

Non-triviality condition

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April 23, 2020

Multinomial resampling: neutral case

Lemma 1. *For all $N \geq 2$, for all t ,*

$$\mathbb{E} \left[c_N(t) \middle| \mathbf{w} = \left(\frac{1}{N}, \dots, \frac{1}{N} \right) \right] = \frac{1}{N}.$$

Proof.

$$\begin{aligned} \mathbb{E} [c_N(t) | \mathbf{w} = (1/N, \dots, 1/N)] &= \frac{1}{(N)_2} \sum_{i=1}^N \mathbb{E} \left[(\nu_t^{(i)})_2 \mid \mathbf{w} = (1/N, \dots, 1/N) \right] \\ &= \frac{1}{(N)_2} \sum_{i=1}^N (N)_2 \left(\frac{1}{N} \right)^2 = \sum_{i=1}^N \frac{1}{N^2} = \frac{1}{N} \end{aligned}$$

□

Lemma 2. *For all $N \geq 4$, for all t ,*

$$\mathbb{E} \left[(c_N(t))^2 \middle| \mathbf{w} = \left(\frac{1}{N}, \dots, \frac{1}{N} \right) \right] = \frac{N+2}{N^3}.$$

Proof.

$$\begin{aligned} \mathbb{E} [(c_N(t))^2 | \mathbf{w} = (1/N, \dots, 1/N)] &= \frac{1}{(N)_2^2} \sum_{i=1}^N \sum_{j=1}^N \mathbb{E} \left[(\nu_t^{(i)})_2 (\nu_t^{(j)})_2 \middle| \mathbf{w} = (1/N, \dots, 1/N) \right] \\ &= \frac{1}{(N)_2^2} \left\{ \sum_{i=1}^N \sum_{j \neq i}^N \mathbb{E} \left[(\nu_t^{(i)})_2 (\nu_t^{(j)})_2 \middle| \mathbf{w} = (1/N, \dots, 1/N) \right] + \sum_{i=1}^N \mathbb{E} \left[(\nu_t^{(i)})_2^2 \middle| \mathbf{w} = (1/N, \dots, 1/N) \right] \right\} \\ &= \frac{1}{(N)_2^2} \left\{ \sum_{i=1}^N \sum_{j \neq i}^N \mathbb{E} \left[(\nu_t^{(i)})_2 (\nu_t^{(j)})_2 \middle| \mathbf{w} = (1/N, \dots, 1/N) \right] + \sum_{i=1}^N \mathbb{E} \left[(\nu_t^{(i)})_4 + 4(\nu_t^{(i)})_3 + 2(\nu_t^{(i)})_2 \middle| \mathbf{w} = (1/N, \dots, 1/N) \right] \right\} \\ &= \frac{1}{(N)_2^2} \left\{ \sum_{i=1}^N \sum_{j \neq i}^N (N)_4 \left(\frac{1}{N} \right)^2 \left(\frac{1}{N} \right)^2 + \sum_{i=1}^N \left((N)_4 \left(\frac{1}{N} \right)^4 + 4(N)_3 \left(\frac{1}{N} \right)^3 + 2(N)_2 \left(\frac{1}{N} \right)^2 \right) \right\} \\ &= \frac{1}{(N)_2^2} \left\{ N(N-1)(N)_4 \frac{1}{N^4} + N(N)_4 \frac{1}{N^4} + 4N(N)_3 \frac{1}{N^3} + 2N(N)_2 \frac{1}{N^2} \right\} \\ &= \frac{(N-2)(N-3)}{N^4} + \frac{(N-2)(N-3)}{N^4(N-1)} + \frac{4(N-2)}{N^3(N-1)} + \frac{2}{N^2(N-1)} \\ &= \frac{1}{N^4(N-1)} [(N-2)(N-3)(N-1+1) + 4N(N-2) + 2N^2] \\ &= \frac{1}{N^3(N-1)} [N^2 - 5N + 6 + 4N - 8 + 2N] = \frac{N^2 + N - 2}{N^3(N-1)} = \frac{(N+2)(N-1)}{N^3(N-1)} = \frac{N+2}{N^3}. \end{aligned}$$

□

Lemma 3. For all $N \geq 4$, for all t ,

$$\mathbb{P} \left[c_N(t) > \frac{2}{N^2} \middle| \mathbf{w} = \left(\frac{1}{N}, \dots, \frac{1}{N} \right) \right] \geq \left(1 - \frac{2}{N} \right)^2 \frac{N}{N+2}.$$

Proof. We apply the Paley-Zygmund inequality,

$$\mathbb{P} [c_N(t) > \theta \mathbb{E}[c_N(t) | \mathbf{w} = (1/N, \dots, 1/N)] | \mathbf{w} = (1/N, \dots, 1/N)] \geq (1 - \theta)^2 \frac{\mathbb{E}[c_N(t) | \mathbf{w} = (1/N, \dots, 1/N)]^2}{\mathbb{E}[(c_N(t))^2 | \mathbf{w} = (1/N, \dots, 1/N)]}.$$

Setting $\theta = 2/N$ and using Lemmata 1-2,

$$\mathbb{P} \left[c_N(t) > \frac{2}{N^2} \middle| \mathbf{w} = (1/N, \dots, 1/N) \right] \geq \left(1 - \frac{2}{N} \right)^2 \frac{(1/N)^2}{(N+2)/N^3} = \left(1 - \frac{2}{N} \right)^2 \frac{N}{N+2}.$$

□

Theorem 1. In the neutral case (i.e. when all weights are equal at every time step) with multinomial resampling, there exists N_0 such that for all $N > N_0$, for all finite t , $\mathbb{P}[\tau_N(t) = \infty] = 0$.

Proof. Let us rewrite the event of interest in a different way.

$$\begin{aligned} \mathbb{P}[\tau_N(t) = \infty] = 0 &\Leftrightarrow \mathbb{P}[\tau_N(t) < \infty] = 1 \\ &\Leftrightarrow \mathbb{P} \left[\min \left\{ s > 1 : \sum_{r=1}^s c_N(r) < t \right\} < \infty \right] = 1 \\ &\Leftrightarrow \mathbb{P} \left[\exists s < \infty : \sum_{r=1}^s c_N(r) < t \right] = 1 \end{aligned}$$

It is sufficient to show that, for all $N > N_0$, $c_N(r)$ is bounded away from zero infinitely often in r . We consider the sequence of events $E_r := \{c_N(r) > 2/N^2\}$ for $r \in \mathbb{N}$. In the neutral case, the resampled family sizes at each generation are independent, hence the events E_r are independent. By the second Borel-Cantelli lemma, E_r occurs infinitely often if $\sum_{r=1}^{\infty} \mathbb{P}(E_r) = \infty$. A lower bound on $\mathbb{P}(E_r)$ is given in Lemma 3. For any fixed $N \geq 4$, the bound is strictly positive and constant in r , so the Borel-Cantelli condition is satisfied, thus we conclude that E_r occurs infinitely often. Hence, taking $N_0 = 3$, we have that $\mathbb{P}[\tau_N(t) = \infty] = 0$ for all $N > N_0$ and all finite t , as required. □

Multinomial resampling: non-neutral case

Lemma 4. For all $N \geq 2$, for all t , for any weight vector (w_1, \dots, w_N) ,

$$\mathbb{P} \left[c_N(t) > \frac{2}{N^2} \middle| \mathbf{w} = (w_1, \dots, w_N) \right] \geq \mathbb{P} \left[c_N(t) > \frac{2}{N^2} \middle| \mathbf{w} = \left(\frac{1}{N}, \dots, \frac{1}{N} \right) \right].$$

That is, the probability of having at least one merger is minimised by the vector of equal weights.

Proof. Fix arbitrary t and $N \geq 2$. Firstly notice that

$$\begin{aligned} \mathbb{P} \left[c_N(t) > \frac{2}{N^2} \middle| \mathbf{w} = (w_1, \dots, w_N) \right] &= 1 - \mathbb{P}[c_N(t) = 0 \mid \mathbf{w} = (w_1, \dots, w_N)] \\ &= 1 - \mathbb{P}[\nu_t^{(1:N)} = (1, \dots, 1) \mid \mathbf{w} = (w_1, \dots, w_N)]. \end{aligned}$$

Since, conditional on the weights, $\nu_t^{(1:N)} \sim \text{Multinomial}(N, (w_1, \dots, w_N))$, the probability of interest is

$$\mathbb{P}[\nu_t^{(1:N)} = (1, \dots, 1) \mid \mathbf{w} = (w_1, \dots, w_N)] = N! \prod_{i=1}^N w_i.$$

We will show that the global maximum of this function on the simplex \mathcal{S}_{N-1} is attained at $\mathbf{w} = (1/N, \dots, 1/N)$. This weight vector will therefore minimise the probability of the complementary event, implying the desired result.

First, since we are working on the simplex, we encode the constraint $\sum w_i = 1$ by rewriting the function to optimise as

$$f(\mathbf{w}) := \prod_{i=1}^N w_i = \left(1 - \sum_{j=1}^{N-1} w_j\right) \prod_{i=1}^{N-1} w_i$$

where we have also dropped the constant positive factor $N!$. Now, for every $k \in \{1, \dots, N-1\}$, we solve

$$\frac{\partial f(\mathbf{w})}{\partial w_k} = \left(1 - w_k - \sum_{j=1}^{N-1} w_j\right) \prod_{i \neq k}^{N-1} w_i = 0.$$

The product over $i \neq k$ is constant for each k , so this reduces to solving

$$w_k = 1 - \sum_{j=1}^{N-1} w_j = w_N$$

for all k . The unique solution is $w_1 = w_2 = \dots = w_N = 1/N$.

To verify that this critical point is a maximum, we calculate the Hessian H :

$$H_{kl}(\mathbf{w}) = \begin{cases} -2 \prod_{i \neq k}^{N-1} w_i & k = l \\ \left(1 - w_k - w_l - \sum_{j=1}^{N-1} w_j\right) \prod_{i \neq k, l}^{N-1} w_i & k \neq l \end{cases}$$

$$H_{kl}((1/N, \dots, 1/N)) = \begin{cases} -2 \left(\frac{1}{N}\right)^{N-2} & k = l \\ -\left(\frac{1}{N}\right)^{N-2} & k \neq l \end{cases}$$

and show that H is negative semi-definite: for any $\mathbf{x} \in \mathbb{R}^{N-1}$,

$$\begin{aligned} \mathbf{x}^T H \mathbf{x} &= \sum_{k=1}^{N-1} \left[-2 \left(\frac{1}{N}\right)^{N-2} x_k^2 - \sum_{l \neq k}^{N-1} \left(\frac{1}{N}\right)^{N-2} x_k x_l \right] = \left(\frac{1}{N}\right)^{N-2} \left[-\sum_{k=1}^{N-1} 2x_k^2 - \sum_{k=1}^{N-1} \sum_{l \neq k}^{N-1} x_k x_l \right] \\ &= \left(\frac{1}{N}\right)^{N-2} \left[-\sum_{k=1}^{N-1} x_k^2 - \sum_{k=1}^{N-1} \sum_{l=1}^{N-1} x_k x_l \right] = \left(\frac{1}{N}\right)^{N-2} \left[-\sum_{k=1}^{N-1} x_k^2 - \left(\sum_{k=1}^{N-1} x_k\right)^2 \right] \leq 0. \end{aligned}$$

□