Weak convergence proof (in progress)

Suzie Brown

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Theorem 1. *Let $\nu_t^{(1:N)}$ denote the offspring numbers in an interacting particle system satisfying the standing assumption and such that, for any N sufficiently large, for all finite t, $\mathbb{P}\{\tau_N(t) = \infty\} = 0$. Suppose that there exists a deterministic sequence $(b_N)_{N \geq 1}$ such that $\lim_{N\to\infty} b_N = 0$ and*

$$\frac{1}{(N)_3} \sum_{i=1}^{N} \mathbb{E}_t\{(\nu_t^{(i)})_3\} \leq b_N \frac{1}{(N)_2} \sum_{i=1}^{N} \mathbb{E}_t\{(\nu_t^{(i)})_2\} \tag{1}$$

for all N, uniformly in $t \geq 1$. Then the rescaled genealogical process $(G_{\tau_N(t)}^{(n,N)})_{t\geq 0}$ converges weakly to Kingman's n-coalescent as $N \to \infty$.

Proof. Define $p_t := \max_{\xi \in E}\{1 - p_{\xi\xi}(t)\} = 1 - p_{\Delta\Delta}(t)$, where Δ denotes the trivial partition of $\{1, \dots, n\}$ into singletons. For a proof that the maximum is attained at $\xi = \Delta$, see Lemma 1. Following Möhle (1999), we now construct the two-dimensional Markov process $(Z_t, S_t)_{t\in\mathbb{N}}$ with transition probabilities

$$\mathbb{P}(Z_t = j, S_t = \eta \mid Z_{t-1} = i, S_{t-1} = \xi) = \begin{cases} 1 - p_t & \text{if } j = i \text{ and } \xi = \eta \\ p_{\xi\xi}(t) + p_t - 1 & \text{if } j = i+1 \text{ and } \xi = \eta \\ p_{\xi\eta}(t) & \text{if } j = i+1 \text{ and } \xi \neq \eta \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The construction is such that the marginal (S_t) has the same distribution as the genealogical process of interest, and (Z_t) has jumps at all the times (S_t) does plus some extra jumps. (The definition of p_t ensures that the probability in the second case is non-negative, attaining the value zero when $\xi = \Delta$.)

Denote by $0 = T_0^{(N)} < T_1^{(N)} < \dots$ the jump times of the rescaled process $(Z_{\tau_N(t)})_{t\geq 0}$, and $\omega_i^{(N)} := T_i^{(N)} - T_{i-1}^{(N)}$ the corresponding holding times ($i \in \mathbb{N}$).

... $\square$

Lemma 1. $\max_{\xi\in E}(1 - p_{\xi\xi}(t)) = 1 - p_{\Delta\Delta}(t)$.

Proof. Consider any $\xi \in E$ consisting of k blocks ($1 \leq k \leq n-1$), and any $\xi' \in E$ consisting of $k+1$ blocks. From the definition of $p_{\xi\eta}(t)$ (Koskela et al., 2018, Equation (1)),

$$p_{\xi\xi}(t) = \frac{1}{(N)_k} \sum_{\substack{i_1,\dots,i_k \\ \text{all distinct}}} \nu_t^{(i_1)} \cdots \nu_t^{(i_k)}. \tag{3}$$

Similarly,

$$\begin{aligned} p_{\xi'\xi'}(t) &= \frac{1}{(N)_{k+1}} \sum_{\substack{i_1,\dots,i_k,i_{k+1} \\ \text{all distinct}}} \nu_t^{(i_1)} \cdots \nu_t^{(i_k)} \nu_t^{(i_{k+1})} \\ &= \frac{1}{(N)_k (N-k)} \sum_{\substack{i_1,\dots,i_k \\ \text{all distinct}}} \left\{ \nu_t^{(i_1)} \cdots \nu_t^{(i_k)} \sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^{N} \nu_t^{(i_{k+1})} \right\}. \end{aligned} \tag{4}$$

Discarding the zero summands,

$$p_{\xi'\xi'}(t) = \frac{1}{(N)_k(N-k)} \sum_{\substack{i_1,\dots,i_k \\ \text{all distinct:} \\ \nu_t^{(i_1)},\dots,\nu_t^{(i_k)}>0}} \left\{ \nu_t^{(i_1)} \cdots \nu_t^{(i_k)} \sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^{N} \nu_t^{(i_{k+1})} \right\}. \tag{5}$$

The inner sum is

$$\sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^{N} \nu_t^{(i_{k+1})} = \left\{ \sum_{i=1}^{N} \nu_t^{(i)} - \sum_{i\in\{i_1,\dots,i_k\}} \nu_t^{(i)} \right\} \leq N-k \tag{6}$$

since $\nu_t^{(i_1)},\dots,\nu_t^{(i_k)}$ are all at least 1. Hence

$$p_{\xi'\xi'}(t) \leq \frac{N-k}{(N)_k(N-k)} \sum_{\substack{i_1,\dots,i_k \\ \text{all distinct:} \\ \nu_t^{(i_1)},\dots,\nu_t^{(i_k)}>0}} \nu_t^{(i_1)} \cdots \nu_t^{(i_k)} = p_{\xi\xi}(t). \tag{7}$$

Thus $p_{\xi\xi}(t)$ is decreasing in the number of blocks of ξ, and is therefore minimised by taking $\xi = \Delta$, which achieves the maximum n blocks. This choice in turn maximises $1 - p_{\xi\xi}(t)$, as required. □

Lemma 2. *For any* $0 < t < \infty$,

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1 - p_r) \right] = e^{-\alpha_n t} \tag{8}$$

where $\alpha_n := n(n-1)/2$.

Proof. The strategy is to find upper and lower bounds on $\mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right]$, both of which converge to $e^{-\alpha_n t}$.

Lower Bound

From Brown et al. (2020, Equation (14)), taking $\xi = \Delta$, we have

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \alpha_n(1 + O(N^{-1}))\left[c_N(t) + B_n D_N(t)\right] \tag{9}$$

where $B_n > 0$ and the $O(N^{-1})$ term does not depend on t. In particular,

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \frac{N^{n-2}}{(N-2)_{n-2}} \alpha_n c_N(t) - \frac{N^{n-3}}{(N-3)_{n-3}} B_n D_N(t). \tag{10}$$

Since $D_N(s) \leq c_N(s)$ for all s (Koskela et al., 2018, p.9), a sufficient condition for the bound to be positive is

$$E_t := \left\{ c_N(t) < \frac{(N-3)_{n-3}}{N^{n-3}} \left(\alpha_n \left(1 + \frac{2}{N-2}\right) + B_n \right)^{-1} \right\}. \tag{11}$$

Hence, by a multinomial expansion,

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1-p_r) &\geq \prod_{r=1}^{\tau_N(t)} \left\{1 - \alpha_n(1+O(N^{-1}))\left[c_N(r) + B_n D_N(r)\right]\right\} \times \prod_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r} \\
&= \left(1 + \sum_{k=1}^{\tau_N(t)} \sum_{\substack{r_1<\dots<r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^{k} \left\{-\alpha_n(1+O(N^{-1}))\left[c_N(r_j) + B_n D_N(r_j)\right]\right\} \right) \times \prod_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r} \\
&= \prod_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r} + \sum_{k=1}^{\tau_N(t)} \left\{-\alpha_n(1+O(N^{-1}))\right\}^k \left(\prod_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r} \right) \sum_{\substack{r_1<\dots<r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^{k} \left\{c_N(r_j) + B_n D_N(r_j)\right\}.
\end{aligned} \tag{12}$$

Taking expectations,

$$
\begin{aligned}
\mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right] &\geq \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}\mathbb{1}_{E_r}\right] \\
&\quad + \mathbb{E}\left[\sum_{k=1}^{\infty}\left\{-\alpha_n(1+O(N^{-1}))\right\}^k \mathbb{1}_{\{k\leq\tau_N(t)\}}\mathbb{1}_{\bigcap E_r}\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)}\prod_{j=1}^{k}\{c_N(r_j)+B_nD_N(r_j)\}\right] \\
&= \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)}E_r\right] \\
&\quad + \sum_{k=1}^{\infty}\left\{-\alpha_n(1+O(N^{-1}))\right\}^k \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)}\prod_{j=1}^{k}\{c_N(r_j)+B_nD_N(r_j)\}\middle| k\leq\tau_N(t), \bigcap_{r=1}^{\tau_N(t)}E_r\right] \\
&\quad \times \mathbb{P}\left[k\leq\tau_N(t), \bigcap_{r=1}^{\tau_N(t)}E_r\right]. \qquad (13)
\end{aligned}
$$

Swapping the expectation with the infinite sum is justified by the dominated convergence theorem, the calculations for which are almost identical to the invocation of Tannery's theorem in equations (21)–(26).

We want to show that the conditional expectation on the right converges to $t^k/k!$, for reasons that will become clear. The strategy is to upper and lower bound this expectation by quantities that converge to $t^k/k!$.

First the lower bound. Fix $k\leq\tau_N(t)$, so that the sum is non-empty. From Koskela et al. (2018, Equation (8)),

$$
\begin{aligned}
\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)}\prod_{j=1}^{k}\{c_N(r_j)+B_nD_N(r_j)\} &\geq \sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)}\prod_{j=1}^{k}c_N(r_j) \\
&\geq \frac{1}{k!}\left(\sum_{s=1}^{\tau_N(t)}c_N(s)\right)^k - \frac{1}{k!}\binom{k}{2}\left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right)\left(\sum_{s=1}^{\tau_N(t)}c_N(s)\right)^{k-2} \\
&\geq \frac{1}{k!}t^k - \frac{1}{k!}\binom{k}{2}(t+1)^{k-2}\left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right) \qquad (14)
\end{aligned}
$$

by the definition of τ_N. Then

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)}\prod_{j=1}^{k}\{c_N(r_j)+B_nD_N(r_j)\}\middle| k\leq\tau_N(t), \bigcap_{r=1}^{\tau_N(t)}E_r\right] \\
&\quad\geq \frac{1}{k!}t^k - \frac{1}{k!}\binom{k}{2}(t+1)^{k-2}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)}c_N(s)^2\middle| k\leq\tau_N(t), \bigcap_{r=1}^{\tau_N(t)}E_r\right] \\
&\quad= \frac{1}{k!}t^k - \frac{1}{k!}\binom{k}{2}(t+1)^{k-2}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)}c_N(s)^2\mathbb{1}_{\{k\leq\tau_N(t)\}}\mathbb{1}_{\{\bigcap_{r=1}^{\tau_N(t)}E_r\}}\right]\left(\mathbb{P}\left[k\leq\tau_N(t), \bigcap_{r=1}^{\tau_N(t)}E_r\right]\right)^{-1} \\
&\quad\geq \frac{1}{k!}t^k - \frac{1}{k!}\binom{k}{2}(t+1)^{k-2}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right]\left(\mathbb{P}\left[k\leq\tau_N(t), \bigcap_{r=1}^{\tau_N(t)}E_r\right]\right)^{-1} \longrightarrow \frac{1}{k!}t^k \qquad (15)
\end{aligned}
$$

as $N\to\infty$ using Brown et al. (2020, Equation (5)) and Lemma 3.

Now for the upper bound. We start with a multinomial expansion and some manipulations of the sums:

$$
\begin{aligned}
\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)} \prod_{j=1}^{k}\{c_N(r_j)+B_nD_N(r_j)\} &= \frac{1}{k!}\sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \prod_{j=1}^{k}\{c_N(r_j)+B_nD_N(r_j)\}\\
&= \frac{1}{k!}\sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \sum_{\mathcal{I}\subseteq\{1,\dots,k\}} (B_n)^{k-|\mathcal{I}|}\left\{\prod_{i\in\mathcal{I}} c_N(r_i)\right\}\left\{\prod_{j\notin\mathcal{I}} D_N(r_j)\right\}\\
&= \frac{1}{k!}\sum_{\mathcal{I}\subseteq\{1,\dots,k\}} (B_n)^{k-|\mathcal{I}|} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \left\{\prod_{i\in\mathcal{I}} c_N(r_i)\right\}\left\{\prod_{j\notin\mathcal{I}} D_N(r_j)\right\}\\
&= \frac{1}{k!}\sum_{I=0}^{k}\binom{k}{I}(B_n)^{k-I} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \left\{\prod_{i=1}^{I} c_N(r_i)\right\}\left\{\prod_{j=I+1}^{k} D_N(r_j)\right\}\\
&= \frac{1}{k!}\sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \left\{\prod_{i=1}^{k} c_N(r_i)\right\}\\
&\quad + \frac{1}{k!}\sum_{I=0}^{k-1}\binom{k}{I}(B_n)^{k-I} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \left\{\prod_{i=1}^{I} c_N(r_i)\right\}\left\{\prod_{j=I+1}^{k} D_N(r_j)\right\}. \qquad (16)
\end{aligned}
$$

Then, using that $D_N(s)\le c_N(s)$ for all s, along with the definition of τ_N,

$$
\begin{aligned}
&\frac{1}{k!}\sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \left\{\prod_{i=1}^{k} c_N(r_i)\right\} + \frac{1}{k!}\sum_{I=0}^{k-1}\binom{k}{I}(B_n)^{k-I} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \left\{\prod_{i=1}^{I} c_N(r_i)\right\}\left\{\prod_{j=I+1}^{k} D_N(r_j)\right\}\\
&\quad\le \frac{1}{k!}\left(\sum_{r=1}^{\tau_N(t)} c_N(r)\right)^{k} + \frac{1}{k!}\sum_{I=0}^{k-1}\binom{k}{I}(B_n)^{k-I} \sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)} \left\{\prod_{i=1}^{k-1} c_N(r_i)\right\} D_N(r_k)\\
&\quad\le \frac{1}{k!}\{t+c_N(\tau_N(t))\}^k + \frac{1}{k!}\sum_{I=0}^{k-1}\binom{k}{I}(B_n)^{k-I}\left\{\sum_{\substack{r_1\neq\cdots\neq r_{k-1}\\ \text{all distinct}}}^{\tau_N(t)} \prod_{i=1}^{k-1} c_N(r_i)\right\}\left(\sum_{r_k=1}^{\tau_N(t)} D_N(r_k)\right)\\
&\quad\le \frac{1}{k!}\{t+c_N(\tau_N(t))\}^k + \frac{1}{k!}\sum_{I=0}^{k-1}\binom{k}{I}(B_n)^{k-I}\left(\sum_{r=1}^{\tau_N(t)} c_N(r)\right)^{k-1}\left(\sum_{r=1}^{\tau_N(t)} D_N(r)\right)\\
&\quad\le \frac{1}{k!}\{t+c_N(\tau_N(t))\}^k + \frac{1}{k!}\sum_{I=0}^{k-1}\binom{k}{I}(B_n)^{k-I}(t+1)^{k-1}\left(\sum_{r=1}^{\tau_N(t)} D_N(r)\right). \qquad (17)
\end{aligned}
$$

Taking expectations,

$$\begin{aligned}
&\mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k\\=1}}^{\tau_N(t)}\prod_{j=1}^{k}\{c_N(r_j)+B_nD_N(r_j)\}\middle| k\le\tau_N(t),\bigcap_{r=1}^{\tau_N(t)}E_r\right]\\
&\le \frac{1}{k!}\mathbb{E}\left[\{t+c_N(\tau_N(t))\}^k\middle| k\le\tau_N(t),\bigcap_{r=1}^{\tau_N(t)}E_r\right]\\
&\quad+\frac{1}{k!}\sum_{I=0}^{k-1}\binom{k}{I}(B_n)^{k-I}(t+1)^{k-1}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)}D_N(r)\middle| k\le\tau_N(t),\bigcap_{r=1}^{\tau_N(t)}E_r\right]\\
&=\frac{1}{k!}\mathbb{E}\left[\{t+c_N(\tau_N(t))\}^k\,\mathbb{1}_{\{k\le\tau_N(t)\}}\mathbb{1}_{\{\bigcap E_r\}}\right]\left(\mathbb{P}\left[k\le\tau_N(t),\bigcap_{r=1}^{\tau_N(t)}E_r\right]\right)^{-1}\\
&\quad+\frac{1}{k!}\sum_{I=0}^{k-1}\binom{k}{I}(B_n)^{k-I}(t+1)^{k-1}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)}D_N(r)\mathbb{1}_{\{k\le\tau_N(t)\}}\mathbb{1}_{\{\bigcap E_r\}}\right]\left(\mathbb{P}\left[k\le\tau_N(t),\bigcap_{r=1}^{\tau_N(t)}E_r\right]\right)^{-1}\\
&\le\left(\frac{1}{k!}\mathbb{E}\left[\{t+c_N(\tau_N(t))\}^k\right]+\frac{1}{k!}\sum_{I=0}^{k-1}\binom{k}{I}(B_n)^{k-I}(t+1)^{k-1}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)}D_N(r)\right]\right)\left(\mathbb{P}\left[k\le\tau_N(t),\bigcap_{r=1}^{\tau_N(t)}E_r\right]\right)^{-1}\\
&\longrightarrow\frac{1}{k!}t^k.
\end{aligned}\tag{18}$$

The limit follows from Lemma 3 and Brown et al. (2020, Equations (3),(4)) along with the fact that, since $c_N(s)\in[0,1]$ for all s, $\mathbb{E}[c_N(s)^k]\le\mathbb{E}[c_N(s)]$ for all $k\ge1$, and the expansion

$$\mathbb{E}\left[\{t+c_N(\tau_N(t))\}^k\right]=\sum_{i=0}^{k}\binom{k}{i}t^i\,\mathbb{E}\left[c_N(\tau_N(t))^{k-i}\right]\longrightarrow t^k \tag{19}$$

which holds for any fixed k.

Combining the upper and lower limits, we conclude that, for any fixed $k\le\tau_N(t)$,

$$\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k\\=1}}^{\tau_N(t)}\prod_{j=1}^{k}\{c_N(r_j)+B_nD_N(r_j)\}\middle| k\le\tau_N(t),\bigcap_{r=1}^{\tau_N(t)}E_r\right]=\frac{1}{k!}t^k. \tag{20}$$

Now, starting with (13), we have

$$\begin{aligned}
\lim_{N\to\infty}\mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right]&\ge\lim_{N\to\infty}\mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)}E_r\right]\\
&+\lim_{N\to\infty}\sum_{k=1}^{\infty}\left\{-\alpha_n(1+O(N^{-1}))\right\}^k\mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k\\=1}}^{\tau_N(t)}\prod_{j=1}^{k}\{c_N(r_j)+B_nD_N(r_j)\}\middle| k\le\tau_N(t),\bigcap_{r=1}^{\tau_N(t)}E_r\right]\mathbb{P}\left[k\le\tau_N(t),\bigcap_{r=1}^{\tau_N(t)}E_r\right]
\end{aligned}\tag{21}$$

Since $\tau_N(t)\to\infty$, some care must be taken when exchanging the limit with the sum. We will use Tannery's theorem (a special case of dominated convergence) to show that this is okay. Let

$$a_k(N):=\{-\alpha_n(1+O(N^{-1}))\}^k\mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k\\=1}}^{\tau_N(t)}\prod_{j=1}^{k}\{c_N(r_j)+B_nD_N(r_j)\}\middle| k\le\tau_N(t),\bigcap_{r=1}^{\tau_N(t)}E_r\right]\mathbb{P}\left[k\le\tau_N(t),\bigcap_{r=1}^{\tau_N(t)}E_r\right]. \tag{22}$$

We know from (20) and Lemma 3 that

$$\lim_{N\to\infty} a_k(N) = (-\alpha_n)^k \frac{1}{k!} t^k. \tag{23}$$

Furthermore, using (18),

$$\begin{aligned}
|a_k(N)| &\le \{\alpha_n(1+O(N^{-1}))\}^k \left(\frac{1}{k!}(t+1)^k + \frac{1}{k!}\sum_{I=0}^{k-1}\binom{k}{I}(B_n)^{k-I}(t+1)^{k-1}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} c_N(r)\right]\right) \times 1 \\
&\le \{\alpha_n(1+O(N^{-1}))\}^k \left(\frac{1}{k!}(t+1)^k + \frac{1}{k!}\sum_{I=0}^{k-1}\binom{k}{I}(B_n)^{k-I}(t+1)^{k}\right) \\
&= \{\alpha_n(1+O(N^{-1}))\}^k \frac{1}{k!}(t+1)^k \left(1 + \sum_{I=0}^{k-1}\binom{k}{I}(B_n)^{k-I}\right) \\
&= \{\alpha_n(1+O(N^{-1}))\}^k \frac{1}{k!}(t+1)^k (1+B_n)^k =: M_k. \qquad (24)
\end{aligned}$$

Now

$$\sum_{k=0}^{\infty} M_k = \exp\left\{\alpha_n(1+O(N^{-1}))(t+1)(1+B_n)\right\} < \infty \tag{25}$$

so we can apply Tannery's theorem:

$$\lim_{N\to\infty}\sum_{k=1}^{\infty} a_k(N) = \sum_{k=1}^{\infty}\lim_{N\to\infty} a_k(N) = e^{-\alpha_n t} - 1 \tag{26}$$

and finally, applying Lemma 3,

$$\lim_{N\to\infty}\mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right] \ge 1 + e^{-\alpha_n t} - 1 = e^{-\alpha_n t}. \tag{27}$$

Upper Bound

From Koskela et al. (2018, Lemma 1 Case 1), taking $\xi = \Delta$, we have

$$1 - p_t = p_{\Delta\Delta}(t) \le 1 - \alpha_n(1+O(N^{-1}))\left[c_N(t) - B_n' D_N(t)\right] \tag{28}$$

where the $O(N^{-1})$ term does not depend on t. A multinomial expansion as in the lower bound yields

$$\prod_{r=1}^{\tau_N(t)}(1-p_r) \le 1 + \sum_{k=1}^{\tau_N(t)}\left\{-\alpha_n(1+O(N^{-1}))\right\}^k \sum_{\substack{r_1<\dots<r_k\\=1}}^{\tau_N(t)}\prod_{j=1}^{k}\left\{c_N(r_j) - B_n' D_N(r_j)\right\}. \tag{29}$$

Analogously to (16), for fixed $k \le \tau_N(t)$ we can write

$$\begin{aligned}
\sum_{\substack{r_1<\dots<r_k\\=1}}^{\tau_N(t)}\prod_{j=1}^{k}\left\{c_N(r_j) - B_n' D_N(r_j)\right\} &= \frac{1}{k!}\sum_{\substack{r_1\neq\dots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)}\left\{\prod_{i=1}^{k} c_N(r_i)\right\} \qquad (30) \\
&\quad + \frac{1}{k!}\sum_{I=0}^{k-1}\binom{k}{I}(-B_n')^{k-I}\sum_{\substack{r_1\neq\dots\neq r_k\\ \text{all distinct}}}\left\{\prod_{i=1}^{I} c_N(r_i)\right\}\left\{\prod_{j=I+1}^{k} D_N(r_j)\right\}.
\end{aligned}$$

We start by dealing with the second term:

$$\frac{1}{k!}\sum_{I=0}^{k-1}\binom{k}{I}(-B_n')^{k-I}\sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}\left\{\prod_{i=1}^{I}c_N(r_i)\right\}\left\{\prod_{j=I+1}^{k}D_N(r_j)\right\}$$
$$=\frac{1}{k!}\sum_{\substack{I=0:\\ k-I\text{ even}}}^{k-1}\binom{k}{I}(B_n')^{k-I}\sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}\left\{\prod_{i=1}^{I}c_N(r_i)\right\}\left\{\prod_{j=I+1}^{k}D_N(r_j)\right\}$$
$$-\frac{1}{k!}\sum_{\substack{I=0:\\ k-I\text{ odd}}}^{k-1}\binom{k}{I}(B_n')^{k-I}\sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}\left\{\prod_{i=1}^{I}c_N(r_i)\right\}\left\{\prod_{j=I+1}^{k}D_N(r_j)\right\}. \tag{31}$$

This is lower bounded by

$$0-\frac{1}{k!}\sum_{\substack{I=0\\ k-I\text{ odd}}}^{k-1}\binom{k}{I}(B_n')^{k-I}(t+1)^{k-1}\left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right) \tag{32}$$

using that $c_N(r), D_N(r)\geq 0$ for all r to bound the even terms below, and arguments from (17) to bound the odd terms above. The same arguments lead to the upper bound

$$\frac{1}{k!}\sum_{\substack{I=0\\ k-I\text{ even}}}^{k-1}\binom{k}{I}(B_n')^{k-I}(t+1)^{k-1}\left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)-0. \tag{33}$$

By Brown et al. (2020, Equation (4)), both bounds have vanishing expectation as $N\to\infty$. We are left with the first term in (30), which is upper bounded by

$$\frac{1}{k!}\sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)}\left\{\prod_{i=1}^{k}c_N(r_i)\right\}\leq\frac{1}{k!}\left(\sum_{s=1}^{\tau_N(t)}c_N(s)\right)^k\leq\frac{1}{k!}\{t+c_N(\tau_N(t))\}^k \tag{34}$$

the expectation of which (conditional on $k\leq\tau_N(t)$; otherwise the sum is empty and has expectation zero) converges to $t^k/k!$ as in (19). We use Koskela et al. (2018, Equation (8)) to construct a lower bound:

$$\frac{1}{k!}\sum_{\substack{r_1\neq\cdots\neq r_k\\ \text{all distinct}}}^{\tau_N(t)}\left\{\prod_{i=1}^{k}c_N(r_i)\right\}\geq\frac{1}{k!}\left(\sum_{s=1}^{\tau_N(t)}c_N(s)\right)^k-\frac{1}{k!}\binom{k}{2}\left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right)\left(\sum_{s=1}^{\tau_N(t)}c_N(s)\right)^{k-2}$$
$$\geq\frac{1}{k!}t^k-\frac{1}{k!}\binom{k}{2}(t+1)^{k-2}\left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right) \tag{35}$$

The expectation of this bound (conditional on $k\leq\tau_N(t)$) also converges to $t^k/k!$, using Brown et al. (2020, Equation (5)). We can now conclude that

$$\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k\\ =1}}^{\tau_N(t)}\prod_{j=1}^{k}\{c_N(r_j)-B_n'D_N(r_j)\}\middle|k\leq\tau_N(t)\right]=\frac{1}{k!}t^k \tag{36}$$

and thus, applying dominated convergence and Tannery's theorem as in the lower bound, along with Lemma 3,

$$\begin{aligned}
&\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right] \\
&\quad \le \lim_{N\to\infty} \mathbb{E}\left[1+\sum_{k=1}^{\tau_N(t)} \left\{-\alpha_n(1+O(N^{-1}))\right\}^k \sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)} \prod_{j=1}^{k} \{c_N(r_j)-B_n' D_N(r_j)\}\right] \\
&\quad = 1+\sum_{k=1}^{\infty} \lim_{N\to\infty}\{-\alpha_n(1+O(N^{-1}))\}^k \lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k\\=1}}^{\tau_N(t)} \prod_{j=1}^{k}\{c_N(r_j)-B_n' D_N(r_j)\}\middle| k\le\tau_N(t)\right] \lim_{N\to\infty}\mathbb{P}\left[k\le\tau_N(t)\right] \\
&\quad = 1+\sum_{k=1}^{\infty}(-\alpha_n)^k \frac{1}{k!} t^k = e^{-\alpha_n t}. \qquad (37)
\end{aligned}$$

We now have upper and lower bounds on $\lim_{N\to\infty}\mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right]$, both of which are equal to $e^{-\alpha_n t}$, and the result follows. ∎

Lemma 3. *For all $k\in\mathbb{N}$, for all $t>0$,*

$$\lim_{N\to\infty}\mathbb{P}\left[k\le\tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r\right]=1 \qquad (38)$$

and consequently

$$\lim_{N\to\infty}\mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} E_r\right]=1 \qquad \textit{and} \qquad \lim_{N\to\infty}\mathbb{P}\left[k\le\tau_N(t)\right]=1. \qquad (39)$$

Proof. We first construct a constant C_1 such that

$$\mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)}\{c_N(r)<C_1\}\right]\le\mathbb{P}\left[k\le\tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r\right] \qquad (40)$$

and

$$\lim_{N\to\infty}\mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)}\{c_N(r)<C_1\}\right]=1. \qquad (41)$$

Any $C_1\le\frac{(N-3)_{n-3}}{N^{n-3}}\left(\alpha_n\left(1+\frac{2}{N-2}\right)+B_n\right)^{-1}$ will suffice to ensure that $\{c_N(r)<C_1\}\subseteq E_r$ for all r. Furthermore, we can write

$$\{\tau_N(t)\ge k\}=\left\{\min\left\{s\ge1:\sum_{r=1}^{s}c_N(r)\ge t\right\}\ge k\right\}=\left\{\sum_{r=1}^{k-1}c_N(r)<t\right\}\supseteq\bigcap_{r=1}^{k-1}\left\{c_N(r)<\frac{t}{k}\right\}. \qquad (42)$$

A suitable choice to satisfy (40) would thus be

$$C_1=\min\left\{\frac{(N-3)_{n-3}}{N^{n-3}}\left(\alpha_n\left(1+\frac{2}{N-2}\right)+B_n\right)^{-1}, \frac{t}{k}\right\}. \qquad (43)$$

Now we show that this choice of C_1 satisfies (41).

$$\begin{aligned}
\mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)}\{c_N(r)<C_1\}\right] &= 1-\mathbb{P}\left[\bigcup_{r=1}^{\tau_N(t)}\{c_N(r)\ge C_1\}\right] = 1-\mathbb{E}\left[\mathbb{1}_{\bigcup\{c_N(r)\ge C_1\}}\right]\ge 1-\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)}\mathbb{1}_{\{c_N(r)\ge C_1\}}\right] \\
&= 1-\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)}\mathbb{E}[\mathbb{1}_{\{c_N(r)\ge C_1\}}\mid\mathcal{F}_{r-1}]\right] = 1-\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)}\mathbb{P}[\{c_N(r)\ge C_1\}\mid\mathcal{F}_{r-1}]\right] \qquad (44)
\end{aligned}$$

where the inequality holds by considering the two possible values of $\mathbb{1}_{\bigcup\{c_N(r)\geq C_1\}}$, and the second line follows from Lemma 4 with $f(r) = \mathbb{1}_{\{c_N(r)\geq C_1\}}$. (To see that this choice of f satisfies the conditions of Lemma 4, note that $\sum_{r=1}^{\tau_N(s)} \mathbb{1}_{\{c_N(r)\geq C_1\}} \leq \sum_{r=1}^{\tau_N(s)} c_N(r)/C_1 \leq (s+1)/C_1$.) Using the generalised Markov inequality,

$$\mathbb{P}[\{c_N(r) \geq C_1\} \mid \mathcal{F}_{r-1}] \leq C_1^{-2}\, \mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}]. \tag{45}$$

Now, using Lemma 4 again, this time with $f(r) = c_N(r)^2$,

$$\begin{aligned}
\mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) < C_1\}\right] &\geq 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} C_1^{-2}\, \mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}]\right] \\
&= 1 - C_1^{-2}\, \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}]\right] \\
&= 1 - C_1^{-2}\, \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} c_N(r)^2\right] \\
&\xrightarrow{N\to\infty} 1 - \min\{(\alpha_n + B_n)^2, k^2/t^2\} \times 0 = 1.
\end{aligned} \tag{46}$$

Clearly $\mathbb{P}\left[k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r\right] \leq \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} E_r\right]$ and $\mathbb{P}\left[k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r\right] \leq \mathbb{P}\left[k \leq \tau_N(t)\right]$, so (39) follows as an immediate corollary. $\square$

The following Lemma is taken from Koskela et al. (2018, Lemma 2), where the function is set to $f(t) = c_N(t)$, but the authors remark that the result holds for other choices of function.

Lemma 4. *Let $(\mathcal{F}_t)$ be the backwards-in-time filtration generated by the offspring counts $\nu_t^{(1:N)}$ at each generation t, and let $f(t)$ be any non-negative deterministic function of $\nu_t^{(1:N)}$ such that for any fixed s, $\sum_{r=1}^{\tau_N(s)} f(r) < \infty$. Then*

$$\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} f(r)\right] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right]. \tag{47}$$

Proof. Define

$$M_s := \sum_{r=1}^{s} \{f(r) - \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\}. \tag{48}$$

It is easy to establish that (M_s) is a martingale with respect to $(\mathcal{F}_s)$, and $M_0 = 0$. Now fix $K > 0$ and note that $\tau_N(t) \wedge K$ is a bounded $\mathcal{F}_t$-stopping time. Hence we can apply the optional stopping theorem:

$$\mathbb{E}[M_{\tau_N(t)\wedge K}] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} \{f(r) - \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\}\right] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} f(r)\right] - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right] = 0. \tag{49}$$

Taking $K \to \infty$ and applying the monotone convergence theorem concludes the proof. $\square$

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