Resampling and genealogies in sequential Monte Carlo algorithms

Susanna Elizabeth Brown

A thesis submitted for the degree of
Doctor of Philosophy in Statistics

University of Warwick, Department of Statistics

January 2021

Contents

List of Figures

List of Tables

Acknowledgements

I would like to thank...

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree.

The work presented (including data generated and data analysis) was carried out by the author except in the cases outlined below:

Parts of this thesis have been published by the author:

Abstract

List of Abbreviations

SMC	sequential Monte Carlo
i.i.d.	independent and identically distributed
MRCA	most recent common ancestor
PRNG	pseudo-random number generator
CDF	cumulative distribution function

Notation and Conventions

$\mathbb{N}$	the natural numbers starting from one, $\{1, 2, \dots\}$
$\mathbb{N}_0$	the natural numbers starting from zero, $\{0, 1, 2, \dots\}$
$[a]$	the set $\{1, 2, \dots, a\}$ where $a \in \mathbb{N}$ also allow $a = 0$ in which case $[a] = \emptyset$?
$\mathcal{S}_k$	the k-dimensional unit simplex $\{x_{1:k+1} \geq 0 : \sum_{i=1}^{k+1} x_i = 1\}$
$(a)_b$	the falling factorial $a(a-1)\cdots(a-b+1)$ where $a \in \mathbb{N}_0, b \in \mathbb{N}$ could even allow $a \in \mathbb{R}$ but I don't think I ever use it in that setting
$\binom{a}{b}$	binomial coefficient where $a, b \in \mathbb{N}_0$, defined to be 0 when $a < b$
$\prod_\emptyset$	the empty product is taken to be 1
$\sum_\emptyset$	the empty sum is taken to be 0, while the sum over an index vector of length zero is the identity operator ?
$\mathcal{F}_t$	the (backward) filtration generated by offspring counts up to time t
$\mathbb{E}$	expectation
$\mathbb{E}_t$	filtered expectation $\mathbb{E}[\cdot \mid \mathcal{F}_{t-1}]$
A^c	denotes the complement of set A
1_N	asymptotic notation for a function that converges to 1 as $N \to \infty$

1 Introduction

2 Background

> Anyone who considers arithmetical methods of producing random digits is, of course, in a state of sin.
>
> JOHN VON NEUMANN

2.1 Sequential Monte Carlo

2.1.1 Motivation

Being Bayesian. SSMs/HMMs. Example(s) of SSM (1D train?).

2.1.2 Inference in SSMs

What quantities do we want to infer? Why is this generally difficult? Filtering, prediction, smoothing, likelihood/normalising constant.

2.1.3 Exact solutions

If the state space model has linear dynamics with Gaussian errors, the posterior distributions of interest are also Gaussian with mean and covariance satisying recursions, implemented by the Kalman filter (Kalman 1960) and Rauch-Tung-Striebel smoother (Rauch, Striebel, and Tung 1965). Recursions are also available for some other conjugate models: see for example Vidoni (1999). Another analytic case occurs if the state space $\mathcal{X}$ is finite, in which case any integrals become finite sums, and the forward-backward algorithm (Baum et al. 1970) yields the exact posteriors. However, if the state space becomes large (albeit finite), exact computation becomes infeasible.

If the model is Gaussian but non-linear, the posterior filtering distributions can be estimated using the *extended Kalman filter* (see for example Jazwinski (2007)), which applies a first-order approximation in order to make use of the Kalman filter. This method performs well on models that are "almost linear". The resulting predictor is only *optimal* when the model is actually linear, in which case the extended Kalman filter coincides with the Kalman filter.

For models that are high-dimensional or highly non-linear or for which gradients are not readily available, the exact Kalman filter updates can be replaced by sample approximations. The *ensemble Kalman filter* (Evensen 1994) uses a Monte Carlo sample from the current time, propagates these points through the transition dynamics, and uses the sample covariance as an estimator of the updated covariance matrix. The means (which are cheaper to evaluate and more stable than the covariances) are still updated using the Kalman filter recursion, based on the estimated covariance. The *unscented Kalman filter* (Wan and Merwe 2000) uses a deterministic sample chosen via the *unscented transformation*, which is then propagated through the non-linear transition to obtain a characterisation of the distribution at the next time step. The sample consists of $2d + 1$ points, where d is the dimension of the state space, and is a sufficient characterisation of a Gaussian distribution. The sample points define a Gaussian approximation to the updated distribution.

In complex or high-dimensional models, such techniques may not be feasible, in which case we must resort to Monte Carlo methods. Markov chain Monte Carlo performs woefully on state space models due to the high dimension of the parameter space and high correlation between dimensions. But we can exploit the sequential nature of the underlying dynamics to decompose the problem into a sequence of inferences of fixed dimension. This is the motivation behind sequential Monte Carlo (SMC).

2.1.4 Feynman-Kac models

Define a generic FK model. Show that this class includes all SSMs. Example of non-SSM that is FK?

2.1.5 Sequential Monte Carlo for Feynman-Kac models

Present generic algorithm. State the SMC estimators of the quantities of interest. Include the dependence diagram and note that the offspring counts are not independent at each time, but can be made so by conditioning on the separatrix $\mathcal{H}$.

Data: $N, T, \mu, (K_t)_{t=1}^T, (g_t)_{t=0}^T$
for $i \in \{1, \dots, N\}$ **do** Sample $X_0^{(i)} \sim \mu(\cdot)$
for $i \in \{1, \dots, N\}$ **do** $w_0^{(i)} \leftarrow \left\{\sum_{j=1}^N g_0(X_0^{(j)})\right\}^{-1} g_0(X_0^{(i)})$
for $t \in \{0, \dots, T-1\}$ **do**
  Sample $a_t^{(1:N)} \sim \textsc{resample}(\{1, \dots, N\}, w_t^{(1:N)})$
  for $i \in \{1, \dots, N\}$ **do** Sample $X_{t+1}^{(i)} \sim K_{t+1}(X_t^{(a_t^{(i)})}, \cdot)$
  for $i \in \{1, \dots, N\}$ **do** $w_{t+1}^{(i)} \leftarrow \left\{\sum_{j=1}^N g_{t+1}(X_t^{(a_t^{(j)})}, X_{t+1}^{(j)})\right\}^{-1} g_{t+1}(X_t^{(a_t^{(i)})}, X_{t+1}^{(i)})$
end

Algorithm 1: Sequential Monte Carlo

Figure 2.1 shows part of the conditional dependence graph implied by Algorithm 1. Our aim is to find a σ-algebra $\mathcal{H}_t$ at each time t that separates the ancestral process (encoded by $a_t^{(1:N)}$) from the filtration $\mathcal{F}_{t-1}$. That is, $a_t^{(1:N)}$ is conditionally independent of $\mathcal{F}_{t-1}$ given $\mathcal{H}_t$. By a D-separation argument (see Verma and Pearl 1988), the nodes highlighted in grey suffice as the generator of $\mathcal{H}_t$. That is, for each t, we take

$$\mathcal{H}_t = \sigma(X_{t-1}^{(1:N)}, X_t^{(1:N)}, w_{t-1}^{(1:N)}, w_t^{(1:N)}).$$

Notice that $\nu_t^{(1:N)}$ can be expressed as a function of $a_t^{(1:N)}$, and as such carries less information.

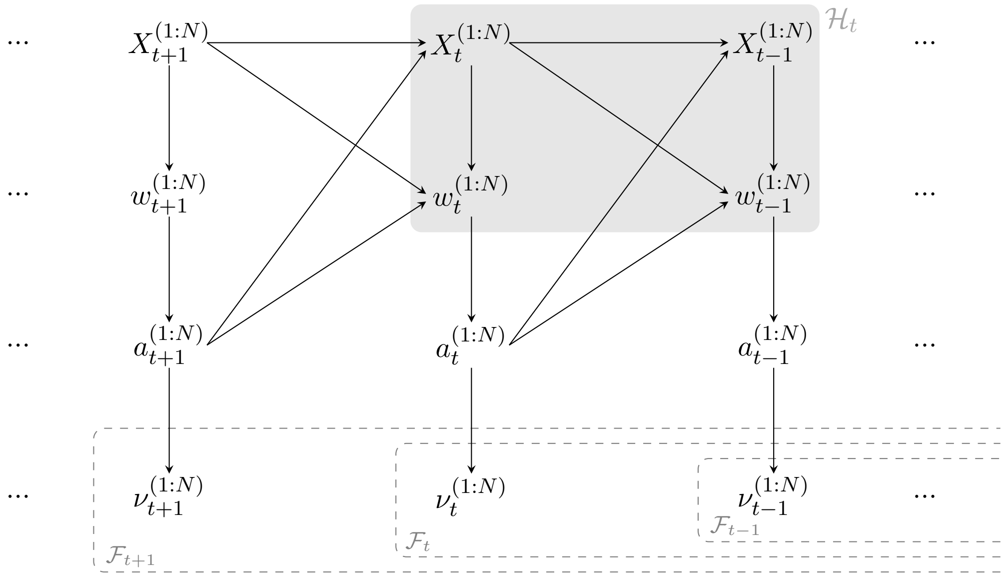


Figure 2.1: Part of the conditional dependence graph implied by Algorithm 1. The direction of time is from left to right. The reverse-time filtration is indicated by the dashed areas. The nodes highlighted in grey generate the separatrix $\mathcal{H}_t$ between $a_t^{(1:N)}$ and $\mathcal{F}_{t-1}$. Use the same shades of grey here as elsewhere

2.1.6 Theoretical justification

How come SMC works? Convergence results (briefly!) e.g. Lp bounds, CLT, stability.

2.2 Coalescent theory

2.2.1 Kingman's coalescent

The Kingman coalescent (Kingman 1982b; Kingman 1982c; Kingman 1982a) is a continuous-time Markov process on the space of partitions of $\mathbb{N}$. For our purposes we need only consider its restriction to $\{1, \ldots, n\}$, termed the n-coalescent (defined below), since we only ever consider finite samples from a population. However, an excellent probabilistic introduction to the Kingman coalescent from the point-of-view of exchangeable random

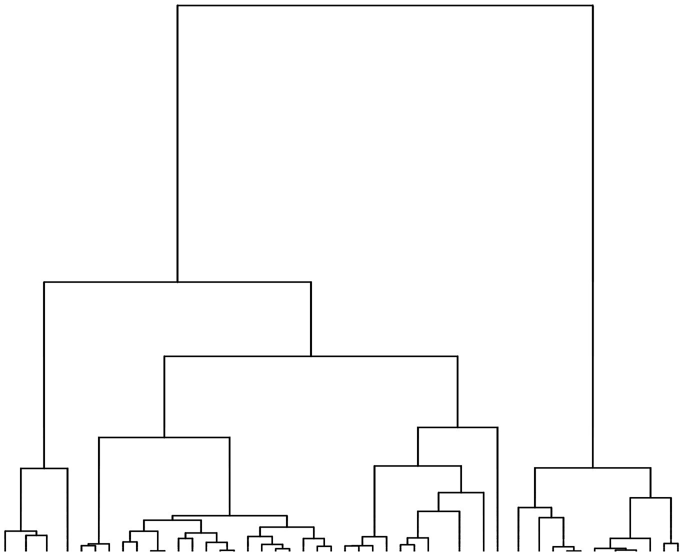


Figure 2.2: A realisation of the n-coalescent with $n = 50$.

partitions can be found in Berestycki (2009, Chapters 1–2). or Wakeley (2009) ? or Durrett (2008) ?

Definition 2.1. The *n-coalescent* is the homogeneous continuous-time Markov process on the set of partitions of $\{1, \ldots, n\}$ with infinitesimal generator Q having entries

$$q_{\xi,\eta} = \begin{cases} 1 & \xi \prec \eta \\ -|\xi|(|\xi|-1)/2 & \xi = \eta \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

where ξ and η are partitions of $\{1, ..., n\}$, $|\xi|$ denotes the number of blocks in ξ, and $\xi \prec \eta$ means that η is obtained from ξ by merging exactly one pair of blocks.

A particularly attractive feature of the n-coalescent is its tractability; its distribution and those of many statistics of interest are available in closed form (Section 2.2.2). It turns out also to be extremely useful as a limiting distribution in population genetics, including the genealogies of a wide range of population models in its domain of attraction (Section 2.2.3).

2.2.2 Properties of Kingman's coalescent

The simplicity of Q allows various properties of the n-coalescent to be studied analytically. Refer to more exhaustive studies of the properties in the literature, e.g. Durrett (2008, Section 1.2). Starting with n blocks, exactly $n-1$ coalescences are required to reach the absorbing state where all blocks have coalesced, known in the population genetics literature as the *most recent common ancestor* (MRCA).

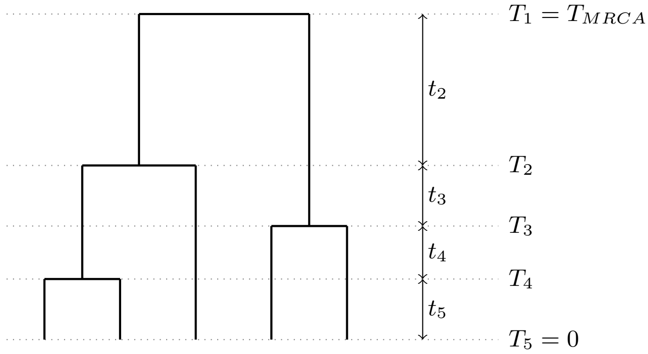


Figure 2.3: Diagram illustrating the definitions of t_i, T_i in the n-coalescent.

Denote by $t_2, t_3 \ldots, t_n$ the waiting times between coalescent events, where t_i is the amount of time for which the coalescent has exactly i distinct lineages (see Figure 2.3). A consequence of Definition 2.1 is that these waiting times are independent and have distributions

$$t_i \sim \operatorname{Exp}\left(\binom{i}{2}\right). \tag{2.2}$$

The partial sum $T_k := \sum_{i=k+1}^{n} t_i$ gives the total time up to the $(n-k)^{th}$ coalescence event, i.e. the first time at which there are only k lineages remaining out of the initial n (see Figure 2.3). The partial sums, being sums of independent Exponential random variables, have HyperExponential distributions.

Refer back to the following three properties later on with reference to their relevance in SMC.

Time to MRCA

Of particular interest is the tree height or time to the most recent common ancestor, $T_{MRCA} := T_1$. With some algebra we find, for instance,

$$\mathbb{E}[T_{MRCA}] = \sum_{i=2}^{n} \mathbb{E}[t_i] = \sum_{i=2}^{n} \frac{2}{i(i-1)} = 2\sum_{i=2}^{n} \left\{\frac{1}{i-1} - \frac{1}{i}\right\} = 2\left(1 - \frac{1}{n}\right) \tag{2.3}$$

and

$$\operatorname{Var}[T_{MRCA}] = \sum_{i=2}^{n} \operatorname{Var}[t_i] = \sum_{i=2}^{n} \left(\frac{2}{i(i-1)}\right)^2. \tag{2.4}$$

The expected tree height converges to 2 as $n \to \infty$, and the variance converges to $4(\pi^2 - 9)/3 \simeq 1.16$. The somewhat surprising fact that the tree height does not diverge with n is a result of the very high rate of coalescence close to the bottom of the tree. This rate is large enough that the full Kingman coalescent (on $\mathbb{N}$) *comes down from infinity*, that is, despite starting with infinitely many blocks, after any positive amount of time these have coalesced into finitely many blocks. Plot mean with sd-ribbon over n for an illustration? SD ribbon isn't the right thing; since we apparently know the actual distribution, plot a high density interval of that. (also for L)

Total branch length

Another quantity of interest is the total branch length, $L := \sum_{i=2}^{n} it_i$. For instance

$$\mathbb{E}[L] = \sum_{i=2}^{n} i\mathbb{E}[t_i] = \sum_{i=2}^{n} \frac{2}{i-1} = \sum_{i=1}^{n-1} \frac{2}{i} \simeq 2\ln(n-1) \tag{2.5}$$

and

$$\mathrm{Var}[L] = \sum_{i=2}^{n} i^2 \,\mathrm{Var}[t_i] = \sum_{i=2}^{n} \frac{4}{(i-1)^2} = \sum_{i=1}^{n-1} \frac{4}{i^2}. \tag{2.6}$$

Note that although the mean total branch length diverges with n, the variance converges to a constant, $4\pi/6 \simeq 6.6$.

Probability that sample MRCA equals population MRCA

One other interesting quantity is the probability that the MRCA of k random lineages coincides with the population MRCA (e.g. Durrett 2008, Theorem 1.7). Denote by S_k the relevant event: that a random sample of k lineages has the same as the MRCA as the population. Consider the two subtrees produced by cutting the tree just below the population MRCA. The sample of k lineages coalesces before the population MRCA if and only if all k sampled leaves lie in just one of these two subtrees. A basic consequence of the exchangeability of the n-coalescent is that, in the limit $N \to \infty$, the proportion of leaves in the left subtree is uniformly distributed on $[0, 1]$. Calling this proportion X, we have

$$\mathbb{P}[S_k^c \mid X = x] = x^k + (1-x)^k$$

Integrating against the distribution of X, the probability of interest is

$$\mathbb{P}[S_k] = 1 - \int_0^1 [x^k + (1-x)^k]dx = \frac{k-1}{k+1}$$

as required.

The above is based on properties of the full Kingman coalescent, but similar results are available for the n-coalescent. Consider now a subsample of size k among n lineages that follow the n-coalescent. Denote by $S_{k,n}$ the event that these k lineages have the same MRCA as all n lineages. This probability of this event is calculated in Saunders, Tavaré, and Watterson (1984, Example 1) and again in Spouge (2014, Equation (3)), in both cases arising as a special case of more general results. A direct proof is given below.

Let X be the number of leaves in the left subtree. So $X \in \{1, \dots, n-1\}$ and, like before, a consequence of exchangeability is that X is uniformly distributed on that set. Now that the total number of branches is finite, we have to count more carefully. Conditional on X

we have

$$\mathbb{P}[S_{k,n}^c \mid X = x] = \left[\binom{x}{k} + \binom{n-x}{k}\right]\binom{n}{k}^{-1}.$$

Integrating against the distribution of X gives

$$\begin{aligned}\mathbb{P}[S_{k,n}] &= 1 - \frac{1}{n-1}\binom{n}{k}^{-1}\sum_{x=1}^{n-1}\left[\binom{x}{k} + \binom{n-x}{k}\right] \\ &= 1 - \frac{1}{n-1}\binom{n}{k}^{-1}\left[\binom{n}{k+1} + \binom{n}{k+1}\right] \\ &= \frac{k-1}{k+1}\frac{n+1}{n-1}\end{aligned}$$

using binomial identities and some algebra. As $n \to \infty$ this agrees with the population-level result above.

2.2.3 Models in population genetics

The Kingman coalescent is the limiting coalescent process (in the large population limit) for a surprisingly wide range of population models. Some important examples of models in Kingman's "domain of attraction" are introduced in this section. Common to all of these models are the following assumptions:

- The population has constant size N
- Reproduction happens in discrete generations
- The offspring distributions are identical at each generation, and independent between generations
- These models are all *neutral*, i.e. the offspring distribution is exchangeable.

As before section/eq ref?, we define offspring counts in terms of parental indices as $\nu_j := |\{i : a_i = j\}|$. Under the assumption of neutrality, it is sufficient to consider only the offspring counts, rather than the parental indices (which generally carry more information). Crucially, in the neutral case, offspring counts carry all the information about the distribution of the genealogy that is contained in the parental indices. From a biological perspective, neutrality encodes the absence of natural selection, i.e. no individual in the population is "fitter" than another.

Wright-Fisher model

The neutral Wright-Fisher model (Fisher 1923; Fisher 1930; Wright 1931) is one of the most studied models in population genetics. At each time step the existing generation dies and is replaced by N offspring. The offspring descend from parents $(a_1, \ldots, a_N)$ which are selected according to

$$a_i \overset{iid}{\sim} \text{Categorical}(\{1, \ldots, N\}, (1/N, \ldots, 1/N)).$$

The joint distribution of the offspring counts is therefore

$$(v_1, \ldots, v_N) \sim \text{Multinomial}(N, (1/N, \ldots, 1/N)).$$

Since the Multinomial distribution is exchangeable, this model is neutral. There are several non-neutral variants of the Wright-Fisher model citations?, but they are typically much less tractable than the neutral one.

Kingman showed in his original papers introducing the Kingman coalescent (Kingman 1982b) that, when time is scaled by a factor of N, genealogies of the neutral Wright-Fisher model converge to the Kingman coalescent as $N \to \infty$.

Cannings model

The neutral Cannings model (Cannings 1974; Cannings 1975) is a more general construction which encompasses the neutral Wright-Fisher model as a special case.

In the Cannings model, the particular offspring distribution is not specified; we only require that it is exchangeable, i.i.d. between generations, and preserves the population size. In particular, the probability of observing offspring counts $(v_1, \ldots, v_N)$ must be invariant under permutations of this vector.

Genealogies of the neutral Cannings model also converge to the Kingman coalescent, under some conditions and a suitable time-scaling which is what?, as $N \to \infty$ (see for example Etheridge 2011, Section 2.2). original reference for this? is not any Kingman 1982 papers, and certainly not Cannings 1974/5 which predates KC

Moran model

The neutral Moran model (Moran 1958), while perhaps less biologically relevant, is mathematically appealing because its simple dynamics make it particularly tractable.

At each time step, an ordered pair of individuals is selected uniformly at random. The first individual in this pair dies (i.e. leaves no offspring in the next generation), while the other reproduces (leaving two offspring). All of the other individuals leave exactly one offspring. This is another special case of the neutral Cannings model, where the offspring distribution is now uniform over all permutations of $(0, 2, 1, 1, \ldots, 1)$. Therefore we know that under a suitable time-scaling, its genealogies converge to the Kingman coalescent. The time scale in this case is N^2, because reproduction happens at a rate N times or is it technically N-1 times? lower than in the Wright-Fisher model. also cite a Moran-specific convergence result: not sure where (it isn't in Kingman 1982* or in Moran 1958 which predates KC)

2.2.4 Particle populations

Much of the population genetics framework transfers readily to the case of SMC. The population is now a population of particles, with each iteration of the SMC algorithm corresponding to a generation, and resampling playing the part of reproduction. In fact, SMC "populations" are in some ways more suited to these population models than actual populations of organisms. The assumptions that the population has constant size N and that reproduction occurs only at discrete generations are satisfied by construction. However, we cannot assume independence between generations: as seen in Figure 2.1, the offspring counts at subsequent generations are not independent without some conditioning. In fact, after marginalising out the information about the positions of the particles, the genealogical process is not even Markovian. Nor is our model neutral: the resampling distribution depends on the weight of each particle (the weight plays the role of fitness in a non-neutral population model).

2.3 Sequential Monte Carlo genealogies

2.3.1 From particles to genealogies

How does the SMC algorithm induce a genealogy? (resampling = parent-child relationship).

2.3.2 Performance

How do genealogies affect performance? Variance (and variance estimation?), storage cost. Ancestral degeneracy.

2.3.3 Mitigating ancestral degeneracy

Low-variance resampling (save details for next section). Adaptive resampling: idea of balancing weight/ancestral degeneracy; rule of thumb for implementing it; when is it effective or not?; necessary changes to our generic SMC algorithm (calculation of weights in particular). Backward sampling: when is it possible to do this?

2.3.4 Asymptotics

Why are large population asymptotics useful? Existing results (path storage, KJJS).

2.4 Resampling

As we have seen, resampling is necessary within SMC to "reset" the weights in order to prevent weight degeneracy. The basic role of a resampling scheme is to map the continuous weights to discrete offspring counts, in some "sensible" way (Definition 2.2). The choice of resampling scheme is explored in detail in this section.

2.4.1 Definition

Definition 2.2. For our purposes, a valid resampling scheme is a stochastic function mapping weights $w_t^{(1:N)} \in \mathcal{S}_{N-1}$ to offspring counts $\nu_t^{(1:N)} \in \{0, \dots, N\}^N$ that satisfies the following properties:

1. the population size is conserved: $\sum_{i=1}^{N} \nu_t^{(i)} = N$ for all N
2. the weights are uniform after resampling: $w_{t+}^{(i)} = 1/N$ for all i
3. the resampling is unbiased: $\mathbb{E}[\nu_t^{(i)} \mid w_t^{(i)}] = N w_t^{(i)}$ for all i.

It is possible to design resampling schemes that violate these properties. For example, a scheme of Liu and Chen (1998) uses the square roots of the weights for resampling, then corrects by setting non-uniform weights after resampling (violating conditions 2 and 3). Fearnhead and Clifford (2003, p.890, point (d)) also appears to resample such that the weights are not uniform after resampling. Resampling different numbers of particles in different iterations (violating condition 1) is of course possible, but we typically have a fixed limit on computational resources, in which case it makes sense to simulate the maximum feasible number of particles N at every iteration. Deterministic resampling schemes (which cannot generally be unbiased, violating condition 3) have been used by some authors. These include schemes based on optimal transport (Reich 2013; Myers et al. 2021; Corenflos et al. 2021) and the importance support points resampling of Huang, Joseph, and Mak (2020). However, the majority of resampling schemes in the literature fit within Definition 2.2, and it is not typically advantageous to violate the properties 1–3.

Within Definition 2.2 there is still a great deal of flexibility. Many different resampling schemes have been proposed in the literature, some of which perform better than others. Section 2.4.2 introduces some important resampling schemes, and their properties are discussed in Section 2.4.3. These are summarised in Table 2.2.

2.4.2 Examples

This whole section was dumped from elsewhere and needs redrafting.

Multinomial resampling

Multinomial resampling (Gordon, Salmond, and Smith 1993; Efron and Tibshirani 1994) is one of the simplest resampling schemes. The parental indices are chosen independently from $\{1, \dots, N\}$, each with probability given by the weight of the corresponding particle $w_t^{(i)}$. That is,

$$a_t^{(1:N)} \sim \text{Categorical}(\{1, \dots, N\}, w_t^{(1:N)}).$$

This implies the joint distribution of the offspring counts is

$$\nu_t^{(1:N)} \overset{d}{=} \text{Multinomial}(N, w_t^{(1:N)}).$$

Abbreviation	Description
`multi`	multinomial resampling
`star`	star resampling
`strat`	stratified resampling
`syst`	systematic resampling
`res-multi`	residual resampling with multinomial residuals
`res-star`	residual resampling with star residuals
`res-strat`	residual resampling with stratified residuals
`res-syst`	residual resampling with systematic residuals
`ssp`	Srinivasan sampling procedure resampling
`branching`	

Table 2.1: Abbreviations for resampling schemes

Note that in this case the parental indices are chosen independently, but the resulting offspring counts are negatively correlated.

A simple way to sample the parental indices is by inversion sampling: divide the unit interval into N disjoint subintervals each of which will correspond to a certain index i and has length equal to the weight $w_t^{(i)}$; then draw N samples $U_i \sim \text{Uniform}(0,1)$ and classify them according to which of these subintervals they fall in. Explicitly, the parental index assigned to child i is the index a_i satisfying

$$\sum_{j=1}^{a_i-1} w_t^{(j)} \leq U_i \leq \sum_{j=1}^{a_i} w_t^{(j)} \tag{2.7}$$

This is illustrated in Figure 2.4a.

Fast implementations of multinomial resampling rely on the U_is being sorted, which speeds up the search step (2.7). Sorting N numbers is an $O(N \log N)$ operation, but in fact we can directly sample the order statistics of a Uniform$[0,1]$ distribution [citations: Chopin and Papaspiliopoulos (2020), and a different (or possibly equivalent) method in Hol, Schön, and Gustafsson (2006)] —explore whether these methods are equivalent. This allows multinomial resampling to be implemented at $O(N)$ cost, with the side-effect that the sampled ancestral indices will be ordered.

Residual resampling

Residual resampling is described in Liu and Chen (1998) and also in Whitley (1994) where it is called "remainder stochastic sampling".

Each particle $X_t^{(i)}$ is deterministically assigned $\lfloor N w_t^{(i)} \rfloor$ offspring and the remaining $R := N - \sum_{i=1}^{N} \lfloor N w_t^{(i)} \rfloor$ offspring are assigned stichastically according to the residual

	support of $\nu_t^{(i)}$ given $k/N \le w_t^{(i)} < (k+1)/N$	stochastic rounding?	degenerate if $w_t^{(1:N)} = (1,\ldots,1)/N$?	sensitive to permutations of weights?	PRNG calls
`multi`	$\{0,\ldots,N\}$	$\times$	$\times$	$\times$	N
`star`	$\{0,N\}$	$\times$	$\times$	$\times$	1
`strat`	$\{k-1,k,k+1,k+2\}$	$\times$	✓	✓	N
`syst`	$\{k,k+1\}$	✓	✓	✓	1
`res-multi`	$\{k,\ldots,N\}$	$\times$	✓	$\times$	$\le N$
`res-star`	$\{k,N\}$	$\times$	✓	$\times$	1
`res-strat`	$\{k,k+1,k+2\}$	$\times$	✓	✓	$\le N$
`res-syst`	$\{k,k+1\}$	✓	✓	✓	1
`ssp`	$\{k,k+1\}$?	✓?	✓?	✓?	?
`branching`	$\{k,k+1\}$?	✓?	✓?	✓?	?

Table 2.2: Properties of resampling schemes

weights

$$r^{(i)} := (Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor)/R.$$

The stochastic part can be done using any of the other basic resampling schemes (e.g. multinomial, stratified, systematic). Most presentations focus on the case where multinomial resampling is used for the residuals, which is by no means the most sensible option. We will explore several different options in what follows.

Stratified resampling

Stratified resampling is introduced in Kitagawa 1996.

The scheme proceeds like Multinomial resampling, except that the Uniform samples that are fed in to do the Categorical sampling are produced in a different way. Instead of sampling N independent numbers from $\mathrm{U}(0,1)$, one number is sampled uniformly from each subinterval of length $1/N$. That is,

$$U_i \sim \text{Uniform}\left(\frac{i-1}{N}, \frac{i}{N}\right).$$

The parents are then assigned as in (2.7). (Of course this means that the offspring distribution is no longer Multinomial, since parental indices are not chosen independently.) This scheme ensures that the samples are “well spread out”, again reducing the probability of randomly losing high-weighted particles. The method is illustrated in Figure 2.4b.

Systematic resampling

Systematic resampling is described in Carpenter, Clifford, and Fearnhead (1999) and also in Whitley (1994) where it is called "stochastic universal sampling".

Like stratified resampling, it constitutes a change to the random number generator for sampling from the Categorical distriubtion. In this scheme, only one Uniform sample is drawn, $U \sim \mathrm{U}(0, 1/N)$, and the other $N-1$ samples are generated deterministically by setting

$$U_i = U + \frac{i-1}{N}$$

for each $i \in \{1, \ldots, N\}$. The parental indices are again selected according to (2.7). The method is illustrated in Figure 2.4c. This scheme again ensures the random numbers are "well spread out", even more so than with stratified resampling.

Systematic resampling is often preferred among practitioners because it is extremely easy to implement and also computationally efficient, requiring only one random number to be generated.

However, this scheme is known to exhibit pathological behaviour in some cases due to its dependence on the ordering of the subintervals (Douc, Cappé, and Moulines 2005). Such behaviour can be avoided by randomly permuting the intervals before sampling, and this is the recommended practice.

Star resampling

For the sake of comparison, we also construct a resampling scheme which is the worst possible (in some sense). Sample

$$a_t \sim \mathrm{Categorical}(\{1, \ldots, N\}, w_t^{(1:N)})$$

and set $a_t^{(i)} = a_t$ for all i. The resulting offspring counts are all equal to zero except for $\nu_t^{(a_t)}$, which is equal to N. This resampling scheme is indeed unbiased, since each offspring count has marginal distribution

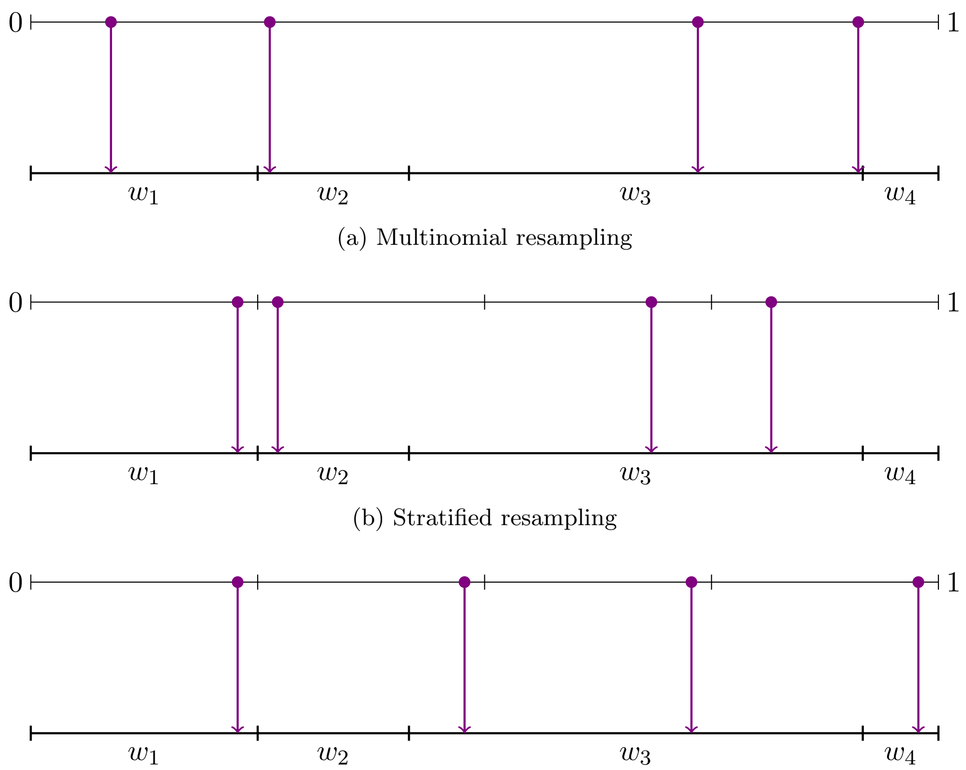
$$\nu_t^{(i)} \mid w_t^{(1:N)} = \begin{cases} 0 & \text{w.p. } 1 - w_t^{(i)} \\ N & \text{w.p. } w_t^{(i)}. \end{cases}$$

We also see these offspring counts have the highest possible marginal variance, subject to $\mathbb{E}[\nu_t^{(i)} \mid w_t^{(i)}] = N w_t^{(i)}$ and $\nu_t^{(i)} \in \{0, \ldots, N\}$.

I call this scheme *star resampling* because the parent-offspring relationships at each iteration form a star graph.

2.4.3 Properties

Low-variance: variance of what? Different criteria/ definitions of optimality. Link back to adaptive resampling: interaction between adaptive and low-variance resampling. Compar-



(a) Multinomial resampling

(b) Stratified resampling

(c) Systematic resampling

Figure 2.4: Inversion sampling to obtain Multinomial offspring counts, where the Uniform variables for inversion are sampled in different ways. For this example $N = 4$ and the weights are $w_{(1:4)} = \frac{1}{N}(1, \frac{2}{3}, 2, \frac{1}{3})$. In each case the samples to be inverted are seeded with the same Uniform$(0, 1)$ samples. (a) Sample N independent Uniform$(0, 1)$ random variables. In this example the sampled offspring counts are $(1, 1, 2, 0)$. (b) The Uniform$(0, 1)$ samples are transformed to Uniform draws from the intervals (0,0.25), (0.25,0.5), (0.5, 0.75), (0.75,1). In this example the sampled offspring counts are $(1, 1, 2, 0)$. (c) Use only the first draw and transform it to a sample from Uniform$(0, 0.25)$. For the subsequent samples, add 0.25 each time to obtain a sample in each interval. In this example the sampled offspring counts are $(1, 0, 2, 1)$.

ison of properties of these, existing results comparing schemes. Implementation considerations. Theoretical justification (or lack of). Mention computational complexity. This section was dumped from elsewhere and most of its subsections need redrafting.

Variance

Some results comparing the Monte Carlo variance associated with different resampling schemes were presented in Douc, Cappé, and Moulines (2005). Their results, plus some others, are presented in Proposition 2.1.

Let $(\mathcal{G}_t)_{t\geq 0}$ be the filtration generated by the particle positions and weights up to and including time t. Let $\tilde{X}_t^{(i)}$ denote position of the ith resampled particle. Of direct interest for the performance of the SMC algorithm is the one-step Monte Carlo variance induced by resampling, that is

$$\rho(\varphi) := \operatorname{Var}\left[\frac{1}{N} \sum_{i=1}^{N} \varphi(\tilde{X}_t^{(i)}) \,\middle|\, \mathcal{G}_t \right] \tag{2.8}$$

where φ is an arbitrary test function.

Proposition 2.1 (Variance of resampling schemes)**.** *Let $\rho_{\texttt{multi}}$ etc. denote the variance (2.8) under the various resampling schemes, as abbreviated in Table 2.1. For any square-integrable ? function φ,*

(a) $\rho_{\texttt{multi}}(\varphi) \geq \rho_{\texttt{res-multi}}(\varphi)$

(b) $\rho_{\texttt{multi}}(\varphi) \geq \rho_{\texttt{strat}}(\varphi)$

(c) $\rho_{\texttt{star}}(\varphi) = N\rho_{\texttt{multi}}(\varphi)$

(d) $\rho_{\texttt{res-star}}(\varphi) \geq \rho_{\texttt{res-multi}}(\varphi) \geq \rho_{\texttt{res-strat}}(\varphi)$

Say which parts were proved in Douc, Cappé, and Moulines (2005) or not.

Proof. **multinomial resampling:** the resampled indices are conditionally i.i.d., so

$$\begin{aligned}
\rho_{\texttt{multi}}(\varphi) &= \operatorname{Var}\left[\frac{1}{N} \sum_{i=1}^{N} \varphi(\tilde{X}_t^{(i)}) \,\middle|\, \mathcal{G}_t \right] = \frac{1}{N} \operatorname{Var}\left[\varphi(\tilde{X}_t^{(i)}) \,\middle|\, \mathcal{G}_t \right] \\
&= \frac{1}{N} \left\{ \mathbb{E}\left[\varphi^2(\tilde{X}_t^{(i)}) \,\middle|\, \mathcal{G}_t \right] - \mathbb{E}\left[\varphi(\tilde{X}_t^{(i)}) \,\middle|\, \mathcal{G}_t \right]^2 \right\} \\
&= \frac{1}{N} \sum_{j=1}^{N} \varphi^2(X_t^{(j)}) \mathbb{P}[\tilde{X}_t^{(i)} = X_t^{(j)} \mid \mathcal{G}_t] - \frac{1}{N} \left\{ \sum_{j=1}^{N} \varphi(X_t^{(j)}) \mathbb{P}[\tilde{X}_t^{(i)} = X_t^{(j)} \mid \mathcal{G}_t] \right\}^2 \\
&= \frac{1}{N} \sum_{j=1}^{N} \varphi^2(X_t^{(j)}) w_t^{(j)} - \frac{1}{N} \left\{ \sum_{j=1}^{N} \varphi(X_t^{(j)}) w_t^{(j)} \right\}^2 .
\end{aligned}$$

residual-multinomial resampling: the Monte Carlo estimate in (2.8) can be decomposed into a sum of conditionally deterministic terms plus a sum of conditionally i.i.d. terms: conditional on $\mathcal{G}_t$,

$$\frac{1}{N}\sum_{i=1}^{N}\varphi(\tilde{X}_t^{(i)}) = \frac{1}{N}\sum_{i=1}^{N}\lfloor Nw_t^{(i)}\rfloor\varphi(X_t^{(i)}) + \frac{1}{N}\sum_{i=1}^{R}\varphi(\hat{X}_t^{(i)})$$

where $\hat{X}_t^{(i)} \sim^{\text{iid}} \text{Multinomial}(R, r^{(1:N)})$. The first sum is conditionally deterministic and hence does not contribute to the Monte Carlo variance (2.8). By a similar calculation to that for multinomial resampling,

$$\begin{aligned}
\rho_{\texttt{res-multi}}(\varphi) &= \text{Var}\left[\frac{1}{N}\sum_{i=1}^{R}\varphi(\hat{X}_t^{(i)}) \,\middle|\, \mathcal{G}_t\right] \\
&= \frac{R}{N^2}\sum_{j=1}^{N}\varphi^2(X_t^{(j)})r^{(j)} - \frac{R}{N^2}\left\{\sum_{j=1}^{N}\varphi(X_t^{(j)})r^{(j)}\right\}^2 \\
&= \frac{1}{N}\sum_{j=1}^{N}\varphi^2(X_t^{(j)})w_t^{(j)} - \frac{1}{N^2}\sum_{j=1}^{N}\varphi^2(X_t^{(j)})\lfloor Nw_t^{(i)}\rfloor - \frac{R}{N^2}\left\{\sum_{j=1}^{N}\varphi(X_t^{(j)})r^{(j)}\right\}^2 .
\end{aligned}$$

By a similar argument, it can be shown that (using the star resampling part below, which should probably be moved to above here...)

$$\rho_{\texttt{res-star}}(\varphi) = R\rho_{\texttt{res-multi}}(\varphi) \geq \rho_{\texttt{res-multi}}(\varphi), \tag{2.9}$$

proving the first inequality in (d).

star resampling: all of the $\tilde{X}_t^{(i)}$s are equal, say $\tilde{X}_t^{(1)} = \cdots = \tilde{X}_t^{(N)} = X_t^{\star}$, so

$$\begin{aligned}
\rho_{\texttt{star}}(\varphi) &= \text{Var}\left[\frac{1}{N}\sum_{i=1}^{N}\varphi(\tilde{X}_t^{(i)}) \,\middle|\, \mathcal{G}_t\right] = \text{Var}\left[\varphi(X_t^{\star}) \mid \mathcal{G}_t\right] \\
&= \mathbb{E}\left[\varphi^2(X_t^{\star}) \mid \mathcal{G}_t\right] - \mathbb{E}\left[\varphi(X_t^{\star}) \mid \mathcal{G}_t\right]^2 \\
&= \sum_{j=1}^{N}\varphi^2(X_t^{(j)})\mathbb{P}[X_t^{\star} = X_t^{(j)} \mid \mathcal{G}_t] - \left\{\sum_{j=1}^{N}\varphi(X_t^{(j)})\mathbb{P}[X_t^{\star} = X_t^{(j)} \mid \mathcal{G}_t]\right\}^2 \\
&= \sum_{j=1}^{N}\varphi^2(X_t^{(j)})w_t^{(j)} - \left\{\sum_{j=1}^{N}\varphi(X_t^{(j)})w_t^{(j)}\right\}^2 \\
&= N\rho_{\texttt{multi}}(\varphi).
\end{aligned}$$

This proves part (c).

... ∎

Can't do a formal variance comparison for syst (and others?).

Negative association

Definition from Gerber, Chopin, and Whiteley (2019). Why is it a good criterion? Which resampling schemes do/don't satisfy it? Also, this could be added as a column in Table 2.2.

Star discrepancy

Something about syst vs. strat vs. mn in terms of star discrepancy (define that). See Hol, Schön, and Gustafsson (2006) for inspiration. Does not appear in Table 2.2 since it only makes sense for resampling schemes based on inversion sampling.

The *star discrepancy* [citation] is a measure of the regularity of a given set of points $u_{1:N}$ in the unit hypercube. For our purposes it is sufficient to define the star discrepancy in one dimension:

$$D^\star(u_1, \dots, u_N) := \sup_{u \in [0,1]} \left| \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\{u_i \leq u\}} - u \right|. \tag{2.10}$$

The quantity inside the supremum is the difference between the empirical CDF of the observed points $u_{1:N}$ and the CDF of the Uniform distribution on $[0, 1]$. The star discrepancy is used in quasi-Monte Carlo, where "low-discrepancy" points are used in place of uniform samples to decrease the variance of Monte Carlo estimates.

We have noted already we haven't actually, but we should that resampling can itself be viewed as a Monte Carlo procedure. From this point-of-view, stratified and systematic resampling are in fact quasi-Monte Carlo versions of multinomial resampling.

Calculate/compare discrepancy for each mn/strat/syst. Can't achieve minimal discrepancy $1/(2N)$ because we need marginally uniform points for unbiasedness (so really this is RQMC).

Support of offspring numbers

Let us consider the support of the marginal offspring distributions in each scheme, given the corresponding weight. Condition on the i^{th} weight lying in the interval $w_t^{(i)} \in [k/N, (k+1)/N]$, but leave the other weights unknown. By considering the best and worst cases for each scheme, we have:

Multinomial: $\nu_t^{(i)} \in \{0, \dots, N\}$

Residual: $\nu_t^{(i)} \in \{k, \dots, k + R\} \subseteq \{k, \dots, N\}$

Stratified: $\nu_t^{(i)} \in \{k - 1, k, k + 1, k + 2\}$

Systematic: $\nu_t^{(i)} \in \{k, k + 1\}$

We see that multinomial resampling allows the possibility of very good particles having 0 offspring, and of very bad particles having N offspring (although the probabilities associated to these events are low). Residual resampling ensures that good particles do not die out, but still allows bad particles to possibly have many offspring. Stratified resampling is more restrictive, although it allows the possibility of a particle with weight $> 1/N$ leaving no offspring. Systematic resampling is more restrictive still, allowing the number of offspring of each particle to vary from its expected value by no more than one.

Permutation invariance

A strange property of stratified and systematic resampling is that they are sensitive to the order in which the subintervals are placed. For example, in Figures 2.4b and 2.4c if the intervals w_2 and w_4 were swapped, the number of offspring assigned to particles 2 and 4 would be swapped in each case. We can also see that because w_1 has weight $\geq 1/N$ and is placed first, it is guaranteed at least one offspring.

This property can lead to pathological behaviour, but is easily avoided by applying a random permutation to the order of the subintervals. Gerber, Chopin, and Whiteley (2019) also propose a variation on systematic resampling that avoids this property.

Sorting

Results from Gerber, Chopin, and Whiteley (2019) about benefits of sorting. What about sorting instead by weights?

Degeneracy under equal weights

Suppose we somehow end up in the situation where all the weights are equal (i.e. $w_t^{(i)} = 1/N$ for all i). In this case, residual resampling will result in a deterministic assignment only: each particle will be assigned one offspring, and there will be no remainder left to assign randomly. This behaviour cannot be avoided, however the event that all weights are equal typically has zero measure.

Stratified and systematic resampling will have the same result: the intervals for sampling will correspond exactly to the weighted subintervals, so no matter which random numbers are sampled, exactly one will fall in each subinterval.

However, for stratified resampling, the formulation of Whitley (1994) avoids this behaviour. He imagines subdivisions of a circle rather than an interval, and then “spins the roulette wheel” around it, which shifts the sampling intervals by a random amount and thus prevents this degeneracy.

Exchangeability

We will call a resampling scheme exchangeable if the resulting distribution of parental indices is invariant under permutations of the children. To put it another way, each child chooses its parent from the same marginal distribution.

It is clear that multinomial resampling is exchangeable since in this case the parental indices are independent and identically distributed. However it is worth noting that some efficient implementations of multinomial sampling may not preserve exchangeability in practice.

Stratified and systematic resampling are clearly not exchangeable since, for instance, child 1 is more likely to choose parent 1 than child N is. However, this is merely a feature of the arbitrary ordering of the sampling steps: exchangeability can easily be reintroduced by applying a random permutation to the vector of parental indices after sampling. The same goes for residual resampling. This property will not appear in Table 2.2 since it depends upon the particular implementation.

Computational complexity

All of the resampling algorithms discussed in Section 2.4.2 can be implemented in $O(N)$ operations. Even `star` and `SSP` and `branching`? If it turns out to differ depending on resampling scheme then include it as a column in Table 2.2. Considering the complexity of each operation, Hol (2004) and Hol, Schön, and Gustafsson (2006) suggest that systematic resampling is fastest because it only requires one pseudo-random number generation, and multinomial resampling is slower than stratified resampling because of the transformations required. Residual resampling is hard to compare directly because a random fraction of the operations are deterministic, so the number of pseudo-random numbers required is less than N. This analysis was backed up by simulation experiments. However, the analysis of per-particle cost is sensitive to the particular implementation of each resampling scheme, the system implementation of pseudo-random number generation and arithmetic operations, and the hardware used.

Optimal resampling

Crisan and Lyons (1999) introduce another resampling scheme based on a branching process, which they show to be optimal in some sense. However, their algorithm is not widely used in practice because it is much more complicated to implement than alternatives like systematic resampling which perform just as well empirically, and share some of its optimality properties (Bain and Crisan 2008). Possibly add an "optimality" column in Table 2.2 containing in what sense a scheme might be considered optimal.

2.4.4 Stochastic rounding

Define stochastic rounding. Resampling schemes contained by this class. General properties for this class (marginal distributions, negative association, minimum-variance).

Definition 2.3. Let $X = (X_1, \dots, X_N)$ be a $\mathbb{R}_+^N$-valued random variable. Then $Y = (Y_1, \dots, Y_N) \in \mathbb{N}^N$ is a *stochastic rounding* of X if each element Y_i takes values

$$Y_i \mid X_i = \begin{cases} \lfloor X_i \rfloor & \text{with probability } 1 - X_i + \lfloor X_i \rfloor \\ \lfloor X_i \rfloor + 1 & \text{with probability } X_i - \lfloor X_i \rfloor. \end{cases}$$

By construction, $\mathbb{E}(Y_i) = X_i$ for each i. Taking X to be N times the vector of particle weights, we can therefore use stochastic rounding to construct a valid resampling scheme, under the further constraint that $Y_1 + \cdots + Y_N = N$. Several ways to enforce this constraint on the joint distribution have been proposed, including systematic resampling, residual resampling with systematic residuals, the branching system of Crisan and Lyons (1997), and the Srinivasan sampling process resampling introduced in Gerber, Chopin, and Whiteley (2019). Properties shared by all SRs: support of offspring counts, degeneracy under equal weights, others?

2.5 Conditional SMC

2.5.1 Particle MCMC

Motivate particle MCMC methods.

The idea behind particle MCMC methods is to use SMC steps within the MCMC updates in a way that improves the mixing properties of the Markov chain. In certain models, generally those including some highly correlated sequential components, this strategy can be very effective.

One popular particle MCMC algorithm is particle marginal Metropolis-Hastings (Andrieu, Doucet, and Holenstein 2010)[Section 2.4.2], a pseudo-marginal MCMC algorithm in which SMC provides an unbiased likelihood estimate with which to compute the Metropolis-Hastings acceptance probability. The following exposition will focus on another particle MCMC algorithm, namely the particle Gibbs sampler (Andrieu, Doucet, and Holenstein 2010)[Section 2.4.3], which is more interesting from the point-of-view of SMC genealogies.

The following scenario illustrates the power of particle MCMC, and is a good model to have in mind as we go on to discuss particle Gibbs and ancestor sampling. Emphasise that the inference itself is not sequential; we are targeting one static posterior distribution, on a fixed time horizon.

2.5.2 Particle Gibbs algorithm

Present particle Gibbs algorithm (for the specific model just introduced?, but note that of course the algorithm is more general). Explain why CSMC is required within particle Gibbs. The following was dumped from elsewhere and NEEDS REDRAFTING.

The scenario we present is a particle Gibbs algorithm for filtering with unknown parameters. The method applies more generally to particle Gibbs (for which the reader is directed to Lindsten and Schön (2013, Chapter 5)), but we find this particular scenario to be simple and instructive. To generalise to Del Moral's SMC framework basically requires just the change of notation $g \to G_t, f \to M_t$.

Consider the following hidden Markov model, parametrised by a constant parameter θ which may be multidimensional. Define the spaces in which theta,X,Y live?

$$\begin{aligned} Y_t \mid X_t &\sim g_\theta(\cdot \mid X_t) \\ X_t \mid X_{t-1} &\sim f_\theta(\cdot \mid X_{t-1}) \\ X_0 &\sim \mu_\theta(\cdot) \\ \theta &\sim p(\cdot) \end{aligned}$$

Add ranges for t in first two lines. Use q instead of f for better congruity with other sections? We work on a fixed time horizon $T \in \mathbb{N}$, which is necessary to implement the particle Gibbs algorithm. Can be artificially imposed by block sampling if doing online-ish inference? The measures corresponding to μ_θ, f_θ and g_θ are assumed to be known and to admit densities, and we are also given a fixed sequence of observations $y_{1:T}$. Either mention that f,g can depend on t, or make this explicit in the notation. Don't really need to include prior on theta; it's not relevant to CSMC step.

Our aim is to generate Monte Carlo samples from the joint distribution of $X_{0:T}$ and θ conditional on $y_{1:T}$. Outside of models admitting closed-form solutions, this is typically the most practical way to draw samples from the marginal distributions of either *theta* or any subset of the states $X_{0:T}$, by marginalising the Monte Carlo samples.

The structure of the model invites Gibbs sampling: alternating between updating θ conditional on $X_{0:T}$, and updating $X_{0:T}$ conditional on θ. "These conditionals are typically much easier to sample from that the corresponding marginals." (due to the dependence structure in the HMM). The θ update consists of sampling from

$$p(\theta \mid x_{0:T}, y_{1:T}) \propto p(\theta)p(x_{0:T}, y_{1:T} \mid \theta),$$

which can be achieved quite easily with a Metropolis–Hastings step. The X update is the 'difficult' part, requiring a sample from

$$p(x_{0:T} \mid \theta, y_{1:T}) =: \gamma_T^\theta(x_{0:T}).$$

Define gamma for general s too:

$$\gamma_s^\theta(x_{0:s}) \propto \mu_\theta(x_0)g_\theta(y_0 \mid x_0)\prod_{r=1}^{s} f_\theta(x_r \mid x_{r-1})g_\theta(y_r \mid x_r).$$

This target distribution is suited to sequential Monte Carlo, and this is where the 'particle' part of particle Gibbs comes in. We update all of the hidden states $X_{0:T}$ in one Gibbs step, which consists of drawing one sample from a particle filter. To target the correct distribution, we use conditional SMC for these updates, conditional on the sample of $X_{0:T}$ from the previous sweep.

Refer back to the CSMC algorithm which I've already written down somewhere, explaining how the inputs to the algorithm correspond to functions/quantities introduced in this setting.

2.5.3 Ancestor sampling

Algorithm (or required changes to generic algorithm). Relation to backward sampling. When can it be implemented? Effect on performance (when is it effective?). Maybe illustrate/motivate with some plots as in the ancestor sampling note. The following was dumped from elsewhere and NEEDS REDRAFTING.

Ancestor sampling was first suggested by Nick Whiteley in the discussion on Andrieu, Doucet, and Holenstein (2010). Its contribution is to reduce autocorrelation between samples obtained using the particle Gibbs algorithm (Andrieu, Doucet, and Holenstein 2010). Proper way to cite discussion of paper?

Ancestral degeneracy leads to poor mixing

Particle Gibbs runs into problems when the time horizon T is large compared to the number of particles N. T is determined by the application at hand, and N is limited by computational resources, so we may not be able to control their relative size. The source of the problem is ancestral degeneracy. We know that in standard SMC algorithms this problem exists and its effect is to increase the variance of our Monte Carlo estimates. In particle Gibbs, the N simulated trajectories are not used to estimate anything; only one trajectory is sampled at each step, and becomes one state a Markov chain Monte Carlo estimate. Ancestral degeneracy now has a less direct effect: it causes the Markov chain to mix slowly.

To see why, take a look at Figure 2.5. In Figure 2.5a we have just completed the r^{th} Gibbs sweep, sampling $\theta[r]$ and $x_{0:T}[r]$. For the $(r+1)^{th}$ sweep, we take as immortal trajectory $x^*_{0:T} = x_{0:T}[r]$, and run conditional SMC. Due to ancestral degeneracy, many of the resulting trajectories coalesce, and since the immortal trajectory must survive across the whole time window, they tend to coalesce onto the immortal trajectory (Figure 2.5b). Now we obtain the next sample $x_{0:T}[r+1]$ by sampling a trajectory among the N we have just simulated. Whichever one we choose, it has a high amount of overlap with the immortal trajectory, i.e. the previous sample $x_{0:T}[r]$. This behaviour tends to repeat at every iteration, meaning the early X coordinates are getting 'stuck' (rarely being updated). This is clearly a problem for the mixing of the Markov chain. Another way to explain this is that the variables defining the immortal trajectory (indices and states) are never

110 *Backward Simulation in Particle MCMC*

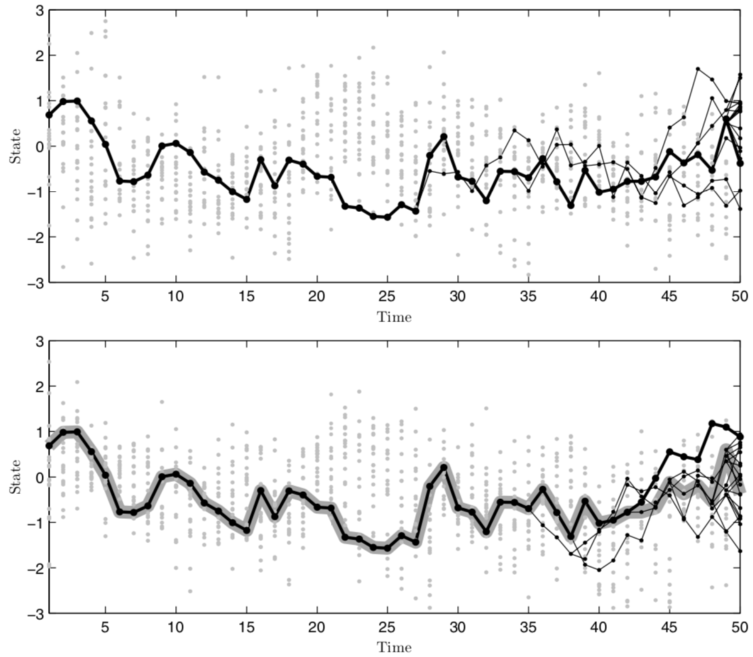


Fig. 5.3 Particle system generated by CSMC at iterations r (top) and $r+1$ (bottom) of the PG sampler. The dots show the particle positions, the thin black lines show the ancestral dependence of the particles and the thick black lines show the sampled trajectories $x_{1:T}[r]$ and $x_{1:T}[r+1]$, respectively. In the bottom pane, the thick gray line illustrates the conditioned path at iteration $r+1$, which by construction equals $x_{1:T}[r]$. Note that, due to path degeneracy, the particles shown as gray dots are not reachable by tracing any of the ancestral lineages from time T and back.

Figure 2.5: PLACEHOLDER. Copied from Lindsten and Schön (2013).

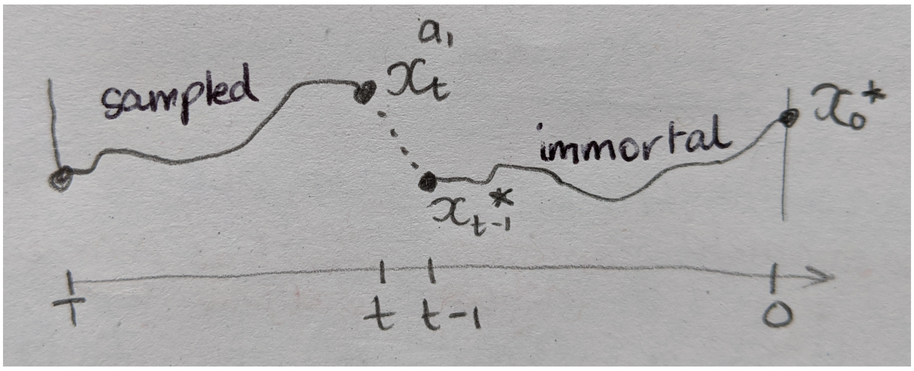


Figure 2.6: PLACEHOLDER. Interpretation of a resampling weight for the immortal offspring.

refreshed during the Gibbs sweep - this is the explanation given in Lindsten Ch5. It renders the particle Gibbs algorithm impractical for any such model where the time horizon T is too large: either we must run the Markov chain for longer, or increase the number of particles N in the conditional SMC step, neither of which is feasible on a limited computational budget.

The solution: ancestor sampling

An effective solution (where it is possible to implement it) was proposed by Nick Whiteley and is known as ancestor sampling. It consists of a simple modification to the resampling step within the conditional SMC algorithm. In the basic CSMC algorithm, at each time step the particles are resampled by multinomial resampling according to their weights. That is, at each time t, each non-immortal offspring is assigned a parent as so:

$$\mathbb{P}[a_t^{(j)} = i] \propto w_t^{(i)},$$

while the immortal offspring is deterministically assigned to the immortal parent.

In ancestor sampling we do the same thing, except that the immortal particle is also resampled, rather than being deterministically assigned:

$$\mathbb{P}[a_t^{(j)} = i] \propto \begin{cases} w_t^{(i)} & \text{non-immortal particles} \\ w_t^{(i)} \frac{\gamma_T^\theta((x_{0:t-1}^{(i)}, x_{t:T}^*))}{\gamma_{t-1}^\theta(x_{0:t-1}^{(i)})} & \text{immortal particle.} \end{cases}$$

The ratio of γs can be interpreted as the conditional probability of the trajectory continuing with $x_{t:T}^*$ given is starts with $x_{0:t-1}^{(i)}$ (see Figure 2.6). Using the structure of the hidden Markov model defined earlier, we can rewrite the ratio

$$\frac{\gamma_T^\theta((x_{0:t-1}^{(i)}, x_{t:T}^*))}{\gamma_{t-1}^\theta(x_{0:t-1}^{(i)})} \propto f_\theta(x_t^* \mid x_{t-1}^{(i)}) g_\theta(y_t \mid x_t^*) \prod_{s=t}^{T} f_\theta(x_r^* \mid x_{r-1}^*) g_\theta(y_r \mid x_r^*) \propto f_\theta(x_t^* \mid x_{t-1}^{(i)}).$$

Should the first propto actually be equality? So it looks like it should also be pretty easy to implement ancestor sampling for our model. Write down the pseudocode to prove it? The only catch is that we need to be able to evaluate f_θ pointwise, whereas in the basic algorithm we only need to draw samples from f_θ. This will rule out ancestor sampling in some applications.

Why ancestor sampling works

Ancestor sampling is backward sampling, but only for the immortal trajectory. (It isn't possible to do backward sampling during the forward sweep for any except the immortal trajectory - see that we can't evaluate the required γs without knowing the future states, which are known only for the immortal trajectory.) We know that backward sampling (on all trajectories, in a separate backward sweep) eradicates ancestral degeneracy. But we've only backward-sampled one trajectory, leaving the other $N-1$ trajectories to do their coalescing thing.

The important point is that ancestor sampling does not prevent ancestral degeneracy (it mitigates it a tiny bit like $1/N$). Ancestral degeneracy is pretty much as severe as ever; the difference is that the trajectories no longer coalesce preferentially onto the immortal trajectory. There is no longer an immortal trajectory to coalesce onto. An illustration of this can be seen in Figure 2.7.

Remember, the problem with ancestral degeneracy in particle Gibbs was that it induced strong autocorrelation among consecutive samples of $X_{0:T}$. It isn't a problem that the trajectories coalesce, as long as the thing they coalesce to is able to move, readily exploring the state space. This is achieved with ancestor sampling.

Why is ancestor sampling even a valid thing to do (i.e. still targetting the right thing)? Extended state space argument?(I think there is one in Lindsten Ch5). No need to go into it here unless there is a simple intuitive explanation to put the reader's mind at rest.

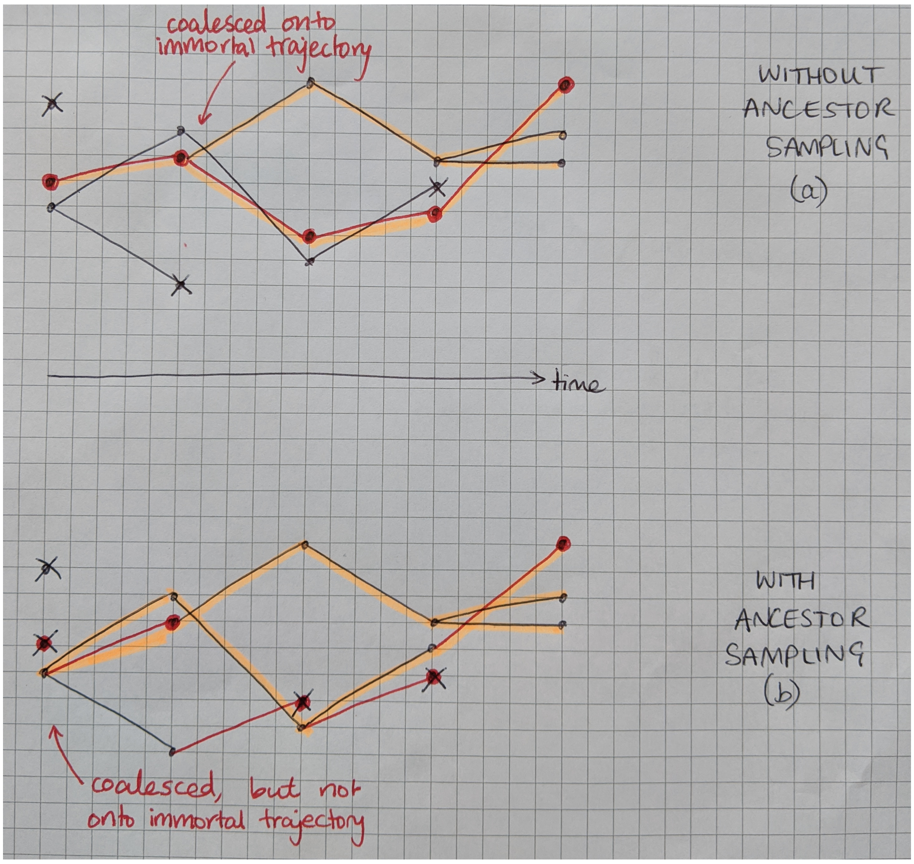


Figure 2.7: PLACEHOLDER. Illustration of how ancestor sampling prevents coalescence onto the immortal trajectory.

3 Limits

3.1 Encoding genealogies

3.1.1 The genealogical process

Before we can analyse genealogies, we need a way to encode them. The encoding will only include the information relevant to the sample genealogy, namely which lineages coalesce at which times. Information about particle positions and "killed" particles is ignored.

Let $\mathcal{P}_n$ be the space of partitions on $\{1, \dots, n\}$. For convenience, we now label time in reverse, so the terminal particles are at time 0, their parents are at time 1, and so on. Consider a sample of n terminal particles among a total of N particles, and label the sampled particles $1, \dots, n$. The genealogical process $(G_t^{(n,N)})_{t \in \mathbb{N}_0}$ for this sample is the $\mathcal{P}_n$-valued stochastic process such that labels i and j are in the same block of the partition $G_t^{(n,N)}$ if and only if terminal particles i and j have a common ancestor at time t (i.e. t generations back).

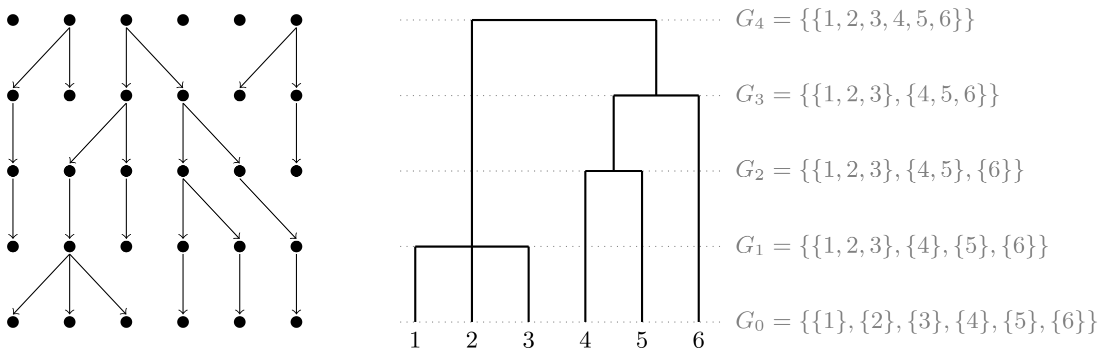
A formulation where $G_t^{(n,N)}$ takes values in the space of equivalence relations from $[n]$ to $[n]$ is sometimes used; interpreting partition blocks as equivalence classes, this formulation is equivalent to ours.

The initial (time 0) value of the process is the partition of singletons $G_0^{(n,N)} = \{\{1\}, \dots, \{n\}\}$, since all of the terminal particles are separate. The only possible non-identity transitions are those that merge some blocks of the partition; this encodes the coalescence of the corresponding lineages. The trivial partition $\{\{1, \dots, n\}\}$ is therefore an absorbing state, corresponding to all lineages in the sample having coalesced (i.e. the MRCA has been reached). The construction of the genealogical process from the resampling indices is illustrated in Figure 3.1.

3.1.2 Time scale

In order to get a continuous limit, we scale time by a function $\tau_N(\cdot)$. In the population genetics literature, a deterministic time scale can be used [citations] and/or this will have been mentioned already in pop gen example models (Section 2.2.3), whereas in our case τ_N depends on the offspring counts and is therefore random. To define the time scale we first define the pair merger rate

$$c_N(t) := \frac{1}{(N)_2} \sum_{i=1}^{N} (\nu_t^{(i)})_2. \tag{3.1}$$



(a) Resampling relationships

(b) Genealogy of terminal particles

Figure 3.1: Illustration of how the sample genealogy is encoded. (a) Relationships induced by resampling with six particles over four iterations. (b) The genealogy of the terminal particles, labelled with the value of the genealogical process G_t at each time.

This is the probability, conditional on $\nu_t^{(1:N)}$, that a randomly chosen pair of lineages in generation t merges exactly one generation back. To achieve the limiting pair merger rate of 1, as in the n-coalescent, we rescale time by the generalised inverse

$$\tau_N(t) := \inf\left\{ s \geq 1 : \sum_{r=1}^{s} c_N(r) \geq t \right\}. \tag{3.2}$$

The function τ_N maps continuous to discrete time, providing the link between the discrete-time SMC dynamics and the continuous-time Kingman limit. We will also need the following quantity, which is an upper bound on the rate of multiple mergers (three or more lineages merging, or two or more simultaneous pairwise mergers):

$$D_N(t) := \frac{1}{N(N)_2} \sum_{i=1}^{N} (\nu_t^{(i)})_2 \left\{ \nu_t^{(i)} + \frac{1}{N} \sum_{j \neq i} (\nu_t^{(j)})^2 \right\}. \tag{3.3}$$

Some basic properties are given in Proposition 3.1.

Proposition 3.1 (Properties of c_N)**.** *For all $t \in \mathbb{N}$, $t' > s' > 0$,*

(a) $c_N(t), D_N(t) \in [0,1]$

(b) $D_N(t) \leq c_N(t)$

(c) $c_N(t)^2 \leq c_N(t)$

(d) $$t' \leq \sum_{r=1}^{\tau_N(t')} c_N(r) \leq t' + 1.$$

(e) $$t' - s' - 1 \leq \sum_{r=\tau_N(s')+1}^{\tau_N(t')} c_N(r) \leq t' - s' + 1.$$

Proof. **(a)** $c_N(t)$ and $D_N(t)$ are clearly non-negative. Both are maximised when one of the offspring counts is equal to N and the rest are zero, in which case $c_N(t) = D_N(t) = 1$.
(b) As outlined in Koskela et al. (2018, p.9),

$$\begin{aligned} D_N(t) &:= \frac{1}{(N)_2} \sum_{i=1}^{N} (\nu_t^{(i)})_2 \frac{1}{N} \left\{ \nu_t^{(i)} + \frac{1}{N} \sum_{j \neq i}^{N} (\nu_t^{(j)})^2 \right\} \\ &\leq \frac{1}{(N)_2} \sum_{i=1}^{N} (\nu_t^{(i)})_2 \frac{1}{N} \left\{ \nu_t^{(i)} + \frac{1}{N} \sum_{j \neq i}^{N} N \nu_t^{(j)} \right\} \\ &= \frac{1}{(N)_2} \sum_{i=1}^{N} (\nu_t^{(i)})_2 \frac{1}{N} \left\{ \sum_{j=1}^{N} \nu_t^{(j)} \right\} \leq \frac{1}{(N)_2} \sum_{i=1}^{N} (\nu_t^{(i)})_2 = c_N(t). \end{aligned}$$

(c) is immediate given (a).
(d) follows directly from the definition of τ_N in (3.2).
(e) Writing

$$\sum_{r=\tau_N(s')+1}^{\tau_N(t')} c_N(r) = \sum_{r=1}^{\tau_N(t')} c_N(r) - \sum_{r=1}^{\tau_N(s')} c_N(r),$$

the result follows by applying (d) to both sums. ■

3.1.3 Transition probabilities

Introduce $p_{\xi\eta}$. Present expression for that (or at least for $p_{\xi\xi}$), and hence the bounds on it that will be used later (keeping big-O terms explicit where possible).

Let $\mathcal{P}_n$ be the space of partitions of $\{1, \ldots, n\}$, and denote by Δ the partition of singletons $\{\{1\}, \ldots, \{n\}\}$. For any $\xi, \eta \in \mathcal{P}_n$ and $t \in \mathbb{N}$, let $p_{\xi\eta}(t)$ denote the conditional transition probabilities of the genealogical process given $\nu_t^{(1:N)}$ ($t \in \mathbb{N}$, $\xi, \eta \in \mathcal{P}_n$). The transition probability $p_{\xi\eta}(t)$ can only be non-zero when η can be obtained from ξ by merging some blocks of ξ. Ordering the blocks by their least element, denote by b_i the number of blocks of ξ that merge to form block i in η ($i \in \{1, \dots, |\eta|\}$). Hence $b_1 + \dots + b_{|\eta|} = |\xi|$. Then the transition probability is given by

$$p_{\xi\eta}(t) := \frac{1}{(N)_{|\xi|}} \sum_{\substack{i_1 \neq \dots \neq i_{|\eta|} \\ =1}}^{N} (\nu_t^{(i_1)})_{b_1} \cdots (\nu_t^{(i_{|\eta|})})_{b_{|\eta|}}. \tag{3.4}$$

We will only need to work directly with the *identity* transition probabilities $p_{\xi\xi}(t)$. Upper and lower bounds on these probabilities are presented in Propositions 3.2 and 3.3.

Proposition 3.2 (Lower bound on identity transition probabilities)**.** *Let $\xi \in \mathcal{P}_n$, $N > 2$. Then*

$$p_{\xi\xi}(t) \geq 1 - \binom{|\xi|}{2} \frac{N^{n-2}}{(N-2)_{n-2}} \left[c_N(t) + B_{|\xi|} D_N(t)\right]$$

where $B_{|\xi|} = K(|\xi|-1)!(|\xi|-2)\exp(2\sqrt{2(|\xi|-2)})$ for some $K > 0$ that does not depend on $|\xi|$.

For weak convergence proof, refer to this proposition but rewrite the inequality using $\xi = \Delta$ and α_n, to provide a local target for cross-referencing. Similarly for UB.

Proof. We have the following expression for $p_{\xi\xi}(t)$, by subtracting all possible non-identity transitions (the omitted $k = |\xi|$ term would count identity transitions):

$$p_{\xi\xi}(t) = 1 - \frac{1}{(N)_{|\xi|}} \sum_{k=1}^{|\xi|-1} \sum_{\substack{b_1 \geq \dots \geq b_k = 1 \\ b_1 + \dots + b_k = |\xi|}}^{|\xi|} \frac{|\xi|!}{\prod_{j=1}^{|\xi|} (j!)^{\kappa_j} \kappa_j!} \sum_{\substack{i_1 \neq \dots \neq i_k = 1 \\ \text{all distinct}}}^{N} (\nu_t^{(i_1)})_{b_1} \dots (\nu_t^{(i_k)})_{b_k},$$

where $\kappa_i = |\{j : b_j = i\}|$ is the multiplicity of mergers of size i (κ_1 counts non-merger events, and we have the identity $\kappa_1 + 2\kappa_2 + \dots + |\xi|\kappa_{|\xi|} = |\xi|$). The combinatorial factor is the number of partitions of a sequence of length $|\xi|$ having κ_j subsequences of length j for each j (Fu 2006, Equation (11)).

We separate the $k = |\xi|-1$ term (which counts single pair mergers), for which $(b_1, b_2, \dots, b_{|\xi|-1}) = (2, 1, \dots, 1)$ and

$$\frac{|\xi|!}{\prod_{j=1}^{|\xi|} (j!)^{\kappa_j} \kappa_j!} = \binom{|\xi|}{2}.$$

For the remaining terms we use

$$\frac{|\xi|!}{\prod_{j=1}^{|\xi|} (j!)^{\kappa_j} \kappa_j!} \leq |\xi|!.$$

Thus

$$p_{\xi\xi}(t) \geq 1 - \frac{1}{(N)_{|\xi|}} \binom{|\xi|}{2} \sum_{\substack{i_1 \neq \ldots \neq i_{|\xi|-1} = 1 \\ \text{all distinct}}}^{N} (\nu_t^{(i_1)})_2 \nu_t^{(i_2)} \ldots \nu_t^{(i_{|\xi|-1})}$$
$$- \frac{1}{(N)_{|\xi|}} \sum_{k=1}^{|\xi|-1} \sum_{\substack{b_1 \geq \ldots \geq b_k = 1 \\ b_1 + \ldots + b_k = |\xi|}}^{|\xi|} |\xi|! \sum_{\substack{i_1 \neq \ldots \neq i_k = 1 \\ \text{all distinct}}}^{N} (\nu_t^{(i_1)})_{b_1} \ldots (\nu_t^{(i_k)})_{b_k}$$

Now, for the $k = |\xi| - 1$ term we use the bound

$$\sum_{i_1 \neq \ldots \neq i_{|\xi|-1} = 1}^{N} (\nu_t^{(i_1)})_2 \nu_t^{(i_2)} \ldots \nu_t^{(i_{|\xi|-1})} \leq N^{|\xi|-2} \sum_{i=1}^{N} (\nu_t^{(i)})_2$$

while for the other terms we have (similarly to Koskela et al. 2018, Lemma 1 Case 3)

$$\sum_{\substack{i_1 \neq \ldots \neq i_k = 1 \\ \text{all distinct}}}^{N} (\nu_t^{(i_1)})_{b_1} \ldots (\nu_t^{(i_k)})_{b_k} \leq \sum_{i=1}^{N} (\nu_t^{(i)})_2 \left(N^{|\xi|-2} - \sum_{\substack{j_1 \neq \ldots \neq j_{|\xi|-2} = 1 \\ \text{all distinct and } \neq i}}^{N} \nu_t^{(j_1)} \ldots \nu_t^{(j_{|\xi|-2})} \right)$$
$$\leq \sum_{i=1}^{N} (\nu_t^{(i)})_2 \left\{ N^{|\xi|-2} - (N - \nu_t^{(i)})^{|\xi|-2} + \binom{|\xi|-2}{2} \sum_{j \neq i} (\nu_t^{(j)})^2 \left(\sum_{k \neq i} \nu_t^{(k)} \right)^{|\xi|-4} \right\}$$
$$\leq \sum_{i=1}^{N} (\nu_t^{(i)})_2 \left\{ (|\xi| - 2) \nu_t^{(i)} N^{|\xi|-3} + \binom{|\xi|-2}{2} \sum_{j \neq i} (\nu_t^{(j)})^2 N^{|\xi|-4} \right\},$$

where the last step uses $(N - x)^b \geq N^b - bxN^{b-1}$ for $x \leq N$, $b \geq 0$. Hence

$$p_{\xi\xi}(t) \geq 1 - \frac{1}{(N)_{|\xi|}} \binom{|\xi|}{2} N^{|\xi|-2} \sum_{i=1}^{N} (\nu_t^{(i)})_2$$
$$- \frac{N^{|\xi|-3}}{(N)_{|\xi|}} |\xi|! \sum_{k=1}^{|\xi|-1} \sum_{\substack{b_1 \geq \ldots \geq b_k = 1 \\ b_1 + \ldots + b_k = |\xi|}}^{|\xi|} \sum_{i=1}^{N} (\nu_t^{(i)})_2 \left\{ (|\xi| - 2) \nu_t^{(i)} + \binom{|\xi|-2}{2} \frac{1}{N} \sum_{j \neq i} (\nu_t^{(j)})^2 \right\}.$$

The summands in the last line are independent of k, b_i, and the number of terms in the sums over k and $b_1, \ldots, b_k$ is bounded by $\gamma_{|\xi|-2}(|\xi| - 2)$, where γ_n is the number of integer partitions of n. By Hardy and Ramanujan (1918, Section 2), $\gamma_n < Ke^{2\sqrt{2n}}/n$ for a constant

$K > 0$ independent of n. Thus, for $|\xi| > 2$,

$$\begin{aligned}
p_{\xi\xi}(t) \geq 1 &- \frac{N^{|\xi|-2}}{(N-2)_{|\xi|-2}}\binom{|\xi|}{2}c_N(t) \\
&- \frac{N^{|\xi|-2}}{(N-2)_{|\xi|-2}}K\exp(2\sqrt{2(|\xi|-2)})|\xi|!\frac{1}{N(N)_2} \\
&\qquad \sum_{i=1}^{N}(\nu_t^{(i)})_2\left\{(|\xi|-2)\nu_t^{(i)} + \binom{|\xi|-2}{2}\frac{1}{N}\sum_{j\neq i}(\nu_t^{(j)})^2\right\} \\
\geq 1 &- \frac{N^{|\xi|-2}}{(N-2)_{|\xi|-2}}\binom{|\xi|}{2}c_N(t) \\
&- \frac{N^{|\xi|-2}}{(N-2)_{|\xi|-2}}K\exp(2\sqrt{2(|\xi|-2)})|\xi|!\binom{|\xi|-1}{2}D_N(t) \\
\geq 1 &- \frac{N^{|\xi|-2}}{(N-2)_{|\xi|-2}}\binom{|\xi|}{2}\left[c_N(t) + B_{|\xi|}D_N(t)\right]
\end{aligned}$$

where

$$\begin{aligned}
B_{|\xi|} &= \binom{|\xi|}{2}^{-1}K\exp(2\sqrt{2(|\xi|-2)})|\xi|!\binom{|\xi|-1}{2} \\
&= K(|\xi|-1)!(|\xi|-2)\exp(2\sqrt{2(|\xi|-2)}).
\end{aligned}$$

When $|\xi| \leq 2$, there are no terms with $k \leq |\xi|-2$, and the result is immediate. ∎

Proposition 3.3 (Upper bound on identity transition probabilities)**.** *Let $\xi \in \mathcal{P}_n$, $N >$ ... some threshold. Then*

$$p_{\xi\xi}(t) \leq 1 - \binom{|\xi|}{2}\frac{N^{n-2}}{(N-2)_{n-2}}\left[c_N(t) - B'_{|\xi|}D_N(t)\right]$$

where $B'_{|\xi|} = \binom{|\xi|-1}{2}$.

Proof. The proof follows Koskela et al. (2018, Proof of Lemma 1 Case 1) but with the terms in N kept explicit. (where possible/only some of them?)

... ∎

3.2 An existing limit theorem

State KJJS theorem. Discuss the conditions in detail. Give outline of proof.

3.3 A new limit theorem

State our limit theorem. Give intuition for the new condition. Compare to KJJS: why our conditions might be considered “weaker” (Moran model example, and whatever else we said to our referee/in the BJJK article); our condition is easier to check (as demonstrated in later corollaries).

Theorem 3.1. *Let $\nu_t^{(1:N)}$ denote the offspring numbers in an IPS satisfying the standing assumption and such that, for any N sufficiently large, $\mathbb{P}[\tau_N(t) = \infty] = 0$ for all finite t. Suppose that there exists a deterministic sequence $(b_N)_{N\geq 1}$ such that $\lim_{N\to\infty} b_N = 0$ and*

$$\frac{1}{(N)_3}\sum_{i=1}^{N}\mathbb{E}_t[(\nu_t^{(i)})_3] \leq b_N \frac{1}{(N)_2}\sum_{i=1}^{N}\mathbb{E}_t[(\nu_t^{(i)})_2] \tag{3.5}$$

for all N, uniformly in $t \geq 1$. Then the rescaled genealogical process $(G^{(n,N)}_{\tau_N(t)})_{t\geq 0}$ converges in the sense of finite-dimensional distributions to Kingman's n-coalescent as $N \to \infty$.

3.3.1 Proof of theorem

Proof that KJJS conditions are implied by ours. Modification of KJJS proof (or even write out a complete proof?) using weaker bound on $p_{\xi\xi}$ (that bound should have been stated and proved already in transition probabilities section).

4 Applications

Earth's crammed with heaven,
And every common bush afire with God,
But only he who sees takes off his shoes;
The rest sit round and pluck blackberries.

Elizabeth Barrett Browning

4.1 Multinomial resampling

This is the easy-to-analyse scheme, because conditionally i.i.d., and was presented in KJJS already. Now (with our simpler conditions) it is easier to show.

4.2 Stratified resampling

Corollary 4.1. *Consider an SMC algorithm using stratified resampling, such that the standing assumption is satisfied. Assume that there exists a constant $a \in [1, \infty)$ such that for all x, x', t,*

$$\frac{1}{a} \leq g_t(x, x') \leq a. \tag{4.1}$$

Assume that $\mathbb{P}[\tau_N(t) = \infty] = 0$ for all finite t. Let $(G_t^{(n,N)})_{t \geq 0}$ denote the genealogy of a random sample of n terminal particles from the output of the algorithm when the total number of particles used is N. Then, for any fixed n, the time-scaled genealogy $(G_{\tau_N(t)}^{(n,N)})_{t \geq 0}$ converges to Kingman's n-coalescent as $N \to \infty$, in the sense of finite-dimensional distributions.

Proof. Recall that the sequence of σ-algebras

$$\mathcal{H}_t := \sigma(X_{t-1}^{(1:N)}, X_t^{(1:N)}, w_{t-1}^{(1:N)}, w_t^{(1:N)}) \tag{4.2}$$

are such that $\nu_t^{(1:N)}$ is conditionally independent of the filtration $\mathcal{F}_{t-1}$ given $\mathcal{H}_t$. With stratified resampling, conditional on the weights each offspring count almost surely takes one of four values: $\nu_t^{(i)} \in \{\lfloor Nw_t^{(i)} \rfloor - 1, \lfloor Nw_t^{(i)} \rfloor, \lfloor Nw_t^{(i)} \rfloor + 1, \lfloor Nw_t^{(i)} \rfloor + 2\}$. Denote $p_j^{(i)} := \mathbb{P}[\nu_t^{(i)} = \lfloor Nw_t^{(i)} \rfloor + j \mid \mathcal{H}_t]$ for $j = -1, 0, 1, 2$. Now

$$\begin{aligned}\mathbb{E}[(\nu_t^{(i)})_2 \mid \mathcal{H}_t] = p_{-1}^{(i)}(\lfloor Nw_t^{(i)} \rfloor - 1)_2 + p_0^{(i)}(\lfloor Nw_t^{(i)} \rfloor)_2 + p_1^{(i)}(\lfloor Nw_t^{(i)} \rfloor + 1)_2 \\ + p_2^{(i)}(\lfloor Nw_t^{(i)} \rfloor + 2)_2\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}[(\nu_t^{(i)})_3 \mid \mathcal{H}_t] &= p_{-1}^{(i)}(\lfloor Nw_t^{(i)} \rfloor - 1)_3 + p_0^{(i)}(\lfloor Nw_t^{(i)} \rfloor)_3 + p_1^{(i)}(\lfloor Nw_t^{(i)} \rfloor + 1)_3 \\ &\qquad + p_2^{(i)}(\lfloor Nw_t^{(i)} \rfloor + 2)_3 \\ &= p_{-1}^{(i)}(\lfloor Nw_t^{(i)} \rfloor - 3)(\lfloor Nw_t^{(i)} \rfloor - 1)_2 + p_0^{(i)}(\lfloor Nw_t^{(i)} \rfloor - 2)(\lfloor Nw_t^{(i)} \rfloor)_2 \\ &\qquad + p_1^{(i)}(\lfloor Nw_t^{(i)} \rfloor - 1)(\lfloor Nw_t^{(i)} \rfloor + 1)_2 + p_2^{(i)}\lfloor Nw_t^{(i)} \rfloor(\lfloor Nw_t^{(i)} \rfloor + 2)_2 \\ &\leq \lfloor Nw_t^{(i)} \rfloor\{p_0^{(i)}(\lfloor Nw_t^{(i)} \rfloor)_2 + p_1^{(i)}(\lfloor Nw_t^{(i)} \rfloor + 1)_2\} \\ &= \lfloor Nw_t^{(i)} \rfloor \mathbb{E}[(\nu_t^{(i)})_2 \mid \mathcal{H}_t] \\ &\leq a^2 \mathbb{E}[(\nu_t^{(i)})_2 \mid \mathcal{H}_t]\end{aligned}$$

The last line uses the almost sure bound $w_t^{(i)} \leq a^2/N$ which follows from (4.1) along with the form of the weights in Algorithm 1. Note that some terms in the above expressions may be equal to zero when $w_t^{(i)}$ is small enough, but the bound always holds nonetheless. Since the above holds for all i, applying the tower rule we have

$$\frac{1}{(N)_3} \sum_{i=1}^{N} \mathbb{E}_t[(\nu_t^{(i)})_3] \leq \frac{a^2}{N-2} \frac{1}{(N)_2} \sum_{i=1}^{N} \mathbb{E}_t[(\nu_t^{(i)})_2]$$

satisfying (3.5) with $b_N := a^2/(N-2) \to 0$. The result then follows by applying Theorem 3.1. ■

Lemma 4.1. *Consider an SMC algorithm using stratified resampling. Suppose that*

$$\varepsilon \leq q_t(x, x') \leq \varepsilon^{-1}$$

uniformly in x, x' for some $\varepsilon \in (0, 1]$, and that there exist $\zeta > 0$ and $\delta \in (0, 1)$ such that

$$\mathbb{P}[\max_i w_t^{(i)} - \min_i w_t^{(i)} \geq 2\delta/N \mid \mathcal{F}_{t-1}] \geq \zeta$$

for infinitely many t. Then, for all $N > 1$, $\mathbb{P}[\tau_N(t) = \infty] = 0$ for all finite t.

Finish constructing the proof.

Proof. It is sufficient to prove, under the conditions, that

$$\sum_{r=0}^{\infty} \mathbb{P}[c_N(r) > 2/N^2 \mid \mathcal{F}_{r-1}] = \infty. \tag{4.3}$$

By the filtered Borel-Cantelli lemma (see for example Durrett 2019, Theorem 4.3.4), this implies that, almost surely, $c_N(r) > 2/N^2$ for infinitely many r. Thus, for any $t < \infty$,

$$\mathbb{P}[\tau_N(t) < \infty] = \mathbb{P}\left[\exists s < \infty : \sum_{r=1}^{s} c_N(r) \geq t\right] = 1 \tag{4.4}$$

which is the desired result.

Now we have to prove (4.3). Firstly,

$$\begin{aligned}\mathbb{P}[c_N(t) \leq 2/N^2 \mid \mathcal{H}_t] &= \mathbb{P}[\nu_t^{(i)} = 1 \, \forall i \in \{1, \dots, N\} \mid \mathcal{H}_t] \\ &\leq \mathbb{P}[\nu_t^{(i^\star)} = 1 \mid \mathcal{H}_t],\end{aligned}$$

where $i^\star := \operatorname{argmax}_i \{w_t^{(i)}\}$ (but note that the inequality holds when $i^\star$ is taken to be any particular index).

... ■

4.3 Stochastic rounding

Corollary 4.2. *Consider an SMC algorithm using any stochastic rounding as its resampling scheme, such that the standing assumption is satisfied. Assume that there exists a constant $a \in [1, \infty)$ such that for all x, x', t,*

$$\frac{1}{a} \leq g_t(x, x') \leq a. \tag{4.5}$$

Assume that $\mathbb{P}[\tau_N(t) = \infty] = 0$ for all finite t. Let $(G_t^{(n,N)})_{t \geq 0}$ denote the genealogy of a random sample of n terminal particles from the output of the algorithm when the total number of particles used is N. Then, for any fixed n, the time-scaled genealogy $(G_{\tau_N(t)}^{(n,N)})_{t \geq 0}$ converges to Kingman's n-coalescent as $N \to \infty$, in the sense of finite-dimensional distributions.

By the way, does the lack of conditions of q_t here imply that we do not even need the transition kernels to admit densities?

Proof version 2. We can apply exactly the proof of Corollary 4.1, except that stochastic rounding is more restrictive than stratified resampling, so that the only possible offspring counts (almost surely, conditional on weights) are $\nu_t^{(i)} \in \{\lfloor N w_t^{(i)} \rfloor, \lfloor N w_t^{(i)} \rfloor + 1\}$. We simply set $p_{-1}^{(i)} = p_2^{(i)} = 0$ in the proof of Corollary 4.1 to see that

$$\frac{1}{(N)_3}\sum_{i=1}^{N}\mathbb{E}_t[(\nu_t^{(i)})_3] \leq \frac{a^2}{N-2}\frac{1}{(N)_2}\sum_{i=1}^{N}\mathbb{E}_t[(\nu_t^{(i)})_2]$$

as required. The result then follows by applying Theorem 3.1. ■

We can also show, under additional conditions, that the assumption $\mathbb{P}[\tau_N(t) = \infty] = 0$ for all finite t holds.

Lemma 4.2. *Consider an SMC algorithm using any stochastic rounding as its resampling scheme. Suppose that*

$$\varepsilon \leq q_t(x, x') \leq \varepsilon^{-1}$$

uniformly in x, x' for some $\varepsilon \in (0, 1]$, and that there exist $\zeta > 0$ and $\delta \in (0, 1)$ such that

$$\mathbb{P}[\max_i w_t^{(i)} - \min_i w_t^{(i)} \geq 2\delta/N \mid \mathcal{F}_{t-1}] \geq \zeta$$

for infinitely many t. Then, for all $N > 1$, $\mathbb{P}[\tau_N(t) = \infty] = 0$ for all finite t.

The extra condition that the transition densities are bounded above and away from zero ... say something intelligent. The second condition is required to ensure that, at least infinitely often, the weights are not equal to $(1, \dots, 1)/N$, since stochastic rounding is degenerate under equal weights and this will cause the time scale to explode. It is hardly conceivable that any real SMC algorithm would fail to satisfy this condition, which effectively ensures that the weights cannot be "too well-behaved".

If I find a more elegant proof for e.g. stratified, I can use the techniques to simplify this one a bit too.

Proof. Let $\mathcal{H}_t$ be defined as in (4.2). The first step is to show that whenever $\max_i w_t^{(i)} \geq (1+\delta)/N$, $\mathbb{P}[c_N(t) > 2/N^2 \mid \mathcal{H}_t] = \mathbb{P}[c_N(t) \neq 0 \mid \mathcal{H}_t]$ is bounded below uniformly in t. For this purpose we need consider only weight vectors such that $w_t^{(i)} \in (0, 2/N)$ for all i; otherwise $\mathbb{P}[c_N(t) \neq 0 \mid \mathcal{H}_t] = 1$ by the definition of stochastic rounding.

Denote $\mathcal{S}_{N-1}^{\delta} = \{w^{(1:N)} \in \mathcal{S}_{N-1} : \forall i,\, 0 < w^{(i)} < 2/N;\, \max_i w^{(i)} \geq (1+\delta)/N\}$ for any $\delta \in (0,1)$, where $\mathcal{S}_k$ denotes the k-dimensional probability simplex. Fix arbitrary $w_t^{(1:N)} \in \mathcal{S}_{N-1}^{\delta}$. Set $i^\star = \arg\max_i w_t^{(i)}$ and denote $\mathcal{I} = \{i \in \{1, \dots, N\} : w^{(i)} > 1/N\}$. Since all weights are in $(0, 2/N)$, for $i \in \mathcal{I}, \nu_t^{(i)} \in \{1, 2\}$ and for $i \notin \mathcal{I}, \nu_t^{(i)} \in \{0, 1\}$; and since the offspring counts must sum to N, we can write

$$
\begin{aligned}
\mathbb{P}[c_N(t) \leq 2/N^2 \mid \mathcal{H}_t] &= \mathbb{P}[\nu_t^{(i)} = 1 \, \forall i \in \{1, \dots, N\} \mid \mathcal{H}_t] \\
&= \mathbb{P}[\nu_t^{(i)} = 1 \, \forall i \in \mathcal{I} \mid \mathcal{H}_t] \\
&= \prod_{i \in \mathcal{I}} \mathbb{P}[\nu_t^{(i)} = 1 \mid \nu_t^{(j)} = 1 \, \forall j \in \mathcal{I} : j < i; \mathcal{H}_t] \\
&= \mathbb{P}[\nu_t^{(i^\star)} = 1 \mid \mathcal{H}_t] \prod_{\substack{i \in \mathcal{I} \\ i \neq i^\star}} \mathbb{P}[\nu_t^{(i)} = 1 \mid \nu_t^{(i^\star)} = 1; \nu_t^{(j)} = 1 \, \forall j \in \mathcal{I} : j < i; \mathcal{H}_t] \\
&\leq \mathbb{P}[\nu_t^{(i^\star)} = 1 \mid \mathcal{H}_t]. \qquad (4.6)
\end{aligned}
$$

The final inequality holds with equality when $|\mathcal{I}| = 1$, i.e. the only weight larger than $1/N$ is $w_t^{(i^\star)}$. Thus $\mathbb{P}[c_N(t) > 2/N^2 \mid \mathcal{H}_t]$ is minimised on $\mathcal{S}_{N-1}^{\delta}$ when only one weight is larger than $1/N$, in which case the values of the other weights do not affect this probability.

Define $w_{\delta'} = \{(1, \dots, 1) + \delta' e_{i^\star} - \delta' e_{j^\star}\}/N$ for fixed $i^\star \neq j^\star$ and $\delta' \in (0, 1)$, where e_i denotes the ith canonical basis vector in $\mathbb{R}^N$. As in the proof of Corollary 4.2, define $p_0^{(i)} = \mathbb{P}[\nu_t^{(i)} = \lfloor N w_t^{(i)} \rfloor \mid \mathcal{H}_t]$ and $p_1^{(i)} = \mathbb{P}[\nu_t^{(i)} = \lfloor N w_t^{(i)} \rfloor + 1 \mid \mathcal{H}_t]$. Then from (4.6) we have

$$
\mathbb{P}[c_N(t) > 2/N^2 \mid \mathcal{H}_t, w_t^{(1:N)} = w_{\delta'}] = 1 - \mathbb{P}[\nu_t^{(i^\star)} = 1 \mid \mathcal{H}_t, w_t^{(1:N)} = w_{\delta'}] = p_1^{(i^\star)},
$$

evaluated on $w_{\delta'}$. We will need a lower bound on $p_1^{(i^\star)}$ when $w_t^{(1:N)} = w_{\delta'}$. We first derive expressions for $p_0^{(i)}$ and $p_1^{(i)}$ up to a constant, then use $p_0^{(i)} + p_1^{(i)} = 1$ to get a normalised bound. We have

$$
\begin{aligned}
p_0^{(i)} &= C(1 - N w_t^{(i)} + \lfloor N w_t^{(i)} \rfloor) \\
&\quad \times \sum_{\substack{a_{1:N} \in \{1, \dots, N\}^N : \\ |\{j : a_j = i\}| = \lfloor N w_t^{(i)} \rfloor}} \mathbb{P}\left[a_t^{(1:N)} = a_{1:N} \mid \nu_t^{(i)}, w_t^{(1:N)}\right] \prod_{k=1}^{N} q_{t-1}(X_t^{(a_k)}, X_{t-1}^{(k)}), \\
p_1^{(i)} &= C(N w_t^{(i)} - \lfloor N w_t^{(i)} \rfloor) \\
&\quad \times \sum_{\substack{a_{1:N} \in \{1, \dots, N\}^N : \\ |\{j : a_j = i\}| = \lfloor N w_t^{(i)} \rfloor + 1}} \mathbb{P}\left[a_t^{(1:N)} = a_{1:N} \mid \nu_t^{(i)}, w_t^{(1:N)}\right] \prod_{k=1}^{N} q_{t-1}(X_t^{(a_k)}, X_{t-1}^{(k)}).
\end{aligned}
$$

Applying the bounds on q_t, we have

$$
\begin{aligned}
C(1 - N w_t^{(i)} + \lfloor N w_t^{(i)} \rfloor) \varepsilon^N \leq p_0^{(i)} \leq C(1 - N w_t^{(i)} + \lfloor N w_t^{(i)} \rfloor) \varepsilon^{-N}, \\
C(N w_t^{(i)} - \lfloor N w_t^{(i)} \rfloor) \varepsilon^N \leq p_1^{(i)} \leq C(N w_t^{(i)} - \lfloor N w_t^{(i)} \rfloor) \varepsilon^{-N}
\end{aligned}
$$

from which we construct the normalised bound

$$p_1^{(i)} \geq \frac{(Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor)\varepsilon^N}{(Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor)\varepsilon^{-N} + (1 - Nw_t^{(i)} + \lfloor Nw_t^{(i)} \rfloor)\varepsilon^{-N}} = (Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor)\varepsilon^{2N}.$$

When $w_t^{(1:N)} = w_{\delta'}$, we have $w_t^{(i^\star)} = (1+\delta')/N$, so $p_1^{(i^\star)} \geq \delta'\varepsilon^{2N}$, which is increasing in δ'. We conclude that $\mathbb{P}[c_N(t) > 2/N^2 | \mathcal{H}_t, \max_i w_t^{(i)} \geq (1+\delta)/N] \geq \min_{\delta' \geq \delta} \delta'\varepsilon^{2N} = \delta\varepsilon^{2N}$.

A slight modification of this argument yields $\mathbb{P}[c_N(t) > 2/N^2 | \mathcal{H}_t, \min_i w_t^{(i)} \leq (1-\delta)/N] \geq \delta\varepsilon^{2N}$. Whenever $\max_i w_t^{(i)} - \min_i w_t^{(i)} \geq 2\delta/N$, either $\max_i w_t^{(i)} \geq (1+\delta)/N$ or $\min_i w_t^{(i)} \leq (1-\delta)/N$, so we have $\mathbb{P}[c_N(t) > 2/N^2 | \mathcal{H}_t, \max_i w_t^{(i)} - \min_i w_t^{(i)} \geq 2\delta/N] \geq \delta\varepsilon^{2N}$. Thus

$$\mathbb{P}[c_N(t) > 2/N^2 \mid \mathcal{H}_t] \geq \delta\varepsilon^{2N} \mathbb{1}_{\max_i w_t^{(i)} - \min_i w_t^{(i)} \geq 2\delta/N}.$$

Using the D-separation established in Appendix **??** combined with the tower property, we have

$$\begin{aligned}\mathbb{P}[c_N(t) > 2/N^2 \mid \mathcal{F}_{t-1}] = \mathbb{E}_t\left[\mathbb{P}[c_N(t) > 2/N^2 \mid \mathcal{H}_t, \mathcal{F}_{t-1}]\right] = \mathbb{E}_t\left[\mathbb{P}[c_N(t) > 2/N^2 \mid \mathcal{H}_t]\right] \\ \geq \delta\varepsilon^{2N}\mathbb{P}[\max_i w_t^{(i)} - \min_i w_t^{(i)} \geq 2\delta/N \mid \mathcal{F}_{t-1}],\end{aligned}$$

which is bounded below by $\zeta\delta\varepsilon^{2N}$ for infinitely many t. Hence,

$$\sum_{t=0}^{\infty} \mathbb{P}[c_N(t) > 2/N^2 \mid \mathcal{F}_{t-1}] = \infty.$$

By a filtered version of the second Borel–Cantelli lemma (see for example Durrett 2019, Theorem 4.3.4), this implies that $c_N(t) > 2/N^2$ for infinitely many t, almost surely. This ensures, for all $t < \infty$, that $\mathbb{P}\left[\exists s < \infty : \sum_{r=1}^{s} c_N(r) \geq t\right] = 1$, which by definition of $\tau_N(t)$ is equivalent to $\mathbb{P}[\tau_N(t) = \infty] = 0$. ■

4.4 Residual resampling with stratified residuals

Corollary 4.3. *Consider an SMC algorithm using residual resampling with stratified residuals, such that the standing assumption is satisfied. Assume that there exists a constant $a \in [1, \infty)$ such that for all x, x', t,*

$$\frac{1}{a} \leq g_t(x, x') \leq a. \tag{4.7}$$

Assume that $\mathbb{P}[\tau_N(t) = \infty] = 0$ for all finite t. Let $(G_t^{(n,N)})_{t\geq 0}$ denote the genealogy of a random sample of n terminal particles from the output of the algorithm when the total number of particles used is N. Then, for any fixed n, the time-scaled genealogy $(G_{\tau_N(t)}^{(n,N)})_{t\geq 0}$ converges to Kingman's n-coalescent as $N \to \infty$, in the sense of finite-dimensional distributions.

Proof. We can apply exactly the proof of Corollary 4.1, except that residual-stratified resampling is more restrictive than stratified resampling, so that the only possible offspring counts (almost surely, conditional on weights) are $\nu_t^{(i)} \in \{\lfloor Nw_t^{(i)} \rfloor, \lfloor Nw_t^{(i)} \rfloor + 1, \lfloor Nw_t^{(i)} \rfloor + 2\}$. We simply set $p_{-1}^{(i)} = 0$ in the proof of Corollary 4.1 to see that

$$\frac{1}{(N)_3} \sum_{i=1}^{N} \mathbb{E}_t[(\nu_t^{(i)})_3] \leq \frac{a^2}{N-2} \frac{1}{(N)_2} \sum_{i=1}^{N} \mathbb{E}_t[(\nu_t^{(i)})_2]$$

as required. The result then follows by applying Theorem 3.1. ∎

State and prove finite time-scale lemma.

4.5 Residual resampling with multinomial residuals

If I manage to prove this corollary, it would make this chapter satisfyingly complete :-)

4.6 The worst possible resampling scheme

Remark that this one doesn't converge to KC, but rather to a star-shaped coalescent.

4.7 Conditional SMC

Why CSMC is qualitatively different to, say, standard SMC with multinomial resampling (immortal particle etc.). Reasons for restriction to multinomial resampling, conjecture that limit theorem holds for other schemes in CSMC.

Corollary 4.4. *Consider a conditional SMC algorithm using multinomial resampling, such that the standing assumption is satisfied. Assume there exist constants* $\varepsilon \in (0,1], a \in [1,\infty)$ *and probability density* h *such that for all* x, x', t,

$$\frac{1}{a} \leq g_t(x,x') \leq a, \quad \varepsilon h(x') \leq q_t(x,x') \leq \frac{1}{\varepsilon} h(x'). \tag{4.8}$$

Let $(G_t^{(n,N)})_{t\geq 0}$ *denote the genealogy of a random sample of* n *terminal particles from the output of the algorithm when the total number of particles used is* N*. Then, for any fixed* n*, the time-scaled genealogy* $(G_{\tau_N(t)}^{(n,N)})_{t\geq 0}$ *converges to Kingman's* n*-coalescent as* $N \to \infty$*, in the sense of finite-dimensional distributions.*

Proof. Define the conditioning σ-algebra $\mathcal{H}_t$ as in (4.2). We assume without loss of generality that the immortal particle takes index 1 in each generation. This significantly simplifies the notation, but the same argument holds if the immortal indices are taken to be $a^\star_{(0:T)}$ rather than $(1,\dots,1)$.

The parental indices are conditionally independent, as in standard SMC with multinomial resampling, but we have to treat $i = 1$ as a special case. We have the following conditional law on parental indices

$$\mathbb{P}\left[a_t^{(i)} = a_i \mid \mathcal{H}_t\right] \propto \begin{cases} \mathbb{1}_{a_i=1} & i = 1 \\ w_t^{(a_i)} q_{t-1}(X_t^{(a_i)}, X_{t-1}^{(i)}) & i = 2, \dots, N. \end{cases}$$

The joint conditional law is therefore

$$\mathbb{P}\left[a_t^{(1:N)} = a_{1:N} \mid \mathcal{H}_t\right] \propto \mathbb{1}_{a_1=1} \prod_{i=2}^{N} w_t^{(a_i)} q_{t-1}(X_t^{(a_i)}, X_{t-1}^{(i)}).$$

First we make the following observation, which follows from a balls-in-bins coupling. Assume (4.8). Then for any function $f : \{1, \dots, N\}^N \to \mathbb{R}$ such that (for a fixed i) $f(a_t'^{(1:N)}) \geq f(a_t^{(1:N)})$ whenever $|\{j : a_t'^{(j)} = i\}| \geq |\{j : a_t^{(j)} = i\}|$,

$$\mathbb{E}[f(A_{1,i}^{(1:N)})] \leq \mathbb{E}[f(a_t^{(1:N)}) \mid \mathcal{H}_t] \leq \mathbb{E}[f(A_{2,i}^{(1:N)})] \tag{4.9}$$

where the elements of $A_{1,i}^{(1:N)}, A_{2,i}^{(1:N)}$ are all mutually independent and independent of $\mathcal{F}_\infty$, and distributed according to

$$A_{1,i}^{(j)} \sim \begin{cases} \delta_1 & j = 1 \\ \text{Categorical}\left((\varepsilon/a)^{\mathbb{1}_{i=1} - \mathbb{1}_{i \neq 1}}, \dots, (\varepsilon/a)^{\mathbb{1}_{i=N} - \mathbb{1}_{i \neq N}}\right) & j \neq 1 \end{cases}$$
$$A_{2,i}^{(j)} \sim \begin{cases} \delta_1 & j = 1 \\ \text{Categorical}\left((a/\varepsilon)^{\mathbb{1}_{i=1} - \mathbb{1}_{i \neq 1}}, \dots, (a/\varepsilon)^{\mathbb{1}_{i=N} - \mathbb{1}_{i \neq N}}\right) & j \neq 1 \end{cases}$$

where the vector of probabilities is given up to a constant in the argument of Categorical distributions. We use these random vectors to construct bounds that are independent of $\mathcal{F}_\infty$. Also define the corresponding offspring counts $V_1^{(i)} = |\{j : A_{1,i}^{(j)} = i\}|$, $V_2^{(i)} = |\{j : A_{2,i}^{(j)} = i\}|$, for $i = 1, \dots, N$, which have marginal distributions

$$V_1^{(i)} \overset{d}{=} \mathbb{1}_{i=1} + \text{Binomial}\left(N-1, \frac{\varepsilon/a}{(\varepsilon/a) + (N-1)(a/\varepsilon)}\right),$$
$$V_2^{(i)} \overset{d}{=} \mathbb{1}_{i=1} + \text{Binomial}\left(N-1, \frac{a/\varepsilon}{(a/\varepsilon) + (N-1)(\varepsilon/a)}\right).$$

Now consider the function $f_i(a_t^{(1:N)}) := (\nu_t^{(i)})_2$. We can apply (4.9) to obtain the lower bound

$$\begin{aligned}\frac{1}{(N)_2}\sum_{i=1}^{N}\mathbb{E}[(\nu_t^{(i)})_2 \mid \mathcal{H}_t] \geq \frac{1}{(N)_2}\sum_{i=1}^{N}\mathbb{E}[(V_1^{(i)})_2] &= \frac{1}{(N)_2}\left[\mathbb{E}[(V_1^{(1)})_2] + \sum_{i=2}^{N}\mathbb{E}[(V_1^{(i)})_2]\right] \\
&= \frac{1}{(N)_2}\left[\frac{(N-1)_2(\varepsilon/a)^2}{\{(\varepsilon/a)+(N-1)(a/\varepsilon)\}^2} + \frac{2(N-1)(\varepsilon/a)}{(\varepsilon/a)+(N-1)(a/\varepsilon)}\right. \\
&\qquad \left. + \sum_{i=2}^{N}\frac{(N-1)_2(\varepsilon/a)^2}{\{(\varepsilon/a)+(N-1)(a/\varepsilon)\}^2}\right] \\
&= \frac{1}{(N)_2}\left[\frac{2(N-1)(\varepsilon/a)}{(\varepsilon/a)+(N-1)(a/\varepsilon)} + \sum_{i=1}^{N}\frac{(N-1)_2(\varepsilon/a)^2}{\{(\varepsilon/a)+(N-1)(a/\varepsilon)\}^2}\right]\end{aligned}$$

using the moments of the Binomial distribution (see Mosimann 1962 for example) along with the identity $(X+1)_2 \equiv 2(X)_1 + (X)_2$. This is further bounded by

$$\begin{aligned}\frac{1}{(N)_2}\sum_{i=1}^{N}\mathbb{E}[(\nu_t^{(i)})_2 \mid \mathcal{H}_t] &\geq \frac{1}{(N)_2}\left\{\frac{2(N-1)(\varepsilon/a)}{N(a/\varepsilon)} + \frac{(N)_3(\varepsilon/a)^2}{N^2(a/\varepsilon)^2}\right\} \\
&= \frac{1}{N^2}\left\{\frac{2\varepsilon^2}{a^2} + \frac{(N-2)\varepsilon^4}{a^4}\right\}. \qquad (4.10)\end{aligned}$$

Similarly, we derive an upper bound on $f_i(a_t^{(1:N)}) := (\nu_t^{(i)})_3$, this time using the identity $(X+1)_3 \equiv 3(X)_2 + (X)_3$:

$$\begin{aligned}\frac{1}{(N)_3}\sum_{i=1}^{N}\mathbb{E}[(\nu_t^{(i)})_3 \mid \mathcal{H}_t] &\leq \frac{1}{(N)_3}\left[\mathbb{E}[(V_2^{(1)})_3] + \sum_{i=2}^{N}\mathbb{E}[(V_2^{(i)})_3]\right] \\
&\leq \frac{1}{(N)_3}\left[\frac{3(N-1)_2(a/\varepsilon)^2}{\{(a/\varepsilon)+(N-1)(\varepsilon/a)\}^2} + \sum_{i=1}^{N}\frac{(N-1)_3(a/\varepsilon)^3}{\{(a/\varepsilon)+(N-1)(\varepsilon/a)\}^3}\right] \\
&\leq \frac{1}{(N)_3}\left\{\frac{3(N-1)_2(a/\varepsilon)^2}{N^2(\varepsilon/a)^2} + \frac{(N)_4(a/\varepsilon)^3}{N^3(\varepsilon/a)^3}\right\} \\
&= \frac{1}{(N)_3}\left\{\frac{3(N-1)_2}{N^2}\frac{a^4}{\varepsilon^4} + \frac{(N)_4}{N^3}\frac{a^6}{\varepsilon^6}\right\} \\
&= \frac{1}{N^3}\left\{\frac{3a^4}{\varepsilon^4} + \frac{(N-3)a^6}{\varepsilon^6}\right\}.\end{aligned}$$

We apply the tower property and conditional independence as in Corollary 4.2, upper bounding the ratio by

$$\begin{aligned}\frac{\frac{1}{(N)_3}\sum_{i=1}^{N}\mathbb{E}_t[(\nu_t^{(i)})_3]}{\frac{1}{(N)_2}\sum_{i=1}^{N}\mathbb{E}_t[(\nu_t^{(i)})_2]} &\leq \frac{N^2}{N^3}\frac{\frac{3a^4}{\varepsilon^4}+\frac{(N-3)a^6}{\varepsilon^6}}{\frac{2\varepsilon^2}{a^2}+\frac{(N-2)\varepsilon^4}{a^4}} \leq \frac{1}{N}\frac{a^6}{\varepsilon^6}\frac{3+(N-3)a^2/\varepsilon^2}{2+(N-2)\varepsilon^2/a^2} \\
&\leq \frac{1}{N}\frac{a^6}{\varepsilon^6}\left\{\frac{3}{2} + \frac{N-3}{N-2}\frac{a^4}{\varepsilon^4}\right\} \leq \frac{1}{N}\left\{\frac{3a^6}{2\varepsilon^6} + \frac{a^{10}}{\varepsilon^{10}}\right\} =: b_N \underset{N\to\infty}{\longrightarrow} 0.\end{aligned}$$

Thus (3.5) is satisfied. It remains to show that, for N sufficiently large, $\mathbb{P}[\tau_N(t) = \infty] = 0$

for all finite t, a technicality which is proved in Lemma 4.3. Applying Theorem 3.1 gives the result. ■

Lemma 4.3. *Consider a conditional SMC algorithm using multinomial resampling, satisfying the standing assumption and (4.8). Then, for all $N > 2$, $\mathbb{P}[\tau_N(t) = \infty] = 0$ for all finite t.*

Proof. Since $c_N(t) \in [0, 1]$ almost surely and has strictly positive expectation, for any fixed N the distribution of $c_N(t)$ with given expectation that maximises $\mathbb{P}[c_N(t) = 0 \mid \mathcal{F}_{t-1}]$ is two atoms, at 0 and 1 respectively. To ensure the correct expectation, the atom at 1 should have mass $\mathbb{P}[c_N(t) = 1 \mid \mathcal{F}_{t-1}] = \mathbb{E}_t[c_N(t)]$, which is bounded below by (4.10). If $c_N(t) > 0$ then $c_N(t) \geq 2/(N)_2 > 2/N^2$. Hence, in general $\mathbb{P}[c_N(t) > 2/N^2 \mid \mathcal{F}_{t-1}] \geq \mathbb{E}_t[c_N(t)]$. Applying (4.10), we have for any finite N,

$$\sum_{t=0}^{\infty} \mathbb{P}[c_N(t) > 2/N^2 \mid \mathcal{F}_{t-1}] \geq \sum_{t=0}^{\infty} \mathbb{E}_t[c_N(t)] \geq \sum_{t=0}^{\infty} \frac{1}{N^2} \left\{ \frac{2\varepsilon^2}{a^2} + \frac{(N-2)\varepsilon^4}{a^4} \right\} = \infty$$

By an argument analogous to the conclusion of Lemma 4.2, $\mathbb{P}[\tau_N(t) = \infty] = 0$ for all $t < \infty$. ■

4.7.1 Effect of ancestor sampling

Argue that ancestor sampling removes bias towards assigning offspring to immortal line, and leaves exactly the same genealogy as standard SMC with multinomial resampling. The following was dumped from elsewhere and NEEDS REDRAFTING.

Ancestor sampling interrupts the immortal trajectory, splitting it into disjoint pieces (see Figure 2.7b). So perhaps it doesn't make sense to talk about genealogies, now that we don't have a strictly coalescing process evolving backwards in time.

Probably need some conditioning e.g. on $\mathcal{H}_t$ in the following equations. But let's look at this resampling scheme backwards in time anyway. With the time indices now reversed, and the notation made consistent with our work on genealogies, we have

$$\mathbb{P}[a_t^{(j)} = a_i] \propto \begin{cases} w_t^{(a_i)} q_{t-1}(X_t^{(a_i)}, X_{t-1}^{(j)}) & \text{non-immortal particles} \\ w_t^{(a_i)} q_{t-1}(X_t^{(a_i)}, X_{t-1}^{*}) & \text{immortal particle.} \end{cases}$$

Might be neater to avoid bringing in j at all, and write $a_t^{(i)} = a_i$ etc. But when j is the index of the immortal particle, $X_{t-1}^{(j)} = X_{t-1}^{*}$, so the reverse-time view makes no distinction between the immortal particle and others in terms of how their parents are chosen. We have

$$\mathbb{P}[a_t^{(j)} = a_i] \propto w_t^{(a_i)} q_{t-1}(X_t^{(a_i)}, X_{t-1}^{(j)})$$

for all particles. This is the same probability we get with multinomial resampling in standard SMC.

This viewpoint gives us another intuition about the effect of ancestor sampling: the immortal particle is no longer specially favoured when it comes to producing offspring. In the basic algorithm, the immortal parent produces on average one more offspring at each generation than any other parent (and must produce at least one offspring). It follows that the immortal line cannot die out (is immortal), and also that it is a more likely site for coalescence events (i.e. producing at least two offspring) than any other parent. With ancestor sampling, the immortal parent loses this advantage; it is treated just like the other parents. It is no longer a particularly favourable site for coalescing, so the trajectory onto which everyone coalesces might just as well be any of the other $N-1$ trajectories. This prevents the Markov chain getting stuck as in Figure 2.5.

5 Weak Convergence

> At the age of twenty-one he wrote a treatise upon the Binomial Theorem, which has had a European vogue. On the strength of it he won the Mathematical Chair at one of our smaller universities, and had, to all appearances, a most brilliant career before him.
>
> Sherlock Holmes

Some motivation/discussion about weak convergence: why it is more useful than FDDs, that the following theorem has the same conditions as the FDDs one...

We start by defining a suitable metric space. Let $\mathcal{P}_n$ be the space of partitions of $\{1, \ldots, n\}$. Denote by $\mathcal{X}$ the set of all functions mapping $[0, \infty)$ to $\mathcal{P}_n$ that are right-continuous with left limits. (Our rescaled genealogical process $(\mathcal{G}^{(n,N)}_{\tau_N(t)})_{t\geq 0}$ and our encoding of the n-coalescent are piecewise-constant functions mapping time $t \in [0, \infty)$ to partitions, and thus live in the space $\mathcal{X}$.) Finally, equip the space $\mathcal{P}_n$ with the zero-one metric,

$$\rho(\xi, \eta) = 1 - \delta_{\xi\eta} := \begin{cases} 0 & \text{if } \xi = \eta \\ 1 & \text{otherwise} \end{cases} \tag{5.1}$$

for any $\xi, \eta \in \mathcal{P}_n$.

Theorem 5.1. *Let $\nu_t^{(1:N)}$ denote the offspring numbers in an interacting particle system satisfying the standing assumption and such that, for any N sufficiently large, for all finite t, $\mathbb{P}\{\tau_N(t) = \infty\} = 0$. Suppose that there exists a deterministic sequence $(b_N)_{N\in\mathbb{N}}$ such that $\lim_{N\to\infty} b_N = 0$ and*

$$\frac{1}{(N)_3} \sum_{i=1}^{N} \mathbb{E}_t\{(\nu_t^{(i)})_3\} \leq b_N \frac{1}{(N)_2} \sum_{i=1}^{N} \mathbb{E}_t\{(\nu_t^{(i)})_2\} \tag{5.2}$$

for all N, uniformly in $t \geq 1$. Then the rescaled genealogical process $(G^{(n,N)}_{\tau_N(t)})_{t\geq 0}$ converges weakly in $(\mathcal{X}, \rho)$ to Kingman's n-coalescent as $N \to \infty$.

Proof. The structure of the proof follows Möhle (1999), albeit with considerable technical complication due to the dependence between generations (non-neutrality) in our model. Is this the main/only source of complication? Since we already have convergence of the finite-dimensional distributions (Theorem ?? refers to a previous chapter not yet written), strengthening this to weak convergence requires relative compactness of the sequence of processes $\{(G^{(n,N)}_{\tau_N(t)})_{t\geq 0}\}_{N\in\mathbb{N}}$.

Ethier and Kurtz (2009, Chapter 3, Corollary 7.4) provides a necessary and sufficient condition for relative compactness: $\mathcal{P}_n$ is finite and therefore complete and separable, and the sample paths of $(G^{(n,N)}_{\tau_N(t)})_{t\geq 0}$ live in $\mathcal{X}$, so the conditions of the corollary are satisfied. The corollary states that the sequence of processes $\{(G^{(n,N)}_{\tau_N(t)})_{t\geq 0}\}_{N\in\mathbb{N}}$ is relatively compact if and only if the following two conditions hold:

1. For every $\varepsilon > 0$, $t \geq 0$ there exists a compact set $\Gamma \subseteq \mathcal{P}_n$ such that

$$\liminf_{N\to\infty} \mathbb{P}[G^{(n,N)}_{\tau_N(t)} \in \Gamma] \geq 1 - \varepsilon \tag{5.3}$$

2. For every $\varepsilon > 0$, $t > 0$ there exists $\delta > 0$ such that

$$\liminf_{N\to\infty} \mathbb{P}[\omega(G^{(n,N)}_{\tau_N(\cdot)}, \delta, t) < \varepsilon] \geq 1 - \varepsilon \tag{5.4}$$

   where ω is the modulus of continuity:

$$\omega(G^{(n,N)}_{\tau_N(\cdot)}, \delta, t) := \inf \max_{i\in[K]} \sup_{u,v\in[T_{i-1},T_i)} \rho\left(G^{(n,N)}_{\tau_N(u)}, G^{(n,N)}_{\tau_N(v)}\right) \tag{5.5}$$

   with the infimum taken over all partitions of the form $0 = T_0 < T_1 < \cdots < T_{K-1} < t \leq T_K$ such that $\min_{i\in[K]}(T_i - T_{i-1}) > \delta$. Clarify that such a partition with any K is valid, i.e. K is not fixed.

In our case, Condition 1 is satisfied automatically with $\Gamma = \mathcal{P}_n$, since $\mathcal{P}_n$ is finite and hence compact. Intuitively, Condition 2 ensures that the jumps of the process are well-separated. In our case where ρ is the zero-one metric, we see that $\rho(G^{(n,N)}_{\tau_N(u)}, G^{(n,N)}_{\tau_N(v)})$ is equal to 1 if there is a jump between times u and v, and 0 otherwise. Taking the supremum and maximum then indicates whether there is a jump inside any of the intervals of the given partition; this can only be equal to zero if all of the jumps up to time t occur exactly at the times $T_0, \ldots, T_K$. The infimum over all allowed partitions, then, can only be equal to zero if no two jumps occur less than δ (unscaled) time apart, because of the restriction we placed on these partitions.

The proof is concentrated on proving Condition 2. To do this, we use a coupling with another process that contains all of the jumps of the genealogical process, with the addition of some extra jumps. This process is constructed in such a way that it can be shown to satisfy Condition 2, and hence so does the genealogical process.

Define $p_t := \max_{\xi\in\mathcal{P}_n}\{1 - p_{\xi\xi}(t)\} = 1 - p_{\Delta\Delta}(t)$, where Δ denotes the trivial partition of singletons $\{\{1\}, \ldots, \{n\}\}$. For a proof that the maximum is attained at $\xi = \Delta$, see Lemma

5.1. Following Möhle (1999), we now construct the two-dimensional conditionally on $\mathcal{F}$? Markov process $(Z_t, S_t)_{t\in\mathbb{N}_0}$ on $\mathbb{N}_0 \times \mathcal{P}_n$ with transition probabilities

$$\mathbb{P}[Z_t = j, S_t = \eta \mid Z_{t-1} = i, S_{t-1} = \xi] = \begin{cases} 1 - p_t & \text{if } j = i \text{ and } \xi = \eta \\ p_{\xi\xi}(t) + p_t - 1 & \text{if } j = i+1 \text{ and } \xi = \eta \\ p_{\xi\eta}(t) & \text{if } j = i+1 \text{ and } \xi \neq \eta \\ 0 & \text{otherwise} \end{cases} \tag{5.6}$$

and initial state $Z_0 = 0$, $S_0 = \Delta$. The construction is such that the marginal (S_t) has the same distribution as the genealogical process of interest, and (Z_t) has jumps at all the times (S_t) does plus some extra jumps. (The definition of p_t ensures that the probability in the second case is non-negative, attaining the value zero when $\xi = \Delta$.) And the transition probabilities (jump times) of Z do not depend on the current state.

Denote by $0 = T_0^{(N)} < T_1^{(N)} < \dots$ the jump times of the rescaled process $(Z_{\tau_N(t)})_{t\geq 0}$, and by $\varpi_i^{(N)} := T_i^{(N)} - T_{i-1}^{(N)}$ the corresponding holding times.

Suppose that for some $t > 0$, there exists $m \in \mathbb{N}$ and $\delta > 0$ such that $\varpi_i^{(N)} > \delta$ for all $i \in \{1, \dots, m\}$, and $T_m^{(N)} \geq t$. Then $K_N := \min\{i : T_i^{(N)} \geq t\}$ is well-defined with $1 \leq K_N \leq m$, and $T_1^{(N)}, \dots, T_{K_N}^{(N)}$ form a partition of the form required for Condition 2. Indeed $(Z_{\tau_N(\cdot)})$ is constant on every interval $[T_{i-1}^{(N)}, T_i^{(N)})$ by construction, so $\omega((Z_{\tau_N(\cdot)}), \delta, t) = 0$. We therefore have that for each $m \in \mathbb{N}$ and $\delta > 0$,

$$\mathbb{P}[\omega((Z_{\tau_N(\cdot)}), \delta, t) < \varepsilon] \geq \mathbb{P}[T_m^{(N)} \geq t, \varpi_i^{(N)} > \delta \, \forall i \in \{1, \dots, m\}]. \tag{5.7}$$

Thus a sufficient condition for Condition 2 is: for any $\varepsilon > 0$, $t > 0$, there exist $m \in \mathbb{N}$, $\delta > 0$ such that

$$\liminf_{N\to\infty} \mathbb{P}[T_m^{(N)} \geq t, \varpi_i^{(N)} > \delta \, \forall i \in \{1, \dots, m\}] \geq 1 - \varepsilon. \tag{5.8}$$

Since $T_m^{(N)} = \varpi_1^{(N)} + \dots + \varpi_m^{(N)}$, there is a positive correlation between $T_m^{(N)}$ and each of the $\varpi_i^{(N)}$, so

$$\begin{aligned} \mathbb{P}[T_m^{(N)} \geq & t, \varpi_i^{(N)} > \delta \, \forall i \in \{1, \dots, m\}] \\ &= \mathbb{P}[T_m^{(N)} \geq t \mid \varpi_i^{(N)} > \delta \, \forall i \in \{1, \dots, m\}] \, \mathbb{P}[\varpi_i^{(N)} > \delta \, \forall i \in \{1, \dots, m\}] \\ &\geq \mathbb{P}[T_m^{(N)} \geq t] \, \mathbb{P}[\varpi_i^{(N)} > \delta \, \forall i \in \{1, \dots, m\}]. \end{aligned} \tag{5.9}$$

Due to Lemma 5.2, the limiting distributions of $\varpi_i^{(N)}$ are i.i.d. $\mathrm{Exp}(\alpha_n)$, so

$$\liminf_{N\to\infty} \mathbb{P}[\varpi_i^{(N)} > \delta \, \forall i \in \{1, \dots, m\}] = (e^{-\alpha_n \delta})^m \tag{5.10}$$

and

$$\liminf_{N\to\infty} \mathbb{P}[T_m^{(N)} \geq t] = \liminf_{N\to\infty} \mathbb{P}[\varpi_1^{(N)} + \cdots + \varpi_m^{(N)} \geq t] = e^{-\alpha_n \delta} \sum_{i=0}^{m-1} \frac{(\alpha_n t)^i}{i!}. \tag{5.11}$$

using the series expansion for the Erlang cumulative distribution function. citation? Hence

$$\liminf_{N\to\infty} \mathbb{P}[T_m^{(N)} \geq t, \varpi_i^{(N)} > \delta \, \forall i \in \{1, \ldots, m\}] \geq (e^{-\alpha_n \delta})^{m+1} \sum_{i=0}^{m-1} \frac{(\alpha_n t)^i}{i!}, \tag{5.12}$$

which can be made $\geq 1 - \varepsilon$ by taking m sufficiently large and δ sufficiently small. Since this argument applies for any ε and t, (5.8) and hence Condition 2 is satisfied, and the proof is complete. ■

Lemma 5.1. $\max_{\xi \in \mathcal{P}_n} (1 - p_{\xi\xi}(t)) = 1 - p_{\Delta\Delta}(t)$.

Proof. Consider any $\xi \in E$ consisting of k blocks ($1 \leq k \leq n-1$), and any $\xi' \in E$ consisting of $k+1$ blocks. From the definition of $p_{\xi\eta}(t)$ (Koskela et al. 2018, Equation (1)),

$$p_{\xi\xi}(t) = \frac{1}{(N)_k} \sum_{\substack{i_1, \ldots, i_k \\ \text{all distinct}}} \nu_t^{(i_1)} \cdots \nu_t^{(i_k)}. \tag{5.13}$$

Similarly,

$$\begin{aligned} p_{\xi'\xi'}(t) &= \frac{1}{(N)_{k+1}} \sum_{\substack{i_1, \ldots, i_k, i_{k+1} \\ \text{all distinct}}} \nu_t^{(i_1)} \cdots \nu_t^{(i_k)} \nu_t^{(i_{k+1})} \\ &= \frac{1}{(N)_k (N-k)} \sum_{\substack{i_1, \ldots, i_k \\ \text{all distinct}}} \left\{ \nu_t^{(i_1)} \cdots \nu_t^{(i_k)} \sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^{N} \nu_t^{(i_{k+1})} \right\}. \end{aligned} \tag{5.14}$$

Discarding the zero summands,

$$p_{\xi'\xi'}(t) = \frac{1}{(N)_k (N-k)} \sum_{\substack{i_1, \ldots, i_k \\ \text{all distinct:} \\ \nu_t^{(i_1)}, \ldots, \nu_t^{(i_k)} > 0}} \left\{ \nu_t^{(i_1)} \cdots \nu_t^{(i_k)} \sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^{N} \nu_t^{(i_{k+1})} \right\}. \tag{5.15}$$

The inner sum is

$$\sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^{N} \nu_t^{(i_{k+1})} = \left\{ \sum_{i=1}^{N} \nu_t^{(i)} - \sum_{i \in \{i_1, \ldots, i_k\}} \nu_t^{(i)} \right\} \leq N - k \tag{5.16}$$

since $\nu_t^{(i_1)}, \dots, \nu_t^{(i_k)}$ are all at least 1. Hence

$$p_{\xi'\xi'}(t) \leq \frac{N-k}{(N)_k (N-k)} \sum_{\substack{i_1,\dots,i_k \\ \text{all distinct:} \\ \nu_t^{(i_1)},\dots,\nu_t^{(i_k)}>0}} \nu_t^{(i_1)} \cdots \nu_t^{(i_k)} = p_{\xi\xi}(t). \tag{5.17}$$

Thus $p_{\xi\xi}(t)$ is decreasing in the number of blocks of ξ, and is therefore minimised by taking $\xi = \Delta$, which achieves the maximum n blocks. This choice in turn maximises $1 - p_{\xi\xi}(t)$, as required. ■

Lemma 5.2. *The finite-dimensional distributions of $\varpi_1^{(N)}, \varpi_2^{(N)}, \dots$ converge as $N \to \infty$ to those of $\varpi_1, \varpi_2, \dots$, where the ϖ_i are independent* $\mathrm{Exp}(\alpha_n)$ *distributed random variables.*

Proof. There is a continuous bijection between the jump times $T_1^{(N)}, T_2^{(N)}, \dots$ and the holding times $\varpi_1^{(N)}, \varpi_2^{(N)}, \dots$, so convergence of the holding times to $\varpi_1, \varpi_2, \dots$ is equivalent to convergence of the jump times to $T_1, T_2, \dots$, where $T_i := \varpi_1 + \cdots + \varpi_i$. We will work with the jump times, following the structure of Möhle (1999, Lemma 3.2).

The idea is to prove by induction that, for any $k \in \mathbb{N}$ and $t_1, \dots, t_k > 0$,

$$\lim_{N\to\infty} \mathbb{P}[T_1^{(N)} \leq t_1, \dots, T_k^{(N)} \leq t_k] = \mathbb{P}[T_1 \leq t_1, \dots, T_k \leq t_k]. \tag{5.18}$$

Take the basis case $k = 1$. Then

$$\mathbb{P}[T_1 \leq t] = \mathbb{P}[\varpi_1 \leq t] = 1 - e^{-\alpha_n t} \tag{5.19}$$

and $T_1^{(N)} > t$ if and only if Z has no jumps up to time t: Expectation appears by tower property to remove (implicit) conditioning in transition probabilities?

$$\mathbb{P}[T_1^{(N)} > t] = \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1 - p_r)\right]. \tag{5.20}$$

Lemma 5.6 shows that this probability converges to $e^{-\alpha_n t}$ as required.

For the induction step, assume that (5.18) holds for some k. We have the following decomposition:

$$\mathbb{P}[T_1^{(N)} \leq t_1, \dots, T_{k+1}^{(N)} \leq t_{k+1}] = \mathbb{P}[T_1^{(N)} \leq t_1, \dots, T_k^{(N)} \leq t_k] - \mathbb{P}[T_1^{(N)} \leq t_1, \dots, T_k^{(N)} \leq t_k, T_{k+1}^{(N)} > t_{k+1}] \tag{5.21}$$

The first term on the RHS converges to $\mathbb{P}[T_1 \leq t_1, \dots, T_k \leq t_k]$ by the induction hypothesis, and it remains to show that

$$\lim_{N\to\infty} \mathbb{P}[T_1^{(N)} \leq t_1, \dots, T_k^{(N)} \leq t_k, T_{k+1}^{(N)} > t_{k+1}] = \mathbb{P}[T_1 \leq t_1, \dots, T_k \leq t_k, T_{k+1} > t_{k+1}]. \tag{5.22}$$

As shown in Möhle (1999), the RHS

$$\mathbb{P}[T_1 \le t_1, \dots, T_k \le t_k, T_{k+1} > t_{k+1}] = \alpha_n^k e^{-\alpha_n t} \sum_{\substack{i_1 \le \cdots \le i_{k-1} \\ \in \{0,\dots,k\}: \\ i_j \ge j \forall j}} \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}. \tag{5.23}$$

The event on the LHS can be written (Möhle 1999)

$$\mathbb{P}[T_1^{(N)} \le t_1, \dots, T_k^{(N)} \le t_k, T_{k+1}^{(N)} > t_{k+1}] = \mathbb{E}\left[\sum_{\substack{r_1 < \cdots < r_k: \\ r_i \le \tau_N(t_i) \forall i}} \left(\prod_{i=1}^{k} p_{r_i} \right) \left(\prod_{\substack{r=1 \\ \notin \{r_1,\dots,r_k\}}}^{\tau_N(t)} (1 - p_r) \right) \right], \tag{5.24}$$

that is, there are jumps at some times $r_1, \dots, r_k$ and identity transitions at all other times. Due to Lemmata 5.7 and 5.8, this probability converges to the correct limit. This completes the induction. ■

5.1 Bounds on sum-products

Lemma 5.3. *Fix $t > 0$, $l \in \mathbb{N}$.*

(a) $$\sum_{s_1 \ne \cdots \ne s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \le (t+1)^l$$

(b) $$t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \binom{l}{2} (t+1)^{l-2} \le \sum_{s_1 \ne \cdots \ne s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \le t^l + c_N(\tau_N(t))(t+1)^l$$

Proof. **(a)** It is a true fact that

$$\sum_{s_1 \ne \cdots \ne s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \le \left(\sum_{s=0}^{\tau_N(t)} c_N(s) \right)^l, \tag{5.25}$$

as can be seen by considering the multinomial expansion of the RHS. By definition of τ_N,

$$t \le \sum_{s=0}^{\tau_N(t)} c_N(s) \le t + 1, \tag{5.26}$$

hence

$$\sum_{s_1 \ne \cdots \ne s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \le (t+1)^l. \tag{5.27}$$

(b) As pointed out in Koskela et al. (2018, Equation (8)),

$$\sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \geq \left(\sum_{s=0}^{\tau_N(t)} c_N(s) \right)^l - \binom{l}{2} \left(\sum_{s=0}^{\tau_N(t)} c_N(s)^2 \right) \left(\sum_{s=0}^{\tau_N(t)} c_N(s) \right)^{l-2}. \tag{5.28}$$

Substituting (5.26) into the RHS of (5.28) yields the lower bound.

For the upper bound we have

$$\sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \leq \left(\sum_{s=0}^{\tau_N(t)} c_N(s) \right)^l \leq \left(\sum_{s=0}^{\tau_N(t)-1} c_N(s) + c_N(\tau_N(t)) \right)^l \leq \left[t + c_N(\tau_N(t)) \right]^l, \tag{5.29}$$

again using the definition of τ_N. A binomial expansion yields

$$\left[t + c_N(\tau_N(t)) \right]^l = t^l + \sum_{i=0}^{l-1} \binom{l}{i} t^i c_N(\tau_N(t))^{l-i} = t^l + c_N(\tau_N(t)) \sum_{i=0}^{l-1} \binom{l}{i} t^i c_N(\tau_N(t))^{l-1-i}, \tag{5.30}$$

then since $c_N(s) \leq 1$ for all s,

$$\sum_{i=0}^{l-1} \binom{l}{i} t^i c_N(\tau_N(t))^{l-1-i} \leq \sum_{i=0}^{l-1} \binom{l}{i} t^i \leq (t+1)^l. \tag{5.31}$$

Putting this together yields the upper bound. ■

Lemma 5.4. *Fix $t > 0$, $l \in \mathbb{N}$. Let B be a positive constant which may depend on n.*

$$\sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} \left[c_N(s_j) + B D_N(s_j) \right] \leq \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1+B)^l. \tag{5.32}$$

Proof. We start with a binomial expansion:

$$\begin{aligned} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} \left[c_N(s_j) + B D_N(s_j) \right] &= \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \sum_{\mathcal{I} \subseteq [l]} B^{l-|\mathcal{I}|} \left(\prod_{i \in \mathcal{I}} c_N(s_i) \right) \left(\prod_{j \notin \mathcal{I}} D_N(s_j) \right) \\ &= \sum_{\mathcal{I} \subseteq [l]} B^{l-|\mathcal{I}|} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \left(\prod_{i \in \mathcal{I}} c_N(s_i) \right) \left(\prod_{j \notin \mathcal{I}} D_N(s_j) \right) \end{aligned} \tag{5.33}$$

where $[l] := \{1, \dots, l\}$. Since the sum is over all permutations of $s_1, \dots, s_l$, we may arbitrarily choose an ordering for $\{1, \dots, l\}$ such that $\mathcal{I} = \{1, \dots, |\mathcal{I}|\}$:

$$\sum_{\mathcal{I} \subseteq [l]} B^{l-|\mathcal{I}|} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \left(\prod_{i \in \mathcal{I}} c_N(s_i) \right) \left(\prod_{j \notin \mathcal{I}} D_N(s_j) \right) = \sum_{I=0}^{l} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right) \tag{5.34}$$

Separating the term $I = l$,

$$\sum_{I=0}^{l} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right)$$
$$= \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) + \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right). \tag{5.35}$$

In the second term on the RHS, there is always at least one D_N term, and $c_N(s) \geq D_N(s)$ for all s (Koskela et al. 2018, p.9), so we can write

$$\begin{aligned}
\sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right) &\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{l-1} c_N(s_i) \right) D_N(s_l) \\
&\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \left(\sum_{s_1 \neq \cdots \neq s_{l-1}}^{\tau_N(t)} \prod_{i=1}^{l-1} c_N(s_i) \right) \sum_{s_l=1}^{\tau_N(t)} D_N(\\
&\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} (t+1)^{l-1} \sum_{s=1}^{\tau_N(t)} D_N(s)
\end{aligned} \tag{5.36}$$

using (5.25) and (5.26). Finally, by the Binomial Theorem,

$$\sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} (t+1)^{l-1} \sum_{s=1}^{\tau_N(t)} D_N(s) \leq \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1+B)^l, \tag{5.37}$$

which, together with (5.35), concludes the proof. ■

Lemma 5.5. *Fix $t > 0$, $l \in \mathbb{N}$. Let B be a positive constant which may depend on n.*

$$\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - B D_N(s_j)] \geq \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1+B)^l. \tag{5.38}$$

Proof. A binomial expansion and subsequent manipulation as in (5.33)–(5.35) gives

$$
\begin{aligned}
\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - BD_N(s_j)] &= \sum_{\mathcal{I}\subseteq[l]} (-B)^{l-|\mathcal{I}|} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i\in\mathcal{I}} c_N(s_i)\right)\left(\prod_{j\notin\mathcal{I}} D_N(s_j)\right) \\
&= \sum_{I=0}^{l} \binom{l}{I} (-B)^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right)\left(\prod_{j=I+1}^{l} D_N(s_j)\right) \\
&= \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) + \sum_{I=0}^{l-1} \binom{l}{I} (-B)^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right)\left(\prod_{j=I+1}^{l} D_N(s_j)\right) \\
&\geq \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) - \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right)\left(\prod_{j=I+1}^{l} D_N(s_j)\right)
\end{aligned}
\tag{5.39}
$$

where the last inequality just multiplies some positive terms by -1. Then (5.36)–(5.37) can be applied directly (noting that an upper bound on negative terms gives a lower bound overall):

$$
-\sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right)\left(\prod_{j=I+1}^{l} D_N(s_j)\right) \geq -\left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)(t+1)^{l-1}(1+B)^{l}
\tag{5.40}
$$

which concludes the proof. ∎

5.2 Main components of weak convergence

Lemma 5.6 (Basis step)**.** *For any* $0 < t < \infty$,

$$
\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1-p_r)\right] = e^{-\alpha_n t}
\tag{5.41}
$$

where $\alpha_n := n(n-1)/2$.

Proof. We start by showing that $\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right] \leq e^{-\alpha_n t}$.
From Koskela et al. (2018, Lemma 1 Case 1), taking $\xi = \Delta$, we have for each r

$$
1 - p_r = p_{\Delta\Delta}(r) \leq 1 - \alpha_n(1 + O(N^{-1}))\left[c_N(r) - B'_n D_N(r)\right]
\tag{5.42}
$$

where the $O(N^{-1})$ term does not depend on r. When $N \geq 3$, a sufficient condition to ensure the bound in (5.42) is non-negative is that the event

$$
E_N^1(r) := \left\{ c_N(r) \leq \frac{(N-2)_{n-2}}{\alpha_n N^{n-2}} \right\}
\tag{5.43}
$$

occurs. We will also need to control the sign of $c_N(r) - B_n' D_N(r)$, for which we define the event

$$E_N^2(r) := \left\{ c_N(r) \geq B_n' D_N(r) \right\}, \tag{5.44}$$

and we define $E_N^1 := \bigcap_{r=1}^{\tau_N(t)} E_N^1(r)$ and $E_N^2 := \bigcap_{r=1}^{\tau_N(t)} E_N^2(r)$. Then

$$1 - p_r = p_{\Delta\Delta}(r) \leq 1 - \alpha_n(1 + O(N^{-1})) \left[c_N(r) - B_n' D_N(r) \right] \mathbb{1}_{E_N^1 \cap E_N^2}. \tag{5.45}$$

Applying a multinomial expansion and then separating the positive and negative terms,

$$\begin{aligned} \prod_{r=1}^{\tau_N(t)} (1 - p_r) &\leq 1 + \sum_{l=1}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} \left[c_N(s_j) - B_n' D_N(s_j) \right] \mathbb{1}_{E_N^1 \cap E_N^2} \\ &= 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} \left[c_N(s_j) - B_n' D_N(s_j) \right] \mathbb{1}_{E_N^1 \cap E_N^2} \\ &\quad - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} \left[c_N(s_j) - B_n' D_N(s_j) \right] \mathbb{1}_{E_N^1 \cap E_N^2}. \end{aligned} \tag{5.46}$$

This is further bounded by applying Lemma 5.5 and then both bounds of Lemma 5.3(b):

$$\begin{aligned} \prod_{r=1}^{\tau_N(t)} (1 - p_r) &\leq 1 + \Bigg\{ \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \\ &\quad - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left[\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1 + B_n')^l \right] \\ &\leq 1 + \Bigg\{ \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left\{ t^l + c_N(\tau_N(t))(t+1)^l \right\} \\ &\quad - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left[t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \binom{l}{2} (t+1)^{l-2} \right] \\ &\quad - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1 + B_n')^l \Bigg\} \mathbb{1}_{E_N^1 \cap E_N^2}. \end{aligned} \tag{5.47}$$

Collecting some terms,

$$
\begin{aligned}
\prod_{r=1}^{\tau_N(t)}(1-p_r) \le 1 &+ \sum_{l=1}^{\tau_N(t)}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l \mathbb{1}_{E_N^1\cap E_N^2} + c_N(\tau_N(t)) \sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)} \alpha_n^l(1+O(N^{-1}))\frac{1}{l!}(t+1)^l \\
&+ \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right) \sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)} \alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\binom{l}{2}(t+1)^{l-2} \\
&+ \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right) \sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)} \alpha_n^l(1+O(N^{-1}))\frac{1}{l!}(t+1)^{l-1}(1+B_n')^l \\
\le 1 &+ \sum_{l=1}^{\infty}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l \mathbb{1}_{\{\tau_N(t)\ge l\}}\mathbb{1}_{E_N^1\cap E_N^2} + c_N(\tau_N(t))\exp[\alpha_n(1+O(N^{-1}))(t+ \\
&+ \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right)\frac{1}{2}\alpha_n^2 \exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
&+ \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')]. \qquad (5.48)
\end{aligned}
$$

Now, taking the expectation and limit, then applying Brown et al. (2021, Equations (3.3)–(3.5)), and Lemmata 5.11, 5.12 and 5.14 to deal with the indicators,

$$
\begin{aligned}
\lim_{N\to\infty}\mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right] &\le 1 + \sum_{l=1}^{\infty}(-\alpha_n)^l\frac{1}{l!}t^l \lim_{N\to\infty}\mathbb{P}\left[\{\tau_N(t)\ge l\}\cap E_N^1\cap E_N^2\right] + \lim_{N\to\infty}\mathbb{E}\left[c_N(\tau_N(t))\right]\exp \\
&\quad + \lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right]\frac{1}{2}\alpha_n^2\exp[\alpha_n(t+1)] \\
&\quad + \lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} D_N(s)\right]\exp[\alpha_n(t+1)(1+B_n')] \\
&= 1 + \sum_{l=1}^{\infty}(-\alpha_n)^l\frac{1}{l!}t^l = e^{-\alpha_n t}. \qquad (5.49)
\end{aligned}
$$

Passing the limit and expectation inside the infinite sum is justified by dominated convergence and Fubini.

It remains to show the corresponding lower bound $\lim_{N\to\infty}\mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right] \ge e^{-\alpha_n t}$. From Brown et al. (2021, Equation (3.14)), taking $\xi = \Delta$, we have

$$
1-p_t = p_{\Delta\Delta}(t) \ge 1-\alpha_n(1+O(N^{-1}))\left[c_N(t)+B_n D_N(t)\right] \qquad (5.50)
$$

where $B_n > 0$ and the $O(N^{-1})$ term does not depend on t. In particular,

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \frac{N^{n-2}}{(N-2)_{n-2}} \alpha_n [c_N(t) + B_n D_N(t)]. \tag{5.51}$$

Since $D_N(s) \leq c_N(s)$ for all s (Koskela et al. 2018, p.9), a sufficient condition for this bound to be non-negative is

$$E_N^3(r) := \left\{ c_N(r) \leq \frac{(N-2)_{n-2}}{N^{n-2}} \alpha_n^{-1} (1 + B_n)^{-1} \right\}, \tag{5.52}$$

and we again define $E_N^3 := \bigcap_{r=1}^{\tau_N(t)} E_N^3(r)$. We now apply a multinomial expansion to the product, and split into positive and negative terms:

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1 - p_r) &\geq \left\{ 1 + \sum_{l=1}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) + B_n D_N(s_j)] \right\} \mathbb{1}_{E_N^3} \\
&= \left\{ 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) + B_n D_N(s_j)] \right. \\
&\qquad \left. - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) + B_n D_N(s_j)] \right\} \mathbb{1}_{E_N^3}
\end{aligned} \tag{5.53}$$

This is further bounded by applying Lemma 5.4 and both bounds in Lemma 5.3(b):

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1 - p_r) &\geq \left\{ 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \right. \\
&\qquad - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left[\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1 + B_n)^l \right] \\
&\geq \left\{ 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left[t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \binom{l}{2} (t+1)^{l-2} \right] \right. \\
&\qquad - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left[t^l + c_N(\tau_N(t))(t+1)^l + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1 + B_n
\end{aligned} \tag{5.54}$$

Collecting terms,

$$
\begin{aligned}
\prod_{r=1}^{\tau_N(t)}(1-p_r) \geq \sum_{l=0}^{\tau_N(t)}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_{E_N^3} &- \left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right)\sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\binom{l}{2}(t+1)^{l-} \\
&- c_N(\tau_N(t))\sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}(t+1)^l \\
&- \left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)\sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}(t+1)^{l-1}(1+B_n)^l \\
\geq \sum_{l=0}^{\infty}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_{E_N^3}\mathbb{1}_{\{\tau_N(t)\geq l\}} &- \left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right)\frac{1}{2}\alpha_n^2\exp[\alpha_n(1+O(N^{-1}))(t \\
&- c_N(\tau_N(t))\exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
&- \left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n)].
\end{aligned} \tag{5.55}
$$

Now, taking the expectation and limit, and applying Brown et al. (2021, Equations (3.3)–(3.5)) to show that all but the first sum vanish, and Lemmata 5.12 and 5.11 to show that $\lim_{N\to\infty}\mathbb{P}[\{\tau_N(t)\geq l\}\cap E_N^3]=1$,

$$
\begin{aligned}
\lim_{N\to\infty}\mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right] &\geq \sum_{l=0}^{\infty}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\lim_{N\to\infty}\mathbb{P}\left[\{\tau_N(t)\geq l\}\cap E_N^3\right] \\
&\quad - \lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right]\frac{1}{2}\alpha_n^2\exp[\alpha_n(t+1)] \\
&\quad - \lim_{N\to\infty}\mathbb{E}\left[c_N(\tau_N(t))\right]\exp[\alpha_n(t+1)] \\
&\quad - \lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)}D_N(s)\right]\exp[\alpha_n(t+1)(1+B_n)] \\
&= \sum_{l=0}^{\infty}(-\alpha_n)^l\frac{1}{l!}t^l = e^{-\alpha_n t}.
\end{aligned} \tag{5.56}
$$

Again, passing the limit and expectation inside the infinite sum is justified by dominated convergence and Fubini. Combining the upper and lower bounds in (5.49) and (5.56) respectively concludes the proof. ■

Lemma 5.7 (Induction step upper bound). *Fix $k \in \mathbb{N}$, $i_0 := 0$, $i_k := k$. For any sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_k \leq t$,*

$$\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\right] \leq \alpha_n^k e^{-\alpha_n t} \sum_{\substack{i_1\leq\cdots\leq i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\geq j\forall j}} \prod_{j=1}^{k} \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}. \tag{5.57}$$

Proof. We use the bound on $(1-p_r)$ from (5.42) and apply a multinomial expansion, defining as in (5.43) and (5.44) respectively the sequences of events E_N^1 and E_N^2 which ensure the bounds are non-negative:

$$\begin{aligned}
\prod_{\substack{r=1\\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)} (1-p_r) \leq & \prod_{\substack{r=1\\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)} \left\{1-\alpha_n(1+O(N^{-1}))[c_N(r)-B_n'D_N(r)]\mathbb{1}_{E_N^1\cap E_N^2}\right\} \\
&= 1+\sum_{l=1}^{\tau_N(t)-k}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}\sum_{\substack{s_1\neq\cdots\neq s_l\\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)}\prod_{j=1}^{l}[c_N(s_j)-B_n'D_N(s_j)]\mathbb{1}_{E_N^1\cap E_N^2} \\
&= 1+\sum_{l=1}^{\tau_N(t)-k}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}[c_N(s_j)-B_n'D_N(s_j)]\mathbb{1}_{E_N^1\cap E_N^2} \\
&\quad -\sum_{l=1}^{\tau_N(t)-k}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}\sum_{\substack{s_1\neq\cdots\neq s_l\\ \exists i,i':s_i=r_{i'}}}\prod_{j=1}^{l}[c_N(s_j)-B_n'D_N(s_j)]\mathbb{1}_{E_N^1\cap E_N^2}.
\end{aligned} \tag{5.58}$$

The penultimate line above is exactly the expansion we had in the basis step (5.46), except for the limit on l, and as such following the same arguments gives a bound analogous to that in (5.48):

$$\begin{aligned}
1+\sum_{l=1}^{\tau_N(t)-k}&(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}[c_N(s_j)-B_n'D_N(s_j)]\mathbb{1}_{E_N^1\cap E_N^2} \\
&\leq 1+\sum_{l=1}^{\tau_N(t)-k}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_{E_N^1\cap E_N^2}+c_N(\tau_N(t))\exp[\alpha_n(1 \\
&\quad +\left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right)\frac{1}{2}\alpha_n^2\exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
&\quad +\left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')].
\end{aligned} \tag{5.59}$$

For the last line of (5.58),

$$
\begin{aligned}
&- \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} \sum_{\substack{s_1 \neq \cdots \neq s_l \\ \exists i,i': s_i = r_{i'}}} \prod_{j=1}^{l} \{c_N(s_j) - B_n' D_N(s_j)\} \mathbb{1}_{E_N^1 \cap E_N^2} \\
&\qquad \leq \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \sum_{\substack{s_1 \neq \cdots \neq s_l \\ \exists i,i': s_i = r_{i'}}} \prod_{j=1}^{l} \{c_N(s_j) + B_n' D_N(s_j)\} \\
&\qquad \leq \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \sum_{\substack{s_1 \neq \cdots \neq s_l \\ \exists i,i': s_i = r_{i'}}} (1+B_n')^l \prod_{j=1}^{l} c_N(s_j) \\
&\qquad \leq \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1+O(N^{-1})) \frac{1}{(l-1)!} \sum_{s_1 \in \{r_1,\ldots,r_k\}} \sum_{s_2 \neq \cdots \neq s_l}^{\tau_N(t)} (1+B_n')^l \prod_{j=1}^{l} c_N(\\
&\qquad = \sum_{s \in \{r_1,\ldots,r_k\}} c_N(s) \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1+O(N^{-1})) \frac{1}{(l-1)!} (1+B_n')^l \sum_{s_1 \neq \cdots \neq s_{l-1}}^{\tau_N(t)} \\
&\qquad \leq \sum_{j=1}^{k} c_N(r_j) \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1+O(N^{-1})) \frac{1}{(l-1)!} (1+B_n')^l (t+1)^{l-1} \\
&\qquad \leq \left(\sum_{j=1}^{k} c_N(r_j) \right) \alpha_n (1+B_n') \exp[\alpha_n (1+O(N^{-1}))(1+B_n')(t+1)],
\end{aligned}
\tag{5.60}
$$

where the penultimate inequality uses Lemma 5.3(a). Putting these together, we have

$$
\begin{aligned}
\prod_{\substack{r=1 \\ \notin \{r_1,\ldots,r_k\}}}^{\tau_N(t)} (1-p_r) \leq 1 &+ \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_{E_N^1 \cap E_N^2} + c_N(\tau_N(t)) \exp[\alpha_n (1+O(N^{-1}))(t+1 \\
&+ \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n (1+O(N^{-1}))(t+1)] \\
&+ \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n (1+O(N^{-1}))(t+1)(1+B_n')] \\
&+ \left(\sum_{j=1}^{k} c_N(r_j) \right) \alpha_n (1+B_n') \exp[\alpha_n (1+O(N^{-1}))(1+B_n')(t+1)].
\end{aligned}
\tag{5.61}
$$

Meanwhile, using the bound on p_r from (5.50) then applying a modification of Lemma 5.4

where the sum is over ordered indices rather than distinct indices,

$$\begin{aligned}\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\prod_{i=1}^{k}p_{r_i} &\le \alpha_n^k(1+O(N^{-1}))\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\prod_{i=1}^{k}[c_N(r_i)+B_nD_N(r_i)]\\ &\le \alpha_n^k(1+O(N^{-1}))\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\prod_{i=1}^{k}c_N(r_i)+\left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)\alpha_n^k(1+O(N^{-1}))(t+1)^{k-1}(1\end{aligned} \tag{5.62}$$

A more liberal (but simpler) bound can be arrived at thus:

$$\begin{aligned}\prod_{i=1}^{k}p_{r_i} &\le \alpha_n^k(1+O(N^{-1}))\prod_{i=1}^{k}[c_N(r_i)+B_nD_N(r_i)]\\ &\le \alpha_n^k(1+O(N^{-1}))\prod_{i=1}^{k}c_N(r_i)(1+B_n)\\ &\le \alpha_n^k(1+O(N^{-1}))(1+B_n)^k\prod_{i=1}^{k}c_N(r_i)\end{aligned} \tag{5.63}$$

which, using Lemma 5.3(a), also leads to the deterministic bound

$$\begin{aligned}\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\prod_{i=1}^{k}p_{r_i} &\le \alpha_n^k(1+O(N^{-1}))(1+B_n)^k\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\prod_{i=1}^{k}c_N(r_i)\\ &\le \alpha_n^k(1+O(N^{-1}))(1+B_n)^k\frac{1}{k!}\sum_{r_1\ne\cdots\ne r_k}^{\tau_N(t)}\prod_{i=1}^{k}c_N(r_i)\\ &\le \alpha_n^k(1+O(N^{-1}))(1+B_n)^k\frac{1}{k!}(t+1)^k.\end{aligned} \tag{5.64}$$

Combining (5.61) with the other product, the expression inside the expectation in (5.57) is bounded above by

$$\begin{aligned}
&\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\\
&\quad\le \left\{1+\sum_{l=1}^{\tau_N(t)-k}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_{E_N^1\cap E_N^2}\right\}\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\prod_{i=1}^{k}p_{r_i}\\
&\qquad+\left\{c_N(\tau_N(t))\exp[\alpha_n(1+O(N^{-1}))(t+1)]+\left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right)\frac{1}{2}\alpha_n^2\exp[\alpha_n(1+O(N^{-1}))(\right.\\
&\qquad\qquad\left.+\left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')]\right\}\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\prod_{i=}\\
&\qquad+\exp[\alpha_n(1+O(N^{-1}))(1+B_n')(t+1)]\alpha_n(1+B_n')\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\sum_{j=1}^{k}c_N(r_j)\prod_{i=1}^{k}p_{r_i}.
\end{aligned}\tag{5.65}$$

Applying the various bounds (5.62)–(5.64), we have

$$\begin{aligned}
&\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\\
&\quad\le \alpha_n^k(1+O(N^{-1}))\left\{1+\sum_{l=1}^{\tau_N(t)-k}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_{E_N^1\cap E_N^2}\right\}\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\prod_{i=1}^{k}c_N(r_i)\\
&\qquad+\left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)\alpha_n^k(1+O(N^{-1}))(t+1)^{k-1}(1+B_n)^k\sum_{l=0}^{\tau_N(t)}(\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\\
&\qquad+\left\{c_N(\tau_N(t))\exp[\alpha_n(1+O(N^{-1}))(t+1)]+\left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right)\frac{1}{2}\alpha_n^2\exp[\alpha_n(1+O(N^{-1}))(\right.\\
&\qquad\qquad\left.+\left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')]\right\}\alpha_n^k(1+O(N^{-1}))(1+B\\
&\qquad+\exp[\alpha_n(1+B_n')(t+1)]\alpha_n(1+B_n')\alpha_n^k(1+O(N^{-1}))(1+B_n)^k\\
&\qquad\qquad\times\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\sum_{j=1}^{k}c_N(r_j)\prod_{i=1}^{k}c_N(r_i).
\end{aligned}\tag{5.66}$$

Upon taking the expectation and limit, we have

$$
\begin{aligned}
&\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\left(\prod_{i=1}^{k}p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)}(1-p_r)\right)\right]\\
&\quad\leq \alpha_n^k\lim_{N\to\infty}\mathbb{E}\left[\left(1+\sum_{l=1}^{\tau_N(t)-k}(-\alpha_n)^l\frac{1}{l!}t^l\mathbb{1}_{E_N^1\cap E_N^2}\right)\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\prod_{i=1}^{k}c_N(r_i)\right]\\
&\qquad+\lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)}D_N(s)\right]\alpha_n^k(t+1)^{k-1}(1+B_n)^k\exp[\alpha_n t]\\
&\qquad+\left\{\lim_{N\to\infty}\mathbb{E}\left[c_N(\tau_N(t))\right]\exp[\alpha_n(t+1)]+\lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right]\frac{1}{2}\alpha_n^2\exp[\alpha_n(t+1)]\right.\\
&\qquad\qquad\left.+\lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)}D_N(s)\right]\exp[\alpha_n(t+1)(1+B_n')]\right\}\alpha_n^k(1+B_n)^k\frac{1}{k!}(t+1)^k\\
&\qquad+\exp[\alpha_n(1+B_n')(t+1)]\alpha_n^{k+1}(1+B_n')(1+B_n)^k\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\sum_{j=1}^{k}c_N(r_j)\prod_{i=1}^{k}c_N(r_i)\right]
\end{aligned}
\tag{5.67}
$$

The middle terms vanish due to Brown et al. (2021, Equations (3.3)–(3.5)) and the expression becomes

$$
\begin{aligned}
&\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\left(\prod_{i=1}^{k}p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)}(1-p_r)\right)\right]\leq\alpha_n^k\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\prod_{i=1}^{k}c_N(r_i)\right]\\
&\qquad+\alpha_n^k\sum_{l=1}^{\infty}(-\alpha_n)^l\frac{1}{l!}t^l\lim_{N\to\infty}\mathbb{E}\left[\mathbb{1}_{\{\tau_N(t)\geq k+l\}}\mathbb{1}_{E_N^1\cap E_N^2}\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\prod_{i=1}^{k}c_N(r_i)\right]\\
&\qquad+\exp[\alpha_n(1+B_n')(t+1)]\alpha_n^{k+1}(1+B_n')(1+B_n)^k\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\sum_{j=1}^{k}c_N(r_j)\prod_{i=1}^{k}c_N(r_i)\right]
\end{aligned}
\tag{5.68}
$$

where passing the limit and expectation inside the infinite sum is justified by dominated convergence and Fubini; see Lemma 5.16. To simplify the last line,

$$\begin{aligned}
\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \sum_{j=1}^{k} c_N(r_j)\prod_{i=1}^{k} c_N(r_i) &\le \frac{1}{k!}\sum_{r_1\neq\cdots\neq r_k}^{\tau_N(t)} \sum_{j=1}^{k} c_N(r_j)\prod_{i=1}^{k} c_N(r_i) \\
&= \frac{1}{k!}\sum_{r_1\neq\cdots\neq r_k}^{\tau_N(t)} \sum_{j=1}^{k} c_N(r_j)^2\prod_{i\neq j} c_N(r_i) \\
&\le \frac{1}{k!}\sum_{j=1}^{k}\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \sum_{r_1\neq\cdots\neq r_{k-1}}^{\tau_N(t)} \prod_{i=1}^{k-1} c_N(r_i) \\
&\le \frac{1}{(k-1)!}\sum_{s=1}^{\tau_N(t)} c_N(s)^2(t+1)^{k-1}, \qquad (5.69)
\end{aligned}$$

using Lemma 5.3(a) for the final inequality. Hence

$$\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \sum_{s\in\{r_1,\ldots,r_k\}} c_N(s)\prod_{i=1}^{k} c_N(r_i)\right] \le \frac{1}{(k-1)!}(t+1)^{k-1}\lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right] = 0 \qquad (5.70)$$

by Brown et al. (2021, Equation (3.5)). By Lemmata 5.12, 5.11 and 5.14, $\lim_{N\to\infty}\mathbb{P}[\{\tau_N(t)\ge k+l\}\cap E_N^1\cap E_N^2] = 1$, so we can apply Lemma 5.9 to the remaining expectations in (5.68), yielding

$$\begin{aligned}
\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\right] &\le \alpha_n^k\sum_{l=0}^{\infty}(-\alpha_n)^l\frac{1}{l!}t^l \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^{k}\frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} \\
&= \alpha_n^k e^{-\alpha_n t}\sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^{k}\frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} \qquad (5.71)
\end{aligned}$$

as required. ■

Lemma 5.8 (Induction step lower bound)**.** *Fix $k \in \mathbb{N}$, $i_0 := 0$, $i_k := k$. For any sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_k \leq t$,*

$$\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\right] \geq \alpha_n^k e^{-\alpha_n t} \sum_{\substack{i_1\leq\cdots\leq i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\geq j\forall j}} \prod_{j=1}^{k} \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}. \tag{5.72}$$

Proof. Firstly,

$$\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right) \geq \sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right)\left(\prod_{r=1}^{\tau_N(t)} (1-p_r)\right). \tag{5.73}$$

Now the second product does not depend on $r_1,\dots,r_k$, and we can use the lower bound from (5.55):

$$\begin{aligned}\prod_{r=1}^{\tau_N(t)} (1-p_r) \geq \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1}))\frac{1}{l!}t^l \mathbb{1}_{E_N^3} - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right)\frac{1}{2}\alpha_n^2 \exp[\alpha_n(1+O(N^{-1}))(t+1)]\\ - c_N(\tau_N(t))\exp[\alpha_n(1+O(N^{-1}))(t+1)]\\ - \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n)]\end{aligned} \tag{5.74}$$

where E_N^3 is defined as in (5.52). We will also need an upper bound on this product, which is formed from (5.48) with a further deterministic bound:

$$\begin{aligned}\prod_{r=1}^{\tau_N(t)} (1-p_r) &\leq \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1}))\frac{1}{l!}t^l \mathbb{1}_{\{\tau_N(t)\geq l\}}\mathbb{1}_{E_N^1\cap E_N^2} + c_N(\tau_N(t))\exp[\alpha_n(1+O(N^{-1}))(t+1)\\ &\quad + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right)\frac{1}{2}\alpha_n^2 \exp[\alpha_n(1+O(N^{-1}))(t+1)]\\ &\quad + \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')]\\ &\leq \exp[\alpha_n(1+O(N^{-1}))t] + \exp[\alpha_n(1+O(N^{-1}))(t+1)]\\ &\quad + \frac{1}{2}\alpha_n^2(t+1)\exp[\alpha_n(1+O(N^{-1}))(t+1)] + (t+1)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B\\ &\leq \left(2+\frac{\alpha_n^2(t+1)}{2}\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)] + (t+1)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+\end{aligned} \tag{5.75}$$

Now let us consider the remaining sum-product on the RHS of (5.73). We use the same bound on p_r as in (5.42):

$$p_r = 1 - p_{\Delta\Delta}(r) \geq \alpha_n(1 + O(N^{-1})) \left[c_N(r) - B_n' D_N(r)\right] \tag{5.76}$$

where the $O(N^{-1})$ term does not depend on r. When N is large enough for the factor of $(1 + O(N^{-1}))$ to be non-negative, the condition that the bound in (5.76) is non-negative holds on the event E_N^2 that was defined in (5.44). Then

$$\prod_{i=1}^{k} p_{r_i} \geq \alpha_n^k(1 + O(N^{-1})) \prod_{i=1}^{k} \left[c_N(r_i) - B_n' D_N(r_i)\right] \mathbb{1}_{E_N^2}. \tag{5.77}$$

Applying a modification of Lemma 5.5 where the sum is over ordered indices rather than distinct indices,

$$\begin{aligned}\sum_{\substack{r_1<\cdots<r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right) &\geq \alpha_n^k(1 + O(N^{-1})) \sum_{\substack{r_1<\cdots<r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} \left[c_N(r_i) - B_n' D_N(r_i)\right] \mathbb{1}_{E_N^2} \\ &\geq \alpha_n^k(1 + O(N^{-1})) \left\{ \sum_{\substack{r_1<\cdots<r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i) \mathbb{1}_{E_N^2} - \frac{1}{k!} \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right) (t+1)^{k-1}(1 + \right.\end{aligned} \tag{5.78}$$

The above expression is already split into positive and negative terms; a lower bound on (5.73) can be formed by multiplying the positive terms by the lower bound (5.74) and the negative terms by the upper bound (5.75). Thus

$$
\begin{aligned}
&\sum_{\substack{r_1<\cdots<r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^{k} p_{r_i} \right) \left(\prod_{\substack{r=1 \\ \notin \{r_1,\ldots,r_k\}}}^{\tau_N(t)} (1-p_r) \right) \\
&\quad \geq \alpha_n^k (1+O(N^{-1})) \sum_{\substack{r_1<\cdots<r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i) \mathbb{1}_{E_N^2} \Bigg\{ \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_{E_N^3} \\
&\qquad - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n (1+O(N^{-1}))(t+1)] \\
&\qquad - c_N(\tau_N(t)) \exp[\alpha_n (1+O(N^{-1}))(t+1)] \\
&\qquad - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n (1+O(N^{-1}))(t+1)(1+B_n)] \Bigg\} \\
&\quad - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \alpha_n^k (1+O(N^{-1})) \frac{1}{k!} (t+1)^{k-1} (1+B_n')^k \Bigg\{ \\
&\qquad \left(2 + \frac{\alpha_n^2 (t+1)}{2} \right) \exp[\alpha_n (1+O(N^{-1}))(t+1)] \\
&\qquad + (t+1) \exp[\alpha_n (1+O(N^{-1}))(t+1)(1+B_n')] \Bigg\}.
\end{aligned}
\tag{5.79}
$$

Due to Brown et al. (2021, Equations (3.3)–(3.5)), all but the first line on the RHS of the above have vanishing expectation, leaving

$$
\begin{aligned}
&\lim_{N\to\infty} \mathbb{E} \left[\sum_{\substack{r_1<\cdots<r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^{k} p_{r_i} \right) \left(\prod_{\substack{r=1 \\ \notin \{r_1,\ldots,r_k\}}}^{\tau_N(t)} (1-p_r) \right) \right] \\
&\quad \geq \lim_{N\to\infty} \mathbb{E} \left[\alpha_n^k (1+O(N^{-1})) \sum_{\substack{r_1<\cdots<r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i) \mathbb{1}_{E_N^2} \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l \mathbb{1} \right. \\
&\quad = \alpha_n^k \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l \lim_{N\to\infty} \mathbb{E} \left[\mathbb{1}_{\{\tau_N(t) \geq l\}} \mathbb{1}_{E_N^2 \cap E_N^3} \sum_{\substack{r_1<\cdots<r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i) \right].
\end{aligned}
\tag{5.80}
$$

Passing the limit and expectation inside the infinite sum is justified by dominated convergence and Fubini; see Lemma 5.16. Lemmata 5.11 and 5.14 establish that $\lim_{N\to\infty} \mathbb{P}[E_N^2 \cap E_N^3] = 1$ and Lemma 5.12 deals with the other indicator. We can therefore apply

Lemma 5.9 to conclude that

$$\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\right] \ge \alpha_n^k \sum_{l=0}^{\infty}(-\alpha_n)^l \frac{1}{l!}t^l \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^{k} \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}$$
$$= \alpha_n^k e^{-\alpha_n t} \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^{k} \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} \tag{5.81}$$

as required. ∎

Lemma 5.9. *Fix $k\in\mathbb{N}$, $i_0 := 0$, $i_k := k$. Let E_N be a sequence of events such that $\lim_{N\to\infty}\mathbb{P}[E_N] = 1$. Then for any sequence of times $0 = t_0 \le t_1 \le \cdots \le t_k \le t$,*

$$\lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_{E_N} \sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^{k} c_N(r_i)\right] = \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^{k} \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}. \tag{5.82}$$

Proof. As pointed out by Möhle (1999, p. 460), the sum-product on the left hand side can be expanded as

$$\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^{k} c_N(r_i) = \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^{k} \frac{1}{(i_j-i_{j-1})!} \sum_{\substack{r_{i_{j-1}+1}\ne\cdots\ne r_{i_j}\\ =\tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i). \tag{5.83}$$

By a modification of the upper bound in Lemma 5.3(b) where the lower limit of the sum is a general time rather than 1,

$$\sum_{\substack{r_{i_{j-1}+1}\ne\cdots\ne r_{i_j}\\ =\tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) \le (t_j-t_{j-1})^{i_j-i_{j-1}} + c_N(\tau_N(t_j))(t_j-t_{j-1}+1)^{i_j-i_{j-1}} \tag{5.84}$$

Now, taking the product on the outside,

$$
\begin{aligned}
\prod_{j=1}^{k} \frac{1}{(i_j - i_{j-1})!} \sum_{\substack{r_{i_{j-1}+1} \neq \cdots \neq r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) &\leq \prod_{j=1}^{k} \left\{ \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} + c_N(\tau_N(t_j)) \frac{(t_j - t_{j-1} + 1)^{i_j - i_{j-}}}{(i_j - i_{j-1})!} \right. \\
&\leq \prod_{j=1}^{k} \left\{ \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} + c_N(\tau_N(t_j)) (t_j - t_{j-1} + 1)^{i_j - i_{j-}} \right. \\
&= \sum_{\mathcal{I} \subseteq [k]} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} c_N(\tau_N(t_j)) (t_j - t_{j-1} \right. \\
&= \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \\
&\quad + \sum_{\mathcal{I} \subset [k]} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} c_N(\tau_N(t_j)) (t_j - \right. \\
&\leq \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \\
&\quad + \sum_{\mathcal{I} \subset [k]} \left(\prod_{j \in \mathcal{I}} t^{i_j - i_{j-1}} \right) \left(\prod_{j \notin \mathcal{I}} c_N(\tau_N(t_j)) (t+1)^{i_j - i_{j-1}} \right) \\
&\leq \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} + \sum_{\mathcal{I} \subset [k]} c_N(\tau_N(t_{j^\star(\mathcal{I})})) \prod_{j=1}^{k} (t+1)^{i_j -} \\
&= \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} + \sum_{\mathcal{I} \subset [k]} c_N(\tau_N(t_{j^\star(\mathcal{I})})) (t+1)^k
\end{aligned}
\tag{5.85}
$$

where, say, $j^\star(\mathcal{I}) := \min\{j \notin \mathcal{I}\}$. Now we are in a position to evaluate the limit in (5.82):

$$
\begin{aligned}
\lim_{N \to \infty} \mathbb{E} \left[\mathbb{1}_{E_N} \sum_{\substack{r_1 < \cdots < r_k : \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i) \right] &\leq \lim_{N \to \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \cdots < r_k : \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i) \right] \\
&\leq \sum_{\substack{i_1 \leq \cdots \leq i_{k-1} \\ \in \{0, \ldots, k\} : \\ i_j \geq j \forall j}} \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} + \sum_{\substack{i_1 \leq \cdots \leq i_{k-1} \\ \in \{0, \ldots, k\} : \\ i_j \geq j \forall j}} \sum_{\mathcal{I} \subset [k]} \lim_{N \to \infty} \mathbb{E} \left[c_N(\tau_N(t_{j^\star(\mathcal{I})})) \right] (t+1)^k \\
&= \sum_{\substack{i_1 \leq \cdots \leq i_{k-1} \\ \in \{0, \ldots, k\} : \\ i_j \geq j \forall j}} \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}
\end{aligned}
\tag{5.86}
$$

using Brown et al. (2021, Equation (3.3)).

For the corresponding lower bound, by a modification of the lower bound in Lemma 5.3(b) where the lower limit of the sum is a general time rather than 1,

$$\begin{aligned}\sum_{\substack{r_{i_{j-1}+1}\neq\cdots\neq r_{i_j}\\=\tau_N(t_{j-1})+1}}^{\tau_N(t_j)}\prod_{i=i_{j-1}+1}^{i_j}c_N(r_i) &\geq (t_j-t_{j-1})^{i_j-i_{j-1}}-\binom{i_j-i_{j-1}}{2}\left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)}c_N(s)^2\right)(t_j-t_{j-1}+1)^{i_j-}\\ &\geq (t_j-t_{j-1})^{i_j-i_{j-1}}-(i_j-i_{j-1})!\left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)}c_N(s)^2\right)(t_j-t_{j-1}+1)^{i_j-}\end{aligned} \tag{5.87}$$

Define the events

$$E_N^4(j)=\left\{\left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)}c_N(s)^2\right)\leq\frac{1}{(i_j-i_{j-1})!}\left(\frac{t_j-t_{j-1}}{t_j-t_{j-1}+1}\right)^{i_j-i_{j-1}}\right\}, \tag{5.88}$$

which is sufficient to ensure the j^{th} term in the following product is non-negative, and define $E_N^4 := \bigcap_{j=1}^k E_N^4(j)$. (If $t_j = t_{j-1}$ then $E_N^4(j)$ has probability one automatically; otherwise the constant on the right is strictly positive and so satisfies the conditions of

Lemma 5.13.) Now, taking a product over j,

$$\begin{aligned}
&\prod_{j=1}^{k} \frac{1}{(i_j - i_{j-1})!} \sum_{\substack{r_{i_{j-1}+1} \neq \cdots \neq r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) \\
&\geq \prod_{j=1}^{k} \left\{ \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} - \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{i_j - i_{j-1} - 2} \right\} \mathbb{1}_{E_N^4} \\
&= \sum_{\mathcal{I} \subseteq [k]} (-1)^{k-|\mathcal{I}|} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{i_j - i_{j-1} - 2} \right) \mathbb{1}_{E_N^4} \\
&= \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E_N^4} \\
&\quad + \sum_{\mathcal{I} \subset [k]} (-1)^{k-|\mathcal{I}|} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{i_j - i_{j-1} - 2} \right) \mathbb{1} \\
&\geq \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E_N^4} \\
&\quad - \sum_{\mathcal{I} \subset [k]} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{i_j - i_{j-1} - 2} \right) \\
&\geq \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E_N^4} \\
&\quad - \sum_{\mathcal{I} \subset [k]} \left(\prod_{j \in \mathcal{I}} t^{i_j - i_{j-1}} \right) \left(\prod_{j \notin \mathcal{I}} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t+1)^{i_j - i_{j-1} - 2} \right) \\
&\geq \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E_N^4} \\
&\quad - \sum_{\mathcal{I} \subset [k]} \left(\sum_{s=\tau_N(t_{j^\star(\mathcal{I})-1})+1}^{\tau_N(t_{j^\star(\mathcal{I})})} c_N(s)^2 \right) \left(\prod_{j \in \mathcal{I}} t^{i_j - i_{j-1}} \right) \left(\prod_{j \notin \mathcal{I}} (t+1)^{i_j - i_{j-1} - 1} \right) \\
&\geq \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E_N^4} - \sum_{\mathcal{I} \subset [k]} \left(\sum_{s=\tau_N(t_{j^\star(\mathcal{I})-1})+1}^{\tau_N(t_{j^\star(\mathcal{I})})} c_N(s)^2 \right) \prod_{j=1}^{k} (t+1)^{i_j - i_{j-1}} \\
&= \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E_N^4} - \sum_{\mathcal{I} \subset [k]} \left(\sum_{s=\tau_N(t_{j^\star(\mathcal{I})-1})+1}^{\tau_N(t_{j^\star(\mathcal{I})})} c_N(s)^2 \right) (t+1)^k, \qquad (5.89)
\end{aligned}$$

where again we have arbitrarily set $j^\star(\mathcal{I}) := \min\{j \notin \mathcal{I}\}$. We can now evaluate the limit:

$$
\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_{E_N} \sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^k c_N(r_i)\right] &\ge \lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_{E_N\cap E_N^4} \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}\right] \\
&\quad - \lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_{E_N} \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\ge j\forall j}} \sum_{\mathcal{I}\subset[k]} \left(\sum_{s=\tau_N(t_{j^\star(\mathcal{I})-1})+1}^{\tau_N(t_{j^\star(\mathcal{I})})} c_N(s)^2\right)(t+1
\right. \\
&\ge \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} \lim_{N\to\infty} \mathbb{E}[\mathbb{1}_{E_N\cap E_N^4}] \\
&\quad - \lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\ge j\forall j}} \sum_{\mathcal{I}\subset[k]} \left(\sum_{s=\tau_N(t_{j^\star(\mathcal{I})-1})+1}^{\tau_N(t_{j^\star(\mathcal{I})})} c_N(s)^2\right)(t+1)^k\right] \\
&= \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} \lim_{N\to\infty} \mathbb{P}[E_N\cap E_N^4] \\
&\quad - \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\ge j\forall j}} \sum_{\mathcal{I}\subset[k]} \lim_{N\to\infty} \mathbb{E}\left[\sum_{s=\tau_N(t_{j^\star(\mathcal{I})-1})+1}^{\tau_N(t_{j^\star(\mathcal{I})})} c_N(s)^2\right](t+1)^k \\
&= \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} \qquad (5.90)
\end{aligned}
$$

where for the last equality we use Brown et al. (2021, Equation (3.5)) to show that the second sum vanishes and Lemma 5.13 to show that $\lim_{N\to\infty}\mathbb{P}[E_N\cap E_N^4] = 1$. We have shown that the upper and lower bounds coincide, so the result follows. ■

5.3 Indicators

Lemma 5.10. *Let $(A_N), (B_N)$ be sequences of events. If $\lim_{N\to\infty}\mathbb{P}[A_N] = 1$ and $\lim_{N\to\infty}\mathbb{P}[B_N] = 1$ then $\lim_{N\to\infty}\mathbb{P}[A_N\cap B_N] = 1$.*

The above might be so obvious as to go unstated, but it is very important because it means we don't have to deal with intersections of dependent events! Here is a little proof just to be sure:

Proof.

$$\begin{aligned}
&\lim_{N\to\infty} \mathbb{P}[A_N] = 1 \textbf{ and } \lim_{N\to\infty} \mathbb{P}[B_N] = 1 \\
\Leftrightarrow &\lim_{N\to\infty} \mathbb{P}[A_N^c] = 0 \textbf{ and } \lim_{N\to\infty} \mathbb{P}[B_N^c] = 0 \\
\Rightarrow &\lim_{N\to\infty} \{\mathbb{P}[A_N^c] + \mathbb{P}[B_N^c]\} = 0 \\
\Rightarrow &\lim_{N\to\infty} \mathbb{P}[A_N^c \cup B_N^c] = 0 \\
\Leftrightarrow &\lim_{N\to\infty} \mathbb{P}[A_N \cap B_N] = 1.
\end{aligned} \tag{5.91}$$

The only part of this argument that I find potentially controversial is going from the third to the fourth line, which is an application of the sandwich theorem (since $0 \leq \mathbb{P}[A_N^c \cup B_N^c] \leq \mathbb{P}[A_N^c] + \mathbb{P}[B_N^c]$). ■

Lemma 5.11. *Let $K > 0$ be a constant which may depend on n, N but not on r, such that $K^{-2} = O(1)$ as $N \to \infty$. Define the events $E_N(r) := \{c_N(r) < K\}$ and denote $E_N := \bigcap_{r=1}^{\tau_N(t)} E_N(r)$. Then $\lim_{N\to\infty} \mathbb{P}[E_N] = 1$.*

Proof.

$$\begin{aligned}
\mathbb{P}[E_N] = 1 - \mathbb{P}[E_N^c] = 1 - \mathbb{P}\left[\bigcup_{r=1}^{\tau_N(t)} E_N^c(r)\right] &= 1 - \mathbb{E}\left[\mathbb{1}_{\bigcup E_N^c(r)}\right] \geq 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{1}_{E_N^c(r)}\right] \\
= 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}\left[\mathbb{1}_{E_N^c(r)} \mid \mathcal{F}_{r-1}\right]\right] &= 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{P}\left[E_N^c(r) \mid \mathcal{F}_{r-1}\right]\right]
\end{aligned} \tag{5.92}$$

where for the second line we apply Lemma 5.15 with $f(r) = \mathbb{1}_{E_N^c(r)}$. By the generalised Markov inequality,

$$\mathbb{P}[E_N^c(r) \mid \mathcal{F}_{r-1}] = \mathbb{P}[c_N(r) \geq K \mid \mathcal{F}_{r-1}] \leq K^{-2}\mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}]. \tag{5.93}$$

Substituting this into (5.92) and applying Lemma 5.15 again, this time with $f(r) = c_N(r)^2$,

$$\mathbb{P}[E_N] \geq 1 - K^{-2}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}]\right] = 1 - K^{-2}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} c_N(r)^2\right]. \tag{5.94}$$

Applying Brown et al. (2021, Equation (3.5)), the limit is

$$\lim_{N\to\infty} \mathbb{P}[E_N] = 1 - O(1) \times 0 = 1 \tag{5.95}$$

as required. ■

Lemma 5.12. *Fix $t > 0$. For any $l \in \mathbb{N}$, $\lim_{N\to\infty} \mathbb{P}[\tau_N(t) \geq l] = 1$.*

Proof. We can replace the event $\{\tau_N(t) \geq l\}$ with an event of the form of E_N in Lemma 5.11:

$$\{\tau_N(t) \geq l\} = \left\{ \min \left\{ s \geq 1 : \sum_{r=1}^{s} c_N(r) \geq t \right\} \geq l \right\} = \left\{ \sum_{r=1}^{l-1} c_N(r) < t \right\} \supseteq \bigcap_{r=1}^{l-1} \left\{ c_N(r) < \frac{t}{l} \right\} \supseteq \bigcap_{r=1}^{\tau_N(t)} \Big\{ c_N \tag{5.96}$$

Hence

$$\lim_{N\to\infty} \mathbb{P}[\tau_N(t) \geq l] \geq \lim_{N\to\infty} \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \left\{ c_N(r) < \frac{t}{l} \right\} \right] = 1 \tag{5.97}$$

by applying Lemma 5.11 with $K = t/l$. ■

Lemma 5.13. *Fix $k \in \mathbb{N}$, a sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_k \leq t$, and let $K_1, \ldots, K_k$ be strictly positive constants. Define the events*

$$E_N := \bigcap_{j=1}^{k} \left\{ \sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \leq K_j \right\}. \tag{5.98}$$

Then $\lim_{N\to\infty} \mathbb{P}[E_N] = 1$.

Proof.

$$\mathbb{P}[E_N] = 1 - \mathbb{P}[E_N^c] = 1 - \mathbb{P}\left[\bigcup_{j=1}^{k} \left\{ \sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 > K_j \right\} \right] \geq 1 - \sum_{j=1}^{k} \mathbb{P}\left[\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \geq K_j \right]. \tag{5.99}$$

Applying Markov's inequality,

$$\mathbb{P}[E_N] \geq 1 - \sum_{j=1}^{k} K_j^{-1} \mathbb{E}\left[\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right] \underset{N\to\infty}{\longrightarrow} 1 - \sum_{j=1}^{k} O(1) \times 0 = 1 \tag{5.100}$$

by Brown et al. (2021, Equation (3.5)). ■

Lemma 5.14. *Fix $t > 0$. Let K be a constant not depending on N, r, but which may depend on n.*

$$\lim_{N\to\infty} \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) \geq K D_N(r)\} \right] = 1. \tag{5.101}$$

Proof.

$$\begin{aligned}
\mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) \geq KD_N(r)\}\right] &\geq \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) > KD_N(r)\}\right] \\
&= 1 - \mathbb{P}\left[\bigcup_{r=1}^{\tau_N(t)} \{c_N(r) \leq KD_N(r)\}\right] \\
&= 1 - \mathbb{E}\left[\mathbb{1}_{\bigcup\{c_N(r)\leq KD_N(r)\}}\right] \\
&\geq 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{1}_{\{c_N(r)\leq KD_N(r)\}}\right] \\
&= 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{P}[c_N(r) \leq KD_N(r) \mid \mathcal{F}_{r-1}]\right] \qquad (5.102)
\end{aligned}$$

where the final inequality is an application of Lemma 5.15 with $f(r) = \mathbb{1}_{\{c_N(r)\leq KD_N(r)\}}$.

Fix $0 < \varepsilon < K^{-1}/2$ and assume $N > \max\{\varepsilon^{-1}, (K^{-1} - 2\varepsilon)^{-1}\}$. For each r, i define the event $A_i(r) := \{\nu_r^{(i)} \leq N\varepsilon\}$. Conditional on $\mathcal{F}_{r-1}$, we have

$$D_N(r) = \frac{1}{N(N)_2}\sum_{i=1}^{N}(\nu_r^{(j)})_2\left[\nu_r^{(i)} + \frac{1}{N}\sum_{j\neq i}(\nu_r^{(i)})^2\right]\mathbb{1}_{A_i^c(r)} + \frac{1}{N(N)_2}\sum_{i=1}^{N}(\nu_r^{(i)})_2\left[\nu_r^{(i)} + \frac{1}{N}\sum_{j\neq i}(\nu_r^{(j)})^2\right]\mathbb{1}_{A_i(r)} \qquad (5.103)$$

For the first term,

$$\frac{1}{N(N)_2}\sum_{i=1}^{N}(\nu_r^{(i)})_2\left[\nu_r^{(i)} + \frac{1}{N}\sum_{j\neq i}(\nu_r^{(j)})^2\right]\mathbb{1}_{A_i^c(r)} \leq \sum_{i=1}^{N}\mathbb{1}_{A_i^c(r)}. \qquad (5.104)$$

For the second term,

$$\begin{aligned}
\frac{1}{N(N)_2}\sum_{i=1}^{N}(\nu_r^{(i)})_2\left[\nu_r^{(i)} + \frac{1}{N}\sum_{j\neq i}(\nu_r^{(j)})^2\right]\mathbb{1}_{A_i(r)} &\leq \frac{1}{N(N)_2}\sum_{i=1}^{N}(\nu_r^{(i)})_2\nu_r^{(i)}\mathbb{1}_{A_i(r)} + \frac{1}{N^2(N)_2}\sum_{i=1}^{N}(\nu_r^{(i)})_2\sum_{j=1}^{N} \\
&\leq \frac{1}{N}c_N(r)N\varepsilon + \frac{1}{N^2(N)_2}\sum_{i=1}^{N}(\nu_r^{(i)})_2\sum_{j=1}^{N}(\nu_r^{(j)})_2\mathbb{1}_{A_i(r)} \\
&\quad + \frac{1}{N^2(N)_2}\sum_{i=1}^{N}(\nu_r^{(i)})_2\sum_{j=1}^{N}(\nu_r^{(j)})\mathbb{1}_{A_i(r)} \\
&\leq \varepsilon c_N(r) + \frac{1}{N^2}\sum_{i=1}^{N}\nu_r^{(i)}N\varepsilon c_N(r) + \frac{1}{N^2}c_N(r)N \\
&= c_N(r)\left(2\varepsilon + \frac{1}{N}\right). \qquad (5.105)
\end{aligned}$$

Altogether we have

$$D_N(r) \le c_N(r)\left(2\varepsilon + \frac{1}{N}\right) + \sum_{i=1}^{N} \mathbb{1}_{A_i^c(r)}. \tag{5.106}$$

Hence, still conditional on $\mathcal{F}_{r-1}$,

$$\begin{aligned}
\{c_N(r) \le K D_N(r)\} &\subseteq \left\{ c_N(r) \le K c_N(r)(2\varepsilon + N^{-1}) + K \sum_{i=1}^{N} \mathbb{1}_{A_i^c(r)} \right\} \\
&= \left\{ K^{-1} - 2\varepsilon - \frac{1}{N} \le \sum_{i=1}^{N} \frac{\mathbb{1}_{A_i^c(r)}}{c_N(r)} \right\}
\end{aligned} \tag{5.107}$$

where the ratio $\mathbb{1}_{A_i^c(r)}/c_N(r)$ is well-defined because

$$A_i^c(r) \Rightarrow c_N(r) := \frac{1}{(N)_2} \sum_{j=1}^{N} (\nu_r^{(j)})_2 \ge \frac{1}{(N)_2} (\nu_r^{(i)})_2 \ge \frac{\varepsilon(N\varepsilon - 1)}{N-1} \ge \varepsilon\left(\varepsilon - \frac{1}{N}\right) > 0. \tag{5.108}$$

Hence by Markov's inequality (the conditions on ε, N ensuring the constant is always strictly positive),

$$\begin{aligned}
\mathbb{P}\left[c_N(r) \le K D_N(r) \mid \mathcal{F}_{r-1}\right] &\le \mathbb{P}\left[\sum_{i=1}^{N} \mathbb{1}_{A_i^c(r)} \ge \left(K^{-1} - 2\varepsilon - \frac{1}{N}\right) \varepsilon \left(\varepsilon - \frac{1}{N}\right) \middle| \mathcal{F}_{r-1} \right] \\
&\le \frac{1}{\left(K^{-1} - 2\varepsilon - \frac{1}{N}\right)\varepsilon\left(\varepsilon - \frac{1}{N}\right)} \mathbb{E}\left[\sum_{i=1}^{N} \mathbb{1}_{A_i^c(r)} \middle| \mathcal{F}_{r-1} \right] \\
&\le \frac{1}{\left(K^{-1} - 2\varepsilon - \frac{1}{N}\right)\varepsilon\left(\varepsilon - \frac{1}{N}\right)} \mathbb{E}\left[\sum_{i=1}^{N} \frac{(\nu_r^{(i)})_3}{(N\varepsilon)_3} \middle| \mathcal{F}_{r-1} \right] \\
&\le \frac{1}{\left(K^{-1} - 2\varepsilon - \frac{1}{N}\right)\varepsilon\left(\varepsilon - \frac{1}{N}\right)} \mathbb{E}\left[\frac{N(N)_2}{(N\varepsilon)_3} D_N(r) \middle| \mathcal{F}_{r-1} \right].
\end{aligned} \tag{5.109}$$

Applying Lemma 5.15 once more, with $f(r) = D_N(r)$,

$$\begin{aligned}
\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{P}[c_N(r) \le K D_N(r) \mid \mathcal{F}_{r-1}] \right] &\le \frac{1}{\left(K^{-1} - 2\varepsilon - \frac{1}{N}\right)\varepsilon\left(\varepsilon - \frac{1}{N}\right)} \frac{N(N)_2}{(N\varepsilon)_3} \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[D_N(r) \mid \mathcal{F}_{r-1}] \right] \\
&= \frac{1}{\left(K^{-1} - 2\varepsilon - \frac{1}{N}\right)\varepsilon\left(\varepsilon - \frac{1}{N}\right)} \frac{N(N)_2}{(N\varepsilon)_3} \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} D_N(r) \right] \\
&\xrightarrow[N\to\infty]{} \frac{1}{(K^{-1} - 2\varepsilon)\varepsilon^5} \times 0 = 0.
\end{aligned} \tag{5.110}$$

Substituting this back into (5.102) concludes the proof. ■

5.4 Other useful results

The following Lemma is taken from Koskela et al. (2018, Lemma 2), where the function is set to $f(r) = c_N(r)$, but the authors remark that the result holds for other choices of function.

Lemma 5.15. *Fix $t > 0$. Let $(\mathcal{F}_r)$ be the backwards-in-time filtration generated by the offspring counts $\nu_r^{(1:N)}$ at each generation r, and let $f(r)$ be any deterministic function of $\nu_r^{(1:N)}$ that is non-negative and bounded. In particular, for all r there exists $B < \infty$ such that $0 \leq f(r) \leq B$. Then*

$$\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} f(r)\right] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right]. \tag{5.111}$$

Proof. Define

$$M_s := \sum_{r=1}^{s} \{f(r) - \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\}. \tag{5.112}$$

It is easy to establish that (M_s) is a martingale with respect to $(\mathcal{F}_s)$, and $M_0 = 0$. Now fix $K \geq 1$ and note that $\tau_N(t) \wedge K$ is a bounded $\mathcal{F}_t$-stopping time. Hence we can apply the optional stopping theorem:

$$\mathbb{E}[M_{\tau_N(t)\wedge K}] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} \{f(r) - \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\}\right] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} f(r)\right] - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right] = \tag{5.113}$$

Since this holds for all $K \geq 1$,

$$\lim_{K\to\infty} \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} f(r)\right] = \lim_{K\to\infty} \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right]. \tag{5.114}$$

The monotone convergence theorem allows the limit to pass inside the expectation on each side (since increasing K can only increase each sum, by possibly adding non-negative terms). Hence

$$\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} f(r)\right] = \mathbb{E}\left[\lim_{K\to\infty} \sum_{r=1}^{\tau_N(t)\wedge K} f(r)\right] = \mathbb{E}\left[\lim_{K\to\infty} \sum_{r=1}^{\tau_N(t)\wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right] \tag{5.115}$$

which concludes the proof. ∎

There are a few instances where Fubini's Theorem and the Dominated Convergence Theorem are needed in order to pass a limit and expectation through an infinite sum. Now we verify that the conditions of these theorems indeed hold. This result, analogous to that in Koskela et al. (2018, Appendix), is used once in Lemma 5.7 at (5.67) and once in Lemma 5.8 at (5.80).

Lemma 5.16. *For any fixed $t > 0$,*

$$\mathbb{E}\left[\sum_{l=0}^{\infty}\left|(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l \sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\prod_{i=1}^{k}c_N(r_i)\right|\right] < \infty. \tag{5.116}$$

Proof.

$$\mathbb{E}\left[\sum_{l=0}^{\infty}\left|(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l \sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\prod_{i=1}^{k}c_N(r_i)\right|\right] \leq \mathbb{E}\left[\sum_{l=0}^{\infty}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}t^l(t+1)^k\right]$$
$$= \mathbb{E}[\exp\{\alpha_n t(1+O(N^{-1}))\}(t+1)^k] = \exp\{\alpha_n t(1+O(N^{-1}))\}(t+1)^k \tag{5.117}$$

■

5.5 Dependency graph

Missing links since this graph was updated:

- Lemma 5.3(a) is used three times in Lemma 5.7, but not anywhere else.
- Lemma 5.3 in the current dependency graph is really referring to Lemma 5.3(b)
- Lemma 5.16 is used in Lemmata 5.8 and 5.7.
- Lemma 5.10 is used in Lemmata 5.6, 5.7, 5.8 and 5.9

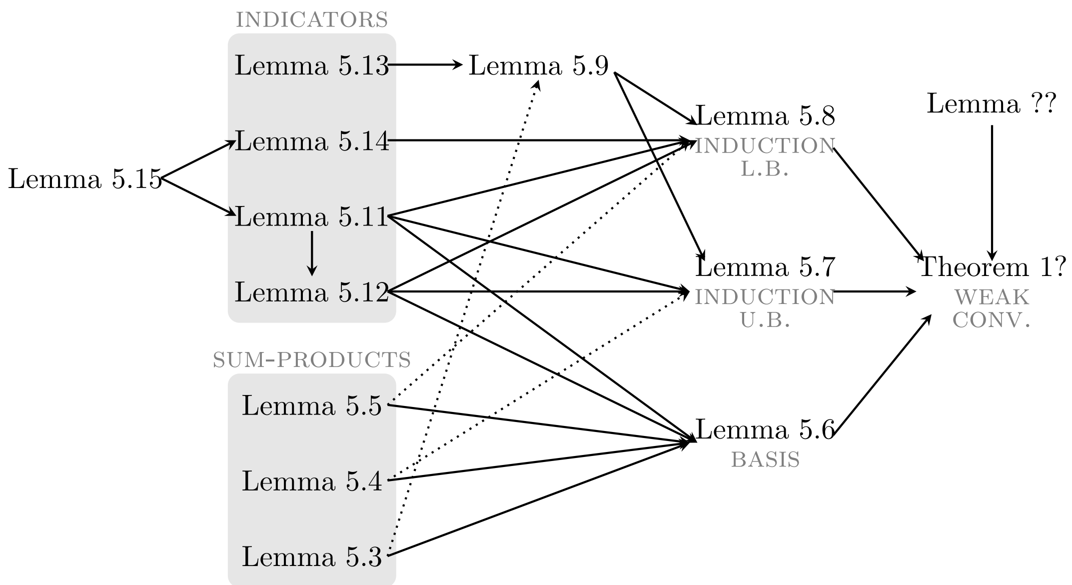


Figure 5.1: Graph showing dependencies between the lemmata used to prove weak convergence. Dotted arrows indicate dependence via a slight modification of the preceding lemma.

6 Discussion

Bibliography

[1] Christophe Andrieu, Arnaud Doucet, and Roman Holenstein. "Particle Markov Chain Monte Carlo Methods". In: *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 72.3 (2010), pp. 269–342.

[2] Alan Bain and Dan Crisan. *Fundamentals of Stochastic Filtering*. Springer, 2008.

[3] Leonard E. Baum et al. "A Maximization Technique Occurring in the Statistical Analysis of Probabilistic Functions of Markov Chains". In: *The Annals of Mathematical Statistics* 41 (1970), pp. 164–171.

[4] Nathanaël Berestycki. *Recent Progress in Coalescent Theory*. 0909.3985. ArXiv, 2009.

[5] Suzie Brown et al. "Simple Conditions for Convergence of Sequential Monte Carlo Genealogies with Applications". In: *Electronic Journal of Probability* 26.1 (2021), pp. 1–22. ISSN: 1083-6489. DOI: 10.1214/20-EJP561.

[6] C. Cannings. "The Latent Roots of Certain Markov Chains Arising in Genetics: A New Approach, I. Haploid Models". In: *Advances in Applied Probability* 6.2 (1974), pp. 260–290.

[7] C. Cannings. "The Latent Roots of Certain Markov Chains Arising in Genetics: A New Approach, II. Further Haploid Models". In: *Advances in Applied Probability* 7.2 (1975), pp. 264–282.

[8] James Carpenter, Peter Clifford, and Paul Fearnhead. "Improved Particle Filter for Nonlinear Problems". In: *IEE Proceedings — Radar, Sonar and Navigation* 146.1 (1999), pp. 2–7.

[9] Nicolas Chopin and Omiros Papaspiliopoulos. *An Introduction to Sequential Monte Carlo*. Springer, 2020.

[10] Adrien Corenflos et al. *Differentiable Particle Filtering via Entropy-Regularized Optimal Transport*. Tech. rep. 2102.07850. ArXiv, 2021.

[11] Dan Crisan and Terry Lyons. "Nonlinear Filtering and Measure-Valued Processes". In: *Probability Theory and Related Fields* 109.2 (1997), pp. 217–244.

[12] Dan Crisan and Terry Lyons. "A Particle Approximation of the Solution of the Kushner–Stratonovitch Equation". In: *Probability Theory and Related Fields* 115.4 (1999), pp. 549–578.

[13] Randal Douc, Olivier Cappé, and Eric Moulines. "Comparison of Resampling Schemes for Particle Filtering". In: *Proceedings of the 4th International Symposium on Image and Signal Processing and Analysis.* IEEE. 2005, pp. 64–69.

[14] Richard Durrett. *Probability Models for DNA Sequence Evolution.* Springer Science & Business Media, 2008.

[15] Richard Durrett. *Probability: Theory and Examples.* 5th ed. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2019. DOI: 10.1017/9781108591034.

[16] Bradley Efron and Robert J. Tibshirani. *An Introduction to the Bootstrap.* CRC press, 1994.

[17] Alison Etheridge. *Some Mathematical Models from Population Genetics: École D'Été de Probabilités de Saint-Flour XXXIX-2009.* Springer, 2011.

[18] Stewart N. Ethier and Thomas G. Kurtz. *Markov Processes: Characterization and Convergence.* John Wiley & Sons, 2009.

[19] Geir Evensen. "Sequential Data Assimilation with a Nonlinear Quasi-Geostrophic Model Using Monte Carlo Methods to Forecast Error Statistics". In: *Journal of Geophysical Research: Oceans* 99.C5 (1994), pp. 10143–10162.

[20] Paul Fearnhead and Peter Clifford. "On-line Inference for Hidden Markov Models via Particle Filters". In: *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 65.4 (2003), pp. 887–899.

[21] Ronald Aylmer Fisher. "On the Dominance Ratio". In: *Proceedings of the Royal Society of Edinburgh* 42 (1923), pp. 321–341.

[22] Ronald Aylmer Fisher. "The Distribution of Gene Ratios for Rare Mutations". In: *Proceedings of the Royal Society of Edinburgh* 50 (1930), pp. 205–220.

[23] Y.-X. Fu. "Exact Coalescent for the Wright–Fisher Model". In: *Theoretical Population Biology* 69 (4 2006), pp. 385–394.

[24] Mathieu Gerber, Nicolas Chopin, and Nick Whiteley. "Negative Association, Ordering and Convergence of Resampling Methods". In: *The Annals of Statistics* 47.4 (2019), pp. 2236–2260.

[25] Neil J. Gordon, David J. Salmond, and Adrian F. M. Smith. "Novel Approach to Nonlinear/Non-Gaussian Bayesian State Estimation". In: *IEE Proceedings F (Radar and Signal Processing).* Vol. 140. 2. IET. 1993, pp. 107–113.

[26] Godfrey Harold Hardy and Srinivasa Aaiyangar Ramanujan. "Asymptotic Formulae in Combinatory Analysis". In: *Proceedings of the London Mathematical Society* s2-17.1 (1918), pp. 75–115.

[27] Jeroen D. Hol. "Resampling in Particle Filters". LiTH-ISY-EX-ET-0283-2004. Internship report. Linköping University, 2004.

[28] Jeroen D. Hol, Thomas B. Schön, and Fredrik Gustafsson. “On Resampling Algorithms for Particle Filters”. In: *Nonlinear Statistical Signal Processing Workshop*. IEEE. 2006, pp. 79–82.

[29] Chaofan Huang, V. Roshan Joseph, and Simon Mak. *Population Quasi-Monte Carlo*. Tech. rep. 2012.13769. ArXiv, 2020.

[30] Andrew H. Jazwinski. *Stochastic Processes and Filtering Theory*. Courier Corporation, 2007.

[31] Rudolph Emil Kalman. “A New Approach to Linear Filtering and Prediction Problems”. In: *Journal of Basic Engineering* 82.1 (1960), pp. 35–45.

[32] John F. C. Kingman. “Exchangeability and the Evolution of Large Populations”. In: *Proceedings of the International Conference on Exchangeability in Probability and Statistics, Rome, 6th-9th April, 1981, in Honour of Professor Bruno de Finetti*. North-Holland, Amsterdam, 1982.

[33] John F. C. Kingman. “On the Genealogy of Large Populations”. In: *Journal of Applied Probability* 19.A (1982), pp. 27–43.

[34] John F. C. Kingman. “The Coalescent”. In: *Stochastic Processes and Their Applications* 13.3 (1982), pp. 235–248.

[35] Genshiro Kitagawa. “Monte Carlo Filter and Smoother for Non-Gaussian Nonlinear State Space Models”. In: *Journal of Computational and Graphical Statistics* 5.1 (1996), pp. 1–25.

[36] Jere Koskela et al. *Asymptotic genealogies of interacting particle systems with an application to sequential Monte Carlo*. Mathematics e-print 1804.01811. ArXiv, 2018.

[37] Fredrik Lindsten and Thomas B Schön. “Backward Simulation Methods for Monte Carlo Statistical Inference”. In: *Foundations and Trends in Machine Learning* 6.1 (2013), pp. 1–143.

[38] Jun S Liu and Rong Chen. “Sequential Monte Carlo Methods for Dynamic Systems”. In: *Journal of the American Statistical Association* 93.443 (1998), pp. 1032–1044.

[39] Martin Möhle. “Weak Convergence to the Coalescent in Neutral Population Models”. In: *Journal of Applied Probability* 36.2 (1999), pp. 446–460.

[40] Patrick Alfred Pierce Moran. “Random Processes in Genetics”. In: *Mathematical Proceedings of the Cambridge Philosophical Society*. Vol. 54. 1. Cambridge University Press, 1958, pp. 60–71.

[41] James E. Mosimann. “On the Compound Multinomial Distribution, the Multivariate β-Distribution, and Correlations among Proportions”. In: *Biometrika* 49.1/2 (1962), pp. 65–82.

[42] Aaron Myers et al. “Sequential Ensemble Transform for Bayesian Inverse Problems”. In: *Journal of Computational Physics* 427 (2021), p. 110055.

[43] Herbert E. Rauch, C. T. Striebel, and F. Tung. “Maximum Likelihood Estimates of Linear Dynamic Systems”. In: *AIAA Journal* 3.8 (1965), pp. 1445–1450.

[44] Sebastian Reich. “A Nonparametric Ensemble Transform Method for Bayesian Inference”. In: *SIAM Journal on Scientific Computing* 35.4 (2013), A2013–A2024.

[45] Ian W. Saunders, Simon Tavaré, and G. A. Watterson. “On the Genealogy of Nested Subsamples from a Haploid Population”. In: *Advances in Applied Probability* (1984), pp. 471–491.

[46] John L. Spouge. “Within a Sample from a Population, the Distribution of the Number of Descendants of a Subsample’s Most Recent Common Ancestor”. In: *Theoretical Population Biology* 92 (2014), pp. 51–54.

[47] Thomas Verma and Judea Pearl. “Causal Networks: Semantics and Expressiveness”. In: *Proceedings of the 4th Workshop on Uncertainty in Artificial Intelligence*. Minneapolis, MN, Mountain View, CA, 1988, pp. 352–359.

[48] Paolo Vidoni. “Exponential Family State Space Models Based on a Conjugate Latent Process”. In: *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 61.1 (1999), pp. 213–221.

[49] John Wakeley. *Coalescent Theory: An Introduction*. Roberts & Co. Publishers, 2009.

[50] Eric A. Wan and Rudolph van der Merwe. “The Unscented Kalman Filter for Nonlinear Estimation”. In: *Proceedings of the IEEE 2000 Adaptive Systems for Signal Processing, Communications, and Control Symposium*. IEEE. 2000, pp. 153–158.

[51] Darrell Whitley. “A Genetic Algorithm Tutorial”. In: *Statistics and Computing* 4.2 (1994), pp. 65–85.

[52] Sewall Wright. “Evolution in Mendelian Populations”. In: *Genetics* 16.2 (1931), pp. 97–159.