Stratified resampling: towards a finite time-scale

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The aim of this note is to calculate (bounds on) the probabilities of the four different outcomes that are possible for the marginal offspring count $\nu_t^{(i)}$ conditional on $w_t^{(i)}$, under stratified resampling. Ultimately, these bounds will be used to prove that the time-scale is finite (i.e. $\mathbb{P}[\tau_N(t) = \infty] = 0$ for all finite t) under stratified resampling, assuming the additional constraints:

- that the weights are bounded away from the degenerate case $(1, \dots, 1)/N$ in some way
- that the transition densities $q_t(x, x')$ are uniformly bounded above and away from zero.

Once the probabilities are calculated (in the following), a proof similar to the one used to prove the finite time-scale condition for stochastic rounding will prove the same for stratified resampling. The corresponding probabilities and hence finite time-scale proof for residual-stratified resampling should follow relatively easily by applying the bounds here within the residual resampling set-up.

Let's do it!

Consider the marginal distribution of one offspring count $\nu_t^{(i)}$ conditional on the corresponding weight $w_t^{(i)}$. Henceforth we drop from the notation the dependence on t and i, which are to be considered fixed throughout the following. As we have already seen, the possible values of ν are restricted conditional on w to $\{\lfloor Nw \rfloor - 1, \lfloor Nw \rfloor, \lfloor Nw \rfloor + 1, \lfloor Nw \rfloor + 2\}$. Denote by p_i the conditional probability $\mathbb{P}[\nu = \lfloor Nw \rfloor + i \mid w]$, for $i = -1, 0, 1, 2$.

We consider first a specific case (of particular interest for the finite time-scale proof) where $w = (1+\delta)/N$. First let's look at $p_0 = \mathbb{P}[\nu = 1 \mid w = (1+\delta)/N]$. Thinking about the inversion sampling schematic, the resampling probabilities will depend upon where the length-w interval falls with respect to the length-$(1/N)$ intervals for sampling. We split the possibilities into three cases (Figure 1).

Case 1

In this case, one offspring is assigned almost surely from the interval that is entirely overlapping. Thus p_0 is just the probability that the partially overlapping interval does not contribute a second offspring to ν. Hence,

$$p_0 = \left(\frac{1}{N} - \frac{\delta}{N} \right) \div \frac{1}{N} = 1 - \delta. \tag{1}$$

Case 2

In this case, one offspring is assigned almost surely from the interval that is entirely overlapping. Thus p_0 is just the probability that neither of the partially overlapping intervals contributes a second offspring to ν. The lengths are such that $\delta_L + \delta_R = \delta$. We have

$$p_0 = (1 - \delta_L)(1 - \delta_R) = 1 - \delta + \delta_L \delta_R. \tag{2}$$

Noting that $\delta_L \delta_R \leq \delta^2/4 \leq \delta/4$, we conclude that

$$p_0 \in \left[1 - \delta, 1 - \frac{3\delta}{4} \right]. \tag{3}$$

When $\delta_L \in \{0, \delta\}$, this case collapses to Case 1, which is consistent with the bounds derived here.

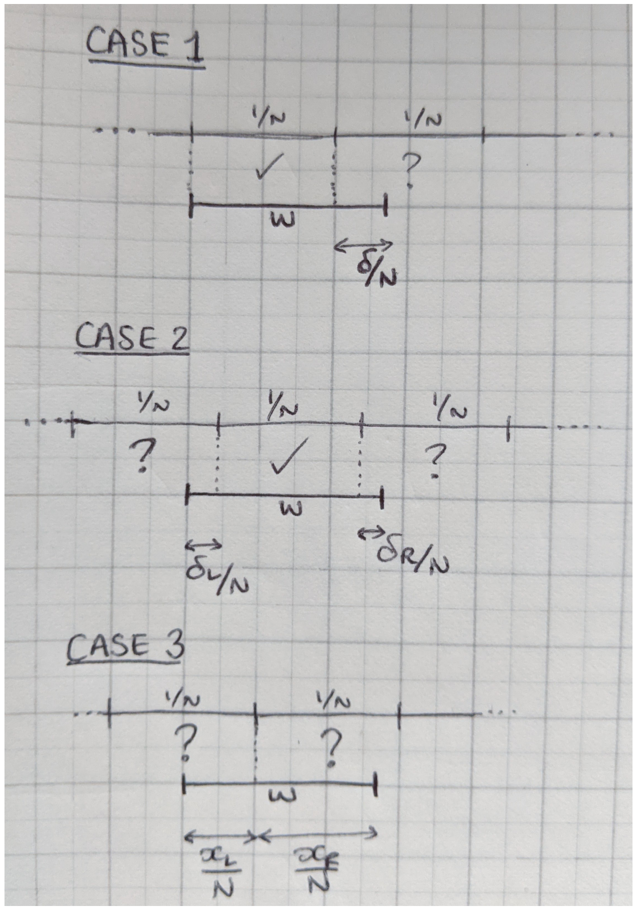

Figure 1: Sketch illustrating the difference between Cases 1–3.

Case 3

Here p_0 is the probability that exactly one of the partially overlapping intervals contributes an offspring to ν. The lengths are such that $x_L + x_R = 1 + \delta$, and also $x_L, x_R \in [\delta, 1]$ (otherwise we would be in Case 2). We have

$$p_0 = x_L(1 - x_R) + x_R(1 - x_L) = 1 + \delta - 2x_L x_R. \tag{4}$$

Notice that $\delta \leq x_L x_R \leq (1 + \delta)^2/4$, hence

$$p_0 \in \left[\frac{1 - \delta^2}{2}, 1 - \delta\right]. \tag{5}$$

When $\delta_L \in \{\delta, 1\}$, this case collapses to Case 1, which is consistent with the bounds derived here.

Altogether

Overall, then, we have the bounds

$$p_0 \in \left[\frac{1 - \delta}{2}, 1 - \frac{3\delta}{4}\right]. \tag{6}$$