# Weak convergence proof v.2 (neater) (in progress)

Suzie Brown

January 25, 2021

## Bounds on sum-products

**Lemma 1.**

$$t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right)\binom{l}{2}(t+1)^{l-2} \leq \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \leq t^l + c_N(\tau_N(t))(t+1)^l. \tag{1}$$

*Proof.* As pointed out in Koskela et al. (2018, Equation (8)),

$$\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \geq \left(\sum_{s=0}^{\tau_N(t)} c_N(s)\right)^l - \binom{l}{2}\left(\sum_{s=0}^{\tau_N(t)} c_N(s)^2\right)\left(\sum_{s=0}^{\tau_N(t)} c_N(s)\right)^{l-2}. \tag{2}$$

By definition of $\tau_N$,

$$t \leq \sum_{s=0}^{\tau_N(t)} c_N(s) \leq t+1. \tag{3}$$

Substituting these bounds into the RHS of (2) yields the lower bound.

It is a true fact that

$$\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \leq \left(\sum_{s=0}^{\tau_N(t)} c_N(s)\right)^l, \tag{4}$$

as can be seen by considering the multinomial expansion of the RHS. This is further bounded by

$$\left(\sum_{s=0}^{\tau_N(t)} c_N(s)\right)^l \leq \left(\sum_{s=0}^{\tau_N(t)-1} c_N(s) + c_N(\tau_N(t))\right)^l \leq \left[t + c_N(\tau_N(t))\right]^l, \tag{5}$$

again using the definition of $\tau_N$. A binomial expansion yields

$$\left[t + c_N(\tau_N(t))\right]^l = t^l + \sum_{i=0}^{l-1}\binom{l}{i} t^i c_N(\tau_N(t))^{l-i} = t^l + c_N(\tau_N(t))\sum_{i=0}^{l-1}\binom{l}{i} t^i c_N(\tau_N(t))^{l-1-i}, \tag{6}$$

then since $c_N(s) \leq 1$ for all $s$,

$$\sum_{i=0}^{l-1}\binom{l}{i} t^i c_N(\tau_N(t))^{l-1-i} \leq \sum_{i=0}^{l-1}\binom{l}{i} t^i \leq (t+1)^l. \tag{7}$$

Putting this together yields the upper bound. ∎

**Lemma 2.** *Let $B$ be a positive constant which may depend on $n$.*

$$\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) + BD_N(s_j)] \leq \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) + \left( \sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1}(1+B)^l. \tag{8}$$

*Proof.* We start with a binomial expansion:

$$\begin{aligned} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) + BD_N(s_j)] &= \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \sum_{\mathcal{I} \subseteq [l]} B^{l-|\mathcal{I}|} \left( \prod_{i \in \mathcal{I}} c_N(s_i) \right) \left( \prod_{j \notin \mathcal{I}} D_N(s_j) \right) \\ &= \sum_{\mathcal{I} \subseteq [l]} B^{l-|\mathcal{I}|} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left( \prod_{i \in \mathcal{I}} c_N(s_i) \right) \left( \prod_{j \notin \mathcal{I}} D_N(s_j) \right) \end{aligned} \tag{9}$$

where $[l] := \{1, \dots, l\}$. Since the sum is over all permutations of $r_1, \dots, r_l$, we may arbitrarily choose an ordering for $\{1, \dots, l\}$ such that $\mathcal{I} = \{1, \dots, |\mathcal{I}|\}$:

$$\sum_{\mathcal{I} \subseteq [l]} B^{l-|\mathcal{I}|} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left( \prod_{i \in \mathcal{I}} c_N(s_i) \right) \left( \prod_{j \notin \mathcal{I}} D_N(s_j) \right) = \sum_{I=0}^{l} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left( \prod_{i=1}^{I} c_N(s_i) \right) \left( \prod_{j=I+1}^{l} D_N(s_j) \right). \tag{10}$$

Separating the term $I = l$,

$$\begin{aligned} \sum_{I=0}^{l} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left( \prod_{i=1}^{I} c_N(s_i) \right) \left( \prod_{j=I+1}^{l} D_N(s_j) \right) &= \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \\ &+ \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left( \prod_{i=1}^{I} c_N(s_i) \right) \left( \prod_{j=I+1}^{l} D_N(s_j) \right). \end{aligned} \tag{11}$$

In the second line, there is always at least one $D_N$ term, and $c_N(s) \leq D_N(s)$ for all $s$, so we can write

$$\begin{aligned} \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left( \prod_{i=1}^{I} c_N(s_i) \right) \left( \prod_{j=I+1}^{l} D_N(s_j) \right) &\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left( \prod_{i=1}^{l-1} c_N(s_i) \right) D_N(s_l) \\ &\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \left( \sum_{s_1 \neq \cdots \neq s_{l-1}}^{\tau_N(t)} \prod_{i=1}^{l-1} c_N(s_i) \right) \sum_{s_l=1}^{\tau_N(t)} D_N(s_l) \\ &\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} (t+1)^{l-1} \sum_{s=1}^{\tau_N(t)} D_N(s) \end{aligned} \tag{12}$$

using (4) and (3). Finally, by the Binomial Theorem,

$$\sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} (t+1)^{l-1} \sum_{s=1}^{\tau_N(t)} D_N(s) \leq \left( \sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1}(1+B)^l, \tag{13}$$

which, together with (11), concludes the proof. ∎

**Lemma 3.** *Let $B$ be a positive constant which may depend on $n$.*

$$\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - BD_N(s_j)] \geq \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) - \left( \sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1}(1+B)^l. \tag{14}$$

*Proof.* A binomial expansion and subsequent manipulation as in (9)–(11) gives

$$\begin{aligned}
\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - BD_N(s_j)] &= \sum_{\mathcal{I}\subseteq[l]} (-B)^{l-|\mathcal{I}|} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i\in\mathcal{I}} c_N(s_i)\right) \left(\prod_{j\notin\mathcal{I}} D_N(s_j)\right) \\
&= \sum_{I=0}^{l} \binom{l}{I} (-B)^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right) \left(\prod_{j=I+1}^{l} D_N(s_j)\right) \\
&= \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) + \sum_{I=0}^{l-1} \binom{l}{I} (-B)^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right) \left(\prod_{j=I+1}^{l} D_N(s_j)\right) \\
&\geq \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) - \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right) \left(\prod_{j=I+1}^{l} D_N(s_j)\right)
\end{aligned} \tag{15}$$

where the last inequality just multiplies some positive terms by $-1$. Then (12)–(13) can be applied directly (noting that an upper bound on negative terms gives a lower bound overall):

$$-\sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right) \left(\prod_{j=I+1}^{l} D_N(s_j)\right) \geq \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right) (t+1)^{l-1}(1+B)^l \tag{16}$$

which concludes the proof. ■

# Main components of weak convergence

**Lemma 4** (Basis step)**.** *For any* $0 < t < \infty$,

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1-p_r)\right] = e^{-\alpha_n t} \tag{17}$$

*where* $\alpha_n := n(n-1)/2$.

*Proof.* We start by showing that $\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right] \leq e^{-\alpha_n t}$.
From Koskela et al. (2018, Lemma 1 Case 1), taking $\xi = \Delta$, we have

$$1 - p_t = p_{\Delta\Delta}(t) \leq 1 - \alpha_n(1 + O(N^{-1}))\,[c_N(t) - B'_n D_N(t)] \tag{18}$$

where the $O(N^{-1})$ term does not depend on $t$. Applying a multinomial expansion and then separating the positive and negative terms,

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1-p_r) &\leq 1 + \sum_{l=1}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - B'_n D_N(s_j)] \\
&= 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - B'_n D_N(s_j)] \\
&\quad - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - B'_n D_N(s_j)]\,.
\end{aligned} \tag{19}$$

This is further bounded by applying Lemma 3 and then both bounds of Lemma 1:

$$
\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1-p_r) \leq 1 &+ \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \\
&- \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \left\{ \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) - \left( \sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1+B_n')^l \right\} \\
\leq 1 &+ \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \left\{ t^l + c_N(\tau_N(t))(t+1)^l \right\} \\
&- \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \left\{ t^l - \left( \sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \binom{l}{2} (t+1)^{l-2} - \left( \sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1+B_n')^l \right\}.
\end{aligned}
\tag{20}
$$

A bit of tidying up and we have

$$
\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1-p_r) \leq & \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l + c_N(\tau_N(t)) \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} (t+1)^l \\
&+ \left( \sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \binom{l}{2} (t+1)^{l-2} \\
&+ \left( \sum_{s=1}^{\tau_N(t)} D_N(s) \right) \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} (t+1)^{l-1} (1+B_n')^l \\
\leq & \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l + c_N(\tau_N(t)) \exp[\alpha_n (1+O(N^{-1}))(t+1)] \\
&+ \left( \sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n (1+O(N^{-1}))(t+1)] \\
&+ \left( \sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n (1+O(N^{-1}))(t+1)(1+B_n')].
\end{aligned}
\tag{21}
$$

Now, taking the expectation and limit, and applying Brown et al. (2021, Equations (3.3)–(3.5)) and Lemma 10,

$$
\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\left[ \prod_{r=1}^{\tau_N(t)} (1-p_r) \right] \leq & \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l \lim_{N\to\infty} \mathbb{P}\left[ \tau_N(t) \geq l \right] + \lim_{N\to\infty} \mathbb{E}\left[ c_N(\tau_N(t)) \right] \exp[\alpha_n (t+1)] \\
&+ \lim_{N\to\infty} \mathbb{E}\left[ \sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right] \frac{1}{2} \alpha_n^2 \exp[\alpha_n (t+1)] \\
&+ \lim_{N\to\infty} \mathbb{E}\left[ \sum_{s=1}^{\tau_N(t)} D_N(s) \right] \exp[\alpha_n (t+1)(1+B_n')] \\
= & \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l = e^{-\alpha_n t}.
\end{aligned}
\tag{22}
$$

It remains to show that $\lim_{N\to\infty} \mathbb{E}\left[ \prod_{r=1}^{\tau_N(t)} (1-p_r) \right] \geq e^{-\alpha_n t}$.

From Brown et al. (2021, Equation (3.14)), taking $\xi = \Delta$, we have

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \alpha_n(1 + O(N^{-1}))\left[c_N(t) + B_n D_N(t)\right] \tag{23}$$

where $B_n > 0$ and the $O(N^{-1})$ term does not depend on $t$. In particular,

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \frac{N^{n-2}}{(N-2)_{n-2}}\alpha_n c_N(t) - \frac{N^{n-3}}{(N-3)_{n-3}} B_n D_N(t). \tag{24}$$

Since $D_N(s) \leq c_N(s)$ for all $s$ (Koskela et al., 2018, p.9), a sufficient condition for this bound to be non-negative is

$$E_r := \left\{ c_N(r) \leq \frac{(N-3)_{n-3}}{N^{n-3}} \left( \alpha_n \left(1 + \frac{2}{N-2}\right) + B_n \right)^{-1} \right\}, \tag{25}$$

and we define $E := \bigcap_{r=1}^{\tau_N(t)} E_r$. We now apply a multinomial expansion to the product, and split into positive and negative terms:

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1 - p_r) &\geq \left\{ 1 + \sum_{l=1}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} \left[c_N(s_j) + B_n D_N(s_j)\right] \right\} \mathbb{1}_E \\
&= \left\{ 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} \left[c_N(s_j) + B_n D_N(s_j)\right] \right. \\
&\qquad \left. - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} \left[c_N(s_j) + B_n D_N(s_j)\right] \right\} \mathbb{1}_E
\end{aligned} \tag{26}$$

This is further bounded by applying Lemma 2 and both bounds in Lemma 1:

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1 - p_r) &\geq \left\{ 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \right. \\
&\qquad \left. - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left[ \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) + \left( \sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1 + B_n)^l \right] \right\} \mathbb{1}_E \\
&\geq \left\{ 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left[ t^l - \left( \sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \binom{l}{2} (t+1)^{l-2} \right] \right. \\
&\qquad \left. - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left[ t^l + c_N(\tau_N(t))(t+1)^l + \left( \sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1 + B_n)^l \right] \right\} \mathbb{1}_E.
\end{aligned} \tag{27}$$

Tidying things up,

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)}(1-p_r) \geq & \sum_{l=0}^{\tau_N(t)}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_E - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right)\sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)} \alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\binom{l}{2}(t+1)^{l-2} \\
& - c_N(\tau_N(t))\sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)} \alpha_n^l(1+O(N^{-1}))\frac{1}{l!}(t+1)^l \\
& - \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)\sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)} \alpha_n^l(1+O(N^{-1}))\frac{1}{l!}(t+1)^{l-1}(1+B_n)^l \\
\geq & \sum_{l=0}^{\tau_N(t)}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_E - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right)\frac{1}{2}\alpha_n^2\exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
& - c_N(\tau_N(t))\exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
& - \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n)]. 
\end{aligned} \tag{28}$$

Now, taking the expectation and limit, and applying Brown et al. (2021, Equations (3.3)–(3.5)) to show that all but the first sum vanish, and Lemmata 10 and 9 to show that $\lim_{N\to\infty}\mathbb{P}[\{\tau_N(t)\geq l\}\cap E]=1$,

$$\begin{aligned}
\lim_{N\to\infty}\mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right] \geq & \sum_{l=0}^{\infty}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\lim_{N\to\infty}\mathbb{P}\left[\{\tau_N(t)\geq l\}\cap E\right] \\
& - \lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right]\frac{1}{2}\alpha_n^2\exp[\alpha_n(t+1)] \\
& - \lim_{N\to\infty}\mathbb{E}\left[c_N(\tau_N(t))\right]\exp[\alpha_n(t+1)] \\
& - \lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} D_N(s)\right]\exp[\alpha_n(t+1)(1+B_n)] \\
= & \sum_{l=0}^{\infty}(-\alpha_n)^l\frac{1}{l!}t^l = e^{-\alpha_n t}.
\end{aligned} \tag{29}$$

Combining the upper and lower bounds in (22) and (29) respectively concludes the proof. ∎

**Lemma 5** (Induction step upper bound)**.** *Fix $k\in\mathbb{N}$, $i_0:=0$, $i_k:=k$. For any sequence of times $0=t_0\leq t_1\leq\cdots\leq t_k\leq t$,*

$$\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\left(\prod_{i=1}^{k}p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)}(1-p_r)\right)\right] \leq \alpha_n^k\sum_{l=0}^{\infty}\frac{1}{l!}(-\alpha_n)^l t^l \sum_{\substack{i_1\leq\cdots\leq i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\geq j\forall j}}\prod_{j=1}^{k}\frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}. \tag{30}$$

*Proof.* We use the bound from (18) and apply a multinomial expansion:

$$
\begin{aligned}
\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r) &\le \prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} \left\{1-\alpha_n(1+O(N^{-1}))[c_N(r)-B_n' D_N(r)]\right\} \\
&= 1 + \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} \sum_{\substack{s_1\neq\dots\neq s_l\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} \prod_{j=1}^{l} \{c_N(s_j)-B_n' D_N(s_j)\} \\
&= 1 + \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} \sum_{s_1\neq\dots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} \{c_N(s_j)-B_n' D_N(s_j)\} \\
&\quad - \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} \sum_{\substack{s_1\neq\dots\neq s_l\\ \exists i,i':s_i=r_{i'}}} \prod_{j=1}^{l} \{c_N(s_j)-B_n' D_N(s_j)\}.
\end{aligned} \tag{31}
$$

The penultimate line above is exactly the expansion we had in the basis step (19), and as such we have the following bound from (21):

$$
\begin{aligned}
1 + \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} \sum_{s_1\neq\dots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} \{c_N(s_j)-B_n' D_N(s_j)\} \\
\le \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l + c_N(\tau_N(t)) \exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
+ \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right) \frac{1}{2}\alpha_n^2 \exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
+ \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right) \exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')].
\end{aligned} \tag{32}
$$

For the last line of (31), Is it definitely okay to pull out the sum over $s_1$? Is some combinatorial correction needed?

$$
\begin{aligned}
-\sum_{l=1}^{\tau_N(t)-k}(-\alpha_n)^l(1+O(N^{-1}))\sum_{\substack{s_1<\cdots<s_l\\ \exists i,i':s_i=r_{i'}}}\prod_{j=1}^{l}\{c_N(s_j)-B_n'D_N(s_j)\}&\\
\le \sum_{l=1}^{\tau_N(t)-k}\alpha_n^l(1+O(N^{-1}))\sum_{\substack{s_1<\cdots<s_l\\ \exists i,i':s_i=r_{i'}}}\prod_{j=1}^{l}\{c_N(s_j)+B_n'D_N(s_j)\}&\\
\le \sum_{l=1}^{\tau_N(t)-k}\alpha_n^l(1+O(N^{-1}))\sum_{\substack{s_1<\cdots<s_l\\ \exists i,i':s_i=r_{i'}}}(1+B_n')^l\prod_{j=1}^{l}c_N(s_j)&\\
\le \sum_{l=1}^{\tau_N(t)-k}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\sum_{s_1\in\{r_1,\dots,r_k\}}\sum_{s_2\neq\cdots\neq s_l}^{\tau_N(t)}(1+B_n')^l\prod_{j=1}^{l}c_N(s_j)&\\
=\sum_{s\in\{r_1,\dots,r_k\}}c_N(s)\sum_{l=1}^{\tau_N(t)-k}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}(1+B_n')^l\sum_{s_1\neq\cdots\neq s_{l-1}}^{\tau_N(t)}\prod_{j=1}^{l-1}c_N(s_j)&\\
\le \sum_{s\in\{r_1,\dots,r_k\}}c_N(s)\sum_{l=1}^{\tau_N(t)-k}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}(1+B_n')^l(t+1)^{l-1}&\\
\le \left(\sum_{s\in\{r_1,\dots,r_k\}}c_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(1+B_n')(t+1)].& \qquad (33)
\end{aligned}
$$

Putting these together, we have

$$
\begin{aligned}
\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)}(1-p_r)\le \sum_{l=0}^{\tau_N(t)}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l+c_N(\tau_N(t))\exp[\alpha_n(1+O(N^{-1}))(t+1)]&\\
+\left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right)\frac{1}{2}\alpha_n^2\exp[\alpha_n(1+O(N^{-1}))(t+1)]&\\
+\left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')]&\\
+\left(\sum_{s\in\{r_1,\dots,r_k\}}c_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(1+B_n')(t+1)].& \qquad (34)
\end{aligned}
$$

Meanwhile, using the bound on $p_r$ from (23) and applying a modification of Lemma 2,

$$
\begin{aligned}
\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\prod_{i=1}^{k}p_{r_i}\le \alpha_n^k(1+O(N^{-1}))\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\prod_{i=1}^{k}[c_N(r_i)+B_nD_N(r_i)]&\\
\le \alpha_n^k(1+O(N^{-1}))\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\prod_{i=1}^{k}c_N(r_i)+\left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)\alpha_n^k(1+O(N^{-1}))(t+1)^{k-1}(1+B_n)^k.&
\end{aligned}
$$

(35)

A more liberal bound can be arrived at thus:

$$\begin{aligned}\prod_{i=1}^{k} p_{r_i} &\le \alpha_n^k (1+O(N^{-1})) \prod_{i=1}^{k} [c_N(r_i) + B_n D_N(r_i)] \\ &\le \alpha_n^k (1+O(N^{-1})) \prod_{i=1}^{k} c_N(r_i)(1+B_n) \\ &\le \alpha_n^k (1+O(N^{-1}))(1+B_n)^k \prod_{i=1}^{k} c_N(r_i) \end{aligned} \tag{36}$$

and this also leads to the deterministic bound

$$\begin{aligned}\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^{k} p_{r_i} &\le \alpha_n^k (1+O(N^{-1}))(1+B_n)^k \sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^{k} c_N(r_i) \\ &\le \alpha_n^k (1+O(N^{-1}))(1+B_n)^k \frac{1}{k!} \sum_{r_1\neq\cdots\neq r_k}^{\tau_N(t)} \prod_{i=1}^{k} c_N(r_i) \\ &\le \alpha_n^k (1+O(N^{-1}))(1+B_n)^k \frac{1}{k!}(t+1)^k. \end{aligned} \tag{37}$$

The expression inside the expectation in (30) is bounded above by

$$\begin{aligned}&\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right) \left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right) \le \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l \sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^{k} p_{r_i} \\ &\quad + \Bigg\{ c_N(\tau_N(t)) \exp[\alpha_n(1+O(N^{-1}))(t+1)] + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right) \frac{1}{2}\alpha_n^2 \exp[\alpha_n(1+O(N^{-1}))(t+1)] \\ &\qquad + \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right) \exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')] \Bigg\} \sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^{k} p_{r_i} \\ &\quad + \exp[\alpha_n(1+O(N^{-1}))(1+B_n')(t+1)] \sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \sum_{s\in\{r_1,\dots,r_k\}} c_N(s) \prod_{i=1}^{k} p_{r_i}. \end{aligned} \tag{38}$$

Applying the various bounds (35)–(37), we have

$$\begin{aligned}&\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right) \left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right) \le \alpha_n^k (1+O(N^{-1})) \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l \sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^{k} c_N(r_i) \\ &\quad + \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right) \alpha_n^k (1+O(N^{-1}))(t+1)^{k-1}(1+B_n)^k \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l \\ &\quad + \Bigg\{ c_N(\tau_N(t)) \exp[\alpha_n(1+O(N^{-1}))(t+1)] + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right) \frac{1}{2}\alpha_n^2 \exp[\alpha_n(1+O(N^{-1}))(t+1)] \\ &\qquad + \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right) \exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')] \Bigg\} \alpha_n^k (1+O(N^{-1}))(1+B_n)^k \frac{1}{k!}(t+1)^k \\ &\quad + \exp[\alpha_n(1+O(N^{-1}))(1+B_n')(t+1)] \alpha_n^k (1+O(N^{-1}))(1+B_n)^k \sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \sum_{s\in\{r_1,\dots,r_k\}} c_N(s) \prod_{i=1}^{k} c_N(r_i). \end{aligned} \tag{39}$$

Upon taking the expectation and limit, some terms vanish due to Brown et al. (2021, Equations (3.3–(3.5)) and the expression simplifies to

$$\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\right] \le \alpha_n^k \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l \lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_{\{\tau_N(t)\ge l\}} \sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^{k} c_N(r_i)\right]$$
$$+ \exp[\alpha_n(1+B_n')(t+1)]\alpha_n^k(1+B_n)^k \lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \sum_{s\in\{r_1,\dots,r_k\}} c_N(s)\prod_{i=1}^{k} c_N(r_i)\right]. \tag{40}$$

To simplify the second line,

$$\begin{aligned}
\sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \sum_{s\in\{r_1,\dots,r_k\}} c_N(s)\prod_{i=1}^{k} c_N(r_i) &\le \frac{1}{k!}\sum_{r_1\ne\dots\ne r_k}^{\tau_N(t)} \sum_{s\in\{r_1,\dots,r_k\}} c_N(s)\prod_{i=1}^{k} c_N(r_i)\\
&\le \frac{1}{k!}\sum_{r_1\ne\dots\ne r_k}^{\tau_N(t)} \sum_{j=1}^{k} c_N(r_j)\prod_{i=1}^{k} c_N(r_i)\\
&= \frac{1}{k!}\sum_{r_1\ne\dots\ne r_k}^{\tau_N(t)} \sum_{j=1}^{k} c_N(r_j)^2\prod_{i\ne j} c_N(r_i)\\
&\le \frac{1}{k!}\sum_{j=1}^{k}\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \sum_{r_1\ne\dots\ne r_{k-1}}^{\tau_N(t)} \prod_{i=1}^{k-1} c_N(r_i)\\
&\le \frac{1}{(k-1)!}\sum_{s=1}^{\tau_N(t)} c_N(s)^2 (t+1)^{k-1}
\end{aligned} \tag{41}$$

hence

$$\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \sum_{s\in\{r_1,\dots,r_k\}} c_N(s)\prod_{i=1}^{k} c_N(r_i)\right] \le \frac{1}{(k-1)!}(t+1)^{k-1} \lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right] = 0. \tag{42}$$

By Lemma 10, $\lim_{N\to\infty}\mathbb{P}[\tau_N(t)\ge l]=1$, so we can apply Lemma 7 to the remaining expectation in (40), yielding

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\right] &\le \alpha_n^k \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l \lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_{\{\tau_N(t)\ge l\}} \sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^{k} c_N(r_i)\right]\\
&= \alpha_n^k \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l \sum_{\substack{i_1\le\dots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^{k} \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}\\
&= \alpha_n^k e^{-\alpha_n t} \sum_{\substack{i_1\le\dots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^{k} \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}
\end{aligned} \tag{43}$$

as required. ■

**Lemma 6** (Induction step lower bound)**.** *Fix $k \in \mathbb{N}$, $i_0 := 0$, $i_k := k$. For any sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_k \leq t$,*

$$\lim_{N\to\infty} \mathbb{E}\left[ \sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq \tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right) \left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\right] \geq \alpha_n^k \sum_{l=0}^{\infty} \frac{1}{l!}(-\alpha_n)^l t^l \sum_{\substack{i_1\leq\cdots\leq i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\geq j\forall j}} \prod_{j=1}^{k} \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}. \tag{44}$$

*Proof.* Firstly,

$$\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq \tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right) \left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right) \geq \sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq \tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right) \left(\prod_{r=1}^{\tau_N(t)} (1-p_r)\right). \tag{45}$$

Now the second product does not depend on $r_1,\dots,r_k$, and we can use the lower bound from (28):

$$\begin{aligned}\prod_{r=1}^{\tau_N(t)}(1-p_r) \geq{}& \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1}))\frac{1}{l!}t^l \mathbb{1}_E - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right)\frac{1}{2}\alpha_n^2 \exp[\alpha_n(1+O(N^{-1}))(t+1)]\\ &- c_N(\tau_N(t))\exp[\alpha_n(1+O(N^{-1}))(t+1)]\\ &- \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n)]. \end{aligned}\tag{46}$$

We will also need an upper bound on this product, which is formed from (21) with a further deterministic bound:

$$\begin{aligned}\prod_{r=1}^{\tau_N(t)}(1-p_r) \leq{}& \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1}))\frac{1}{l!}t^l + c_N(\tau_N(t))\exp[\alpha_n(1+O(N^{-1}))(t+1)]\\ &+ \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right)\frac{1}{2}\alpha_n^2 \exp[\alpha_n(1+O(N^{-1}))(t+1)]\\ &+ \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')]\\ \leq{}& \exp[\alpha_n(1+O(N^{-1}))t] + \exp[\alpha_n(1+O(N^{-1}))(t+1)]\\ &+ \frac{1}{2}\alpha_n^2(t+1)\exp[\alpha_n(1+O(N^{-1}))(t+1)] + (t+1)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')]\\ \leq{}& \left(2+\frac{\alpha_n^2(t+1)}{2}\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)] + (t+1)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')] \end{aligned}\tag{47}$$

where $E$ is defined as in (25). Now let us consider the remaining sum-product. We use the same bound on $p_r$ as in (18):

$$p_r = 1 - p_{\Delta\Delta}(t) \geq \alpha_n(1+O(N^{-1}))\left[c_N(r) - B_n' D_N(r)\right]. \tag{48}$$

A sufficient condition to ensure this bound is non-negative is given in the event

$$E_r' := \{c_N(r) \geq B_n' D_N(r)\} \tag{49}$$

and we define $E' := \bigcap_{r=1}^{\tau_N(t)} E'_r$. Applying a multinomial expansion followed by a result similar to Lemma 3,

$$\begin{aligned}
\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^k p_{r_i}\right) &\geq \alpha_n^k(1+O(N^{-1})) \sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^k \left[c_N(r) - B'_n D_N(r)\right] \mathbb{1}_{E'} \\
&\geq \alpha_n^k(1+O(N^{-1})) \left\{ \sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^k c_N(r_i)\mathbb{1}_{E'} - \frac{1}{k!}\left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)(t+1)^{k-1}(1+B'_n)^k \right\}.
\end{aligned} \tag{50}$$

The above expression is split into even and odd terms; a lower bound on (45) can be formed by multiplying the even terms by the lower bound (46) and the odd terms by the upper bound (47). Thus

$$\begin{aligned}
\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^k p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right) \geq \alpha_n^k(1+O(N^{-1})) &\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^k c_N(r_i)\mathbb{1}_{E'} \Bigg\{ \\
&\sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_E \\
&- \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right)\frac{1}{2}\alpha_n^2\exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
&- c_N(\tau_N(t))\exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
&- \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n)]\Bigg\} \\
- \alpha_n^k(1+O(N^{-1}))\frac{1}{k!}&\left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)(t+1)^{k-1}(1+B'_n)^k\Bigg\{ \\
&\left(2+\frac{\alpha_n^2(t+1)}{2}\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
&+ (t+1)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B'_n)]\Bigg\}.
\end{aligned} \tag{51}$$

Due to Brown et al. (2021, Equations (3.3)–(3.5)), all of the negative terms have vanishing expectation, leaving

$$\begin{aligned}
\lim_{N\to\infty}\mathbb{E}&\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^k p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\right] \\
&\geq \alpha_n^k(1+O(N^{-1})) \sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^k c_N(r_i)\mathbb{1}_{E'} \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_E \\
&= \alpha_n^k\sum_{l=0}^{\infty}(-\alpha_n)^l\frac{1}{l!}t^l \lim_{N\to\infty}\mathbb{E}\left[\mathbb{1}_{\{\tau_N(t)\ge l\}}\mathbb{1}_{E\cap E'} \sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^k c_N(r_i)\right]
\end{aligned} \tag{52}$$

Lemma 9 establishes that $\lim_{N\to\infty}\mathbb{P}[E\cap E'] = 1$ and Lemma 10 deals with the other indicator. We can therefore apply Lemma 7 to conclude that

$$\begin{aligned}\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\left(\prod_{i=1}^k p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)}(1-p_r)\right)\right] &\ge \alpha_n^k\sum_{l=0}^{\infty}(-\alpha_n)^l\frac{1}{l!}t^l\sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}}\prod_{j=1}^k\frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}\\ &=\alpha_n^k e^{-\alpha_n t}\sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}}\prod_{j=1}^k\frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}\end{aligned}\tag{53}$$

as required. ∎

**Lemma 7.** *Fix $l,k\in\mathbb{N}$, $i_0:=0$, $i_k:=k$. Let $E$ be any event independent of $r_1,\dots,r_k$ such that $\lim_{N\to\infty}\mathbb{P}[E]=1$. Then for any sequence of times $0=t_0\le t_1\le\cdots\le t_k\le t$,*

$$\lim_{N\to\infty}\mathbb{E}\left[\mathbb{1}_E\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\prod_{i=1}^k c_N(r_i)\right]=\sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}}\prod_{j=1}^k\frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}.\tag{54}$$

*Proof.* As pointed out by Möhle (1999, p. 460), the sum-product on the left hand side can be expanded as

$$\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\prod_{i=1}^k c_N(r_i)=\sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}}\prod_{j=1}^k\sum_{\substack{r_{i_{j-1}+1}<\cdots<r_{i_j}\\ =\tau_N(t_{j-1})+1}}^{\tau_N(t_j)}\prod_{i=i_{j-1}+1}^{i_j}c_N(r_i).\tag{55}$$

Moreover,

$$\sum_{\substack{r_{i_{j-1}+1}<\cdots<r_{i_j}\\ =\tau_N(t_{j-1})+1}}^{\tau_N(t_j)}\prod_{i=i_{j-1}+1}^{i_j}c_N(r_i)=\frac{1}{(i_j-i_{j-1})!}\sum_{\substack{r_{i_{j-1}+1}\ne\cdots\ne r_{i_j}\\ =\tau_N(t_{j-1})+1}}^{\tau_N(t_j)}\prod_{i=i_{j-1}+1}^{i_j}c_N(r_i).\tag{56}$$

An argument akin to (4) gives us an upper bound:

$$\begin{aligned}\sum_{\substack{r_{i_{j-1}+1}\ne\cdots\ne r_{i_j}\\ =\tau_N(t_{j-1})+1}}^{\tau_N(t_j)}\prod_{i=i_{j-1}+1}^{i_j}c_N(r_i)\le\left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)}c_N(s)\right)^{i_j-i_{j-1}} &\le[t_j-t_{j-1}+c_N(\tau_N(t_j))]^{i_j-i_{j-1}}\\ &=\sum_{l=0}^{i_j-i_{j-1}}\binom{i_j-i_{j-1}}{l}(t_j-t_{j-1})^l\,[c_N(\tau_N(t_j))]^{i_j-i_{j-1}-l}\\ &=(t_j-t_{j-1})^{i_j-i_{j-1}}\\ &\quad+c_N(\tau_N(t_j))\sum_{l=0}^{i_j-i_{j-1}-1}\binom{i_j-i_{j-1}}{l}(t_j-t_{j-1})^l\,[c_N(\tau_N(t_j))]^{i_j-i_{j-1}-1-l}\\ &\le(t_j-t_{j-1})^{i_j-i_{j-1}}+c_N(\tau_N(t_j))(t_j-t_{j-1}+1)^k,\end{aligned}\tag{57}$$

using in the last line that $c_N \leq 1$ and $0 \leq i_j - i_{j-1} \leq k$. Now, taking the product on the outside,

$$\begin{aligned}
\prod_{j=1}^{k} \sum_{\substack{r_{i_{j-1}+1}<\cdots<r_{i_j} \\ =\tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) &\leq \prod_{j=1}^{k} \left\{ \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} + c_N(\tau_N(t_j)) \frac{(1 + t_j - t_{j-1})^k}{(i_j - i_{j-1})!} \right\} \\
&\leq \prod_{j=1}^{k} \left\{ \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} + c_N(\tau_N(t_j))(1 + t_j - t_{j-1})^k \right\} \\
&= \sum_{\mathcal{I} \subseteq [k]} \left( \prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left( \prod_{j \notin \mathcal{I}} c_N(\tau_N(t_j))(1 + t_j - t_{j-1})^k \right) \\
&= \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \\
&\qquad + \sum_{\mathcal{I} \subset [k]} \left( \prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left( \prod_{j \notin \mathcal{I}} c_N(\tau_N(t_j))(1 + t_j - t_{j-1})^k \right) \\
&\leq \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \\
&\qquad + \sum_{\mathcal{I} \subset [k]} c_N(\tau_N(t_{j^*})) \left( \prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) (1 + t_j - t_{j-1})^{k^2}
\end{aligned} \tag{58}$$

where, say, $j^* := \min\{j \notin \mathcal{I}\}$. Now we are in a position to evaluate the limit in (54):

$$\begin{aligned}
&\lim_{N\to\infty} \mathbb{E}\left[ \mathbb{1}_E \sum_{\substack{r_1<\cdots<r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i) \right] \leq \lim_{N\to\infty} \mathbb{E}\left[ \sum_{\substack{r_1<\cdots<r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i) \right] \\
&\quad \leq \sum_{\substack{i_1 \leq \cdots \leq i_{k-1} \\ \in\{0,\dots,k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \\
&\qquad + \sum_{\substack{i_1 \leq \cdots \leq i_{k-1} \\ \in\{0,\dots,k\}: \\ i_j \geq j \forall j}} \sum_{\mathcal{I} \subset [k]} \lim_{N\to\infty} \mathbb{E}\left[c_N(\tau_N(t_{j^*}))\right] \left( \prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) (1 + t_j - t_{j-1})^{k^2} \\
&\quad = \sum_{\substack{i_1 \leq \cdots \leq i_{k-1} \\ \in\{0,\dots,k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}
\end{aligned} \tag{59}$$

using Brown et al. (2021, Equation (3.3)). For the corresponding lower bound, by a slight modification of (2),

$$\begin{aligned}\sum_{\substack{r_{i_{j-1}+1}\neq\cdots\neq r_{i_j}\\=\tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) &\geq \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)\right)^{i_j-i_{j-1}} \\ &\quad - \binom{i_j-i_{j-1}}{2}\left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2\right)\left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)\right)^{i_j-i_{j-1}-2} \\ &\geq (t_j-t_{j-1})^{i_j-i_{j-1}} - \binom{i_j-i_{j-1}}{2}\left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2\right)(t_j-t_{j-1}+1)^{i_j-i_{j-1}-2} \\ &\geq (t_j-t_{j-1})^{i_j-i_{j-1}} - (i_j-i_{j-1})!\left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2\right)(t_j-t_{j-1}+1)^{k-1}. \end{aligned} \tag{60}$$

Define the event

$$E_j^\star = \left\{\left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2\right) \leq \frac{(t_j-t_{j-1})^{i_j-i_{j-1}-k+1}}{(i_j-i_{j-1})!}\right\}, \tag{61}$$

which is sufficient to ensure the $j^{th}$ term in the following is non-negative, and let $E^\star := \bigcap_{j=1}^k E_j^\star$. Now, taking a product over $j$ as in (58),

$$\begin{aligned}\prod_{j=1}^k \sum_{\substack{r_{i_{j-1}+1}<\cdots<r_{i_j}\\=\tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) &\geq \prod_{j=1}^k \left\{\frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} - \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2\right)(t_j-t_{j-1}+1)^{k-1}\right\}\mathbb{1}_{E^\star} \\ &= \sum_{\mathcal{I}\subseteq[k]}(-1)^{k-|\mathcal{I}|}\left(\prod_{j\in\mathcal{I}}\frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}\right)\left(\prod_{j\notin\mathcal{I}}\left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2\right)(t_j-t_{j-1}+1)^{k-1}\right)\mathbb{1}_{E^\star} \\ &= \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}\mathbb{1}_{E^\star} \\ &\quad + \sum_{\mathcal{I}\subset[k]}(-1)^{k-|\mathcal{I}|}\left(\prod_{j\in\mathcal{I}}\frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}\right)\left(\prod_{j\notin\mathcal{I}}\left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2\right)(t_j-t_{j-1}+1)^{k-1}\right) \\ &\geq \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}\mathbb{1}_{E^\star} \\ &\quad - \sum_{\substack{\mathcal{I}\subset[k]:\\k-|\mathcal{I}|\text{ odd}}}\left(\prod_{j\in\mathcal{I}}\frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}\right)\left(\prod_{j\notin\mathcal{I}}\left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2\right)(t_j-t_{j-1}+1)^{k-1}\right) \\ &\geq \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}\mathbb{1}_{E^\star} \\ &\quad - \sum_{\mathcal{I}\subset[k]}\left(\sum_{s=\tau_N(t_{j^*-1})+1}^{\tau_N(t_{j^*})} c_N(s)^2\right)\left(\prod_{j\in\mathcal{I}}(t_j-t_{j-1})^k\right)\left(\prod_{j\notin\mathcal{I}}(t_j-t_{j-1}+1)^k\right) \\ &\geq \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}\mathbb{1}_{E^\star} - \sum_{\mathcal{I}\subset[k]}\left(\sum_{s=\tau_N(t_{j^*-1})+1}^{\tau_N(t_{j^*})} c_N(s)^2\right)(t_j-t_{j-1}+1)^{k^2}. \end{aligned} \tag{62}$$

We can now evaluate the limit:

$$
\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_E \sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^k c_N(r_i)\right] &\ge \lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_{E\cap E^\star} \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}\right] \\
&\quad - \lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_E \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \sum_{\mathcal{I}\subset[k]} \left(\sum_{s=\tau_N(t_{j^*-1})+1}^{\tau_N(t_{j^*})} c_N(s)^2\right)(t_j-t_{j-1}+1)^{k^2}\right] \\
&\ge \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} \lim_{N\to\infty} \mathbb{E}[\mathbb{1}_{E\cap E^\star}] \\
&\quad - \lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \sum_{\mathcal{I}\subset[k]} \left(\sum_{s=\tau_N(t_{j^*-1})+1}^{\tau_N(t_{j^*})} c_N(s)^2\right)(t_j-t_{j-1}+1)^{k^2}\right] \\
&= \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} \lim_{N\to\infty} \mathbb{P}[E\cap E^\star] \\
&\quad - \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \sum_{\mathcal{I}\subset[k]} \lim_{N\to\infty} \mathbb{E}\left[\sum_{s=\tau_N(t_{j^*-1})+1}^{\tau_N(t_{j^*})} c_N(s)^2\right](t_j-t_{j-1}+1)^{k^2} \\
&= \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}
\end{aligned} \tag{63}
$$

where for the last equality we use Brown et al. (2021, Equation (3.5)) to show that the second sum vanishes and Lemma 11 to show that $\lim_{N\to\infty} \mathbb{P}[E\cap E^\star] = 1$. We have shown that the upper and lower bounds coincide, so the result follows. ■

# Indicators

**Lemma 8.** *Let $A, B$ be events. Sequences of events, really. Dependence on some incremental variable is implicit, also in the limit notation. If $\lim \mathbb{P}[A]=1$ and $\lim \mathbb{P}[B]=1$ then $\lim \mathbb{P}[A\cap B]=1$.*

The above might be so obvious as to go unstated, but it is very important because it means we don't have to deal with intersections of dependent events! Here is a little proof just to be sure:

*Proof.*

$$
\begin{aligned}
&\lim \mathbb{P}[A]=1 \textbf{ and } \lim \mathbb{P}[B]=1 \\
\Leftrightarrow &\lim \mathbb{P}[A^c]=0 \textbf{ and } \lim \mathbb{P}[B^c]=0 \\
\Rightarrow &\lim \{\mathbb{P}[A^c]+\mathbb{P}[B^c]\}=0 \\
\Rightarrow &\lim \mathbb{P}[A^c\cup B^c]=0 \\
\Leftrightarrow &\lim \mathbb{P}[A\cap B]=1.
\end{aligned} \tag{64}
$$

The only part of this argument that I find potentially controversial is going from the third to the fourth line, which is an application of the sandwich theorem (since $0 \le \mathbb{P}[A^c \cup B^c] \le \mathbb{P}[A^c] + \mathbb{P}[B^c]$). ■

**Lemma 9.** *Let $K$ be a constant which may depend on $n, N$ but not on $r$, such that $K^{-2} = O(1)$ as $N \to \infty$. Define the events $E_r := \{c_N(r) < K\}$ and denote $E := \bigcap_{r=1}^{\tau_N(t)} E_r$. Then $\lim_{N\to\infty} \mathbb{P}[E] = 1$.*

*Proof.*

$$\begin{aligned}\mathbb{P}[E] = 1 - \mathbb{P}[E^c] &= 1 - \mathbb{P}\left[\bigcup_{r=1}^{\tau_N(t)} E_r^c\right] = 1 - \mathbb{E}\left[\mathbb{1}_{\bigcup E_r^c}\right] \ge 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r^c}\right] \\ &= 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}\left[\mathbb{1}_{E_r^c} \mid \mathcal{F}_{r-1}\right]\right] = 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{P}\left[E_r^c \mid \mathcal{F}_{r-1}\right]\right] \end{aligned} \tag{65}$$

where for the second line we apply Lemma 13 with $f(r) = \mathbb{1}_{E_r^c}$. By the generalised Markov inequality,

$$\mathbb{P}[E_r^c \mid \mathcal{F}_{r-1}] = \mathbb{P}[c_N(r) \ge K \mid \mathcal{F}_{r-1}] \le \mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}]K^{-2}. \tag{66}$$

Substituting this into (65) and applying Lemma 13 again, this time with $f(r) = c_N(r)^2$,

$$\mathbb{P}[E] \ge 1 - K^{-2}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}]\right] = 1 - K^{-2}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} c_N(r)^2\right]. \tag{67}$$

Applying Brown et al. (2021, Equation (3.5)), the limit is

$$\lim_{N\to\infty} \mathbb{P}[E] = 1 - O(1) \times 0 = 1 \tag{68}$$

as required. ■

**Lemma 10.** *Fix $t > 0$. For any $l \in \mathbb{R}$, $\lim_{N\to\infty} \mathbb{P}[\tau_N(t) \ge l] = 1$ .*

*Proof.*

$$\{\tau_N(t) \ge l\} = \left\{\min\left\{s \ge 1 : \sum_{r=1}^{s} c_N(r) \ge t\right\} \ge l\right\} = \left\{\sum_{r=1}^{l-1} c_N(r) < t\right\} \supseteq \bigcap_{r=1}^{l-1}\left\{c_N(r) < \frac{t}{l}\right\} \supseteq \bigcap_{r=1}^{\tau_N(t)}\left\{c_N(r) < \frac{t}{l}\right\}. \tag{69}$$

Hence

$$\lim_{N\to\infty} \mathbb{P}[\tau_N(t) \ge l] \ge \lim_{N\to\infty} \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)}\left\{c_N(r) < \frac{t}{l}\right\}\right] \tag{70}$$

and the result follows by applying Lemma 9 with $K = t/l$. ■

**Lemma 11.** *Define the event*

$$E^\star := \bigcap_{j=1}^{k}\left\{\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \le \frac{(t_j - t_{j-1})^{i_j - i_{j-1} - k + 1}}{(i_j - i_{j-1})!}\right\}. \tag{71}$$

*Then $\lim_{N\to\infty} \mathbb{P}[E^\star] = 1$.*

*Proof.*

$$\begin{aligned}E^\star &\supseteq \left\{ \sum_{j=1}^{k} \sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \le \sum_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1} - k+1}}{(i_j - i_{j-1})!} \right\} \\ &= \left\{ \sum_{s=1}^{\tau_N(t)} c_N(s)^2 \le \sum_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1} - k+1}}{(i_j - i_{j-1})!} \right\} \\ &\supseteq \left\{ \sum_{s=1}^{\tau_N(t)} c_N(s)^2 \le \frac{1}{k!} \sum_{j=1}^{k} (t_j - t_{j-1})^{i_j - i_{j-1} - k+1} \right\}. \end{aligned} \tag{72}$$

To simplify the RHS further, consider the possible values of $(i_j - i_{j-1} - k + 1) \in \{-k+1, \dots, 1\}$: This simplification isn't necessary for the result, but it makes the expressions less cumbersome later on.

**Case** $(i_j - i_{j-1} - k + 1) < 0$:

$$\sum_{j=1}^{k} (t_j - t_{j-1})^{i_j - i_{j-1} - k+1} \ge \sum_{j=1}^{k} t^{i_j - i_{j-1} - k+1} \ge \sum_{j=1}^{k} t^{-k+1} = kt^{-k+1}. \tag{73}$$

**Case** $(i_j - i_{j-1} - k + 1) = 0$:

$$\sum_{j=1}^{k} (t_j - t_{j-1})^{i_j - i_{j-1} - k+1} = \sum_{j=1}^{k} 1 = k. \tag{74}$$

**Case** $(i_j - i_{j-1} - k + 1) = 1$:

$$\sum_{j=1}^{k} (t_j - t_{j-1})^{i_j - i_{j-1} - k+1} = \sum_{j=1}^{k} (t_j - t_{j-1}) = t_k - t_0 = t. \tag{75}$$

Altogether

$$\sum_{j=1}^{k} (t_j - t_{j-1})^{i_j - i_{j-1} - k+1} \ge \min\{kt^{-k+1}, k, t\} = \min\{kt^{-k+1}, t\} = t \min\{kt^{-k}, 1\}, \tag{76}$$

so

$$E^\star \supseteq \left\{ \sum_{s=1}^{\tau_N(t)} c_N(s)^2 < \frac{t}{k!} \min\{kt^{-k}, 1\} \right\}. \tag{77}$$

Using Markov's inequality,

$$\begin{aligned}\mathbb{P}[E^\star] &\ge \mathbb{P}\left[ \sum_{s=1}^{\tau_N(t)} c_N(s)^2 < \frac{t}{k!} \min\{kt^{-k}, 1\} \right] = 1 - \mathbb{P}\left[ \sum_{s=1}^{\tau_N(t)} c_N(s)^2 \ge \frac{t}{k!} \min\{kt^{-k}, 1\} \right] \\ &\ge 1 - \frac{k!}{t} \max\{1, k^{-1}t^k\} \mathbb{E}\left[ \sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right], \end{aligned} \tag{78}$$

and by Brown et al. (2021, Equation (3.5))

$$\lim_{N\to\infty} \mathbb{P}[E^\star] = 1 - O(1) \times 0 = 1 \tag{79}$$

as required. ∎

**Lemma 12.** *Let $K$ be a constant not depending on $N, r$, but which may depend on $n$.*

$$\lim_{N\to\infty} \mathbb{P}\left[ \bigcap_{r=1}^{\tau_N(t)} \{c_N(r) \ge KD_N(r)\} \right] = 1. \tag{80}$$

*Proof.*

$$\begin{aligned}\mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)}\{c_N(r)\geq KD_N(r)\}\right] &\geq \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)}\{c_N(r)> KD_N(r)\}\right]\\ &= 1-\mathbb{P}\left[\bigcup_{r=1}^{\tau_N(t)}\{c_N(r)\leq KD_N(r)\}\right]\\ &= 1-\mathbb{E}\left[\mathbb{1}_{\bigcup\{c_N(r)\leq KD_N(r)\}}\right]\\ &\geq 1-\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)}\mathbb{1}_{\{c_N(r)\leq KD_N(r)\}}\right]\\ &= 1-\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)}\mathbb{P}[c_N(r)\leq KD_N(r)\mid \mathcal{F}_{r-1}]\right] \qquad (81)\end{aligned}$$

where the final inequality is an application of Lemma 13 with $f(r)=\mathbb{1}_{\{c_N(r)\leq KD_N(r)\}}$.

Fix $0<\varepsilon<K^{-1}/2$ and let $N>\max\{\varepsilon^{-1},(\binom{n-2}{2}-2\varepsilon)^{-1}\}$. For each $r,i$ define the event $A_i(r):=\{\nu_r^{(i)}\leq N\varepsilon\}$. Conditional on $\mathcal{F}_{r-1}$, we have

$$D_N(r)=\frac{1}{N(N)_2}\sum_{i=1}^{N}(\nu_r^{(j)})_2\left[\nu_r^{(i)}+\frac{1}{N}\sum_{j\neq i}(\nu_r^{(i)})^2\right]\mathbb{1}_{A_i(r)^c}+\frac{1}{N(N)_2}\sum_{i=1}^{N}(\nu_r^{(i)})_2\left[\nu_r^{(i)}+\frac{1}{N}\sum_{j\neq i}(\nu_r^{(j)})^2\right]\mathbb{1}_{A_i(r)}. \qquad (82)$$

For the first term,

$$\frac{1}{N(N)_2}\sum_{i=1}^{N}(\nu_r^{(i)})_2\left[\nu_r^{(i)}+\frac{1}{N}\sum_{j\neq i}(\nu_r^{(j)})^2\right]\mathbb{1}_{A_i(r)^c}\leq\sum_{i=1}^{N}\mathbb{1}_{A_i(r)^c}. \qquad (83)$$

For the second term,

$$\begin{aligned}\frac{1}{N(N)_2}\sum_{i=1}^{N}(\nu_r^{(i)})_2\left[\nu_r^{(i)}+\frac{1}{N}\sum_{j\neq i}(\nu_r^{(j)})^2\right]\mathbb{1}_{A_i(r)} &\leq \frac{1}{N(N)_2}\sum_{i=1}^{N}(\nu_r^{(i)})_2\nu_r^{(i)}\mathbb{1}_{A_i(r)}+\frac{1}{N^2(N)_2}\sum_{i=1}^{N}(\nu_r^{(i)})_2\sum_{j=1}^{N}(\nu_r^{(j)})^2\mathbb{1}_{A_i(r)}\\ &\leq \frac{1}{N}c_N(r)N\varepsilon+\frac{1}{N^2(N)_2}\sum_{i=1}^{N}(\nu_r^{(i)})_2\sum_{j=1}^{N}(\nu_r^{(j)})_2\mathbb{1}_{A_i(r)}\\ &\quad+\frac{1}{N^2(N)_2}\sum_{i=1}^{N}(\nu_r^{(i)})_2\sum_{j=1}^{N}(\nu_r^{(j)})\mathbb{1}_{A_i(r)}\\ &\leq \varepsilon c_N(r)+\frac{1}{N^2}\sum_{i=1}^{N}\nu_r^{(i)}N\varepsilon c_N(r)+\frac{1}{N^2}c_N(r)N\\ &= c_N(r)\left(2\varepsilon+\frac{1}{N}\right). \qquad (84)\end{aligned}$$

Hence, conditional on $\mathcal{F}_{r-1}$,

$$\begin{aligned}\{c_N(r)\geq KD_N(r)\} &\supseteq \left\{c_N(r)\leq Kc_N(r)(2\varepsilon+N^{-1})+K\sum_{i=1}^{N}\mathbb{1}_{A_i(r)^c}\right\}\\ &= \left\{K^{-1}-2\varepsilon-\frac{1}{N}\leq\sum_{i=1}^{N}\mathbb{1}_{A_i(r)^c}c_N(r)^{-1}\right\} \qquad (85)\end{aligned}$$

where the ratio $\mathbb{1}_{A_i(r)^c}/c_N(r)$ is well-defined because

$$A_i(r)^c\Rightarrow c_N(r):=\frac{1}{(N)_2}\sum_{j=1}^{N}(\nu_r^{(j)})_2\geq\frac{1}{(N)_2}(\nu_r^{(i)})_2\geq\frac{\varepsilon(N\varepsilon-1)}{N-1}\geq\varepsilon\left(\varepsilon-\frac{1}{N}\right)>0. \qquad (86)$$

Hence by Markov's inequality (the conditions on $\varepsilon, N$ ensuring the constant is always strictly positive),

$$\begin{aligned}
\mathbb{P}\left[c_N(r) \leq K D_N(r) \mid \mathcal{F}_{r-1}\right] &\leq \mathbb{P}\left[\sum_{i=1}^{N} \mathbb{1}_{A_i(r)^c} \geq \left(K^{-1} - 2\varepsilon - \frac{1}{N}\right) \varepsilon \left(\varepsilon - \frac{1}{N}\right) \middle| \mathcal{F}_{r-1}\right] \\
&\leq \frac{1}{\left(K^{-1} - 2\varepsilon - \frac{1}{N}\right) \varepsilon \left(\varepsilon - \frac{1}{N}\right)} \mathbb{E}\left[\sum_{i=1}^{N} \mathbb{1}_{A_i(r)^c} \middle| \mathcal{F}_{r-1}\right] \\
&\leq \frac{1}{\left(K^{-1} - 2\varepsilon - \frac{1}{N}\right) \varepsilon \left(\varepsilon - \frac{1}{N}\right)} \mathbb{E}\left[\sum_{i=1}^{N} \frac{(\nu_r^{(i)})_3}{(N\varepsilon)_3} \middle| \mathcal{F}_{r-1}\right] \\
&\leq \frac{1}{\left(K^{-1} - 2\varepsilon - \frac{1}{N}\right) \varepsilon \left(\varepsilon - \frac{1}{N}\right)} \mathbb{E}\left[\frac{N(N)_2}{(N\varepsilon)_3} D_N(r) \middle| \mathcal{F}_{r-1}\right]. \qquad (87)
\end{aligned}$$

Applying Lemma 13 once more, with $f(r) = D_N(r)$,

$$\begin{aligned}
\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{P}[c_N(r) \leq K D_N(r) \mid \mathcal{F}_{r-1}]\right] &\leq \frac{1}{\left(K^{-1} - 2\varepsilon - \frac{1}{N}\right) \varepsilon \left(\varepsilon - \frac{1}{N}\right)} \frac{N(N)_2}{(N\varepsilon)_3} \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[D_N(r) \mid \mathcal{F}_{r-1}]\right] \\
&= \frac{1}{\left(K^{-1} - 2\varepsilon - \frac{1}{N}\right) \varepsilon \left(\varepsilon - \frac{1}{N}\right)} \frac{N(N)_2}{(N\varepsilon)_3} \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} D_N(r)\right] \\
&\underset{N\to\infty}{\longrightarrow} \frac{1}{(K^{-1} - 2\varepsilon)\varepsilon^5} \times 0 = 0. \qquad (88)
\end{aligned}$$

Substituting this back into (81) concludes the proof. ■

# Other useful results

The following Lemma is taken from Koskela et al. (2018, Lemma 2), where the function is set to $f(t) = c_N(t)$, but the authors remark that the result holds for other choices of function.

**Lemma 13.** *Let $(\mathcal{F}_t)$ be the backwards-in-time filtration generated by the offspring counts $\nu_t^{(1:N)}$ at each generation $t$, and let $f(t)$ be any deterministic function of $\nu_t^{(1:N)}$ that is non-negative and bounded. In particular, for all $t$ there exists $B < \infty$ such that $0 \leq f(t) \leq B$. Then*

$$\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} f(r)\right] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right]. \qquad (89)$$

*Proof.* Define

$$M_s := \sum_{r=1}^{s} \left\{f(r) - \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right\}. \qquad (90)$$

It is easy to establish that $(M_s)$ is a martingale with respect to $(\mathcal{F}_s)$, and $M_0 = 0$. Now fix $K \geq 1$ and note that $\tau_N(t) \wedge K$ is a bounded $\mathcal{F}_t$-stopping time. Hence we can apply the optional stopping theorem:

$$\mathbb{E}[M_{\tau_N(t)\wedge K}] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} \left\{f(r) - \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right\}\right] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} f(r)\right] - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right] = 0. \qquad (91)$$

Since this holds for all $K \geq 1$,

$$\lim_{K\to\infty} \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} f(r)\right] = \lim_{K\to\infty} \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right]. \qquad (92)$$

The monotone convergence theorem allows the limit to pass inside the expectation on each side (since increasing $K$ can only increase each sum, by possibly adding non-negative terms). Hence

$$\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} f(r)\right] = \mathbb{E}\left[\lim_{K\to\infty} \sum_{r=1}^{\tau_N(t)\wedge K} f(r)\right] = \mathbb{E}\left[\lim_{K\to\infty} \sum_{r=1}^{\tau_N(t)\wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right] \tag{93}$$

which concludes the proof. ■

# References


Brown, S., Jenkins, P. A., Johansen, A. M. and Koskela, J. (2021), 'Simple conditions for convergence of sequential Monte Carlo genealogies with applications', *Electronic Journal of Probability* **26**(1), 1–22.

Koskela, J., Jenkins, P. A., Johansen, A. M. and Spanò, D. (2018), Asymptotic genealogies of interacting particle systems with an application to sequential Monte Carlo, Mathematics e-print 1804.01811, ArXiv.

Möhle, M. (1999), 'Weak convergence to the coalescent in neutral population models', *Journal of Applied Probability* **36**(2), 446–460.