Residual resampling in SMC

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The offspring counts are sampled according to:

$$v_t^{(i)} = \lfloor N w_t^{(i)} \rfloor + X_i$$
$$X_i \sim \text{Multinomial}(N-k, (\bar{w}_t^{(1)}, \dots, \bar{w}_t^{(N)}))$$

where $k := \sum_{i=1}^N \lfloor N w_t^{(i)} \rfloor$ is the number of offspring assigned deterministically, and $\bar{w}_t^{(i)} := \frac{N w_t^{(i)} - \lfloor N w_t^{(i)} \rfloor}{N-k}$ are the residual weights. Let us also define the residuals $r_i := N w_t^{(i)} - \lfloor N w_t^{(i)} \rfloor$. So $\sum_{i=1}^N r_i = N - k$.

The coalescence rate is defined as

$$c_N(t) := \frac{1}{(N)_2} \sum_{i=1}^N (v_t^{(i)})_2.$$

We will use $c_N^m(t)$ and $c_N^r(t)$ to denote the coalescence rates with multinomial and residual resampling respectively. The expectation then comes out as

$$\begin{aligned}
\mathbb{E}[(v_t^{(i)})_2 | \mathcal{F}_{t-1}] &= \mathbb{E}[(v_t^{(i)})^2 | \mathcal{F}_{t-1}] - \mathbb{E}[v_t^{(i)} | \mathcal{F}_{t-1}] \\
&= \mathbb{E}[\lfloor N w_t^{(i)} \rfloor^2 | \mathcal{F}_{t-1}] + 2\mathbb{E}[\lfloor N w_t^{(i)} \rfloor r_i | \mathcal{F}_{t-1}] + \mathbb{E}\left[r_i \left(1 - \frac{r_i}{N-k} + r_i \right) | \mathcal{F}_{t-1} \right] - \mathbb{E}[N w_t^{(i)} | \mathcal{F}_{t-1}] \\
&= \mathbb{E}[\lfloor N w_t^{(i)} \rfloor^2 | \mathcal{F}_{t-1}] - \mathbb{E}[\lfloor N w_t^{(i)} \rfloor | \mathcal{F}_{t-1}] + 2\mathbb{E}[\lfloor N w_t^{(i)} \rfloor r_i | \mathcal{F}_{t-1}] + \mathbb{E}\left[r_i^2 \left(1 - \frac{1}{N-k} \right) | \mathcal{F}_{t-1} \right] \\
&= \mathbb{E}[(N w_t^{(i)})^2 | \mathcal{F}_{t-1}] - \mathbb{E}[\lfloor N w_t^{(i)} \rfloor | \mathcal{F}_{t-1}] - \mathbb{E}\left[\frac{r_i^2}{N-k} | \mathcal{F}_{t-1} \right]
\end{aligned}$$

so we get

$$\begin{aligned}
\mathbb{E}[c_N^r(t) | \mathcal{F}_{t-1}] &= \frac{1}{(N)_2} \sum_{i=1}^N \mathbb{E}[(v_t^{(i)})_2 | \mathcal{F}_{t-1}] \\
&= \frac{N}{N-1} \sum_{i=1}^N \mathbb{E}[(w_t^{(i)})^2 | \mathcal{F}_{t-1}] - \frac{1}{(N)_2} \sum_{i=1}^N \mathbb{E}\left[\frac{r_i^2}{N-k} | \mathcal{F}_{t-1} \right] - \frac{1}{(N)_2} \mathbb{E}[k | \mathcal{F}_{t-1}] \\
&= \mathbb{E}[c_N^m(t) | \mathcal{F}_{t-1}] \left(1 + \frac{1}{N-1} \right) - \frac{1}{(N)_2} \mathbb{E}\left[\frac{\sum_{i=1}^N (N w_t^{(i)} - \lfloor N w_t^{(i)} \rfloor)^2}{\sum_{j=1}^N (N w_t^{(j)} - \lfloor N w_t^{(j)} \rfloor)} | \mathcal{F}_{t-1} \right] \\
&\quad - \frac{1}{(N)_2} \mathbb{E}\left[\sum_{i=1}^N \lfloor N w_t^{(i)} \rfloor | \mathcal{F}_{t-1} \right]
\end{aligned}$$

Sanity check:

When the weights are all equal, $w_t^{(i)} \equiv 1/N$, we should have $\mathbb{E}[c_N^r(t) | \mathcal{F}_{t-1}] = 0$ since each particle will have exactly one offspring so it is impossible for any lineages to coalesce. In this case we have $\mathbb{E}[c_N^m(t) | \mathcal{F}_{t-1}] = \sum_{i=1}^N \mathbb{E}[(w_t^{(i)})^2 | \mathcal{F}_{t-1}] = 1/N$ for multinomial resampling. We also have that $N w_t^{(i)} \equiv \lfloor N w_t^{(i)} \rfloor \equiv 1$ and hence $r_i = 0$ and $k = N$. Thus the RHS comes out as

$$\frac{1}{N} \frac{N}{N-1} - 0 - \frac{1}{(N)_2} N = \frac{1}{N-1} - \frac{1}{N-1} = 0$$

as expected.