Weak convergence proof (in progress)

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September 22, 2020

Theorem 1. *Let $\nu_t^{(1:N)}$ denote the offspring numbers in an interacting particle system satisfying the standing assumption and such that, for any N sufficiently large, $\mathbb{P}\{\tau_N(t)=\infty\}=0$ for all finite t. Suppose that there exists a deterministic sequence $(b_N)_{N\geq 1}$ such that $\lim_{N\to\infty} b_N = 0$ and*

$$\frac{1}{(N)_3}\sum_{i=1}^{N}\mathbb{E}_t\{(\nu_t^{(i)})_3\} \leq b_N \frac{1}{(N)_2}\sum_{i=1}^{N}\mathbb{E}_t\{(\nu_t^{(i)})_2\} \tag{1}$$

for all N, uniformly in $t \geq 1$. Then the rescaled genealogical process $(G^{(n,N)}_{\tau_N(t)})_{t\geq 0}$ converges weakly to Kingman's n-coalescent as $N\to\infty$.

Proof. Define $p_t := \max_{\xi\in E}\{1-p_{\xi\xi}(t)\} = 1-p_{\Delta\Delta}(t)$, where Δ denotes the trivial partition of $\{1,\dots,n\}$ into singletons. For a proof that the maximum is attained at $\xi=\Delta$, see Lemma 1. Following Möhle (1999), we now construct the two-dimensional Markov process $(Z_t,S_t)_{t\in\mathbb{N}}$ with transition probabilities

$$\mathbb{P}(Z_t=j, S_t=\eta \mid Z_{t-1}=i, S_{t-1}=\xi) = \begin{cases} 1-p_t & \text{if } j=i \text{ and } \xi=\eta \\ p_{\xi\xi}(t)+p_t-1 & \text{if } j=i+1 \text{ and } \xi=\eta \\ p_{\xi\eta}(t) & \text{if } j=i+1 \text{ and } \xi\neq\eta \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The construction is such that the marginal (S_t) has the same distribution as the genealogical process of interest, and (Z_t) has jumps at all the times (S_t) does plus some extra jumps. (The definition of p_t ensures that the probability in the second case is non-negative, attaining the value zero when $\xi=\Delta$.)

Denote by $0=T_0^{(N)}<T_1^{(N)}<\dots$ the jump times of the rescaled process $(Z_{\tau_N(t)})_{t\geq 0}$, and $\omega_i^{(N)} := T_i^{(N)}-T_{i-1}^{(N)}$ the corresponding holding times ($i\in\mathbb{N}$).

... □

Lemma 1. $\max_{\xi\in E}(1-p_{\xi\xi}(t)) = 1-p_{\Delta\Delta}(t)$.

Proof. Consider any $\xi\in E$ consisting of k blocks ($1\leq k\leq n-1$), and any $\xi'\in E$ consisting of $k+1$ blocks. From the definition of $p_{\xi\eta}(t)$ (Koskela et al., 2018, Equation (1)),

$$p_{\xi\xi}(t) = \frac{1}{(N)_k}\sum_{\substack{i_1,\dots,i_k \\ \text{all distinct}}} \nu_t^{(i_1)}\cdots\nu_t^{(i_k)}. \tag{3}$$

Similarly,

$$\begin{aligned} p_{\xi'\xi'}(t) &= \frac{1}{(N)_{k+1}}\sum_{\substack{i_1,\dots,i_k,i_{k+1} \\ \text{all distinct}}} \nu_t^{(i_1)}\cdots\nu_t^{(i_k)}\nu_t^{(i_{k+1})} \\ &= \frac{1}{(N)_k(N-k)}\sum_{\substack{i_1,\dots,i_k \\ \text{all distinct}}} \left\{ \nu_t^{(i_1)}\cdots\nu_t^{(i_k)} \sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^{N} \nu_t^{(i_{k+1})} \right\}. \end{aligned} \tag{4}$$

Discarding the zero summands,

$$p_{\xi'\xi'}(t) = \frac{1}{(N)_k (N-k)} \sum_{\substack{i_1,\dots,i_k \\ \text{all distinct:} \\ \nu_t^{(i_1)},\dots,\nu_t^{(i_k)}>0}} \left\{ \nu_t^{(i_1)} \cdots \nu_t^{(i_k)} \sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^{N} \nu_t^{(i_{k+1})} \right\}. \tag{5}$$

The inner sum is

$$\sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^{N} \nu_t^{(i_{k+1})} = \left\{ \sum_{i=1}^{N} \nu_t^{(i)} - \sum_{i \in \{i_1,\dots,i_k\}} \nu_t^{(i)} \right\} \leq N-k \tag{6}$$

since $\nu_t^{(i_1)},\dots,\nu_t^{(i_k)}$ are all at least 1. Hence

$$p_{\xi'\xi'}(t) \leq \frac{N-k}{(N)_k (N-k)} \sum_{\substack{i_1,\dots,i_k \\ \text{all distinct:} \\ \nu_t^{(i_1)},\dots,\nu_t^{(i_k)}>0}} \nu_t^{(i_1)} \cdots \nu_t^{(i_k)} = p_{\xi\xi}(t). \tag{7}$$

Thus $p_{\xi\xi}(t)$ is decreasing in the number of blocks of ξ, and is therefore minimised by taking $\xi = \Delta$, which achieves the maximum n blocks. This choice in turn maximises $1 - p_{\xi\xi}(t)$, as required. □

Lemma 2.

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1-p_r) \right] = e^{-\alpha t} \tag{8}$$

where $\alpha := n(n-1)/2$.

Proof. The strategy is to find upper and lower bounds on $\mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right]$, both of which converge to $e^{-\alpha t}$.

Lower Bound

From Brown et al. (2020, Equation (14)), taking $\xi = \Delta$, we have

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \alpha(1 + O(N^{-1})) \left[\frac{B_n}{\alpha} D_N(t) + c_N(t) \right] \tag{9}$$

where $B_n > 0$. Hence, by a multinomial expansion,

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1-p_r) &\geq \prod_{r=1}^{\tau_N(t)} \left\{ 1 - \alpha(1+O(N^{-1}) \left[\frac{B_n}{\alpha} D_N(r) + c_N(r) \right] \right\} \\
&= 1 + \sum_{k=1}^{\infty} \sum_{\substack{r_1<\dots<r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^{k} \left\{ -\alpha(1+O(N^{-1})) \left[\frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right] \right\} \\
&= 1 + \sum_{k=1}^{\infty} \left\{ -\alpha(1+O(N^{-1})) \right\}^k \sum_{\substack{r_1<\dots<r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^{k} \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\}.
\end{aligned} \tag{10}$$

(The empty sum is defined to be zero and the empty product is defined to be one throughout.) Taking expectations,

$$\mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1-p_r) \right] \geq 1 + \sum_{k=1}^{\infty} \left\{ -\alpha(1+O(N^{-1})) \right\}^k \mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^{k} \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\} \right] \tag{11}$$

(the infinite sum has only finitely many non-zero summands, since the inner sum is empty for $k > \tau_N(t)$, which justifies swapping the sum and expectation.) We want to show that the expectation on the right converges to $t^k/k!$, for reasons that will become clear. The strategy is to upper and lower bound this expectation by quantities that converge to $t^k/k!$.

First the lower bound. From Koskela et al. (2018, Equation (8)),

$$\begin{aligned}
\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)} \prod_{j=1}^{k} \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\} &\geq \sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)} \prod_{j=1}^{k} c_N(r_j) \\
&\geq \frac{1}{k!} \left(\sum_{s=1}^{\tau_N(t)} c_N(s) \right)^k - \frac{1}{k!} \binom{k}{2} \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \left(\sum_{s=1}^{\tau_N(t)} c_N(s) \right)^{k-2} \\
&\geq \frac{1}{k!} t^k - \frac{1}{k!} \binom{k}{2} (t+1)^{k-2} \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \qquad (12)
\end{aligned}$$

by the definition of τ_N. Then

$$\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)} \prod_{j=1}^{k} \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\} \right] \geq \frac{1}{k!} t^k - \frac{1}{k!} \binom{k}{2} (t+1)^{k-2} \mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right] \longrightarrow \frac{1}{k!} t^k \qquad (13)$$

as $N \to \infty$ using Brown et al. (2020, Equation (5)), which is shown to hold via lemmata 1 and 3 therein.

Now for the upper bound. We start with a multinomial expansion and some manipulations of the sums:

$$\begin{aligned}
\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)} \prod_{j=1}^{k} \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\} &= \frac{1}{k!} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \prod_{j=1}^{k} \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\} \\
&= \frac{1}{k!} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \sum_{\mathcal{I}\subseteq\{1,\ldots,k\}} \left(\frac{B_n}{\alpha} \right)^{k-|\mathcal{I}|} \left\{ \prod_{i\in\mathcal{I}} c_N(r_i) \right\} \left\{ \prod_{j\notin\mathcal{I}} D_N(r_j) \right\} \\
&= \frac{1}{k!} \sum_{\mathcal{I}\subseteq\{1,\ldots,k\}} \left(\frac{B_n}{\alpha} \right)^{k-|\mathcal{I}|} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i\in\mathcal{I}} c_N(r_i) \right\} \left\{ \prod_{j\notin\mathcal{I}} D_N(r_j) \right\} \\
&= \frac{1}{k!} \sum_{I=0}^{k} \binom{k}{I} \left(\frac{B_n}{\alpha} \right)^{k-I} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^{I} c_N(r_i) \right\} \left\{ \prod_{j=I+1}^{k} D_N(r_j) \right\} \\
&= \frac{1}{k!} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^{k} c_N(r_i) \right\} \\
&\quad + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left(\frac{B_n}{\alpha} \right)^{k-I} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^{I} c_N(r_i) \right\} \left\{ \prod_{j=I+1}^{k} D_N(r_j) \right\}. \qquad (14)
\end{aligned}$$

Then, using that $D_N(s) \leq c_N(s)$ for all s (see Lemma 3 for a proof), along with the definition of τ_N,

$$
\begin{aligned}
&\frac{1}{k!} \sum_{\substack{r_1 \neq \cdots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^{k} c_N(r_i) \right\} + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left(\frac{B_n}{\alpha} \right)^{k-I} \sum_{\substack{r_1 \neq \cdots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^{I} c_N(r_i) \right\} \left\{ \prod_{j=I+1}^{k} D_N(r_j) \right\} \\
&\qquad \leq \frac{1}{k!} \left(\sum_{r=1}^{\tau_N(t)} c_N(r) \right)^k + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left(\frac{B_n}{\alpha} \right)^{k-I} \sum_{\substack{r_1 \neq \cdots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^{k-1} c_N(r_i) \right\} \{ D_N(r_k) \} \\
&\qquad \leq \frac{1}{k!} \{ t + c_N(\tau_N(t)) \}^k + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left(\frac{B_n}{\alpha} \right)^{k-I} \left\{ \sum_{\substack{r_1 \neq \cdots \neq r_{k-1} \\ \text{all distinct}}}^{\tau_N(t)} \prod_{i=1}^{k-1} c_N(r_i) \right\} \left\{ \sum_{r_k=1}^{\tau_N(t)} D_N(r_k) \right\} \\
&\qquad \leq \frac{1}{k!} \{ t + c_N(\tau_N(t)) \}^k + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left(\frac{B_n}{\alpha} \right)^{k-I} \left(\sum_{r=1}^{\tau_N(t)} c_N(r) \right)^{k-1} \left(\sum_{r=1}^{\tau_N(t)} D_N(r) \right) \\
&\qquad \leq \frac{1}{k!} \{ t + c_N(\tau_N(t)) \}^k + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left(\frac{B_n}{\alpha} \right)^{k-I} (t+1)^{k-1} \left(\sum_{r=1}^{\tau_N(t)} D_N(r) \right). \qquad (15)
\end{aligned}
$$

Taking expectations,

$$
\begin{aligned}
\lim_{N \to \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \cdots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^{k} \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\} \right] &\leq \frac{1}{k!} \lim_{N \to \infty} \mathbb{E}[\{ t + c_N(\tau_N(t)) \}^k] \qquad (16) \\
&\quad + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left(\frac{B_n}{\alpha} \right)^{k-I} (t+1)^{k-1} \lim_{N \to \infty} \mathbb{E} \left[\left(\sum_{r=1}^{\tau_N(t)} D_N(r) \right) \right] \\
&= \frac{1}{k!} t^k.
\end{aligned}
$$

The limit follows from Brown et al. (2020, Equations (3),(4)) along with the fact that, since $c_N(s) \in [0,1]$ for all s, $\mathbb{E}[c_N(s)^k] \leq \mathbb{E}[c_N(s)]$ for all $k \geq 1$, and the expansion

$$
\mathbb{E} \left[\frac{1}{k!} \{ t + c_N(\tau_N(t)) \}^k \right] = \mathbb{E} \left[\frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} t^i c_N(\tau_N(t))^{k-i} \right] = \frac{1}{k!} \left\{ t^k + k t^{k-1} \mathbb{E}[c_N(\tau_N(t))] + \dots \right\} \longrightarrow \frac{1}{k!} t^k. \qquad (17)
$$

Combining these upper and lower limits, we conclude that

$$
\lim_{N \to \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \cdots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^{k} \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\} \right] = \frac{1}{k!} t^k \qquad (18)
$$

and thus

$$
1 + \sum_{k=1}^{\infty} \left\{ -\alpha(1 + O(N^{-1})) \right\}^k \mathbb{E} \left[\sum_{\substack{r_1 < \cdots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^{k} \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\} \right] \longrightarrow 1 + \sum_{k=1}^{\infty} (-\alpha)^k \frac{1}{k!} t^k = e^{-\alpha t} \qquad (19)
$$

as $N \to \infty$.

Upper Bound
From Koskela et al. (2018, Lemma 1 Case 1), taking $\xi = \Delta$, we have

$$
1 - p_t = p_{\Delta\Delta}(t) \leq 1 - \alpha(1 + O(N^{-1})) \left[c_N(t) - \binom{n-1}{2} D_N(t) \right]. \qquad (20)
$$

A multinomial expansion as before yields

$$\prod_{r=1}^{\tau_N(t)} (1-p_r) \leq 1 + \sum_{k=1}^{\infty} \left\{-\alpha(1+O(N^{-1}))\right\}^k \sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)} \prod_{j=1}^{k} \left\{ c_N(r_j) - \binom{n-1}{2} D_N(r_j) \right\}. \tag{21}$$

Analogously to (13), we can write

$$\begin{aligned}\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)} \prod_{j=1}^{k} \left\{ c_N(r_j) - \binom{n-1}{2} D_N(r_j) \right\} &= \frac{1}{k!} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^{k} c_N(r_i) \right\} \\ &+ \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left(-\binom{n-1}{2} \right)^{k-I} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}} \left\{ \prod_{i=1}^{I} c_N(r_i) \right\} \left\{ \prod_{j=I+1}^{k} D_N(r_j) \right\}.\end{aligned} \tag{22}$$

We start by dealing with the second term:

$$\begin{aligned}&\frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left(-\binom{n-1}{2} \right)^{k-I} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}} \left\{ \prod_{i=1}^{I} c_N(r_i) \right\} \left\{ \prod_{j=I+1}^{k} D_N(r_j) \right\} \\ &\qquad= \frac{1}{k!} \sum_{\substack{I=0:\\ k-I \text{ even}}}^{k-1} \binom{k}{I} \binom{n-1}{2}^{k-I} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}} \left\{ \prod_{i=1}^{I} c_N(r_i) \right\} \left\{ \prod_{j=I+1}^{k} D_N(r_j) \right\} \\ &\qquad\quad - \frac{1}{k!} \sum_{\substack{I=0:\\ k-I \text{ odd}}}^{k-1} \binom{k}{I} \binom{n-1}{2}^{k-I} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}} \left\{ \prod_{i=1}^{I} c_N(r_i) \right\} \left\{ \prod_{j=I+1}^{k} D_N(r_j) \right\}.\end{aligned} \tag{23}$$

This is lower bounded by

$$0 - \frac{1}{k!} \sum_{\substack{I=0\\ (k-I)\text{odd}}}^{k-1} \binom{k}{I} \binom{n-1}{2}^{k-I} (t+1)^{k-1} \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \tag{24}$$

using that $c_N(r), D_N(r) \geq 0$ for all r to bound the even terms below, and arguments from (14) to bound the odd terms above. The same arguments lead to the upper bound

$$\frac{1}{k!} \sum_{\substack{I=0\\ (k-I)\text{odd}}}^{k-1} \binom{k}{I} \binom{n-1}{2}^{k-I} (t+1)^{k-1} \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) - 0. \tag{25}$$

By Brown et al. (2020, Equation (4)), both bounds have vanishing expectation as $N \to \infty$. We are left with the first term in (18), which is upper bounded by

$$\frac{1}{k!} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^{k} c_N(r_i) \right\} \leq \frac{1}{k!} \left(\sum_{s=1}^{\tau_N(t)} c_N(s) \right)^k \leq \frac{1}{k!} \{t + c_N(\tau_N(t))\}^k \tag{26}$$

the expectation of which converges to $t^k/k!$ as in (16). We use Koskela et al. (2018, Equation (8)) to construct a lower bound:

$$\begin{aligned}\frac{1}{k!} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^{k} c_N(r_i) \right\} &\geq \frac{1}{k!} \left(\sum_{s=1}^{\tau_N(t)} c_N(s) \right)^k - \frac{1}{k!} \binom{k}{2} \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \left(\sum_{s=1}^{\tau_N(t)} c_N(s) \right)^{k-2} \\ &\geq \frac{1}{k!} t^k - \frac{1}{k!} \binom{k}{2} (t+1)^{k-2} \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right)\end{aligned} \tag{27}$$

The expectation of this bound also converges to $t^k/k!$, using Brown et al. (2020, Equation (5)). We can now conclude that

$$\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)} \prod_{j=1}^{k} \left\{c_N(r_j) - \binom{n-1}{2} D_N(r_j)\right\}\right] = \frac{1}{k!}t^k \tag{28}$$

and thus

$$1 + \sum_{k=1}^{\infty} \left\{-\alpha(1+O(N^{-1}))\right\}^k \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)} \prod_{j=1}^{k} \left\{c_N(r_j) - \binom{n-1}{2} D_N(r_j)\right\}\right] \longrightarrow 1 + \sum_{k=1}^{\infty}(-\alpha)^k \frac{1}{k!}t^k = e^{-\alpha t} \tag{29}$$

as $N \to \infty$.

We now have upper and lower bounds on $\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right]$, both of which are equal to $e^{-\alpha t}$, and the result follows. □

Lemma 3. *$D_N(t) \leq c_N(t)$ for all t.*

Proof.

$$\begin{aligned} D_N(t) &:= \frac{1}{(N)_2}\sum_{i=1}^{N}(\nu_t^{(i)})_2 \frac{1}{N}\left\{\nu_t^{(i)} + \frac{1}{N}\sum_{j\neq i}^{N}(\nu_t^{(j)})^2\right\} \leq \frac{1}{(N)_2}\sum_{i=1}^{N}(\nu_t^{(i)})_2 \frac{1}{N}\left\{\nu_t^{(i)} + \frac{1}{N}\sum_{j\neq i}^{N} N\nu_t^{(j)}\right\} \\ &= \frac{1}{(N)_2}\sum_{i=1}^{N}(\nu_t^{(i)})_2 \frac{1}{N}\left\{\sum_{j=1}^{N}\nu_t^{(j)}\right\} \leq \frac{1}{(N)_2}\sum_{i=1}^{N}(\nu_t^{(i)})_2 = c_N(t). \end{aligned} \tag{30}$$

□

References

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