Corollaries 1–3 proofs with details...

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Definition 1. A function f is said to be i-increasing if it is an increasing function in $v_t^{(i)} = |\{j : a_t^{(j)} = i\}|$.

Multinomial resampling

Corollary 1. *Under the time scaling (??), supposing there exist constants $0 < \varepsilon \leq 1 \leq a < \infty$ such that*

$$\frac{1}{a} \leq g_t(x, x') \leq a \tag{1}$$

$$\varepsilon h(x') \leq q_t(x, x') \leq \frac{1}{\varepsilon} h(x'), \tag{2}$$

genealogies of SMC algorithms with multinomial resampling converge to Kingman's n-coalescent in the sense of finite-dimensional distributions as $N \to \infty$.

Lemma 1. *Let $a_t^{(i)}$ be the parental indices from a SMC algorithm with multinomial resampling. For any function f that is i-increasing,*

$$\mathbb{E}[f(\mathbf{a}_t) \mid \mathcal{H}_t] \leq \mathbb{E}[f(\mathbf{A}_1)]$$
$$\mathbb{E}[f(\mathbf{a}_t) \mid \mathcal{H}_t] \geq \mathbb{E}[f(\mathbf{A}_2)]$$

where the elements of $\mathbf{A}_1, \mathbf{A}_2$ are all mutually independent and independent of $\mathcal{F}_\infty$, and distributed according to

$$A_1^{(j)} \sim \text{Categorical}\left(\left(\frac{a}{\varepsilon}\right)^{\mathbb{1}_{\{i=1\}} - \mathbb{1}_{\{i\neq 1\}}}, \dots, \left(\frac{a}{\varepsilon}\right)^{\mathbb{1}_{\{i=N\}} - \mathbb{1}_{\{i\neq N\}}}\right)$$
$$A_2^{(j)} \sim \text{Categorical}\left(\left(\frac{\varepsilon}{a}\right)^{\mathbb{1}_{\{i=1\}} - \mathbb{1}_{\{i\neq 1\}}}, \dots, \left(\frac{\varepsilon}{a}\right)^{\mathbb{1}_{\{i=N\}} - \mathbb{1}_{\{i\neq N\}}}\right),$$

where the arguments of Categorical and Multinomial distributions are given up to a normalising constant here and throughout this document.

Proof. The result follows using the bounds given in equations (1), (2) with a balls-in-bins coupling, and cancelling h from the top and bottom. □

Define the corresponding "family sizes" $V_1^{(i)} := |\{j : A_1^{(j)} = i\}|$ and $V_2^{(i)} := |\{j : A_2^{(j)} = i\}|$ for $i = 1, \dots, N$. The distributions of $\mathbf{A}_1, \mathbf{A}_2$ imply the following:

$$\mathbf{V}_1 \sim \text{Multinomial}\left(N, \left(\frac{a}{\varepsilon}\right)^{\mathbb{1}_{\{i=1\}} - \mathbb{1}_{\{i\neq 1\}}}, \dots, \left(\frac{a}{\varepsilon}\right)^{\mathbb{1}_{\{i=N\}} - \mathbb{1}_{\{i\neq N\}}}\right)$$
$$\mathbf{V}_2 \sim \text{Multinomial}\left(N, \left(\frac{\varepsilon}{a}\right)^{\mathbb{1}_{\{i=1\}} - \mathbb{1}_{\{i\neq 1\}}}, \dots, \left(\frac{\varepsilon}{a}\right)^{\mathbb{1}_{\{i=N\}} - \mathbb{1}_{\{i\neq N\}}}\right),$$

Notice that the function $f_i(\mathbf{a}_t) := (v_t^{(i)})_2$ is i-increasing for each $i = 1, \dots, N$. Applying Lemma 1 and the Multinomial moments formula (Mosimann, 1962), we obtain the following lower bound:

$$\mathbb{E}_t[f_i(\mathbf{a}_t)] \geq \mathbb{E}[f_i(\mathbf{A}_2)] = \mathbb{E}[(V_2^{(i)})_2]$$
$$= \frac{(N)_2 (\varepsilon/a)^2}{[(\varepsilon/a) + (N-1)(a/\varepsilon)]^2} \geq \frac{(N)_2 (\varepsilon/a)^2}{N^2 (a/\varepsilon)^2} = \frac{(N)_2}{N^2} \frac{\varepsilon^4}{a^4}.$$

So we can lower bound the denominator by

$$\mathbb{E}_t[c_N(t)] = \frac{1}{(N)_2}\sum_{i=1}^{N}\mathbb{E}_t[(v_t^{(i)})_2] \geq \frac{N}{(N)_2}\frac{(N)_2}{N^2}\frac{\varepsilon^4}{a^4} = \frac{\varepsilon^4}{Na^4}.$$

To upper bound the numerator, consider the function $f_i(\mathbf{a}_t) := (v_t^{(i)})_3$, which is i-increasing for each $i = 1, \dots, N$. Again using Lemma 1 and (Mosimann, 1962), we obtain the following lower bound:

$$\begin{aligned}\mathbb{E}_t[f_i(\mathbf{a}_t)] &\leq \mathbb{E}[f_i(\mathbf{A}_1)] = \mathbb{E}[(V_1^{(i)})_3]\\ &= \frac{(N)_3(a/\varepsilon)^3}{[(a/\varepsilon)+(N-1)(\varepsilon/a)]^3} \geq \frac{(N)_3(a/\varepsilon)^3}{N^3(\varepsilon/a)^3} = \frac{(N)_3}{N^3}\frac{a^6}{\varepsilon^6}.\end{aligned}$$

and the numerator is therefore bounded above by

$$\frac{1}{(N)_3}\sum_{i=1}^{N}\mathbb{E}_t[(v_t^{(i)})_3] \leq \frac{N}{(N)_3}\frac{(N)_3}{N^3}\frac{\varepsilon^6}{a^6} = \frac{\varepsilon^6}{N^2a^6}.$$

The ratio is therefore bounded above by

$$\frac{\frac{1}{(N)_3}\sum_{i=1}^{N}\mathbb{E}_t[(v_t^{(i)})_3]}{\frac{1}{(N)_2}\sum_{i=1}^{N}\mathbb{E}_t[(v_t^{(i)})_2]} \leq \frac{N}{(N)_3}\frac{(N)_3}{N^3}\frac{\varepsilon^6}{a^6} = \frac{\varepsilon^6}{N^2a^6}\frac{Na^4}{\varepsilon^4} = \frac{a^{10}}{N\varepsilon^{10}} =: b_N \lim_{N\to\infty} 0.$$

We can thus conclude the proof of Corollary 1 by applying Theorem 1.

Conditional SMC with multinomial resampling

We can apply the same technique to tackle conditional SMC, but it requires an adjustment of the bounding distributions. We assume wlog that the immortal particle always takes index 1.

Lemma 2. *Let $a_t^{(i)}$ be the parental indices from a conditional SMC algorithm with multinomial resampling. For any function f that is i-increasing,*

$$\begin{aligned}&\mathbb{E}[f(\mathbf{a}_t) \mid \mathcal{H}_t] \leq \mathbb{E}[f(\mathbf{A}_1)]\\ &\mathbb{E}[f(\mathbf{a}_t) \mid \mathcal{H}_t] \geq \mathbb{E}[f(\mathbf{A}_2)]\end{aligned}$$

where the elements of $\mathbf{A}_1, \mathbf{A}_2$ are all mutually independent and independent of $\mathcal{F}_\infty$, and distributed according to

$$A_1^{(j)} \sim \begin{cases} \delta_1 & j = 1\\ \text{Categorical}\left(\left(\frac{a}{\varepsilon}\right)^{\mathbb{1}_{\{i=1\}}-\mathbb{1}_{\{i\neq 1\}}}, \dots, \left(\frac{a}{\varepsilon}\right)^{\mathbb{1}_{\{i=N\}}-\mathbb{1}_{\{i\neq N\}}}\right) & j \neq 1\end{cases}$$

$$A_2^{(j)} \sim \begin{cases} \delta_1 & j = 1\\ \text{Categorical}\left(\left(\frac{\varepsilon}{a}\right)^{\mathbb{1}_{\{i=1\}}-\mathbb{1}_{\{i\neq 1\}}}, \dots, \left(\frac{\varepsilon}{a}\right)^{\mathbb{1}_{\{i=N\}}-\mathbb{1}_{\{i\neq N\}}}\right) & j \neq 1.\end{cases}$$

As before, we can define the corresponding "family sizes" $V_1^{(i)} := |\{j : A_1^{(j)} = i\}|$ and $V_2^{(i)} := |\{j : A_2^{(j)} = i\}|$ for $i = 1, \dots, N$. They now have the following distributions:

$$\begin{aligned}\mathbf{V}_1 &\overset{d}{=} (1, 0, \dots, 0) + \text{Multinomial}\left(N-1, \left(\frac{a}{\varepsilon}\right)^{\mathbb{1}_{\{i=1\}}-\mathbb{1}_{\{i\neq 1\}}}, \dots, \left(\frac{a}{\varepsilon}\right)^{\mathbb{1}_{\{i=N\}}-\mathbb{1}_{\{i\neq N\}}}\right)\\ \mathbf{V}_2 &\overset{d}{=} (1, 0, \dots, 0) + \text{Multinomial}\left(N-1, \left(\frac{\varepsilon}{a}\right)^{\mathbb{1}_{\{i=1\}}-\mathbb{1}_{\{i\neq 1\}}}, \dots, \left(\frac{\varepsilon}{a}\right)^{\mathbb{1}_{\{i=N\}}-\mathbb{1}_{\{i\neq N\}}}\right).\end{aligned}$$

Now consider again the i-increasing function $f_i(\mathbf{a}_t) := (v_t^{(i)})_2$. In the conditional SMC case, we can apply Lemma 2 to obtain the lower bound

$$\frac{1}{(N)_2}\sum_{i=1}^{N}\mathbb{E}_t[(v_t^{(i)})_2] \geq \frac{1}{(N)_2}\sum_{i=1}^{N}\mathbb{E}[(V_2^{(i)})_2] = \frac{1}{(N)_2}\left[\sum_{i=1}^{N}\frac{(N-1)_2(\varepsilon/a)^2}{[(\varepsilon/a)+(N-2)(a/\varepsilon)]^2} + 2\frac{(N-1)(\varepsilon/a)}{(\varepsilon/a)+(N-2)(a/\varepsilon)}\right]$$

using the Multinomial moments as before, along with the identity $(X+1)_2 \equiv (X)_2 + 2(X)_1$. This is further bounded by

$$\frac{1}{(N)_2}\sum_{i=1}^{N}\mathbb{E}_t[(v_t^{(i)})_2] \geq \frac{1}{(N)_2}\left[\frac{(N)_3(\varepsilon/a)^2}{(N-1)^2(a/\varepsilon)^2} + \frac{2(N-1)(\varepsilon/a)}{(N-1)(a/\varepsilon)}\right] = \frac{1}{(N)_2}\left[\frac{(N)_3}{(N-1)^2}\frac{\varepsilon^4}{a^4} + \frac{2\varepsilon^2}{a^2}\right]$$

Similarly, we derive an upper bound on $f_i(\mathbf{a}_t) := (v_t^{(i)})_3$, yielding

$$\begin{aligned}\frac{1}{(N)_3}\sum_{i=1}^{N}\mathbb{E}_t[(v_t^{(i)})_3] &\leq \frac{1}{(N)_3}\left[\sum_{i=1}^{N}\frac{(N-1)_3(a/\varepsilon)^3}{[(a/\varepsilon)+(N-2)(\varepsilon/a)]^3} + 3\frac{(N-1)_2(a/\varepsilon)^2}{[(a/\varepsilon)+(N-2)(\varepsilon/a)]^2}\right] \\ &\leq \frac{1}{(N)_3}[...]\end{aligned}$$

References

Mosimann, J. E. (1962), 'On the compound multinomial distribution, the multivariate β-distribution, and correlations among proportions', *Biometrika* **49**(1/2), 65–82.