t-1} = \xi) = \begin{cases} 1 - p_t & \text{if } j = i \text{ and } \xi = \eta \\ p_{\xi\xi}(t) + p_t - 1 & \text{if } j = i + 1 \text{ and } \xi = \eta \\ p_{\xi\eta}(t) & \text{if } j = i + 1 \text{ and } \xi \neq \eta \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

The construction is such that the marginal (S_t) has the same distribution as the genealogical process of interest, and (Z_t) has jumps at all the times (S_t) does plus some extra jumps. (The definition of p_t ensures that the probability in the second case is non-negative, attaining the value zero when $\xi = \Delta$.)

Denote by $0 = T_0^{(N)} < T_1^{(N)} < \dots$ the jump times of the rescaled process $(Z_{\tau_N(t)})_{t \geq 0}$, and $\omega_i^{(N)} := T_i^{(N)} - T_{i-1}^{(N)}$ the corresponding holding times ($i \in \mathbb{N}$).

...

□

Lemma 1. $\max_{\xi \in E} (1 - p_{\xi\xi}(t)) = 1 - p_{\Delta\Delta}(t)$.

Proof. Consider any $\xi \in E$ consisting of k blocks ($1 \leq k \leq n - 1$), and any $\xi' \in E$ consisting of $k + 1$ blocks. From the definition of $p_{\xi\xi}(t)$ (Koskela et al., 2018, Equation (1)),

$$p_{\xi\xi}(t) = \frac{1}{(N)_k} \sum_{\substack{i_1, \dots, i_k \\ \text{all distinct}}} \nu_t^{(i_1)} \dots \nu_t^{(i_k)}. \quad (3)$$

Similarly,

$$\begin{aligned} p_{\xi'\xi'}(t) &= \frac{1}{(N)_{k+1}} \sum_{\substack{i_1, \dots, i_{k+1} \\ \text{all distinct}}} \nu_t^{(i_1)} \dots \nu_t^{(i_k)} \nu_t^{(i_{k+1})} \\ &= \frac{1}{(N)_k(N-k)} \sum_{\substack{i_1, \dots, i_k \\ \text{all distinct}}} \left\{ \nu_t^{(i_1)} \dots \nu_t^{(i_k)} \sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^N \nu_t^{(i_{k+1})} \right\}. \end{aligned} \quad (4)$$

Discarding the zero summands,

$$p_{\xi'\xi'}(t) = \frac{1}{(N)_k(N-k)} \sum_{\substack{i_1, \dots, i_k \\ \text{all distinct:} \\ \nu_t^{(i_1)}, \dots, \nu_t^{(i_k)} > 0}} \left\{ \nu_t^{(i_1)} \dots \nu_t^{(i_k)} \sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^N \nu_t^{(i_{k+1})} \right\}. \quad (5)$$

The inner sum is

$$\sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^N \nu_t^{(i_{k+1})} = \left\{ \sum_{i=1}^N \nu_t^{(i)} - \sum_{i \in \{i_1, \dots, i_k\}} \nu_t^{(i)} \right\} \leq N - k \quad (6)$$

since $\nu_t^{(i_1)}, \dots, \nu_t^{(i_k)}$ are all at least 1. Hence

$$p_{\xi'\xi'}(t) \leq \frac{N-k}{(N)_k(N-k)} \sum_{\substack{i_1, \dots, i_k \\ \text{all distinct:} \\ \nu_t^{(i_1)}, \dots, \nu_t^{(i_k)} > 0}} \nu_t^{(i_1)} \dots \nu_t^{(i_k)} = p_{\xi\xi}(t). \quad (7)$$

Thus $p_{\xi\xi}(t)$ is decreasing in the number of blocks of ξ , and is therefore minimised by taking $\xi = \Delta$, which achieves the maximum n blocks. This choice in turn maximises $1 - p_{\xi\xi}(t)$, as required. \square

Lemma 2.

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\prod_{r=1}^{\tau_N(t)} (1 - p_r) \right] = e^{-\alpha t} \quad (8)$$

where $\alpha := n(n-1)/2$.

Proof. The strategy is to find upper and lower bounds on $\mathbb{E} \left[\prod_{r=1}^{\tau_N(t)} (1 - p_r) \right]$, both of which converge to $e^{-\alpha t}$.

Lower Bound

From Brown et al. (2020, Equation (14)), taking $\xi = \Delta$, we have

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \alpha(1 + O(N^{-1})) \left[\frac{B_n}{\alpha} D_N(t) + c_N(t) \right] \quad (9)$$

where $B_n > 0$. Hence, by a multinomial expansion,

$$\begin{aligned} \prod_{r=1}^{\tau_N(t)} (1 - p_r) &\geq \prod_{r=1}^{\tau_N(t)} \left\{ 1 - \alpha(1 + O(N^{-1})) \left[\frac{B_n}{\alpha} D_N(r) + c_N(r) \right] \right\} \\ &= 1 + \sum_{k=1}^{\infty} \sum_{\substack{r_1 < \dots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^k \left\{ -\alpha(1 + O(N^{-1})) \left[\frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right] \right\} \\ &= 1 + \sum_{k=1}^{\infty} \{ -\alpha(1 + O(N^{-1})) \}^k \sum_{\substack{r_1 < \dots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^k \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\}. \end{aligned} \quad (10)$$

(The empty sum is defined to be zero and the empty product is defined to be one throughout.) Taking expectations,

$$\mathbb{E} \left[\prod_{r=1}^{\tau_N(t)} (1 - p_r) \right] \geq 1 + \sum_{k=1}^{\infty} \{ -\alpha(1 + O(N^{-1})) \}^k \mathbb{E} \left[\sum_{\substack{r_1 < \dots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^k \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\} \right] \quad (11)$$

(the infinite sum has only finitely many non-zero summands, since the inner sum is empty for $k > \tau_N(t)$, which justifies swapping the sum and expectation.) We want to show that the expectation on the right converges to $t^k/k!$, for reasons that will become clear. The strategy is to upper and lower bound this expectation by quantities that converge to $t^k/k!$.

First the lower bound. From Koskela et al. (2018, Equation (8)),

$$\begin{aligned}
\sum_{\substack{r_1 < \dots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^k \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\} &\geq \sum_{\substack{r_1 < \dots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^k c_N(r_j) \\
&\geq \frac{1}{k!} \left(\sum_{s=1}^{\tau_N(t)} c_N(s) \right)^k - \frac{1}{k!} \binom{k}{2} \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \left(\sum_{s=1}^{\tau_N(t)} c_N(s) \right)^{k-2} \\
&\geq \frac{1}{k!} t^k - \frac{1}{k!} \binom{k}{2} (t+1)^{k-2} \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right)
\end{aligned} \tag{12}$$

by the definition of τ_N . Then

$$\mathbb{E} \left[\sum_{\substack{r_1 < \dots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^k \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\} \right] \geq \frac{1}{k!} t^k - \frac{1}{k!} \binom{k}{2} (t+1)^{k-2} \mathbb{E} \left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right] \rightarrow \frac{1}{k!} t^k \tag{13}$$

as $N \rightarrow \infty$ using Brown et al. (2020, Equation (5)), which is shown to hold via lemmata 1 and 3 therein.

Now for the upper bound. We start with a multinomial expansion and some manipulations of the sums:

$$\begin{aligned}
\sum_{\substack{r_1 < \dots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^k \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\} &= \frac{1}{k!} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \prod_{j=1}^k \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\} \\
&= \frac{1}{k!} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \sum_{\mathcal{I} \subseteq \{1, \dots, k\}} \left(\frac{B_n}{\alpha} \right)^{k-|\mathcal{I}|} \left\{ \prod_{i \in \mathcal{I}} c_N(r_i) \right\} \left\{ \prod_{j \notin \mathcal{I}} D_N(r_j) \right\} \\
&= \frac{1}{k!} \sum_{\mathcal{I} \subseteq \{1, \dots, k\}} \left(\frac{B_n}{\alpha} \right)^{k-|\mathcal{I}|} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i \in \mathcal{I}} c_N(r_i) \right\} \left\{ \prod_{j \notin \mathcal{I}} D_N(r_j) \right\} \\
&= \frac{1}{k!} \sum_{I=0}^k \binom{k}{I} \left(\frac{B_n}{\alpha} \right)^{k-I} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^I c_N(r_i) \right\} \left\{ \prod_{j=I+1}^k D_N(r_j) \right\} \\
&= \frac{1}{k!} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^k c_N(r_i) \right\} \\
&\quad + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left(\frac{B_n}{\alpha} \right)^{k-I} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^I c_N(r_i) \right\} \left\{ \prod_{j=I+1}^k D_N(r_j) \right\}. \tag{14}
\end{aligned}$$

Then, using that $D_N(s) \leq c_N(s)$ for all s (see Lemma 3 for a proof), along with the definition of τ_N ,

$$\begin{aligned}
& \frac{1}{k!} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^k c_N(r_i) \right\} + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left(\frac{B_n}{\alpha} \right)^{k-I} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^I c_N(r_i) \right\} \left\{ \prod_{j=I+1}^k D_N(r_j) \right\} \\
& \leq \frac{1}{k!} \left(\sum_{r=1}^{\tau_N(t)} c_N(r) \right)^k + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left(\frac{B_n}{\alpha} \right)^{k-I} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^{k-1} c_N(r_i) \right\} \{D_N(r_k)\} \\
& \leq \frac{1}{k!} \{t + c_N(\tau_N(t))\}^k + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left(\frac{B_n}{\alpha} \right)^{k-I} \left\{ \sum_{\substack{r_1 \neq \dots \neq r_{k-1} \\ \text{all distinct}}}^{\tau_N(t)} \prod_{i=1}^{k-1} c_N(r_i) \right\} \left\{ \sum_{r_k=1}^{\tau_N(t)} D_N(r_k) \right\} \\
& \leq \frac{1}{k!} \{t + c_N(\tau_N(t))\}^k + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left(\frac{B_n}{\alpha} \right)^{k-I} \left(\sum_{r=1}^{\tau_N(t)} c_N(r) \right)^{k-1} \left(\sum_{r=1}^{\tau_N(t)} D_N(r) \right) \\
& \leq \frac{1}{k!} \{t + c_N(\tau_N(t))\}^k + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left(\frac{B_n}{\alpha} \right)^{k-I} (t+1)^{k-1} \left(\sum_{r=1}^{\tau_N(t)} D_N(r) \right). \tag{15}
\end{aligned}$$

Taking expectations,

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{r_1 < \dots < r_k}^{\tau_N(t)} \prod_{j=1}^k \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\} \right] & \leq \frac{1}{k!} \lim_{N \rightarrow \infty} \mathbb{E}[\{t + c_N(\tau_N(t))\}^k] \\
& + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left(\frac{B_n}{\alpha} \right)^{k-I} (t+1)^{k-1} \lim_{N \rightarrow \infty} \mathbb{E} \left[\left(\sum_{r=1}^{\tau_N(t)} D_N(r) \right) \right] \\
& = \frac{1}{k!} t^k. \tag{16}
\end{aligned}$$

The limit follows from Brown et al. (2020, Equations (3),(4)) along with the fact that, since $c_N(s) \in [0, 1]$ for all s , $\mathbb{E}[c_N(s)^k] \leq \mathbb{E}[c_N(s)]$ for all $k \geq 1$, and the expansion

$$\mathbb{E} \left[\frac{1}{k!} \{t + c_N(\tau_N(t))\}^k \right] = \mathbb{E} \left[\frac{1}{k!} \sum_{i=0}^k \binom{k}{i} t^i c_N(\tau_N(t))^{k-i} \right] = \frac{1}{k!} \{t^k + k t^{k-1} \mathbb{E}[c_N(\tau_N(t))] + \dots\} \rightarrow \frac{1}{k!} t^k. \tag{17}$$

Combining these upper and lower limits, we conclude that

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{r_1 < \dots < r_k}^{\tau_N(t)} \prod_{j=1}^k \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\} \right] = \frac{1}{k!} t^k \tag{18}$$

and thus

$$1 + \sum_{k=1}^{\infty} \{-\alpha(1 + O(N^{-1}))\}^k \mathbb{E} \left[\sum_{r_1 < \dots < r_k}^{\tau_N(t)} \prod_{j=1}^k \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\} \right] \rightarrow 1 + \sum_{k=1}^{\infty} (-\alpha)^k \frac{1}{k!} t^k = e^{-\alpha t} \tag{19}$$

as $N \rightarrow \infty$.

Upper Bound

From Koskela et al. (2018, Lemma 1 Case 1), taking $\xi = \Delta$, we have

$$1 - p_t = p_{\Delta\Delta}(t) \leq 1 - \alpha(1 + O(N^{-1})) \left[c_N(t) - \binom{n-1}{2} D_N(t) \right]. \tag{20}$$

A multinomial expansion as before yields

$$\prod_{r=1}^{\tau_N(t)} (1 - p_r) \leq 1 + \sum_{k=1}^{\infty} \{-\alpha(1 + O(N^{-1}))\}^k \sum_{\substack{r_1 < \dots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^k \left\{ c_N(r_j) - \binom{n-1}{2} D_N(r_j) \right\}. \quad (21)$$

Analogously to (14), we can write

$$\begin{aligned} \sum_{\substack{r_1 < \dots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^k \left\{ c_N(r_j) - \binom{n-1}{2} D_N(r_j) \right\} &= \frac{1}{k!} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^k c_N(r_i) \right\} \\ &\quad + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left(-\binom{n-1}{2} \right)^{k-I} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}} \left\{ \prod_{i=1}^I c_N(r_i) \right\} \left\{ \prod_{j=I+1}^k D_N(r_j) \right\}. \end{aligned} \quad (22)$$

We start by dealing with the second term:

$$\begin{aligned} \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left(-\binom{n-1}{2} \right)^{k-I} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}} \left\{ \prod_{i=1}^I c_N(r_i) \right\} \left\{ \prod_{j=I+1}^k D_N(r_j) \right\} \\ = \frac{1}{k!} \sum_{\substack{I=0: \\ k-I \text{ even}}}^{k-1} \binom{k}{I} \binom{n-1}{2}^{k-I} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}} \left\{ \prod_{i=1}^I c_N(r_i) \right\} \left\{ \prod_{j=I+1}^k D_N(r_j) \right\} \\ - \frac{1}{k!} \sum_{\substack{I=0: \\ k-I \text{ odd}}}^{k-1} \binom{k}{I} \binom{n-1}{2}^{k-I} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}} \left\{ \prod_{i=1}^I c_N(r_i) \right\} \left\{ \prod_{j=I+1}^k D_N(r_j) \right\}. \end{aligned} \quad (23)$$

This is lower bounded by

$$0 - \frac{1}{k!} \sum_{\substack{I=0 \\ (k-I) \text{ odd}}}^{k-1} \binom{k}{I} \binom{n-1}{2}^{k-I} (t+1)^{k-1} \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \quad (24)$$

using that $c_N(r), D_N(r) \geq 0$ for all r to bound the even terms below, and arguments from (15) to bound the odd terms above. The same arguments lead to the upper bound

$$\frac{1}{k!} \sum_{\substack{I=0 \\ (k-I) \text{ odd}}}^{k-1} \binom{k}{I} \binom{n-1}{2}^{k-I} (t+1)^{k-1} \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) - 0. \quad (25)$$

By Brown et al. (2020, Equation (4)), both bounds have vanishing expectation as $N \rightarrow \infty$. We are left with the first term in (18), which is upper bounded by

$$\frac{1}{k!} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^k c_N(r_i) \right\} \leq \frac{1}{k!} \left(\sum_{s=1}^{\tau_N(t)} c_N(s) \right)^k \leq \frac{1}{k!} \{t + c_N(\tau_N(t))\}^k \quad (26)$$

the expectation of which converges to $t^k/k!$ as in (17). We use Koskela et al. (2018, Equation (8)) to construct a lower bound:

$$\begin{aligned} \frac{1}{k!} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^k c_N(r_i) \right\} &\geq \frac{1}{k!} \left(\sum_{s=1}^{\tau_N(t)} c_N(s) \right)^k - \frac{1}{k!} \binom{k}{2} \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \left(\sum_{s=1}^{\tau_N(t)} c_N(s) \right)^{k-2} \\ &\geq \frac{1}{k!} t^k - \frac{1}{k!} \binom{k}{2} (t+1)^{k-2} \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \end{aligned} \quad (27)$$

The expectation of this bound also converges to $t^k/k!$, using Brown et al. (2020, Equation (5)). We can now conclude that

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{r_1 < \dots < r_k}^{\tau_N(t)} \prod_{j=1}^k \left\{ c_N(r_j) - \binom{n-1}{2} D_N(r_j) \right\} \right] = \frac{1}{k!} t^k \quad (28)$$

and thus

$$1 + \sum_{k=1}^{\infty} \left\{ -\alpha(1 + O(N^{-1})) \right\}^k \mathbb{E} \left[\sum_{r_1 < \dots < r_k}^{\tau_N(t)} \prod_{j=1}^k \left\{ c_N(r_j) - \binom{n-1}{2} D_N(r_j) \right\} \right] \rightarrow 1 + \sum_{k=1}^{\infty} (-\alpha)^k \frac{1}{k!} t^k = e^{-\alpha t} \quad (29)$$

as $N \rightarrow \infty$.

We now have upper and lower bounds on $\lim_{N \rightarrow \infty} \mathbb{E} \left[\prod_{r=1}^{\tau_N(t)} (1 - p_r) \right]$, both of which are equal to $e^{-\alpha t}$, and the result follows. \square

Lemma 3. $D_N(t) \leq c_N(t)$ for all t .

Proof.

$$\begin{aligned} D_N(t) &:= \frac{1}{(N)_2} \sum_{i=1}^N (\nu_t^{(i)})_2 \frac{1}{N} \left\{ \nu_t^{(i)} + \frac{1}{N} \sum_{j \neq i}^N (\nu_t^{(j)})^2 \right\} \leq \frac{1}{(N)_2} \sum_{i=1}^N (\nu_t^{(i)})_2 \frac{1}{N} \left\{ \nu_t^{(i)} + \frac{1}{N} \sum_{j \neq i}^N N \nu_t^{(j)} \right\} \\ &= \frac{1}{(N)_2} \sum_{i=1}^N (\nu_t^{(i)})_2 \frac{1}{N} \left\{ \sum_{j=1}^N \nu_t^{(j)} \right\} \leq \frac{1}{(N)_2} \sum_{i=1}^N (\nu_t^{(i)})_2 = c_N(t). \end{aligned} \quad (30)$$

\square

References

- Brown, S., Jenkins, P. A., Johansen, A. M. and Koskela, J. (2020), ‘Simple conditions for convergence of sequential Monte Carlo genealogies with applications’, *arXiv preprint arXiv:2007.00096*.
- Koskela, J., Jenkins, P. A., Johansen, A. M. and Spanò, D. (2018), ‘Asymptotic genealogies of interacting particle systems with an application to sequential Monte Carlo’, *arXiv preprint arXiv:1804.01811*.
- Möhle, M. (1999), ‘Weak convergence to the coalescent in neutral population models’, *Journal of Applied Probability* **36**(2), 446–460.