

# Weak convergence proof (in progress)

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**Theorem 1.** Let  $\nu_t^{(1:N)}$  denote the offspring numbers in an interacting particle system satisfying the standing assumption and such that, for any  $N$  sufficiently large,  $\mathbb{P}\{\tau_N(t) = \infty\} = 0$  for all finite  $t$ . Suppose that there exists a deterministic sequence  $(b_N)_{N \geq 1}$  such that  $\lim_{N \rightarrow \infty} b_N = 0$  and

$$\frac{1}{(N)_3} \sum_{i=1}^N \mathbb{E}_t\{(\nu_t^{(i)})_3\} \leq b_N \frac{1}{(N)_2} \sum_{i=1}^N \mathbb{E}_t\{(\nu_t^{(i)})_2\} \quad (1)$$

for all  $N$ , uniformly in  $t \geq 1$ . Then the rescaled genealogical process  $(G_{\tau_N(t)}^{(n,N)})_{t \geq 0}$  converges weakly to Kingman's  $n$ -coalescent as  $N \rightarrow \infty$ .

*Proof.* Define  $p_t := \max_{\xi \in E} \{1 - p_{\xi\xi}(t)\} = 1 - p_{\Delta\Delta}(t)$ , where  $\Delta$  denotes the trivial partition of  $\{1, \dots, n\}$  into singletons. For a proof that the maximum is attained at  $\xi = \Delta$ , see Lemma 1. Following Möhle (1999), we now construct the two-dimensional Markov process  $(Z_t, S_t)_{t \in \mathbb{N}}$  with transition probabilities

$$\mathbb{P}(Z_t = j, S_t = \eta \mid Z_{t-1} = i, S_{t-1} = \xi) = \begin{cases} 1 - p_t & \text{if } j = i \text{ and } \xi = \eta \\ p_{\xi\xi}(t) + p_t - 1 & \text{if } j = i + 1 \text{ and } \xi = \eta \\ p_{\xi\eta}(t) & \text{if } j = i + 1 \text{ and } \xi \neq \eta \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

The construction is such that the marginal  $(S_t)$  has the same distribution as the genealogical process of interest, and  $(Z_t)$  has jumps at all the times  $(S_t)$  does plus some extra jumps. (The definition of  $p_t$  ensures that the probability in the second case is non-negative, attaining the value zero when  $\xi = \Delta$ .)

Denote by  $0 = T_0^{(N)} < T_1^{(N)} < \dots$  the jump times of the rescaled process  $(Z_{\tau_N(t)})_{t \geq 0}$ , and  $\omega_i^{(N)} := T_i^{(N)} - T_{i-1}^{(N)}$  the corresponding holding times ( $i \in \mathbb{N}$ ).

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□

**Lemma 1.**  $\max_{\xi \in E} (1 - p_{\xi\xi}(t)) = 1 - p_{\Delta\Delta}(t)$ .

*Proof.* Consider any  $\xi \in E$  consisting of  $k$  blocks ( $1 \leq k \leq n - 1$ ), and any  $\xi' \in E$  consisting of  $k + 1$  blocks. From the definition of  $p_{\xi\xi}(t)$  (Koskela et al., 2018, Equation (1)),

$$p_{\xi\xi}(t) = \frac{1}{(N)_k} \sum_{\substack{i_1, \dots, i_k \\ \text{all distinct}}} \nu_t^{(i_1)} \dots \nu_t^{(i_k)}. \quad (3)$$

Similarly,

$$\begin{aligned} p_{\xi'\xi'}(t) &= \frac{1}{(N)_{k+1}} \sum_{\substack{i_1, \dots, i_{k+1} \\ \text{all distinct}}} \nu_t^{(i_1)} \dots \nu_t^{(i_k)} \nu_t^{(i_{k+1})} \\ &= \frac{1}{(N)_k (N - k)} \sum_{\substack{i_1, \dots, i_k \\ \text{all distinct}}} \left\{ \nu_t^{(i_1)} \dots \nu_t^{(i_k)} \sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^N \nu_t^{(i_{k+1})} \right\}. \end{aligned} \quad (4)$$

Discarding the zero summands,

$$p_{\xi'\xi'}(t) = \frac{1}{(N)_k(N-k)} \sum_{\substack{i_1, \dots, i_k \\ \text{all distinct:} \\ \nu_t^{(i_1)}, \dots, \nu_t^{(i_k)} > 0}} \left\{ \nu_t^{(i_1)} \dots \nu_t^{(i_k)} \sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^N \nu_t^{(i_{k+1})} \right\}. \quad (5)$$

The inner sum is

$$\sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^N \nu_t^{(i_{k+1})} = \left\{ \sum_{i=1}^N \nu_t^{(i)} - \sum_{i \in \{i_1, \dots, i_k\}} \nu_t^{(i)} \right\} \leq N - k \quad (6)$$

since  $\nu_t^{(i_1)}, \dots, \nu_t^{(i_k)}$  are all at least 1. Hence

$$p_{\xi'\xi'}(t) \leq \frac{N-k}{(N)_k(N-k)} \sum_{\substack{i_1, \dots, i_k \\ \text{all distinct:} \\ \nu_t^{(i_1)}, \dots, \nu_t^{(i_k)} > 0}} \nu_t^{(i_1)} \dots \nu_t^{(i_k)} = p_{\xi\xi}(t). \quad (7)$$

Thus  $p_{\xi\xi}(t)$  is decreasing in the number of blocks of  $\xi$ , and is therefore minimised by taking  $\xi = \Delta$ , which achieves the maximum  $n$  blocks. This choice in turn maximises  $1 - p_{\xi\xi}(t)$ , as required.  $\square$

**Lemma 2.**

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[ \prod_{r=1}^{\tau_N(t)} (1 - p_r) \right] = e^{-\alpha t} \quad (8)$$

where  $\alpha := n(n-1)/2$ .

*Proof.* The strategy is to find upper and lower bounds on  $\mathbb{E} \left[ \prod_{r=1}^{\tau_N(t)} (1 - p_r) \right]$ , both of which converge to  $e^{-\alpha t}$ .

**Lower Bound**

From Brown et al. (2020, Equation (14)), taking  $\xi = \Delta$ , we have

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \alpha(1 + O(N^{-1})) \left[ \frac{B_n}{\alpha} D_N(t) + c_N(t) \right] \quad (9)$$

where  $B_n > 0$ . Hence, by a multinomial expansion,

$$\begin{aligned} \prod_{r=1}^{\tau_N(t)} (1 - p_r) &\geq \prod_{r=1}^{\tau_N(t)} \left\{ 1 - \alpha(1 + O(N^{-1})) \left[ \frac{B_n}{\alpha} D_N(r) + c_N(r) \right] \right\} \\ &= 1 + \sum_{k=1}^{\infty} \sum_{\substack{r_1 < \dots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^k \left\{ -\alpha(1 + O(N^{-1})) \left[ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right] \right\} \\ &= 1 + \sum_{k=1}^{\infty} \{ -\alpha(1 + O(N^{-1})) \}^k \sum_{\substack{r_1 < \dots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^k \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\}. \end{aligned} \quad (10)$$

(The empty sum is defined to be zero and the empty product is defined to be one throughout.) Taking expectations,

$$\mathbb{E} \left[ \prod_{r=1}^{\tau_N(t)} (1 - p_r) \right] \geq 1 + \sum_{k=1}^{\infty} \{ -\alpha(1 + O(N^{-1})) \}^k \mathbb{E} \left[ \sum_{\substack{r_1 < \dots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^k \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\} \right] \quad (11)$$

(the infinite sum has only finitely many non-zero summands, since the inner sum is empty for  $k > \tau_N(t)$ , which justifies swapping the sum and expectation.) We want to show that the expectation on the right converges to  $t^k/k!$ , for reasons that will become clear. The strategy is to upper and lower bound this expectation by quantities that converge to  $t^k/k!$ .

First the lower bound. From Koskela et al. (2018, Equation (8)),

$$\begin{aligned}
\sum_{\substack{r_1 < \dots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^k \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\} &\geq \sum_{\substack{r_1 < \dots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^k c_N(r_j) \\
&\geq \frac{1}{k!} \left( \sum_{s=1}^{\tau_N(t)} c_N(s) \right)^k - \frac{1}{k!} \binom{k}{2} \left( \sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \left( \sum_{s=1}^{\tau_N(t)} c_N(s) \right)^{k-2} \\
&\geq \frac{1}{k!} t^k - \frac{1}{k!} \binom{k}{2} (t+1)^{k-2} \left( \sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right)
\end{aligned} \tag{12}$$

by the definition of  $\tau_N$ . Then

$$\mathbb{E} \left[ \sum_{\substack{r_1 < \dots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^k \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\} \right] \geq \frac{1}{k!} t^k - \frac{1}{k!} \binom{k}{2} (t+1)^{k-2} \mathbb{E} \left[ \sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right] \rightarrow \frac{1}{k!} t^k \tag{13}$$

as  $N \rightarrow \infty$  using Brown et al. (2020, Equation (5)), which is shown to hold via lemmata 1 and 3 therein.

Now for the upper bound. We start with a multinomial expansion and some manipulations of the sums:

$$\begin{aligned}
\sum_{\substack{r_1 < \dots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^k \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\} &= \frac{1}{k!} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \prod_{j=1}^k \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\} \\
&= \frac{1}{k!} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \sum_{\mathcal{I} \subseteq \{1, \dots, k\}} \left( \frac{B_n}{\alpha} \right)^{k-|\mathcal{I}|} \left\{ \prod_{i \in \mathcal{I}} c_N(r_i) \right\} \left\{ \prod_{j \notin \mathcal{I}} D_N(r_j) \right\} \\
&= \frac{1}{k!} \sum_{\mathcal{I} \subseteq \{1, \dots, k\}} \left( \frac{B_n}{\alpha} \right)^{k-|\mathcal{I}|} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i \in \mathcal{I}} c_N(r_i) \right\} \left\{ \prod_{j \notin \mathcal{I}} D_N(r_j) \right\} \\
&= \frac{1}{k!} \sum_{I=0}^k \binom{k}{I} \left( \frac{B_n}{\alpha} \right)^{k-I} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^I c_N(r_i) \right\} \left\{ \prod_{j=I+1}^k D_N(r_j) \right\} \\
&= \frac{1}{k!} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^k c_N(r_i) \right\} \\
&\quad + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left( \frac{B_n}{\alpha} \right)^{k-I} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^I c_N(r_i) \right\} \left\{ \prod_{j=I+1}^k D_N(r_j) \right\}. \tag{14}
\end{aligned}$$

Then, using that  $D_N(s) \leq c_N(s)$  for all  $s$  (see Lemma 3 for a proof), along with the definition of  $\tau_N$ ,

$$\begin{aligned}
& \frac{1}{k!} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^k c_N(r_i) \right\} + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left( \frac{B_n}{\alpha} \right)^{k-I} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^I c_N(r_i) \right\} \left\{ \prod_{j=I+1}^k D_N(r_j) \right\} \\
& \leq \frac{1}{k!} \left( \sum_{r=1}^{\tau_N(t)} c_N(r) \right)^k + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left( \frac{B_n}{\alpha} \right)^{k-I} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^{k-1} c_N(r_i) \right\} \{D_N(r_k)\} \\
& \leq \frac{1}{k!} \{t + c_N(\tau_N(t))\}^k + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left( \frac{B_n}{\alpha} \right)^{k-I} \left\{ \sum_{\substack{r_1 \neq \dots \neq r_{k-1} \\ \text{all distinct}}}^{\tau_N(t)} \prod_{i=1}^{k-1} c_N(r_i) \right\} \left\{ \sum_{r_k=1}^{\tau_N(t)} D_N(r_k) \right\} \\
& \leq \frac{1}{k!} \{t + c_N(\tau_N(t))\}^k + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left( \frac{B_n}{\alpha} \right)^{k-I} \left( \sum_{r=1}^{\tau_N(t)} c_N(r) \right)^{k-1} \left( \sum_{r=1}^{\tau_N(t)} D_N(r) \right) \\
& \leq \frac{1}{k!} \{t + c_N(\tau_N(t))\}^k + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left( \frac{B_n}{\alpha} \right)^{k-I} (t+1)^{k-1} \left( \sum_{r=1}^{\tau_N(t)} D_N(r) \right). \tag{15}
\end{aligned}$$

Taking expectations,

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathbb{E} \left[ \sum_{r_1 < \dots < r_k}^{\tau_N(t)} \prod_{j=1}^k \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\} \right] & \leq \frac{1}{k!} \lim_{N \rightarrow \infty} \mathbb{E}[\{t + c_N(\tau_N(t))\}^k] \\
& + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left( \frac{B_n}{\alpha} \right)^{k-I} (t+1)^{k-1} \lim_{N \rightarrow \infty} \mathbb{E} \left[ \left( \sum_{r=1}^{\tau_N(t)} D_N(r) \right) \right] \\
& = \frac{1}{k!} t^k. \tag{16}
\end{aligned}$$

The limit follows from Brown et al. (2020, Equations (3),(4)) along with the fact that, since  $c_N(s) \in [0, 1]$  for all  $s$ ,  $\mathbb{E}[c_N(s)^k] \leq \mathbb{E}[c_N(s)]$  for all  $k \geq 1$ , and the expansion

$$\mathbb{E} \left[ \frac{1}{k!} \{t + c_N(\tau_N(t))\}^k \right] = \mathbb{E} \left[ \frac{1}{k!} \sum_{i=0}^k \binom{k}{i} t^i c_N(\tau_N(t))^{k-i} \right] = \frac{1}{k!} \{t^k + k t^{k-1} \mathbb{E}[c_N(\tau_N(t))] + \dots\} \rightarrow \frac{1}{k!} t^k. \tag{17}$$

Combining these upper and lower limits, we conclude that

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[ \sum_{r_1 < \dots < r_k}^{\tau_N(t)} \prod_{j=1}^k \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\} \right] = \frac{1}{k!} t^k \tag{18}$$

and thus

$$1 + \sum_{k=1}^{\infty} \{-\alpha(1 + O(N^{-1}))\}^k \mathbb{E} \left[ \sum_{r_1 < \dots < r_k}^{\tau_N(t)} \prod_{j=1}^k \left\{ \frac{B_n}{\alpha} D_N(r_j) + c_N(r_j) \right\} \right] \rightarrow 1 + \sum_{k=1}^{\infty} (-\alpha)^k \frac{1}{k!} t^k = e^{-\alpha t} \tag{19}$$

as  $N \rightarrow \infty$ .

### Upper Bound

From Koskela et al. (2018, Lemma 1 Case 1), taking  $\xi = \Delta$ , we have

$$1 - p_t = p_{\Delta\Delta}(t) \leq 1 - \alpha(1 + O(N^{-1})) \left[ c_N(t) - \binom{n-1}{2} D_N(t) \right]. \tag{20}$$

A multinomial expansion as before yields

$$\prod_{r=1}^{\tau_N(t)} (1 - p_r) \leq 1 + \sum_{k=1}^{\infty} \{-\alpha(1 + O(N^{-1}))\}^k \sum_{\substack{r_1 < \dots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^k \left\{ c_N(r_j) - \binom{n-1}{2} D_N(r_j) \right\}. \quad (21)$$

Analogously to (13), we can write

$$\begin{aligned} \sum_{\substack{r_1 < \dots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^k \left\{ c_N(r_j) - \binom{n-1}{2} D_N(r_j) \right\} &= \frac{1}{k!} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^k c_N(r_i) \right\} \\ &\quad + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left( -\binom{n-1}{2} \right)^{k-I} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}} \left\{ \prod_{i=1}^I c_N(r_i) \right\} \left\{ \prod_{j=I+1}^k D_N(r_j) \right\}. \end{aligned} \quad (22)$$

We start by dealing with the second term:

$$\begin{aligned} \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} \left( -\binom{n-1}{2} \right)^{k-I} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}} \left\{ \prod_{i=1}^I c_N(r_i) \right\} \left\{ \prod_{j=I+1}^k D_N(r_j) \right\} \\ = \frac{1}{k!} \sum_{\substack{I=0: \\ k-I \text{ even}}}^{k-1} \binom{k}{I} \binom{n-1}{2}^{k-I} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}} \left\{ \prod_{i=1}^I c_N(r_i) \right\} \left\{ \prod_{j=I+1}^k D_N(r_j) \right\} \\ - \frac{1}{k!} \sum_{\substack{I=0: \\ k-I \text{ odd}}}^{k-1} \binom{k}{I} \binom{n-1}{2}^{k-I} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}} \left\{ \prod_{i=1}^I c_N(r_i) \right\} \left\{ \prod_{j=I+1}^k D_N(r_j) \right\}. \end{aligned} \quad (23)$$

This is lower bounded by

$$0 - \frac{1}{k!} \sum_{\substack{I=0 \\ (k-I) \text{ odd}}}^{k-1} \binom{k}{I} \binom{n-1}{2}^{k-I} (t+1)^{k-1} \left( \sum_{s=1}^{\tau_N(t)} D_N(s) \right) \quad (24)$$

using that  $c_N(r), D_N(r) \geq 0$  for all  $r$  to bound the even terms below, and arguments from (14) to bound the odd terms above. The same arguments lead to the upper bound

$$\frac{1}{k!} \sum_{\substack{I=0 \\ (k-I) \text{ odd}}}^{k-1} \binom{k}{I} \binom{n-1}{2}^{k-I} (t+1)^{k-1} \left( \sum_{s=1}^{\tau_N(t)} D_N(s) \right) - 0. \quad (25)$$

By Brown et al. (2020, Equation (4)), both bounds have vanishing expectation as  $N \rightarrow \infty$ . We are left with the first term in (18), which is upper bounded by

$$\frac{1}{k!} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^k c_N(r_i) \right\} \leq \frac{1}{k!} \left( \sum_{s=1}^{\tau_N(t)} c_N(s) \right)^k \leq \frac{1}{k!} \{t + c_N(\tau_N(t))\}^k \quad (26)$$

the expectation of which converges to  $t^k/k!$  as in (16). We use Koskela et al. (2018, Equation (8)) to construct a lower bound:

$$\begin{aligned} \frac{1}{k!} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^k c_N(r_i) \right\} &\geq \frac{1}{k!} \left( \sum_{s=1}^{\tau_N(t)} c_N(s) \right)^k - \frac{1}{k!} \binom{k}{2} \left( \sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \left( \sum_{s=1}^{\tau_N(t)} c_N(s) \right)^{k-2} \\ &\geq \frac{1}{k!} t^k - \frac{1}{k!} \binom{k}{2} (t+1)^{k-2} \left( \sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \end{aligned} \quad (27)$$

The expectation of this bound also converges to  $t^k/k!$ , using Brown et al. (2020, Equation (5)). We can now conclude that

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[ \sum_{r_1 < \dots < r_k}^{\tau_N(t)} \prod_{j=1}^k \left\{ c_N(r_j) - \binom{n-1}{2} D_N(r_j) \right\} \right] = \frac{1}{k!} t^k \quad (28)$$

and thus

$$1 + \sum_{k=1}^{\infty} \left\{ -\alpha(1 + O(N^{-1})) \right\}^k \mathbb{E} \left[ \sum_{r_1 < \dots < r_k}^{\tau_N(t)} \prod_{j=1}^k \left\{ c_N(r_j) - \binom{n-1}{2} D_N(r_j) \right\} \right] \rightarrow 1 + \sum_{k=1}^{\infty} (-\alpha)^k \frac{1}{k!} t^k = e^{-\alpha t} \quad (29)$$

as  $N \rightarrow \infty$ .

We now have upper and lower bounds on  $\lim_{N \rightarrow \infty} \mathbb{E} \left[ \prod_{r=1}^{\tau_N(t)} (1 - p_r) \right]$ , both of which are equal to  $e^{-\alpha t}$ , and the result follows.  $\square$

**Lemma 3.**  $D_N(t) \leq c_N(t)$  for all  $t$ .

*Proof.*

$$\begin{aligned} D_N(t) &:= \frac{1}{(N)_2} \sum_{i=1}^N (\nu_t^{(i)})_2 \frac{1}{N} \left\{ \nu_t^{(i)} + \frac{1}{N} \sum_{j \neq i}^N (\nu_t^{(j)})^2 \right\} \leq \frac{1}{(N)_2} \sum_{i=1}^N (\nu_t^{(i)})_2 \frac{1}{N} \left\{ \nu_t^{(i)} + \frac{1}{N} \sum_{j \neq i}^N N \nu_t^{(j)} \right\} \\ &= \frac{1}{(N)_2} \sum_{i=1}^N (\nu_t^{(i)})_2 \frac{1}{N} \left\{ \sum_{j=1}^N \nu_t^{(j)} \right\} \leq \frac{1}{(N)_2} \sum_{i=1}^N (\nu_t^{(i)})_2 = c_N(t). \end{aligned} \quad (30)$$

$\square$

## References

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