Weak convergence proof (in progress)

Suzie Brown

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Theorem 1. *Let $\nu_t^{(1:N)}$ denote the offspring numbers in an interacting particle system satisfying the standing assumption and such that, for any N sufficiently large, $\mathbb{P}\{\tau_N(t) = \infty\} = 0$ for all finite t. Suppose that there exists a deterministic sequence $(b_N)_{N\geq 1}$ such that $\lim_{N\to\infty} b_N = 0$ and*

$$\frac{1}{(N)_3} \sum_{i=1}^{N} \mathbb{E}_t\{(\nu_t^{(i)})_3\} \leq b_N \frac{1}{(N)_2} \sum_{i=1}^{N} \mathbb{E}_t\{(\nu_t^{(i)})_2\} \tag{1}$$

for all N, uniformly in $t \geq 1$. Then the rescaled genealogical process $(G_{\tau_N(t)}^{(n,N)})_{t\geq 0}$ converges weakly to Kingman's n-coalescent as $N \to \infty$.

Proof. Define $p_t := \max_{\xi\in E}\{1 - p_{\xi\xi}(t)\} = 1 - p_{\Delta\Delta}(t)$, where Δ denotes the trivial partition of $\{1,\dots,n\}$ into singletons. For a proof that the maximum is attained at $\xi = \Delta$, see Lemma 1. Following Möhle (1999), we now construct the two-dimensional Markov process $(Z_t, S_t)_{t\in\mathbb{N}}$ with transition probabilities

$$\mathbb{P}(Z_t = j, S_t = \eta \mid Z_{t-1} = i, S_{t-1} = \xi) = \begin{cases} 1 - p_t & \text{if } j = i \text{ and } \xi = \eta \\ p_{\xi\xi}(t) + p_t - 1 & \text{if } j = i+1 \text{ and } \xi = \eta \\ p_{\xi\eta}(t) & \text{if } j = i+1 \text{ and } \xi \neq \eta \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The construction is such that the marginal (S_t) has the same distribution as the genealogical process of interest, and (Z_t) has jumps at all the times (S_t) does plus some extra jumps. (The definition of p_t ensures that the probability in the second case is non-negative, attaining the value zero when $\xi = \Delta$.)

Denote by $0 = T_0^{(N)} < T_1^{(N)} < \dots$ the jump times of the rescaled process $(Z_{\tau_N(t)})_{t\geq 0}$, and $\omega_i^{(N)} := T_i^{(N)} - T_{i-1}^{(N)}$ the corresponding holding times ($i \in \mathbb{N}$).

... $\square$

Lemma 1. $\max_{\xi\in E}(1 - p_{\xi\xi}(t)) = 1 - p_{\Delta\Delta}(t)$.

Proof. Consider any $\xi \in E$ consisting of k blocks ($1 \leq k \leq n-1$), and any $\xi' \in E$ consisting of $k+1$ blocks. From the definition of $p_{\xi\eta}(t)$ (Koskela et al., 2018, Equation (1)),

$$p_{\xi\xi}(t) = \frac{1}{(N)_k} \sum_{\substack{i_1,\dots,i_k \\ \text{all distinct}}} \nu_t^{(i_1)} \cdots \nu_t^{(i_k)}. \tag{3}$$

Similarly,

$$\begin{aligned} p_{\xi'\xi'}(t) &= \frac{1}{(N)_{k+1}} \sum_{\substack{i_1,\dots,i_k,i_{k+1} \\ \text{all distinct}}} \nu_t^{(i_1)} \cdots \nu_t^{(i_k)} \nu_t^{(i_{k+1})} \\ &= \frac{1}{(N)_k (N-k)} \sum_{\substack{i_1,\dots,i_k \\ \text{all distinct}}} \left\{ \nu_t^{(i_1)} \cdots \nu_t^{(i_k)} \sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^{N} \nu_t^{(i_{k+1})} \right\}. \end{aligned}$$

Discarding some zero summands,

$$p_{\xi'\xi'}(t) = \frac{1}{(N)_k(N-k)} \sum_{\substack{i_1,\dots,i_k \\ \text{all distinct:} \\ \nu_t^{(i_1)},\dots,\nu_t^{(i_k)}>0}} \left\{ \nu_t^{(i_1)} \cdots \nu_t^{(i_k)} \sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^{N} \nu_t^{(i_{k+1})} \right\}. \tag{4}$$

The inner sum is

$$\sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^{N} \nu_t^{(i_{k+1})} = \left\{ \sum_{i=1}^{N} \nu_t^{(i)} - \sum_{i\in\{i_1,\dots,i_k\}} \nu_t^{(i)} \right\} \leq N-k \tag{5}$$

since $\nu_t^{(i_1)},\dots,\nu_t^{(i_k)}$ are all at least 1. Hence

$$p_{\xi'\xi'}(t) \leq \frac{N-k}{(N)_k(N-k)} \sum_{\substack{i_1,\dots,i_k \\ \text{all distinct:} \\ \nu_t^{(i_1)},\dots,\nu_t^{(i_k)}>0}} \nu_t^{(i_1)} \cdots \nu_t^{(i_k)} = p_{\xi\xi}(t). \tag{6}$$

Thus $p_{\xi\xi}(t)$ is decreasing in the number of blocks of ξ, and is therefore minimised by taking $\xi = \Delta$, which achieves the maximum n blocks. This choice in turn maximises $1 - p_{\xi\xi}(t)$, as required. □

Lemma 2.

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1-p_r)\right] = e^{-\alpha t} \tag{7}$$

where $\alpha := n(n-1)/2$.

Proof.

Lower Bound

From Koskela et al. (2018, Lemma 1 Case 1), taking $\xi = \Delta$, we have

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \alpha(1+O(N^{-1}))\left[\frac{(3n-1)(n-2)}{6N^2} + c_N(t)\right]. \tag{8}$$

Hence, by a multinomial expansion,

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1-p_r) &\geq \prod_{r=1}^{\tau_N(t)} \left\{ 1 - \alpha(1+O(N^{-1})\left[\frac{(3n-1)(n-2)}{6N^2} + c_N(r)\right] \right\} \\
&= 1 + \sum_{k=1}^{\infty} \sum_{\substack{r_1<\dots<r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^{k} \left\{ -\alpha(1+O(N^{-1}))\left[\frac{(3n-1)(n-2)}{6N^2} + c_N(r_j)\right] \right\} \\
&= 1 + \sum_{k=1}^{\infty} \left\{-\alpha(1+O(N^{-1}))\right\}^k \sum_{\substack{r_1<\dots<r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^{k} \left\{ \frac{(3n-1)(n-2)}{6N^2} + c_N(r_j) \right\}
\end{aligned}$$

where the empty sum is taken to be zero. Taking expectations,

$$\mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1-p_r)\right] \geq 1 + \sum_{k=1}^{\infty} \left\{-\alpha(1+O(N^{-1}))\right\}^k \mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^{k} \left\{ \frac{(3n-1)(n-2)}{6N^2} + c_N(r_j) \right\} \right] \tag{9}$$

(the infinite sum has only finitely many non-zero summands, since the inner sum is empty for $k > \tau_N(t)$, which justifies swapping the sum and expectation.) From Koskela et al. (2018, Equation (8)),

$$\begin{aligned}
\sum_{\substack{r_1<\dots<r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^{k} c_N(r_j) &\geq \frac{1}{k!}\left(\sum_{s=1}^{\tau_N(t)} c_N(s)\right)^k - \frac{1}{k!}\binom{k}{2}\left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right)\left(\sum_{s=1}^{\tau_N(t)} c_N(s)\right)^{k-2} \\
&\geq \frac{1}{k!}t^k - \frac{1}{k!}\binom{k}{2}(t+1)^{k-2}\left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right).
\end{aligned}$$

Then

$$\mathbb{E}\left[\frac{1}{k!}t^k - \frac{1}{k!}\binom{k}{2}(t+1)^{k-2}\left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right)\right] = \frac{1}{k!}t^k - \frac{1}{k!}\binom{k}{2}(t+1)^{k-2}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right] \longrightarrow \frac{1}{k!}t^k \tag{10}$$

as $N \to \infty$ using Brown et al. (2020, Equation (5)), via lemmata 1 and 3 therein. Similarly, applying Koskela et al. (2018, Equation (9)),

$$\sum_{\substack{r_1<\dots<r_k\\=1}}^{\tau_N(t)} \prod_{j=1}^{k} c_N(r_j) \leq \frac{1}{k!}(t+1)^k$$

But that sneaky $+1$ ruins the whole thing, because it means we don't have a limit for the expectation on the RHS of (9), which means we can't say anything useful about the limit of the sum as a whole :(

□

References

Brown, S., Jenkins, P. A., Johansen, A. M. and Koskela, J. (2020), 'Simple conditions for convergence of sequential Monte Carlo genealogies with applications', *arXiv preprint arXiv:2007.00096* .

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