

Asymptotic Genealogies of Sequential Monte Carlo Algorithms

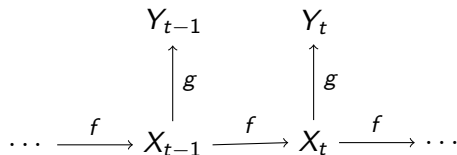
Suzie Brown

26 April 2019

State space models

Hidden process $X_0, \dots, X_T \in \mathcal{X}$

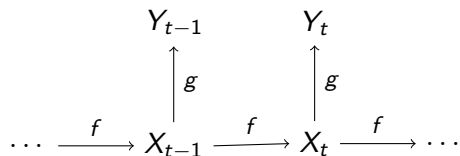
Observable process $Y_0, \dots, Y_T \in \mathcal{Y}$



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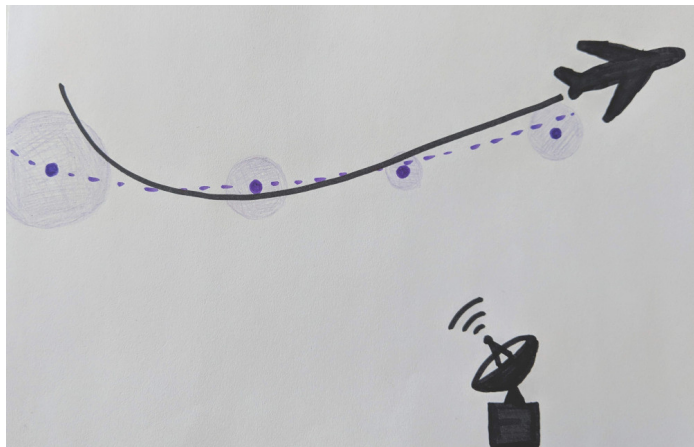


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Target tracking



use noisy radar data to infer position/trajectory of aircraft:

- f models how aircraft moves
- g models uncertainty in radar measurements

Inference problems

Filtering: where is it now? $p(x_t|y_{0:t})$

Prediction: where will it go next? $p(x_{t+1}|y_{0:t})$

Smoothing: where has it been? $p(x_{0:t}|y_{0:t})$

Smoothing is “harder” than filtering/prediction [1].

Deterministic solutions

Kalman filter [2]

Under a linear Gaussian model:

$$X_0 \sim \mathcal{N}(0, \Sigma_0)$$

$$X_{t+1} \mid (X_t = x_t) \sim \mathcal{N}(Ax_t, \Sigma_x)$$

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we can recursively compute filtering distributions. Then a backward pass of RTS smoother [3] provides smoothing distributions.

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Unscented Kalman filter [5]

For highly non-linear Gaussian models, replace the prediction step of Kalman filter by propagating a representative set of points through f .

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Sequential Monte Carlo (SMC) provides general-purpose (stochastic) methods that do not require a tractable model.

Only requires sampling from $f(\cdot|x)$, and pointwise evaluation of $g(y|x)$ up to a normalising constant for each y .

Sequential Monte Carlo

Prior: $p(x_{0:t}) = \mu(x_0) \prod_{i=1}^t f(x_i|x_{i-1})$

Likelihood: $p(y_{0:t}|x_{0:t}) = \prod_{i=0}^t g(y_i|x_i)$

Posterior: $p(x_{0:t}|y_{0:t}) \propto \mu(x_0)g(y_0|x_0) \prod_{i=1}^t f(x_i|x_{i-1})g(y_i|x_i)$

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- Represent posterior distribution at time t with N particles.
- Posterior factorises sequentially, avoiding increase of dimension with T .

Sequential Monte Carlo

Algorithm

After initialisation, iterate these steps:

- **Propagate:** move the particles through the transition f
- **Calculate weights:** weight each particle according to g
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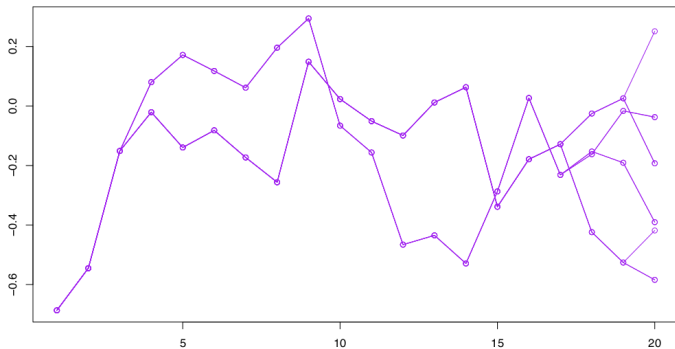
Approximate posterior distribution $p(dx_{0:t}|y_{0:t})$ by the empirical measure of the particles:

$$\hat{p}(dx_{0:t}|y_{0:t}) = \frac{1}{N} \sum_{i=1}^N \delta_{X_{0:t}^{(i)}}(dx_{0:t})$$

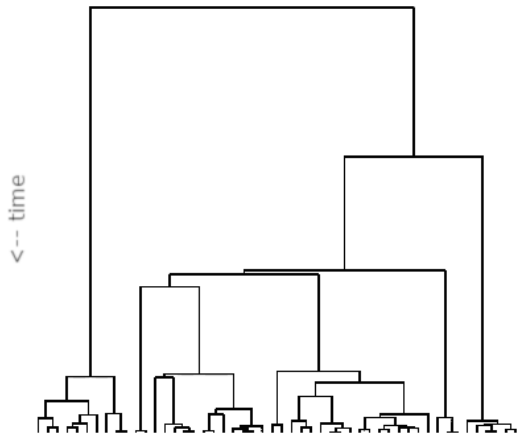
Ancestral degeneracy

For smoothing we need a sample of trajectories.

Resampling means that trajectories of time T particles coalesce backwards in time.

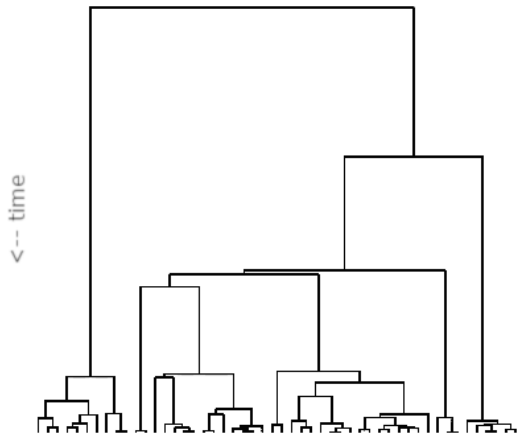


Kingman's coalescent



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This is the limiting coalescent process in many population models as $N \rightarrow \infty$ [7, 8].

Coalescent for SMC

- Analysing the coalescent can help us to understand the performance of various SMC algorithms.
- The limiting coalescent of an SMC algorithm may depend on the resampling mechanism.
- For multinomial resampling, the limiting coalescent is a scaled Kingman coalescent [9].
- What about other (more used) resampling schemes?

Requirements:

- The total number of particles N remains fixed.
- The particles after resampling are equally weighted.
- The resampling scheme is unbiased; that is, the expected number of offspring of each particle i is equal to $Nw_t^{(i)}$.

Resampling

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$$r_t^{(i)} := (Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor)$$

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- Residual resampling yields lower Monte Carlo variance than multinomial resampling [11].
- Residual resampling is widely used by practitioners.
- Analysing the coalescent for residual resampling is a work in progress.

Particle Gibbs [12]

Hidden Markov model where transition depends on a hyperparameter θ :

$$\theta \sim \nu(\cdot)$$

$$X_0 \sim \mu(\cdot)$$

$$X_{t+1} \mid (X_t = x_t) \sim f_\theta(\cdot \mid x_t) \quad t = 0, \dots, T-1$$

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Gibbs sampler: alternately sample from $p(\theta \mid x_{0:t}, y_{0:t})$ and $p(x_{0:t} \mid \theta, y_{0:t})$.

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- SMC is an appropriate method for sampling from $p(x_{0:t} \mid \theta, y_{0:t})$
- To target the correct posterior distribution, need to use *conditional SMC*.

Conditional SMC

- One “immortal” trajectory is conditioned to survive all of the resampling steps.
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- But we have this result:

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- Intuition: as $N \rightarrow \infty$, there is zero probability that an arbitrary sample of fixed size n contains the immortal particle.

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- In progress: investigate by simulations whether this is a good model for finite N .
- In progress: analysis of residual resampling (and other resampling schemes).

References I

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