

Lemma 1.

$$1 - C_{|\xi|} \{1 + O(N^{-1})\} D_N(t) - \binom{|\xi|}{2} \{1 + O(N^{-1})\} c_N(t) \leq p_{\xi\xi}(t),$$

for a constant $C_{|\xi|} > 0$ that does not depend on N .

Proof. Let $\kappa_i := \#\{j : b_j = i\}$ denote the multiplicity of mergers of size i , with the slight abuse of terminology in that κ_1 counts non-merger events. In particular, we have that $\kappa_1 + 2\kappa_2 + \dots + |\xi|\kappa_{|\xi|} = |\xi|$. Now

$$p_{\xi\xi}(t) = 1 - \frac{1}{(N)^{|\xi|}} \sum_{k=1}^{|\xi|-1} \sum_{\substack{b_1 \geq \dots \geq b_k = 1 \\ b_1 + \dots + b_k = |\xi|}} \frac{|\xi|!}{\prod_{j=1}^{|\xi|} (j!)^{\kappa_j} \kappa_j!} \sum_{\substack{i_1 \neq \dots \neq i_k = 1 \\ \text{all distinct}}}^N (\nu_t^{(i_1)})_{b_1} \dots (\nu_t^{(i_k)})_{b_k},$$

because the right hand side subtracts the probabilities of all possible merger events. See (?, eq (11)) for the combinatorial factor. The omitted $k = |\xi|$ summand would correspond to the probability of an identity transition. The non-increasing ordering of (b_1, \dots, b_k) is arbitrary, but without loss of generality: choosing any ordering of the same set of merger sizes would give the same result.

Firstly, we separate out the $k = |\xi| - 1$ term, which covers isolated binary mergers, and note that in that case the only possible b -vector is $(2, 1, \dots, 1)$, for which

$$\frac{|\xi|!}{\prod_{j=1}^{|\xi|} (j!)^{\kappa_j} \kappa_j!} = \frac{|\xi|!}{2!(|\xi| - 2)!} = \binom{|\xi|}{2},$$

and

$$\begin{aligned} & \sum_{i_1 \neq \dots \neq i_{|\xi|-1} = 1}^N (\nu_t^{(i_1)})_2 \nu_t^{(i_2)} \dots \nu_t^{(i_{|\xi|-1})} \\ & \leq \sum_{i=1}^N (\nu_t^{(i)})_2 \left[(N - \nu_t^{(i)})^{|\xi|-2} - \binom{|\xi|-2}{2} \sum_{j \neq i}^N (\nu_t^{(j)})^2 (N - \nu_t^{(i)})^{|\xi|-4} \right] \\ & \leq N^{|\xi|-2} \sum_{i=1}^N (\nu_t^{(i)})_2, \end{aligned}$$

Thus

$$\begin{aligned} p_{\xi\xi}(t) & \geq 1 - \binom{|\xi|}{2} \frac{1 + O(N^{-1})}{(N)_2} \sum_{i=1}^N (\nu_t^{(i)})_2 \\ & \quad - \frac{1}{(N)^{|\xi|}} \sum_{k=1}^{|\xi|-2} \sum_{\substack{b_1 \geq \dots \geq b_k = 1 \\ b_1 + \dots + b_k = |\xi|}} \frac{|\xi|!}{\prod_{j=1}^{|\xi|} (j!)^{\kappa_j} \kappa_j!} \sum_{\substack{i_1 \neq \dots \neq i_k = 1 \\ \text{all distinct}}}^N (\nu_t^{(i_1)})_{b_1} \dots (\nu_t^{(i_k)})_{b_k}. \end{aligned}$$

For the other summands, we have

$$\frac{|\xi|!}{\prod_{j=1}^{|\xi|} (j!)^{\kappa_j} \kappa_j!} \leq |\xi|!$$

and (similarly to Lemma 1, Case 3 in our paper),

$$\begin{aligned}
\sum_{\substack{i_1 \neq \dots \neq i_k=1 \\ \text{all distinct}}}^N (\nu_t^{(i_1)})_{b_1} \dots (\nu_t^{(i_k)})_{b_k} &\leq \sum_{i=1}^N (\nu_t^{(i)})_2 \left\{ N^{|\xi|-2} - \sum_{\substack{j_1 \neq \dots \neq j_{|\xi|-2}=1 \\ \text{all distinct and } \neq i}}^N \nu_t^{(j_1)} \dots \nu_t^{(j_{|\xi|-2})} \right\} \\
&= \sum_{i=1}^N (\nu_t^{(i)})_2 \left\{ N^{|\xi|-2} - (N - \nu_t^{(i)})^{|\xi|-2} + \binom{|\xi|-2}{2} \sum_{j \neq i} (\nu_t^{(j)})^2 \left(\sum_{k \neq i} \nu_t^{(k)} \right)^{|\xi|-4} \right\} \\
&\leq \sum_{i=1}^N (\nu_t^{(i)})_2 \left\{ (|\xi|-2) \nu_t^{(i)} N^{|\xi|-3} + \binom{|\xi|-2}{2} \sum_{j \neq i} (\nu_t^{(j)})^2 N^{|\xi|-4} \right\},
\end{aligned}$$

where the last step uses $(N - x)^b \geq N^b - bxN^{b-1}$. Overall

$$\begin{aligned}
p_{\xi\xi}(t) &\geq 1 - \binom{|\xi|}{2} \frac{1 + O(N^{-1})}{(N)_2} \sum_{i=1}^N (\nu_t^{(i)})_2 \\
&\quad - \frac{N^{|\xi|-3}}{(N)^{|\xi|}} \sum_{k=1}^{|\xi|-2} \sum_{\substack{b_1 \geq \dots \geq b_k=1 \\ b_1 + \dots + b_k = |\xi|}} |\xi|! \sum_{i=1}^N (\nu_t^{(i)})_2 \left\{ (|\xi|-2) \nu_t^{(i)} N^{|\xi|-3} + \binom{|\xi|-2}{2} \sum_{j \neq i} (\nu_t^{(j)})^2 N^{|\xi|-4} \right\}.
\end{aligned}$$

The summand in the third term depends neither on k nor on b_1, \dots, b_k , and the number of terms in those sums is bounded above by $(|\xi|-2)\gamma_{|\xi|-2}$, where γ_n is the number of integer partitions of n . By (? , Section 2), $\gamma_n < Ke^{2\sqrt{2n}}/n$ for a constant $K > 0$ independent of n . Thus

$$\begin{aligned}
p_{\xi\xi}(t) &\geq 1 - \binom{|\xi|}{2} \frac{1 + O(N^{-1})}{(N)_2} \sum_{i=1}^N (\nu_t^{(i)})_2 \\
&\quad - Ke^{2\sqrt{2(|\xi|-2)}} |\xi|! \binom{|\xi|-2}{2} \frac{N^{|\xi|-3}}{(N)^{|\xi|}} \sum_{i=1}^N (\nu_t^{(i)})_2 \left\{ \nu_t^{(i)} + \frac{1}{N} \sum_{j \neq i} (\nu_t^{(j)})^2 \right\} \\
&= 1 - \binom{|\xi|}{2} \{1 + O(N^{-1})\} c_N(t) - C_{|\xi|} \{1 + O(N^{-1})\} D_N(t),
\end{aligned}$$

where $C_{|\xi|} > 0$ depends on $|\xi|$ but not on N . □

In order to use Lemma 1 to remove assumption (6) from (? , Theorem 1), it is necessary to rewrite the argument for the lower bound. The upper bound does not use assumption (6). We do this below.

Proof of Theorem 1 without Assumption (6).

$$\begin{aligned}
\chi_d &\geq \sum_{s_1 < \dots < s_\alpha = \tau_N(t_{d-1})+1}^{\tau_N(t_d)} (\tilde{Q}^\alpha)_{\eta_{d-1}\eta_d} \left\{ \prod_{r=1}^\alpha \left(c_N(s_r) - \binom{n-2}{2} \{1 + O(N^{-1})\} D_N(s_r) \right) \right\} \\
&\quad \times \prod_{\substack{r=\tau_N(t_{d-1})+1 \\ r \neq s_1, \dots, r \neq s_\alpha}}^{\tau_N(t_d)} \left\{ 1 - C_n \{1 + O(N^{-1})\} D_N(r) \right. \\
&\quad \left. - \binom{|\eta_{d-1}| - |\{i : s_i < r\}|}{2} \{1 + O(N^{-1})\} c_N(r) \right\}.
\end{aligned}$$

A multinomial expansion of the product on the last line yields

$$\begin{aligned} \chi_d \geq & \sum_{\beta=0}^{\tau_N(t_d)-\tau_N(t_{d-1})-\alpha} (\tilde{Q}^\alpha)_{\eta_{d-1}\eta_d} \sum_{(\lambda,\mu) \in \Pi_2([\alpha+\beta]):|\lambda|=\alpha} \{1 + O(N^{-1})\}^\beta \\ & \times \sum_{s_1 < \dots < s_{\alpha+\beta} = \tau_N(t_{d-1})+1}^{\tau_N(t_d)} \left\{ \prod_{r \in \lambda} \left[c_N(s_r) - \binom{n-2}{2} \{1 + O(N^{-1})\} D_N(s_r) \right] \right\} \\ & \times \prod_{r \in \mu} \left\{ - \binom{|\eta_{d-1}| - |\{i \in \lambda : i < r\}|}{2} c_N(s_r) - C_n D_N(s_r) \right\}. \end{aligned}$$

Expanding the product over λ gives

$$\begin{aligned} \chi_d \geq & \sum_{\beta=0}^{\tau_N(t_d)-\tau_N(t_{d-1})-\alpha} (\tilde{Q}^\alpha)_{\eta_{d-1}\eta_d} \sum_{(\lambda,\mu,\pi) \in \Pi_3([\alpha+\beta]):|\mu|=\beta} \binom{n-2}{2}^{|\pi|} (-1)^{|\pi|} \{1 + O(N^{-1})\}^{\beta+|\pi|} \\ & \times \sum_{s_1 < \dots < s_{\alpha+\beta} = \tau_N(t_{d-1})+1}^{\tau_N(t_d)} \left\{ \prod_{r \in \lambda} c_N(s_r) \right\} \left\{ \prod_{r \in \pi} D_N(s_r) \right\} \\ & \times \prod_{r \in \mu} \left\{ - \binom{|\eta_{d-1}| - |\{i \in \lambda \cup \pi : i < r\}|}{2} c_N(s_r) - C_n D_N(s_r) \right\}, \end{aligned}$$

and expanding the product over μ results in

$$\begin{aligned} \chi_d \geq & \sum_{\beta=0}^{\tau_N(t_d)-\tau_N(t_{d-1})-\alpha} (\tilde{Q}^\alpha)_{\eta_{d-1}\eta_d} \sum_{(\lambda,\mu,\pi,\sigma) \in \Pi_4([\alpha+\beta]):|\mu|+|\sigma|=\beta} C_n^{|\sigma|} \binom{n-2}{2}^{|\pi|} (-1)^{|\pi|+|\sigma|} \\ & \times \{1 + O(N^{-1})\}^{\beta+|\pi|} \left\{ \prod_{r \in \mu} - \binom{|\eta_{d-1}| - |\{i \in \lambda \cup \pi : i < r\}|}{2} \right\} \\ & \times \sum_{s_1 < \dots < s_{\alpha+\beta} = \tau_N(t_{d-1})+1}^{\tau_N(t_d)} \left\{ \prod_{r \in \lambda \cup \mu} c_N(s_r) \right\} \prod_{r \in \pi \cup \sigma} D_N(s_r). \end{aligned}$$

Via a further multinomial expansion, the lower bound for the k -step transition probability can be written as

$$\begin{aligned} \lim_{N \rightarrow \infty} \mathbb{E} \left[\prod_{d=1}^k \chi_d \right] & \geq \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\beta_1=0}^{\infty} \dots \sum_{\beta_k=0}^{\infty} \sum_{(\lambda_1,\mu_1,\pi_1,\sigma_1) \in \Pi_4([\alpha_1+\beta_1]):|\mu_1|+|\sigma_1|=\beta_1} \dots \right. \\ & \sum_{(\lambda_k,\mu_k,\pi_k,\sigma_k) \in \Pi_4([\alpha_k+\beta_k]):|\mu_k|+|\sigma_k|=\beta_k} C_n^{\sum_{d=1}^k |\sigma_d|} \binom{n-2}{2}^{\sum_{d=1}^k |\pi_d|} \\ & \times (-1)^{\sum_{d=1}^k |\pi_d|+|\sigma_d|} \{1 + O(N^{-1})\}^{|\beta|+\sum_{d=1}^k |\pi_d|} \\ & \times \left\{ \prod_{d=1}^k (\tilde{Q}^{\alpha_d})_{\eta_{d-1}\eta_d} \prod_{r \in \mu_d} - \binom{|\eta_{d-1}| - |\{i \in \lambda_d \cup \pi_d : i < r\}|}{2} \right\} \\ & \times \sum_{s_1^{(1)} < \dots < s_{\alpha_1+\beta_1}^{(1)} = \tau_N(t_0)+1}^{\tau_N(t_1)} \dots \sum_{s_1^{(k)} < \dots < s_{\alpha_k+\beta_k}^{(k)} = \tau_N(t_{k-1})+1}^{\tau_N(t_k)} \\ & \left. \prod_{d=1}^k \mathbb{1}_{\{\tau_N(t_d)-\tau_N(t_{d-1}) \geq \alpha_d+\beta_d\}} \left\{ \prod_{r \in \lambda_d \cup \mu_d} c_N(s_r^{(d)}) \right\} \prod_{r \in \pi_d \cup \sigma_d} D_N(s_r^{(d)}) \right]. \end{aligned}$$

An argument completely analogous to that in (?, Appendix) shows that passing the expectation and the limit through the infinite sums is justified, whereupon the contribution of terms with $\sum_{d=1}^k |\pi_d| + |\sigma_d| > 0$ vanishes. To see why, follow the argument used to show that the contribution of multiple merger trajectories vanishes in the corresponding upper bound in ?. That leaves

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathbb{E} \left[\prod_{d=1}^k \chi_d \right] &\geq \sum_{\beta_1=0}^{\infty} \cdots \sum_{\beta_k=0}^{\infty} \sum_{(\lambda_1, \mu_1) \in \Pi_2([\alpha_1 + \beta_1]): |\mu_1| = \beta_1} \cdots \sum_{(\lambda_k, \mu_k) \in \Pi_2([\alpha_k + \beta_k]): |\mu_k| = \beta_k} \\
&\quad \left\{ \prod_{d=1}^k (\tilde{Q}^{\alpha_d})_{\eta_{d-1}\eta_d} \prod_{r \in \mu_d} - \binom{|\eta_{d-1}| - |\{i \in \lambda_d \cup \pi_d : i < r\}|}{2} \right\} \\
&\quad \times \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{s_1^{(1)} < \dots < s_{\alpha_1 + \beta_1}^{(1)} = \tau_N(t_0) + 1}^{\tau_N(t_1)} \cdots \sum_{s_1^{(k)} < \dots < s_{\alpha_k + \beta_k}^{(k)} = \tau_N(t_{k-1}) + 1}^{\tau_N(t_k)} \right. \\
&\quad \left. \prod_{d=1}^k \mathbb{1}_{\{\tau_N(t_d) - \tau_N(t_{d-1}) \geq \alpha_d + \beta_d\}} \left\{ \prod_{r \in \lambda_d \cup \mu_d} c_N(s_r^{(d)}) \right\} \right]. \tag{1}
\end{aligned}$$

Recall (?, Eq (11)):

$$\sum_{(\lambda, \mu) \in \Pi_2([\alpha + \beta]): |\mu| = \beta} (\tilde{Q}^{\alpha})_{\eta_{d-1}\eta_d} \prod_{r \in \mu} - \binom{|\eta_{d-1}| - |\{i \in \lambda \cup \pi : i < r\}|}{2} = (Q^{\alpha + \beta})_{\eta_{d-1}\eta_d},$$

and note that applying this k times in (1) yields

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathbb{E} \left[\prod_{d=1}^k \chi_d \right] &\geq \sum_{\beta_1=0}^{\infty} \cdots \sum_{\beta_k=0}^{\infty} \left\{ \prod_{d=1}^k (Q^{\alpha_d + \beta_d})_{\eta_{d-1}\eta_d} \right\} \\
&\quad \times \lim_{N \rightarrow \infty} \mathbb{E} \left[\left\{ \prod_{d=1}^k \mathbb{1}_{\{\tau_N(t_d) - \tau_N(t_{d-1}) \geq \alpha_d + \beta_d\}} \right\} \sum_{s_1^{(1)} < \dots < s_{\alpha_1 + \beta_1}^{(1)} = \tau_N(t_0) + 1}^{\tau_N(t_1)} \right. \\
&\quad \left. \cdots \sum_{s_1^{(k)} < \dots < s_{\alpha_k + \beta_k}^{(k)} = \tau_N(t_{k-1}) + 1}^{\tau_N(t_k)} \prod_{d=1}^k \prod_{r \in \lambda_d \cup \mu_d} c_N(s_r^{(d)}) \right].
\end{aligned}$$

We now apply (?, Eq (14)) to those terms with a negative sign ($|\beta|$ odd) and (?, Eq (15)) to those terms with a positive sign ($|\beta|$ even), and obtain

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathbb{E} \left[\prod_{d=1}^k \chi_d \right] &\geq \sum_{\beta_1=0}^{\infty} \cdots \sum_{\beta_k=0}^{\infty} \left\{ \prod_{d=1}^k (Q^{\alpha_d + \beta_d})_{\eta_{d-1}\eta_d} \frac{(t_d - t_{d-1})^{\alpha_d + \beta_d}}{(\alpha_d + \beta_d)!} \right\} \\
&\quad \times \lim_{N \rightarrow \infty} \mathbb{E} \left[\prod_{d=1}^k \mathbb{1}_{\{\tau_N(t_d) - \tau_N(t_{d-1}) \geq \alpha_d + \beta_d\}} \right].
\end{aligned}$$

An invocation of (?, Eq (16)) concludes the proof. \square