Residual resampling in SMC

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May 27, 2021

THIS DOCUMENT IS OBSOLETE. These calculations are reproduced (with corrections) in `resmn_roundup_210526`.

The offspring counts are sampled according to:

$$v_t^{(i)} = \lfloor Nw_t^{(i)} \rfloor + X_i$$
$$X_i \sim \text{Multinomial}(N-k, (\bar{w}_t^{(1)}, \dots, \bar{w}_t^{(N)}))$$

where $k := \sum_{i=1}^N \lfloor Nw_t^{(i)} \rfloor$ is the number of offspring assigned deterministically, and $\bar{w}_t^{(i)} := \frac{Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor}{N-k}$ are the residual weights. Let us also define the residuals $r_i := Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor$. So $\sum_{i=1}^N r_i = N-k$.

The coalescence rate is defined as

$$c_N(t) := \frac{1}{(N)_2} \sum_{i=1}^N (v_t^{(i)})_2.$$

We will use $c_N^m(t)$ and $c_N^r(t)$ to denote the coalescence rates with multinomial and residual resampling respectively. The expectation then comes out as

$$\begin{aligned}
\mathbb{E}[(v_t^{(i)})_2 | \mathcal{F}_{t-1}] &= \mathbb{E}[(v_t^{(i)})^2 | \mathcal{F}_{t-1}] - \mathbb{E}[v_t^{(i)} | \mathcal{F}_{t-1}] \\
&= \mathbb{E}[\lfloor Nw_t^{(i)} \rfloor^2 | \mathcal{F}_{t-1}] + 2\mathbb{E}[\lfloor Nw_t^{(i)} \rfloor r_i | \mathcal{F}_{t-1}] + \mathbb{E}\left[r_i \left(1 - \frac{r_i}{N-k} + r_i\right) | \mathcal{F}_{t-1} \right] - \mathbb{E}[Nw_t^{(i)} | \mathcal{F}_{t-1}] \\
&= \mathbb{E}[\lfloor Nw_t^{(i)} \rfloor^2 | \mathcal{F}_{t-1}] - \mathbb{E}[\lfloor Nw_t^{(i)} \rfloor | \mathcal{F}_{t-1}] + 2\mathbb{E}[\lfloor Nw_t^{(i)} \rfloor r_i | \mathcal{F}_{t-1}] + \mathbb{E}\left[r_i^2 \left(1 - \frac{1}{N-k}\right) | \mathcal{F}_{t-1} \right] \\
&= \mathbb{E}[(Nw_t^{(i)})^2 | \mathcal{F}_{t-1}] - \mathbb{E}[\lfloor Nw_t^{(i)} \rfloor | \mathcal{F}_{t-1}] - \mathbb{E}\left[\frac{r_i^2}{N-k} | \mathcal{F}_{t-1} \right]
\end{aligned}$$

so we get

$$\begin{aligned}
\mathbb{E}[c_N^r(t) | \mathcal{F}_{t-1}] &= \frac{1}{(N)_2} \sum_{i=1}^N \mathbb{E}[(v_t^{(i)})_2 | \mathcal{F}_{t-1}] \\
&= \frac{N}{N-1} \sum_{i=1}^N \mathbb{E}[(w_t^{(i)})^2 | \mathcal{F}_{t-1}] - \frac{1}{(N)_2} \sum_{i=1}^N \mathbb{E}\left[\frac{r_i^2}{N-k} | \mathcal{F}_{t-1} \right] - \frac{1}{(N)_2} \mathbb{E}[k | \mathcal{F}_{t-1}] \\
&= \mathbb{E}[c_N^m(t) | \mathcal{F}_{t-1}] \left(1 + \frac{1}{N-1}\right) - \frac{1}{(N)_2} \mathbb{E}\left[\frac{\sum_{i=1}^N (Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor)^2}{\sum_{j=1}^N (Nw_t^{(j)} - \lfloor Nw_t^{(j)} \rfloor)} \middle| \mathcal{F}_{t-1} \right] \\
&\quad - \frac{1}{(N)_2} \mathbb{E}\left[\sum_{i=1}^N \lfloor Nw_t^{(i)} \rfloor \middle| \mathcal{F}_{t-1} \right] \qquad (1)
\end{aligned}$$

Sanity check:

When the weights are all equal, $w_t^{(i)} \equiv 1/N$, we should have $\mathbb{E}[c_N^r(t)|\mathcal{F}_{t-1}] = 0$ since each particle will have exactly one offspring so it is impossible for any lineages to coalesce. In this case we have $\mathbb{E}[c_N^m(t)|\mathcal{F}_{t-1}] = \sum_{i=1}^N \mathbb{E}[(w_t^{(i)})^2|\mathcal{F}_{t-1}] = 1/N$ for multinomial resampling. We also have that $Nw_t^{(i)} \equiv \lfloor Nw_t^{(i)} \rfloor \equiv 1$ and hence $r_i = 0$ and $k = N$. Thus the RHS comes out as

$$\frac{1}{N}\frac{N}{N-1} - 0 - \frac{1}{(N)_2} N = \frac{1}{N-1} - \frac{1}{N-1} = 0$$

as expected.

We can write it in a different form by combining the second and third terms of (1):

$$
\begin{aligned}
&- \frac{1}{(N)_2}\mathbb{E}\left[\frac{\sum_{i=1}^N (Nw_t^{(i)} - \lfloor Nw_t^{(i)}\rfloor)^2}{\sum_{j=1}^N (Nw_t^{(j)} - \lfloor Nw_t^{(j)}\rfloor)}|\mathcal{F}_{t-1}\right] - \frac{1}{(N)_2}\mathbb{E}\left[\sum_{k=1}^N \lfloor Nw_t^{(k)}\rfloor \middle| \mathcal{F}_{t-1}\right] \\
&=: -\frac{1}{(N)_2}\mathbb{E}[A|\mathcal{F}_{t-1}]
\end{aligned}
$$

Then

$$
\begin{aligned}
A &= \frac{\sum_{i=1}^N (Nw_t^{(i)} - \lfloor Nw_t^{(i)}\rfloor)^2}{\sum_{j=1}^N (Nw_t^{(j)} - \lfloor Nw_t^{(j)}\rfloor)} + \sum_{k=1}^N \lfloor Nw_t^{(k)}\rfloor \\
&= \frac{\sum_{i=1}^N (Nw_t^{(i)} - \lfloor Nw_t^{(i)}\rfloor)^2 + \sum_{i=1}^N \sum_{k=1}^N \lfloor Nw_t^{(k)}\rfloor (Nw_t^{(i)} - \lfloor Nw_t^{(i)}\rfloor)}{\sum_{j=1}^N (Nw_t^{(j)} - \lfloor Nw_t^{(j)}\rfloor)} \\
&=: \frac{A'}{\sum_{j=1}^N (Nw_t^{(j)} - \lfloor Nw_t^{(j)}\rfloor)}
\end{aligned}
$$

Then

$$
\begin{aligned}
A' &= \sum_{i=1}^N (Nw_t^{(i)} - \lfloor Nw_t^{(i)}\rfloor)^2 + \sum_{i=1}^N \sum_{k=1}^N \lfloor Nw_t^{(k)}\rfloor (Nw_t^{(i)} - \lfloor Nw_t^{(i)}\rfloor) \\
&= \sum_{i=1}^N \left\{ \left(Nw_t^{(i)} - \lfloor Nw_t^{(i)}\rfloor\right)^2 + \lfloor Nw_t^{(i)}\rfloor \left(Nw_t^{(i)} - \lfloor Nw_t^{(i)}\rfloor\right) + \sum_{k\neq i} \lfloor Nw_t^{(k)}\rfloor \left(Nw_t^{(i)} - \lfloor Nw_t^{(i)}\rfloor\right) \right\} \\
&= \sum_{i=1}^N \left\{ (Nw_t^{(i)})^2 - Nw_t^{(i)}\lfloor Nw_t^{(i)}\rfloor + \sum_{k\neq i} \lfloor Nw_t^{(k)}\rfloor \left(Nw_t^{(i)} - \lfloor Nw_t^{(i)}\rfloor\right) \right\} \\
&= \sum_{i=1}^N \left\{ \left(Nw_t^{(i)} - \lfloor Nw_t^{(i)}\rfloor\right) \left(Nw_t^{(i)} + \sum_{k\neq i} \lfloor Nw_t^{(k)}\rfloor \right) \right\}
\end{aligned}
$$

So we have

$$
\begin{aligned}
\mathbb{E}[c_N^r(t)|\mathcal{F}_{t-1}] = \mathbb{E}[c_N^m(t)|\mathcal{F}_{t-1}] &\left(1 + \frac{1}{N-1}\right) \\
&- \frac{1}{(N)_2}\mathbb{E}\left[\frac{\sum_{i=1}^N \left(Nw_t^{(i)} - \lfloor Nw_t^{(i)}\rfloor\right)\left(Nw_t^{(i)} + \sum_{k\neq i}\lfloor Nw_t^{(k)}\rfloor\right)}{\sum_{j=1}^N \left(Nw_t^{(j)} - \lfloor Nw_t^{(j)}\rfloor\right)} \middle| \mathcal{F}_{t-1}\right]
\end{aligned}
$$