

Coupling argument for residual resampling

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Consider a population of N particles at time t , having fixed weights $(w_t^{(1)}, \dots, w_t^{(N)})$ respectively. These will be the parents, and we consider assigning to them N offspring which will form the next generation. Let (a_1, \dots, a_N) denote the vector of parental indices of each child, and (v_1, \dots, v_N) the resulting offspring counts for each parent. That is, $v_i = \sum_{j=1}^N \mathbb{I}\{a_j = i\}$. We will use a superscript m or r to denote the multinomial or residual schemes respectively.

In the case of multinomial resampling, we have the following:

$$\begin{aligned} (a_1^m, \dots, a_N^m) &\sim \text{Categorical}(1 : N, w_t^{(1:N)}) \\ (v_1^m, \dots, v_N^m) &\sim \text{Multinomial}(N, w_t^{(1:N)}) \end{aligned}$$

In residual resampling, the first $k := \sum_{i=1}^N \lfloor Nw_t^{(i)} \rfloor$ offspring are assigned deterministically, and the remaining $N - k$ are assigned randomly:

$$(v_1^r, \dots, v_N^r) \stackrel{d}{=} \text{Multinomial}(N - k, w_t^{(1:N)}) + \lfloor Nw_t^{(1:N)} \rfloor$$

This distribution can be achieved by doing multinomial resampling as usual, but then overwriting the parental indices of k offspring (to be chosen at random) with the k deterministic assignments (since the offspring are exchangeable and marginals of the Multinomial/Categorical distribution are Multinomial/Categorical).

The probability that a randomly chosen pair of children share the same parent is, with multinomial resampling:

$$p^m = \sum_{i=1}^N (w_t^{(i)})^2$$

and with residual resampling (by partitioning over the common parent and into the cases of deterministic assignment and random assignment):

$$\begin{aligned} p^r &= \sum_{i=1}^N \left[\frac{N-k}{N} w_t^{(i)} \left(\frac{N-k-1}{N} w_t^{(i)} + \frac{\lfloor Nw_t^{(i)} \rfloor}{N} \right) + \frac{\lfloor Nw_t^{(i)} \rfloor}{N} \left(\frac{N-k}{N} w_t^{(i)} + \frac{k-1}{k} \frac{\lfloor Nw_t^{(i)} \rfloor - 1}{N} \right) \right] \\ &= \sum_{i=1}^N \frac{1}{N^2} \left[\left(w_t^{(i)}(N-k) + \lfloor Nw_t^{(i)} \rfloor \right)^2 - (N-k)(w_t^{(i)})^2 - \frac{k-1}{k} \lfloor Nw_t^{(i)} \rfloor - \frac{1}{k} \lfloor Nw_t^{(i)} \rfloor^2 \right] \\ &\leq \sum_{i=1}^N \frac{1}{N^2} \left((N-k)w_t^{(i)} + \lfloor Nw_t^{(i)} \rfloor \right)^2 \end{aligned}$$