Weak convergence proof v.2

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Bounds on sum-products

Lemma 1. *Fix $t > 0$, $l \in \mathbb{N}$.*

(a) $$\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \le (t+1)^l$$

(b) $$t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \binom{l}{2} (t+1)^{l-2} \le \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \le t^l + c_N(\tau_N(t))(t+1)^l$$

Proof. **(a)** It is a true fact that

$$\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \le \left(\sum_{s=0}^{\tau_N(t)} c_N(s) \right)^l, \tag{1}$$

as can be seen by considering the multinomial expansion of the RHS. By definition of τ_N,

$$t \le \sum_{s=0}^{\tau_N(t)} c_N(s) \le t+1, \tag{2}$$

hence

$$\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \le (t+1)^l. \tag{3}$$

(b) As pointed out in Koskela et al. (2018, Equation (8)),

$$\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \ge \left(\sum_{s=0}^{\tau_N(t)} c_N(s) \right)^l - \binom{l}{2} \left(\sum_{s=0}^{\tau_N(t)} c_N(s)^2 \right) \left(\sum_{s=0}^{\tau_N(t)} c_N(s) \right)^{l-2}. \tag{4}$$

Substituting (2) into the RHS of (4) yields the lower bound.

For the upper bound we have

$$\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \le \left(\sum_{s=0}^{\tau_N(t)} c_N(s) \right)^l \le \left(\sum_{s=0}^{\tau_N(t)-1} c_N(s) + c_N(\tau_N(t)) \right)^l \le \left[t + c_N(\tau_N(t)) \right]^l, \tag{5}$$

again using the definition of τ_N. A binomial expansion yields

$$[t + c_N(\tau_N(t))]^l = t^l + \sum_{i=0}^{l-1} \binom{l}{i} t^i c_N(\tau_N(t))^{l-i} = t^l + c_N(\tau_N(t)) \sum_{i=0}^{l-1} \binom{l}{i} t^i c_N(\tau_N(t))^{l-1-i}, \tag{6}$$

then since $c_N(s) \le 1$ for all s,

$$\sum_{i=0}^{l-1} \binom{l}{i} t^i c_N(\tau_N(t))^{l-1-i} \le \sum_{i=0}^{l-1} \binom{l}{i} t^i \le (t+1)^l. \tag{7}$$

Putting this together yields the upper bound. ∎

Lemma 2. *Fix $t > 0$, $l \in \mathbb{N}$. Let B be a positive constant which may depend on n.*

$$\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) + BD_N(s_j)] \leq \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1}(1+B)^l. \tag{8}$$

Proof. We start with a binomial expansion:

$$\begin{aligned} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) + BD_N(s_j)] &= \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \sum_{\mathcal{I} \subseteq [l]} B^{l-|\mathcal{I}|} \left(\prod_{i \in \mathcal{I}} c_N(s_i) \right) \left(\prod_{j \notin \mathcal{I}} D_N(s_j) \right) \\ &= \sum_{\mathcal{I} \subseteq [l]} B^{l-|\mathcal{I}|} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i \in \mathcal{I}} c_N(s_i) \right) \left(\prod_{j \notin \mathcal{I}} D_N(s_j) \right) \end{aligned} \tag{9}$$

where $[l] := \{1, \ldots, l\}$. Since the sum is over all permutations of $s_1, \ldots, s_l$, we may arbitrarily choose an ordering for $\{1, \ldots, l\}$ such that $\mathcal{I} = \{1, \ldots, |\mathcal{I}|\}$:

$$\sum_{\mathcal{I} \subseteq [l]} B^{l-|\mathcal{I}|} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i \in \mathcal{I}} c_N(s_i) \right) \left(\prod_{j \notin \mathcal{I}} D_N(s_j) \right) = \sum_{I=0}^{l} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right). \tag{10}$$

Separating the term $I = l$,

$$\begin{aligned} &\sum_{I=0}^{l} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right) \\ &\qquad = \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) + \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right). \end{aligned} \tag{11}$$

In the second term on the RHS, there is always at least one D_N term, and $c_N(s) \geq D_N(s)$ for all s (Koskela et al., 2018, p.9), so we can write

$$\begin{aligned} \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right) &\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{l-1} c_N(s_i) \right) D_N(s_l) \\ &\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \left(\sum_{s_1 \neq \cdots \neq s_{l-1}}^{\tau_N(t)} \prod_{i=1}^{l-1} c_N(s_i) \right) \sum_{s_l=1}^{\tau_N(t)} D_N(s_l) \\ &\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} (t+1)^{l-1} \sum_{s=1}^{\tau_N(t)} D_N(s) \end{aligned} \tag{12}$$

using (1) and (2). Finally, by the Binomial Theorem,

$$\sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} (t+1)^{l-1} \sum_{s=1}^{\tau_N(t)} D_N(s) \leq \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1}(1+B)^l, \tag{13}$$

which, together with (11), concludes the proof. ∎

Lemma 3. *Fix $t > 0$, $l \in \mathbb{N}$. Let B be a positive constant which may depend on n.*

$$\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - BD_N(s_j)] \geq \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1}(1+B)^l. \tag{14}$$

Proof. A binomial expansion and subsequent manipulation as in (9)–(11) gives

$$\begin{aligned}
\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} \left[c_N(s_j) - BD_N(s_j)\right] &= \sum_{\mathcal{I}\subseteq[l]} (-B)^{l-|\mathcal{I}|} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i\in\mathcal{I}} c_N(s_i)\right)\left(\prod_{j\notin\mathcal{I}} D_N(s_j)\right) \\
&= \sum_{I=0}^{l} \binom{l}{I} (-B)^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right)\left(\prod_{j=I+1}^{l} D_N(s_j)\right) \\
&= \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) + \sum_{I=0}^{l-1} \binom{l}{I} (-B)^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right)\left(\prod_{j=I+1}^{l} D_N(s_j)\right) \\
&\geq \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) - \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right)\left(\prod_{j=I+1}^{l} D_N(s_j)\right)
\end{aligned} \tag{15}$$

where the last inequality just multiplies some positive terms by -1. Then (12)–(13) can be applied directly (noting that an upper bound on negative terms gives a lower bound overall):

$$-\sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right)\left(\prod_{j=I+1}^{l} D_N(s_j)\right) \geq -\left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)(t+1)^{l-1}(1+B)^l \tag{16}$$

which concludes the proof. ■

Main components of weak convergence

Lemma 4 (Basis step)**.** *For any* $0 < t < \infty$*,*

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1-p_r)\right] = e^{-\alpha_n t} \tag{17}$$

where $\alpha_n := n(n-1)/2$.

Proof. We start by showing that $\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right] \leq e^{-\alpha_n t}$.
From Koskela et al. (2018, Lemma 1 Case 1), taking $\xi = \Delta$, we have for each r

$$1 - p_r = p_{\Delta\Delta}(r) \leq 1 - \alpha_n(1 + O(N^{-1}))\left[c_N(r) - B_n' D_N(r)\right] \tag{18}$$

where the $O(N^{-1})$ term does not depend on r. When $N \geq 3$, a sufficient condition to ensure the bound in (18) is non-negative is that the event

$$E_N^1(r) := \left\{ c_N(r) \leq \frac{(N-2)_{n-2}}{\alpha_n N^{n-2}} \right\} \tag{19}$$

occurs. We will also need to control the sign of $c_N(r) - B_n' D_N(r)$, for which we define the event

$$E_N^2(r) := \left\{ c_N(r) \geq B_n' D_N(r) \right\}, \tag{20}$$

and we define $E_N^1 := \bigcap_{r=1}^{\tau_N(t)} E_N^1(r)$ and $E_N^2 := \bigcap_{r=1}^{\tau_N(t)} E_N^2(r)$. Then

$$1 - p_r = p_{\Delta\Delta}(r) \leq 1 - \alpha_n(1 + O(N^{-1}))\left[c_N(r) - B_n' D_N(r)\right] \mathbb{1}_{E_N^1 \cap E_N^2}. \tag{21}$$

Applying a multinomial expansion and then separating the positive and negative terms,

$$\begin{aligned}\prod_{r=1}^{\tau_N(t)}(1-p_r) &\leq 1+\sum_{l=1}^{\tau_N(t)}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}\left[c_N(s_j)-B_n'D_N(s_j)\right]\mathbb{1}_{E_N^1\cap E_N^2}\\ &= 1+\sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}\left[c_N(s_j)-B_n'D_N(s_j)\right]\mathbb{1}_{E_N^1\cap E_N^2}\\ &\quad -\sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}\left[c_N(s_j)-B_n'D_N(s_j)\right]\mathbb{1}_{E_N^1\cap E_N^2}. \end{aligned} \tag{22}$$

This is further bounded by applying Lemma 3 and then both bounds of Lemma 1(b):

$$\begin{aligned}\prod_{r=1}^{\tau_N(t)}(1-p_r) &\leq 1+\Bigg\{\sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}c_N(s_j)\\ &\quad -\sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\left[\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}c_N(s_j)-\left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)(t+1)^{l-1}(1+B_n')^l\right]\Bigg\}\mathbb{1}_{E_N^1\cap E_N^2}\\ &\leq 1+\Bigg\{\sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\left\{t^l+c_N(\tau_N(t))(t+1)^l\right\}\\ &\quad -\sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\left[t^l-\left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right)\binom{l}{2}(t+1)^{l-2}-\left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)(t+1)^{l-1}(1+B_n')^l\right]\Bigg\}\mathbb{1}_{E_N^1\cap E_N^2} \end{aligned} \tag{23}$$

Collecting some terms,

$$\begin{aligned}\prod_{r=1}^{\tau_N(t)}(1-p_r) &\leq 1+\sum_{l=1}^{\tau_N(t)}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_{E_N^1\cap E_N^2}+c_N(\tau_N(t))\sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}(t+1)^l\\ &\quad +\left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right)\sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\binom{l}{2}(t+1)^{l-2}\\ &\quad +\left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)\sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}(t+1)^{l-1}(1+B_n')^l\\ &\leq 1+\sum_{l=1}^{\infty}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_{\{\tau_N(t)\geq l\}}\mathbb{1}_{E_N^1\cap E_N^2}+c_N(\tau_N(t))\exp[\alpha_n(1+O(N^{-1}))(t+1)]\\ &\quad +\left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right)\frac{1}{2}\alpha_n^2\exp[\alpha_n(1+O(N^{-1}))(t+1)]\\ &\quad +\left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')]. \end{aligned} \tag{24}$$

Now, taking the expectation and limit, then applying Brown et al. (2021, Equations (3.3)–(3.5)), and Lemmata 9, 10 and 12 to deal with the indicators,

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right] \le 1 + \sum_{l=1}^{\infty}(-\alpha_n)^l \frac{1}{l!}t^l \lim_{N\to\infty}\mathbb{P}\left[\{\tau_N(t)\ge l\}\cap E_N^1 \cap E_N^2\right] + \lim_{N\to\infty}\mathbb{E}\left[c_N(\tau_N(t))\right]\exp[\alpha_n(t+1)]$$

$$+ \lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right]\frac{1}{2}\alpha_n^2 \exp[\alpha_n(t+1)]$$

$$+ \lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} D_N(s)\right]\exp[\alpha_n(t+1)(1+B_n')]$$

$$= 1 + \sum_{l=1}^{\infty}(-\alpha_n)^l \frac{1}{l!}t^l = e^{-\alpha_n t}. \tag{25}$$

Passing the limit and expectation inside the infinite sum is justified by dominated convergence and Fubini.

It remains to show the corresponding lower bound $\lim_{N\to\infty}\mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right] \ge e^{-\alpha_n t}$. From Brown et al. (2021, Equation (3.14)), taking $\xi = \Delta$, we have

$$1 - p_t = p_{\Delta\Delta}(t) \ge 1 - \alpha_n(1+O(N^{-1}))\left[c_N(t) + B_n D_N(t)\right] \tag{26}$$

where $B_n > 0$ and the $O(N^{-1})$ term does not depend on t. In particular,

$$1 - p_t = p_{\Delta\Delta}(t) \ge 1 - \frac{N^{n-2}}{(N-2)_{n-2}}\alpha_n[c_N(t) + B_n D_N(t)]. \tag{27}$$

Since $D_N(s) \le c_N(s)$ for all s (Koskela et al., 2018, p.9), a sufficient condition for this bound to be non-negative is

$$E_N^3(r) := \left\{c_N(r) \le \frac{(N-2)_{n-2}}{N^{n-2}}\alpha_n^{-1}(1+B_n)^{-1}\right\}, \tag{28}$$

and we again define $E_N^3 := \bigcap_{r=1}^{\tau_N(t)} E_N^3(r)$. We now apply a multinomial expansion to the product, and split into positive and negative terms:

$$\prod_{r=1}^{\tau_N(t)}(1-p_r) \ge \left\{1 + \sum_{l=1}^{\tau_N(t)}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}\sum_{s_1\ne\cdots\ne s_l}^{\tau_N(t)}\prod_{j=1}^{l}[c_N(s_j)+B_nD_N(s_j)]\right\}\mathbb{1}_{E_N^3}$$

$$= \left\{1 + \sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\sum_{s_1\ne\cdots\ne s_l}^{\tau_N(t)}\prod_{j=1}^{l}[c_N(s_j)+B_nD_N(s_j)]\right.$$

$$\left. - \sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\sum_{s_1\ne\cdots\ne s_l}^{\tau_N(t)}\prod_{j=1}^{l}[c_N(s_j)+B_nD_N(s_j)]\right\}\mathbb{1}_{E_N^3} \tag{29}$$

This is further bounded by applying Lemma 2 and both bounds in Lemma 1(b):

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1-p_r) \geq \Bigg\{ 1 + \sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \\
- \sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \left[\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1+B_n)^l \right] \Bigg\} \mathbb{1}_{E_N^3} \\
\geq \Bigg\{ 1 + \sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \left[t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \binom{l}{2} (t+1)^{l-2} \right] \\
- \sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \left[t^l + c_N(\tau_N(t))(t+1)^l + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1+B_n)^l \right] \Bigg\} \mathbb{1}_{E_N^3}.
\end{aligned} \tag{30}$$

Collecting terms,

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1-p_r) \geq \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_{E_N^3} - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \binom{l}{2} (t+1)^{l-2} \\
- c_N(\tau_N(t)) \sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} (t+1)^l \\
- \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} (t+1)^{l-1} (1+B_n)^l \\
\geq \sum_{l=0}^{\infty} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_{E_N^3} \mathbb{1}_{\{\tau_N(t) \geq l\}} - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n (1+O(N^{-1}))(t+1)] \\
- c_N(\tau_N(t)) \exp[\alpha_n (1+O(N^{-1}))(t+1)] \\
- \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n (1+O(N^{-1}))(t+1)(1+B_n)].
\end{aligned} \tag{31}$$

Now, taking the expectation and limit, and applying Brown et al. (2021, Equations (3.3)–(3.5)) to show that all but the first sum vanish, and Lemmata 10 and 9 to show that $\lim_{N\to\infty} \mathbb{P}[\{\tau_N(t) \geq l\} \cap E_N^3] = 1$,

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1-p_r) \right] \geq \sum_{l=0}^{\infty} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l \lim_{N\to\infty} \mathbb{P}\left[\{\tau_N(t) \geq l\} \cap E_N^3 \right] \\
- \lim_{N\to\infty} \mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right] \frac{1}{2} \alpha_n^2 \exp[\alpha_n (t+1)] \\
- \lim_{N\to\infty} \mathbb{E}\left[c_N(\tau_N(t)) \right] \exp[\alpha_n (t+1)] \\
- \lim_{N\to\infty} \mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} D_N(s) \right] \exp[\alpha_n (t+1)(1+B_n)] \\
= \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l = e^{-\alpha_n t}.
\end{aligned} \tag{32}$$

Again, passing the limit and expectation inside the infinite sum is justified by dominated convergence and Fubini. Combining the upper and lower bounds in (25) and (32) respectively concludes the proof. ■

Lemma 5 (Induction step upper bound). *Fix $k \in \mathbb{N}$, $i_0 := 0$, $i_k := k$. For any sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_k \leq t$,*

$$\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\right] \leq \alpha_n^k e^{-\alpha_n t} \sum_{\substack{i_1\leq\cdots\leq i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\geq j\forall j}} \prod_{j=1}^{k} \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}. \tag{33}$$

Proof. We use the bound on $(1-p_r)$ from (18) and apply a multinomial expansion, defining as in (19) and (20) respectively the sequences of events E_N^1 and E_N^2 which ensure the bounds are non-negative:

$$\begin{aligned}
\prod_{\substack{r=1\\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)} (1-p_r) &\leq \prod_{\substack{r=1\\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)} \left\{1-\alpha_n(1+O(N^{-1}))[c_N(r)-B_n' D_N(r)]\mathbb{1}_{E_N^1\cap E_N^2}\right\} \\
&= 1+\sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1+O(N^{-1}))\frac{1}{l!} \sum_{\substack{s_1\neq\cdots\neq s_l\\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j)-B_n' D_N(s_j)]\mathbb{1}_{E_N^1\cap E_N^2} \\
&= 1+\sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1+O(N^{-1}))\frac{1}{l!} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j)-B_n' D_N(s_j)]\mathbb{1}_{E_N^1\cap E_N^2} \\
&\quad - \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1+O(N^{-1}))\frac{1}{l!} \sum_{\substack{s_1\neq\cdots\neq s_l\\ \exists i,i':s_i=r_{i'}}} \prod_{j=1}^{l} [c_N(s_j)-B_n' D_N(s_j)]\mathbb{1}_{E_N^1\cap E_N^2}.
\end{aligned} \tag{34}$$

The penultimate line above is exactly the expansion we had in the basis step (22), except for the limit on l, and as such following the same arguments gives a bound analogous to that in (24):

$$\begin{aligned}
1+\sum_{l=1}^{\tau_N(t)-k} &(-\alpha_n)^l (1+O(N^{-1}))\frac{1}{l!} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j)-B_n' D_N(s_j)]\mathbb{1}_{E_N^1\cap E_N^2} \\
&\leq 1+\sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1+O(N^{-1}))\frac{1}{l!} t^l \mathbb{1}_{E_N^1\cap E_N^2} + c_N(\tau_N(t)) \exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right) \frac{1}{2}\alpha_n^2 \exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right) \exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')].
\end{aligned} \tag{35}$$

For the last line of (34),

$$
\begin{aligned}
-\sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} \sum_{\substack{s_1 \neq \cdots \neq s_l \\ \exists i,i': s_i = r_{i'}}} \prod_{j=1}^{l} \{c_N(s_j) - B_n' D_N(s_j)\} \mathbb{1}_{E_N^1 \cap E_N^2} \\
&\leq \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \sum_{\substack{s_1 \neq \cdots \neq s_l \\ \exists i,i': s_i = r_{i'}}} \prod_{j=1}^{l} \{c_N(s_j) + B_n' D_N(s_j)\} \\
&\leq \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \sum_{\substack{s_1 \neq \cdots \neq s_l \\ \exists i,i': s_i = r_{i'}}} (1+B_n')^l \prod_{j=1}^{l} c_N(s_j) \\
&\leq \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1+O(N^{-1})) \frac{1}{(l-1)!} \sum_{s_1 \in \{r_1,\dots,r_k\}} \sum_{s_2 \neq \cdots \neq s_l}^{\tau_N(t)} (1+B_n')^l \prod_{j=1}^{l} c_N(s_j) \\
&= \sum_{s \in \{r_1,\dots,r_k\}} c_N(s) \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1+O(N^{-1})) \frac{1}{(l-1)!} (1+B_n')^l \sum_{s_1 \neq \cdots \neq s_{l-1}}^{\tau_N(t)} \prod_{j=1}^{l-1} c_N(s_j) \\
&\leq \sum_{j=1}^{k} c_N(r_j) \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1+O(N^{-1})) \frac{1}{(l-1)!} (1+B_n')^l (t+1)^{l-1} \\
&\leq \left(\sum_{j=1}^{k} c_N(r_j) \right) \alpha_n (1+B_n') \exp[\alpha_n (1+O(N^{-1}))(1+B_n')(t+1)],
\end{aligned} \tag{36}
$$

where the penultimate inequality uses Lemma 1(a). Putting these together, we have

$$
\begin{aligned}
\prod_{\substack{r=1 \\ \notin \{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r) \leq 1 &+ \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_{E_N^1 \cap E_N^2} + c_N(\tau_N(t)) \exp[\alpha_n (1+O(N^{-1}))(t+1)] \\
&+ \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n (1+O(N^{-1}))(t+1)] \\
&+ \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n (1+O(N^{-1}))(t+1)(1+B_n')] \\
&+ \left(\sum_{j=1}^{k} c_N(r_j) \right) \alpha_n (1+B_n') \exp[\alpha_n (1+O(N^{-1}))(1+B_n')(t+1)].
\end{aligned} \tag{37}
$$

Meanwhile, using the bound on p_r from (26) then applying a modification of Lemma 2,

$$
\begin{aligned}
\sum_{\substack{r_1 < \cdots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} p_{r_i} &\leq \alpha_n^k (1+O(N^{-1})) \sum_{\substack{r_1 < \cdots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} [c_N(r_i) + B_n D_N(r_i)] \\
&\leq \alpha_n^k (1+O(N^{-1})) \sum_{\substack{r_1 < \cdots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i) + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \alpha_n^k (1+O(N^{-1}))(t+1)^{k-1}(1+B_n)^k.
\end{aligned} \tag{38}
$$

A more liberal (but simpler) bound can be arrived at thus:

$$
\begin{aligned}
\prod_{i=1}^{k} p_{r_i} &\leq \alpha_n^k (1 + O(N^{-1})) \prod_{i=1}^{k} [c_N(r_i) + B_n D_N(r_i)] \\
&\leq \alpha_n^k (1 + O(N^{-1})) \prod_{i=1}^{k} c_N(r_i)(1 + B_n) \\
&\leq \alpha_n^k (1 + O(N^{-1}))(1 + B_n)^k \prod_{i=1}^{k} c_N(r_i)
\end{aligned} \tag{39}
$$

which, using Lemma 1(a), also leads to the deterministic bound

$$
\begin{aligned}
\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} p_{r_i} &\leq \alpha_n^k (1 + O(N^{-1}))(1 + B_n)^k \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i) \\
&\leq \alpha_n^k (1 + O(N^{-1}))(1 + B_n)^k \frac{1}{k!} \sum_{r_1 \neq \dots \neq r_k}^{\tau_N(t)} \prod_{i=1}^{k} c_N(r_i) \\
&\leq \alpha_n^k (1 + O(N^{-1}))(1 + B_n)^k \frac{1}{k!} (t+1)^k.
\end{aligned} \tag{40}
$$

Combining (37) with the other product, the expression inside the expectation in (33) is bounded above by

$$
\begin{aligned}
\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^{k} p_{r_i} \right) \left(\prod_{\substack{r=1 \\ \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) \right) &\leq \left\{ 1 + \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_{E_N^1 \cap E_N^2} \right\} \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} p_{r_i} \\
&+ \left\{ c_N(\tau_N(t)) \exp[\alpha_n (1 + O(N^{-1}))(t+1)] + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n (1 + O(N^{-1}))(t+1)] \right. \\
&\left. + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n (1 + O(N^{-1}))(t+1)(1 + B_n')] \right\} \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} p_{r_i} \\
&+ \exp[\alpha_n (1 + O(N^{-1}))(1 + B_n')(t+1)] \alpha_n (1 + B_n') \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \sum_{j=1}^{k} c_N(r_j) \prod_{i=1}^{k} p_{r_i}.
\end{aligned} \tag{41}
$$

Applying the various bounds (38)–(40), we have

$$
\begin{aligned}
&\sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right) \left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right) \\
&\qquad \le \alpha_n^k (1+O(N^{-1})) \left\{ 1 + \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_{E_N^1 \cap E_N^2} \right\} \sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^{k} c_N(r_i) \\
&\qquad\quad + \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right) \alpha_n^k (1+O(N^{-1}))(t+1)^{k-1}(1+B_n)^k \sum_{l=0}^{\tau_N(t)} (\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l \\
&\qquad\quad + \Bigg\{ c_N(\tau_N(t)) \exp[\alpha_n(1+O(N^{-1}))(t+1)] + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right) \frac{1}{2}\alpha_n^2 \exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
&\qquad\qquad + \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right) \exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')] \Bigg\} \alpha_n^k (1+O(N^{-1}))(1+B_n)^k \frac{1}{k!}(t+1)^k \\
&\qquad\quad + \exp[\alpha_n(1+B_n')(t+1)]\alpha_n(1+B_n')\alpha_n^k(1+O(N^{-1}))(1+B_n)^k \\
&\qquad\qquad\qquad \times \sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \sum_{j=1}^{k} c_N(r_j) \prod_{i=1}^{k} c_N(r_i). \qquad (42)
\end{aligned}
$$

Upon taking the expectation and limit, we have

$$
\begin{aligned}
&\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right) \left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\right] \le \alpha_n^k \lim_{N\to\infty} \mathbb{E}\left[\left(1 + \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l \frac{1}{l!} t^l \mathbb{1}_{E_N^1\cap E_N^2}\right) \sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^{k} c_N(r_i) \right] \\
&\quad + \lim_{N\to\infty} \mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} D_N(s)\right] \alpha_n^k (t+1)^{k-1} (1+B_n)^k \exp[\alpha_n t] \\
&\quad + \Bigg\{ \lim_{N\to\infty} \mathbb{E}\left[c_N(\tau_N(t))\right] \exp[\alpha_n(t+1)] + \lim_{N\to\infty} \mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right] \frac{1}{2}\alpha_n^2 \exp[\alpha_n(t+1)] \\
&\qquad + \lim_{N\to\infty} \mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} D_N(s)\right] \exp[\alpha_n(t+1)(1+B_n')] \Bigg\} \alpha_n^k (1+B_n)^k \frac{1}{k!}(t+1)^k \\
&\quad + \exp[\alpha_n(1+B_n')(t+1)]\alpha_n^{k+1}(1+B_n')(1+B_n)^k \lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \sum_{j=1}^{k} c_N(r_j) \prod_{i=1}^{k} c_N(r_i) \right]. \qquad (43)
\end{aligned}
$$

The middle terms vanish due to Brown et al. (2021, Equations (3.3)–(3.5)) and the expression becomes

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right) \left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\right] &\le \alpha_n^k \lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^{k} c_N(r_i)\right] \\
&+ \alpha_n^k \sum_{l=1}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l \lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_{\{\tau_N(t)\ge k+l\}} \mathbb{1}_{E_N^1\cap E_N^2} \sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^{k} c_N(r_i)\right] \\
&+ \exp[\alpha_n(1+B_n')(t+1)]\alpha_n^{k+1}(1+B_n')(1+B_n)^k \lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \sum_{j=1}^{k} c_N(r_j) \prod_{i=1}^{k} c_N(r_i)\right], \qquad (44)
\end{aligned}$$

where passing the limit and expectation inside the infinite sum is justified by dominated convergence and Fubini; see Lemma 14. To simplify the last line,

$$\begin{aligned}
\sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \sum_{j=1}^{k} c_N(r_j) \prod_{i=1}^{k} c_N(r_i) &\le \frac{1}{k!} \sum_{r_1\ne\dots\ne r_k}^{\tau_N(t)} \sum_{j=1}^{k} c_N(r_j) \prod_{i=1}^{k} c_N(r_i) \\
&= \frac{1}{k!} \sum_{r_1\ne\dots\ne r_k}^{\tau_N(t)} \sum_{j=1}^{k} c_N(r_j)^2 \prod_{i\ne j} c_N(r_i) \\
&\le \frac{1}{k!} \sum_{j=1}^{k} \sum_{s=1}^{\tau_N(t)} c_N(s)^2 \sum_{r_1\ne\dots\ne r_{k-1}}^{\tau_N(t)} \prod_{i=1}^{k-1} c_N(r_i) \\
&\le \frac{1}{(k-1)!} \sum_{s=1}^{\tau_N(t)} c_N(s)^2 (t+1)^{k-1}, \qquad (45)
\end{aligned}$$

using Lemma 1(a) for the final inequality. Hence

$$\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \sum_{s\in\{r_1,\dots,r_k\}} c_N(s) \prod_{i=1}^{k} c_N(r_i)\right] \le \frac{1}{(k-1)!}(t+1)^{k-1} \lim_{N\to\infty} \mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right] = 0 \qquad (46)$$

by Brown et al. (2021, Equation (3.5)). By Lemmata 10, 9 and 12, $\lim_{N\to\infty} \mathbb{P}[\{\tau_N(t)\ge k+l\}\cap E_N^1 \cap E_N^2] = 1$, so we can apply Lemma 7 to the remaining expectations in (44), yielding

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right) \left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\right] &\le \alpha_n^k \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l \sum_{\substack{i_1\le\dots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^{k} \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} \\
&= \alpha_n^k e^{-\alpha_n t} \sum_{\substack{i_1\le\dots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^{k} \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} \qquad (47)
\end{aligned}$$

as required. ■

Lemma 6 (Induction step lower bound). *Fix $k \in \mathbb{N}$, $i_0 := 0$, $i_k := k$. For any sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_k \leq t$,*

$$\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\right] \geq \alpha_n^k e^{-\alpha_n t} \sum_{\substack{i_1\leq\cdots\leq i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\geq j\forall j}} \prod_{j=1}^{k} \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}. \tag{48}$$

Proof. Firstly,

$$\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right) \geq \sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right)\left(\prod_{r=1}^{\tau_N(t)} (1-p_r)\right). \tag{49}$$

Now the second product does not depend on $r_1,\dots,r_k$, and we can use the lower bound from (31):

$$\begin{aligned}\prod_{r=1}^{\tau_N(t)}(1-p_r) \geq \sum_{l=0}^{\tau_N(t)} &(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l \mathbb{1}_{E_N^3} - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right)\frac{1}{2}\alpha_n^2 \exp[\alpha_n(1+O(N^{-1}))(t+1)]\\ &- c_N(\tau_N(t))\exp[\alpha_n(1+O(N^{-1}))(t+1)]\\ &- \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n)]\end{aligned} \tag{50}$$

where E_N^3 is defined as in (28). We will also need an upper bound on this product, which is formed from (24) with a further deterministic bound:

$$\begin{aligned}\prod_{r=1}^{\tau_N(t)}(1-p_r) &\leq \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l + c_N(\tau_N(t))\exp[\alpha_n(1+O(N^{-1}))(t+1)]\\ &\quad + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right)\frac{1}{2}\alpha_n^2\exp[\alpha_n(1+O(N^{-1}))(t+1)]\\ &\quad + \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')]\\ &\leq \exp[\alpha_n(1+O(N^{-1}))t] + \exp[\alpha_n(1+O(N^{-1}))(t+1)]\\ &\quad + \frac{1}{2}\alpha_n^2(t+1)\exp[\alpha_n(1+O(N^{-1}))(t+1)] + (t+1)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')]\\ &\leq \left(2+\frac{\alpha_n^2(t+1)}{2}\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)] + (t+1)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')].\end{aligned} \tag{51}$$

Now let us consider the remaining sum-product on the RHS of (49). We use the same bound on p_r as in (18):

$$p_r = 1 - p_{\Delta\Delta}(r) \geq \alpha_n(1+O(N^{-1}))\left[c_N(r) - B_n' D_N(r)\right] \tag{52}$$

where the $O(N^{-1})$ term does not depend on r. When N is large enough for the factor of $(1+O(N^{-1}))$ to be non-negative, the condition that the bound in (52) is non-negative holds on the event E_N^2 that was defined in (20). Then

$$\prod_{i=1}^{k} p_{r_i} \geq \alpha_n^k(1+O(N^{-1}))\prod_{i=1}^{k}\left[c_N(r_i) - B_n' D_N(r_i)\right]\mathbb{1}_{E_N^2}. \tag{53}$$

Applying a modification of Lemma 3,

$$\begin{aligned}
\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^k p_{r_i}\right) &\ge \alpha_n^k(1+O(N^{-1})) \sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^k \left[c_N(r_i) - B_n' D_N(r_i)\right] \mathbb{1}_{E_N^2} \\
&\ge \alpha_n^k(1+O(N^{-1})) \left\{ \sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^k c_N(r_i)\mathbb{1}_{E_N^2} - \frac{1}{k!}\left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)(t+1)^{k-1}(1+B_n')^k \right\}.
\end{aligned} \tag{54}$$

The above expression is already split into positive and negative terms; a lower bound on (49) can be formed by multiplying the positive terms by the lower bound (50) and the negative terms by the upper bound (51). Thus

$$\begin{aligned}
&\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^k p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right) \\
&\quad\ge \alpha_n^k(1+O(N^{-1})) \sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^k c_N(r_i)\mathbb{1}_{E_N^2} \left\{ \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_{E_N^3} \right. \\
&\qquad\qquad - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right)\frac{1}{2}\alpha_n^2\exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
&\qquad\qquad - c_N(\tau_N(t))\exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
&\qquad\qquad \left. - \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n)] \right\} \\
&\quad - \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)\alpha_n^k(1+O(N^{-1}))\frac{1}{k!}(t+1)^{k-1}(1+B_n')^k \left\{ \vphantom{\frac{1}{2}} \right. \\
&\qquad\qquad \left(2+\frac{\alpha_n^2(t+1)}{2}\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
&\qquad\qquad \left. + (t+1)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')] \right\}.
\end{aligned} \tag{55}$$

Due to Brown et al. (2021, Equations (3.3)–(3.5)), all but the first line on the RHS of the above have vanishing expectation, leaving

$$\begin{aligned}
&\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^k p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right) \right] \\
&\qquad \ge \lim_{N\to\infty} \mathbb{E}\left[\alpha_n^k(1+O(N^{-1})) \sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^k c_N(r_i)\mathbb{1}_{E_N^2} \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_{E_N^3} \right] \\
&\qquad = \alpha_n^k \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!}t^l \lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_{\{\tau_N(t)\ge l\}}\mathbb{1}_{E_N^2\cap E_N^3} \sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^k c_N(r_i) \right].
\end{aligned} \tag{56}$$

Passing the limit and expectation inside the infinite sum is justified by dominated convergence and Fubini; see Lemma 14. Lemmata 9 and 12 establish that $\lim_{N\to\infty}\mathbb{P}[E_N^2\cap E_N^3] = 1$ and Lemma 10 deals with the other indicator. We can therefore apply Lemma 7 to conclude that

$$\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\left(\prod_{i=1}^{k}p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)}(1-p_r)\right)\right]\ge\alpha_n^k\sum_{l=0}^{\infty}(-\alpha_n)^l\frac{1}{l!}t^l\sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\ge j\forall j}}\prod_{j=1}^{k}\frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}$$

$$=\alpha_n^k e^{-\alpha_n t}\sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\ge j\forall j}}\prod_{j=1}^{k}\frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}\tag{57}$$

as required. ∎

Lemma 7. *Fix $k\in\mathbb{N}$, $i_0:=0$, $i_k:=k$. Let E_N be a sequence of events such that $\lim_{N\to\infty}\mathbb{P}[E_N]=1$. Then for any sequence of times $0=t_0\le t_1\le\cdots\le t_k\le t$,*

$$\lim_{N\to\infty}\mathbb{E}\left[\mathbb{1}_{E_N}\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\prod_{i=1}^{k}c_N(r_i)\right]=\sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\ge j\forall j}}\prod_{j=1}^{k}\frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}.\tag{58}$$

Proof. As pointed out by Möhle (1999, p. 460), the sum-product on the left hand side can be expanded as

$$\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\prod_{i=1}^{k}c_N(r_i)=\sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\ge j\forall j}}\prod_{j=1}^{k}\frac{1}{(i_j-i_{j-1})!}\sum_{\substack{r_{i_{j-1}+1}\ne\cdots\ne r_{i_j}\\ =\tau_N(t_{j-1})+1}}^{\tau_N(t_j)}\prod_{i=i_{j-1}+1}^{i_j}c_N(r_i).\tag{59}$$

By a modification of the upper bound in Lemma 1(b),

$$\sum_{\substack{r_{i_{j-1}+1}\ne\cdots\ne r_{i_j}\\ =\tau_N(t_{j-1})+1}}^{\tau_N(t_j)}\prod_{i=i_{j-1}+1}^{i_j}c_N(r_i)\le(t_j-t_{j-1})^{i_j-i_{j-1}}+c_N(\tau_N(t_j))(t_j-t_{j-1}+1)^{i_j-i_{j-1}}\tag{60}$$

Now, taking the product on the outside,

$$
\begin{aligned}
\prod_{j=1}^{k} \frac{1}{(i_j - i_{j-1})!} \sum_{\substack{r_{i_{j-1}+1} \neq \cdots \neq r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) &\leq \prod_{j=1}^{k} \left\{ \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} + c_N(\tau_N(t_j)) \frac{(t_j - t_{j-1}+1)^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right\} \\
&\leq \prod_{j=1}^{k} \left\{ \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} + c_N(\tau_N(t_j)) (t_j - t_{j-1}+1)^{i_j - i_{j-1}} \right\} \\
&= \sum_{\mathcal{I} \subseteq [k]} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} c_N(\tau_N(t_j)) (t_j - t_{j-1}+1)^{i_j - i_{j-1}} \right) \\
&= \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \\
&\quad + \sum_{\mathcal{I} \subset [k]} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} c_N(\tau_N(t_j)) (t_j - t_{j-1}+1)^{i_j - i_{j-1}} \right) \\
&\leq \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \\
&\quad + \sum_{\mathcal{I} \subset [k]} \left(\prod_{j \in \mathcal{I}} t^{i_j - i_{j-1}} \right) \left(\prod_{j \notin \mathcal{I}} c_N(\tau_N(t_j)) (t+1)^{i_j - i_{j-1}} \right) \\
&\leq \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} + \sum_{\mathcal{I} \subset [k]} c_N(\tau_N(t_{j^\star(\mathcal{I})})) \prod_{j=1}^{k} (t+1)^{i_j - i_{j-1}} \\
&= \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} + \sum_{\mathcal{I} \subset [k]} c_N(\tau_N(t_{j^\star(\mathcal{I})})) (t+1)^k
\end{aligned} \tag{61}
$$

where, say, $j^\star(\mathcal{I}) := \min\{j \notin \mathcal{I}\}$. Now we are in a position to evaluate the limit in (58):

$$
\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_{E_N} \sum_{\substack{r_1 < \cdots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i) \right] &\leq \lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1 < \cdots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i) \right] \\
&\leq \sum_{\substack{i_1 \leq \cdots \leq i_{k-1} \\ \in \{0,\dots,k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} + \sum_{\substack{i_1 \leq \cdots \leq i_{k-1} \\ \in \{0,\dots,k\}: \\ i_j \geq j \forall j}} \sum_{\mathcal{I} \subset [k]} \lim_{N\to\infty} \mathbb{E}\left[c_N(\tau_N(t_{j^\star(\mathcal{I})})) \right] (t+1)^k \\
&= \sum_{\substack{i_1 \leq \cdots \leq i_{k-1} \\ \in \{0,\dots,k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}
\end{aligned} \tag{62}
$$

using Brown et al. (2021, Equation (3.3)).

For the corresponding lower bound, by a modification of the lower bound in Lemma 1(b),

$$
\begin{aligned}
\sum_{\substack{r_{i_{j-1}+1} \neq \cdots \neq r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) &\geq (t_j - t_{j-1})^{i_j - i_{j-1}} - \binom{i_j - i_{j-1}}{2} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1}+1)^{i_j - i_{j-1}-2} \\
&\geq (t_j - t_{j-1})^{i_j - i_{j-1}} - (i_j - i_{j-1})! \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1}+1)^{i_j - i_{j-1}-2}.
\end{aligned} \tag{63}
$$

Define the event

$$E_N^4(j) = \left\{ \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) \leq \frac{1}{(i_j - i_{j-1})!} \left(\frac{t_j - t_{j-1}}{t_j - t_{j-1} + 1} \right)^{i_j - i_{j-1}} \right\}, \tag{64}$$

which is sufficient to ensure the j^{th} term in the following product is non-negative, and define $E_N^4 := \bigcap_{j=1}^k E_N^4(j)$. (If $t_j = t_{j-1}$ then $E_N^4(j)$ has probability one automatically; otherwise the constant on the right is strictly positive and so satisfies the conditions of Lemma 11.) Now, taking a product over j,

$$\begin{aligned}
&\prod_{j=1}^k \frac{1}{(i_j - i_{j-1})!} \sum_{\substack{r_{i_{j-1}+1} \neq \cdots \neq r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) \\
&\geq \prod_{j=1}^k \left\{ \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} - \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{i_j - i_{j-1} - 2} \right\} \mathbb{1}_{E_N^4} \\
&= \sum_{\mathcal{I} \subseteq [k]} (-1)^{k - |\mathcal{I}|} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{i_j - i_{j-1} - 2} \right) \mathbb{1}_{E_N^4} \\
&= \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E_N^4} \\
&\qquad + \sum_{\mathcal{I} \subset [k]} (-1)^{k - |\mathcal{I}|} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{i_j - i_{j-1} - 2} \right) \mathbb{1}_{E_N^4} \\
&\geq \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E_N^4} \\
&\qquad - \sum_{\mathcal{I} \subset [k]} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{i_j - i_{j-1} - 2} \right) \\
&\geq \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E_N^4} \\
&\qquad - \sum_{\mathcal{I} \subset [k]} \left(\prod_{j \in \mathcal{I}} t^{i_j - i_{j-1}} \right) \left(\prod_{j \notin \mathcal{I}} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t+1)^{i_j - i_{j-1} - 2} \right) \\
&\geq \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E_N^4} \\
&\qquad - \sum_{\mathcal{I} \subset [k]} \left(\sum_{s=\tau_N(t_{j^\star(\mathcal{I})-1})+1}^{\tau_N(t_{j^\star(\mathcal{I})})} c_N(s)^2 \right) \left(\prod_{j \in \mathcal{I}} t^{i_j - i_{j-1}} \right) \left(\prod_{j \notin \mathcal{I}} (t+1)^{i_j - i_{j-1} - 1} \right) \\
&\geq \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E_N^4} - \sum_{\mathcal{I} \subset [k]} \left(\sum_{s=\tau_N(t_{j^\star(\mathcal{I})-1})+1}^{\tau_N(t_{j^\star(\mathcal{I})})} c_N(s)^2 \right) \prod_{j=1}^k (t+1)^{i_j - i_{j-1}} \\
&= \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E_N^4} - \sum_{\mathcal{I} \subset [k]} \left(\sum_{s=\tau_N(t_{j^\star(\mathcal{I})-1})+1}^{\tau_N(t_{j^\star(\mathcal{I})})} c_N(s)^2 \right) (t+1)^k,
\end{aligned} \tag{65}$$

where again we have arbitrarily set $j^\star(\mathcal{I}) := \min\{j \notin \mathcal{I}\}$. We can now evaluate the limit:

$$
\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_{E_N} \sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^{k} c_N(r_i)\right] &\ge \lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_{E_N\cap E_N^4} \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^{k} \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}\right] \\
&\quad - \lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_{E_N} \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \sum_{\mathcal{I}\subset[k]} \left(\sum_{s=\tau_N(t_{j^\star(\mathcal{I})-1})+1}^{\tau_N(t_{j^\star(\mathcal{I})})} c_N(s)^2\right)(t+1)^k\right] \\
&\ge \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^{k} \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} \lim_{N\to\infty} \mathbb{E}[\mathbb{1}_{E_N\cap E_N^4}] \\
&\quad - \lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \sum_{\mathcal{I}\subset[k]} \left(\sum_{s=\tau_N(t_{j^\star(\mathcal{I})-1})+1}^{\tau_N(t_{j^\star(\mathcal{I})})} c_N(s)^2\right)(t+1)^k\right] \\
&= \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^{k} \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} \lim_{N\to\infty} \mathbb{P}[E_N\cap E_N^4] \\
&\quad - \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \sum_{\mathcal{I}\subset[k]} \lim_{N\to\infty} \mathbb{E}\left[\sum_{s=\tau_N(t_{j^\star(\mathcal{I})-1})+1}^{\tau_N(t_{j^\star(\mathcal{I})})} c_N(s)^2\right](t+1)^k \\
&= \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^{k} \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} \qquad (66)
\end{aligned}
$$

where for the last equality we use Brown et al. (2021, Equation (3.5)) to show that the second sum vanishes and Lemma 11 to show that $\lim_{N\to\infty} \mathbb{P}[E_N \cap E_N^4] = 1$. We have shown that the upper and lower bounds coincide, so the result follows. ■

Indicators

Lemma 8. *Let $(A_N), (B_N)$ be sequences of events. If $\lim_{N\to\infty} \mathbb{P}[A_N] = 1$ and $\lim_{N\to\infty} \mathbb{P}[B_N] = 1$ then $\lim_{N\to\infty} \mathbb{P}[A_N \cap B_N] = 1$.*

The above might be so obvious as to go unstated, but it is very important because it means we don't have to deal with intersections of dependent events! Here is a little proof just to be sure:

Proof.

$$\begin{aligned}
&\lim_{N\to\infty}\mathbb{P}[A_N]=1 \textbf{ and } \lim_{N\to\infty}\mathbb{P}[B_N]=1\\
\Leftrightarrow &\lim_{N\to\infty}\mathbb{P}[A_N^c]=0 \textbf{ and } \lim_{N\to\infty}\mathbb{P}[B_N^c]=0\\
\Rightarrow &\lim_{N\to\infty}\{\mathbb{P}[A_N^c]+\mathbb{P}[B_N^c]\}=0\\
\Rightarrow &\lim_{N\to\infty}\mathbb{P}[A_N^c\cup B_N^c]=0\\
\Leftrightarrow &\lim_{N\to\infty}\mathbb{P}[A_N\cap B_N]=1.
\end{aligned} \tag{67}$$

The only part of this argument that I find potentially controversial is going from the third to the fourth line, which is an application of the sandwich theorem (since $0\le\mathbb{P}[A_N^c\cup B_N^c]\le\mathbb{P}[A_N^c]+\mathbb{P}[B_N^c]$). ■

Lemma 9. *Let $K>0$ be a constant which may depend on n,N but not on r, such that $K^{-2}=O(1)$ as $N\to\infty$. Define the events $E_N(r):=\{c_N(r)<K\}$ and denote $E_N:=\bigcap_{r=1}^{\tau_N(t)}E_N(r)$. Then $\lim_{N\to\infty}\mathbb{P}[E_N]=1$.*

Proof.

$$\begin{aligned}
\mathbb{P}[E_N]=1-\mathbb{P}[E_N^c]=1-\mathbb{P}\left[\bigcup_{r=1}^{\tau_N(t)}E_N^c(r)\right]&=1-\mathbb{E}\left[\mathbb{1}_{\bigcup E_N^c(r)}\right]\ge 1-\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)}\mathbb{1}_{E_N^c(r)}\right]\\
=1-\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)}\mathbb{E}\left[\mathbb{1}_{E_N^c(r)}\mid\mathcal{F}_{r-1}\right]\right]&=1-\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)}\mathbb{P}\left[E_N^c(r)\mid\mathcal{F}_{r-1}\right]\right]
\end{aligned} \tag{68}$$

where for the second line we apply Lemma 13 with $f(r)=\mathbb{1}_{E_N^c(r)}$. By the generalised Markov inequality,

$$\mathbb{P}[E_N^c(r)\mid\mathcal{F}_{r-1}]=\mathbb{P}[c_N(r)\ge K\mid\mathcal{F}_{r-1}]\le K^{-2}\mathbb{E}[c_N(r)^2\mid\mathcal{F}_{r-1}]. \tag{69}$$

Substituting this into (68) and applying Lemma 13 again, this time with $f(r)=c_N(r)^2$,

$$\mathbb{P}[E_N]\ge 1-K^{-2}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)}\mathbb{E}[c_N(r)^2\mid\mathcal{F}_{r-1}]\right]=1-K^{-2}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)}c_N(r)^2\right]. \tag{70}$$

Applying Brown et al. (2021, Equation (3.5)), the limit is

$$\lim_{N\to\infty}\mathbb{P}[E_N]=1-O(1)\times 0=1 \tag{71}$$

as required. ■

Lemma 10. *Fix $t>0$. For any $l\in\mathbb{N}$, $\lim_{N\to\infty}\mathbb{P}[\tau_N(t)\ge l]=1$.*

Proof. We can replace the event $\{\tau_N(t)\ge l\}$ with an event of the form of E in Lemma 9:

$$\{\tau_N(t)\ge l\}=\left\{\min\left\{s\ge 1:\sum_{r=1}^{s}c_N(r)\ge t\right\}\ge l\right\}=\left\{\sum_{r=1}^{l-1}c_N(r)<t\right\}\supseteq\bigcap_{r=1}^{l-1}\left\{c_N(r)<\frac{t}{l}\right\}\supseteq\bigcap_{r=1}^{\tau_N(t)}\left\{c_N(r)<\frac{t}{l}\right\}. \tag{72}$$

Hence

$$\lim_{N\to\infty}\mathbb{P}[\tau_N(t)\ge l]\ge\lim_{N\to\infty}\mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)}\left\{c_N(r)<\frac{t}{l}\right\}\right]=1 \tag{73}$$

by applying Lemma 9 with $K=t/l$. ■

Lemma 11. *Fix $k \in \mathbb{N}$, a sequence of times $0 = t_0 \le t_1 \le \cdots \le t_k \le t$, and let $K_1, \dots, K_k$ be strictly positive constants. Define the events*

$$E_N := \bigcap_{j=1}^{k} \left\{ \sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \le K_j \right\}. \tag{74}$$

Then $\lim_{N\to\infty} \mathbb{P}[E_N] = 1$.

Proof.

$$\mathbb{P}[E_N] = 1 - \mathbb{P}[E_N^c] = 1 - \mathbb{P}\left[\bigcup_{j=1}^{k} \left\{ \sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 > K_j \right\} \right] \ge 1 - \sum_{j=1}^{k} \mathbb{P}\left[\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \ge K_j \right]. \tag{75}$$

Applying Markov's inequality,

$$\mathbb{P}[E_N] \ge 1 - \sum_{j=1}^{k} K_j^{-1} \mathbb{E}\left[\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right] \xrightarrow[N\to\infty]{} 1 - \sum_{j=1}^{k} O(1) \times 0 = 1 \tag{76}$$

by Brown et al. (2021, Equation (3.5)). ■

Lemma 12. *Fix $t > 0$. Let K be a constant not depending on N, r, but which may depend on n.*

$$\lim_{N\to\infty} \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) \ge K D_N(r)\} \right] = 1. \tag{77}$$

Proof.

$$\begin{aligned}
\mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) \ge K D_N(r)\} \right] &\ge \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) > K D_N(r)\} \right] \\
&= 1 - \mathbb{P}\left[\bigcup_{r=1}^{\tau_N(t)} \{c_N(r) \le K D_N(r)\} \right] \\
&= 1 - \mathbb{E}\left[\mathbb{1}_{\bigcup \{c_N(r) \le K D_N(r)\}} \right] \\
&\ge 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{1}_{\{c_N(r) \le K D_N(r)\}} \right] \\
&= 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{P}[c_N(r) \le K D_N(r) \mid \mathcal{F}_{r-1}] \right]
\end{aligned} \tag{78}$$

where the final inequality is an application of Lemma 13 with $f(r) = \mathbb{1}_{\{c_N(r) \le K D_N(r)\}}$.

Fix $0 < \varepsilon < K^{-1}/2$ and assume $N > \max\{\varepsilon^{-1}, (K^{-1} - 2\varepsilon)^{-1}\}$. For each r, i define the event $A_i(r) := \{\nu_r^{(i)} \le N\varepsilon\}$. Conditional on $\mathcal{F}_{r-1}$, we have

$$D_N(r) = \frac{1}{N(N)_2} \sum_{i=1}^{N} (\nu_r^{(j)})_2 \left[\nu_r^{(i)} + \frac{1}{N} \sum_{j \ne i} (\nu_r^{(i)})^2 \right] \mathbb{1}_{A_i^c(r)} + \frac{1}{N(N)_2} \sum_{i=1}^{N} (\nu_r^{(i)})_2 \left[\nu_r^{(i)} + \frac{1}{N} \sum_{j \ne i} (\nu_r^{(j)})^2 \right] \mathbb{1}_{A_i(r)}. \tag{79}$$

For the first term,

$$\frac{1}{N(N)_2} \sum_{i=1}^{N} (\nu_r^{(i)})_2 \left[\nu_r^{(i)} + \frac{1}{N} \sum_{j \ne i} (\nu_r^{(j)})^2 \right] \mathbb{1}_{A_i^c(r)} \le \sum_{i=1}^{N} \mathbb{1}_{A_i^c(r)}. \tag{80}$$

For the second term,

$$\frac{1}{N(N)_2}\sum_{i=1}^N(\nu_r^{(i)})_2\left[\nu_r^{(i)}+\frac{1}{N}\sum_{j\neq i}(\nu_r^{(j)})^2\right]\mathbb{1}_{A_i(r)} \leq \frac{1}{N(N)_2}\sum_{i=1}^N(\nu_r^{(i)})_2\nu_r^{(i)}\mathbb{1}_{A_i(r)}+\frac{1}{N^2(N)_2}\sum_{i=1}^N(\nu_r^{(i)})_2\sum_{j=1}^N(\nu_r^{(j)})^2\mathbb{1}_{A_i(r)}$$

$$\leq \frac{1}{N}c_N(r)N\varepsilon+\frac{1}{N^2(N)_2}\sum_{i=1}^N(\nu_r^{(i)})_2\sum_{j=1}^N(\nu_r^{(j)})_2\mathbb{1}_{A_i(r)}$$

$$+\frac{1}{N^2(N)_2}\sum_{i=1}^N(\nu_r^{(i)})_2\sum_{j=1}^N(\nu_r^{(j)})\mathbb{1}_{A_i(r)}$$

$$\leq \varepsilon c_N(r)+\frac{1}{N^2}\sum_{i=1}^N\nu_r^{(i)}N\varepsilon c_N(r)+\frac{1}{N^2}c_N(r)N$$

$$= c_N(r)\left(2\varepsilon+\frac{1}{N}\right). \tag{81}$$

Altogether we have

$$D_N(r)\leq c_N(r)\left(2\varepsilon+\frac{1}{N}\right)+\sum_{i=1}^N\mathbb{1}_{A_i^c(r)}. \tag{82}$$

Hence, still conditional on $\mathcal{F}_{r-1}$,

$$\{c_N(r)\leq KD_N(r)\}\subseteq\left\{c_N(r)\leq Kc_N(r)(2\varepsilon+N^{-1})+K\sum_{i=1}^N\mathbb{1}_{A_i^c(r)}\right\}$$

$$=\left\{K^{-1}-2\varepsilon-\frac{1}{N}\leq\sum_{i=1}^N\frac{\mathbb{1}_{A_i^c(r)}}{c_N(r)}\right\} \tag{83}$$

where the ratio $\mathbb{1}_{A_i^c(r)}/c_N(r)$ is well-defined because

$$A_i^c(r)\Rightarrow c_N(r):=\frac{1}{(N)_2}\sum_{j=1}^N(\nu_r^{(j)})_2\geq\frac{1}{(N)_2}(\nu_r^{(i)})_2\geq\frac{\varepsilon(N\varepsilon-1)}{N-1}\geq\varepsilon\left(\varepsilon-\frac{1}{N}\right)>0. \tag{84}$$

Hence by Markov's inequality (the conditions on ε, N ensuring the constant is always strictly positive),

$$\mathbb{P}\left[c_N(r)\leq KD_N(r)\mid\mathcal{F}_{r-1}\right]\leq\mathbb{P}\left[\sum_{i=1}^N\mathbb{1}_{A_i^c(r)}\geq\left(K^{-1}-2\varepsilon-\frac{1}{N}\right)\varepsilon\left(\varepsilon-\frac{1}{N}\right)\middle|\mathcal{F}_{r-1}\right]$$

$$\leq\frac{1}{\left(K^{-1}-2\varepsilon-\frac{1}{N}\right)\varepsilon\left(\varepsilon-\frac{1}{N}\right)}\mathbb{E}\left[\sum_{i=1}^N\mathbb{1}_{A_i^c(r)}\middle|\mathcal{F}_{r-1}\right]$$

$$\leq\frac{1}{\left(K^{-1}-2\varepsilon-\frac{1}{N}\right)\varepsilon\left(\varepsilon-\frac{1}{N}\right)}\mathbb{E}\left[\sum_{i=1}^N\frac{(\nu_r^{(i)})_3}{(N\varepsilon)_3}\middle|\mathcal{F}_{r-1}\right]$$

$$\leq\frac{1}{\left(K^{-1}-2\varepsilon-\frac{1}{N}\right)\varepsilon\left(\varepsilon-\frac{1}{N}\right)}\mathbb{E}\left[\frac{N(N)_2}{(N\varepsilon)_3}D_N(r)\middle|\mathcal{F}_{r-1}\right]. \tag{85}$$

Applying Lemma 13 once more, with $f(r)=D_N(r)$,

$$\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)}\mathbb{P}[c_N(r)\leq KD_N(r)\mid\mathcal{F}_{r-1}]\right]\leq\frac{1}{\left(K^{-1}-2\varepsilon-\frac{1}{N}\right)\varepsilon\left(\varepsilon-\frac{1}{N}\right)}\frac{N(N)_2}{(N\varepsilon)_3}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)}\mathbb{E}[D_N(r)\mid\mathcal{F}_{r-1}]\right]$$

$$=\frac{1}{\left(K^{-1}-2\varepsilon-\frac{1}{N}\right)\varepsilon\left(\varepsilon-\frac{1}{N}\right)}\frac{N(N)_2}{(N\varepsilon)_3}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)}D_N(r)\right]$$

$$\underset{N\to\infty}{\longrightarrow}\frac{1}{(K^{-1}-2\varepsilon)\varepsilon^5}\times 0=0. \tag{86}$$

Substituting this back into (78) concludes the proof. ■

Other useful results

The following Lemma is taken from Koskela et al. (2018, Lemma 2), where the function is set to $f(r) = c_N(r)$, but the authors remark that the result holds for other choices of function.

Lemma 13. *Fix $t > 0$. Let $(\mathcal{F}_r)$ be the backwards-in-time filtration generated by the offspring counts $\nu_r^{(1:N)}$ at each generation r, and let $f(r)$ be any deterministic function of $\nu_r^{(1:N)}$ that is non-negative and bounded. In particular, for all r there exists $B < \infty$ such that $0 \leq f(r) \leq B$. Then*

$$\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} f(r)\right] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right]. \tag{87}$$

Proof. Define

$$M_s := \sum_{r=1}^{s} \{f(r) - \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\}. \tag{88}$$

It is easy to establish that (M_s) is a martingale with respect to $(\mathcal{F}_s)$, and $M_0 = 0$. Now fix $K \geq 1$ and note that $\tau_N(t) \wedge K$ is a bounded $\mathcal{F}_t$-stopping time. Hence we can apply the optional stopping theorem:

$$\mathbb{E}[M_{\tau_N(t)\wedge K}] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} \{f(r) - \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\}\right] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} f(r)\right] - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right] = 0. \tag{89}$$

Since this holds for all $K \geq 1$,

$$\lim_{K\to\infty} \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} f(r)\right] = \lim_{K\to\infty} \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right]. \tag{90}$$

The monotone convergence theorem allows the limit to pass inside the expectation on each side (since increasing K can only increase each sum, by possibly adding non-negative terms). Hence

$$\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} f(r)\right] = \mathbb{E}\left[\lim_{K\to\infty} \sum_{r=1}^{\tau_N(t)\wedge K} f(r)\right] = \mathbb{E}\left[\lim_{K\to\infty} \sum_{r=1}^{\tau_N(t)\wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right] \tag{91}$$

which concludes the proof. ∎

There are a few instances where Fubini's Theorem and the Dominated Convergence Theorem are needed in order to pass a limit and expectation through an infinite sum. Now we verify that the conditions of these theorems indeed hold. This result, analogous to that in Koskela et al. (2018, Appendix), is used once in Lemma 5 at (43) and once in Lemma 6 at (56).

Lemma 14. *For any fixed $t > 0$,*

$$\mathbb{E}\left[\sum_{l=0}^{\infty} \left|(-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \sum_{\substack{r_1 < \cdots < r_k : \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i)\right|\right] < \infty. \tag{92}$$

Proof.

$$\begin{aligned}\mathbb{E}\left[\sum_{l=0}^{\infty} \left|(-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \sum_{\substack{r_1 < \cdots < r_k : \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i)\right|\right] &\leq \mathbb{E}\left[\sum_{l=0}^{\infty} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} t^l (t+1)^k\right] \\ &= \mathbb{E}[\exp\{\alpha_n t(1 + O(N^{-1}))\}(t+1)^k] = \exp\{\alpha_n t(1 + O(N^{-1}))\}(t+1)^k < \infty. \end{aligned} \tag{93}$$

∎

Dependency graph

Missing links since this graph was updated:

- Lemma 1(a) is used three times in Lemma 5, but not anywhere else.
- Lemma 1 in the current dependency graph is really referring to Lemma 1(b)
- Lemma 14 is used in Lemmata 6 and 5.
- Lemma 8 is used in Lemmata 4, 5, 6 and 7

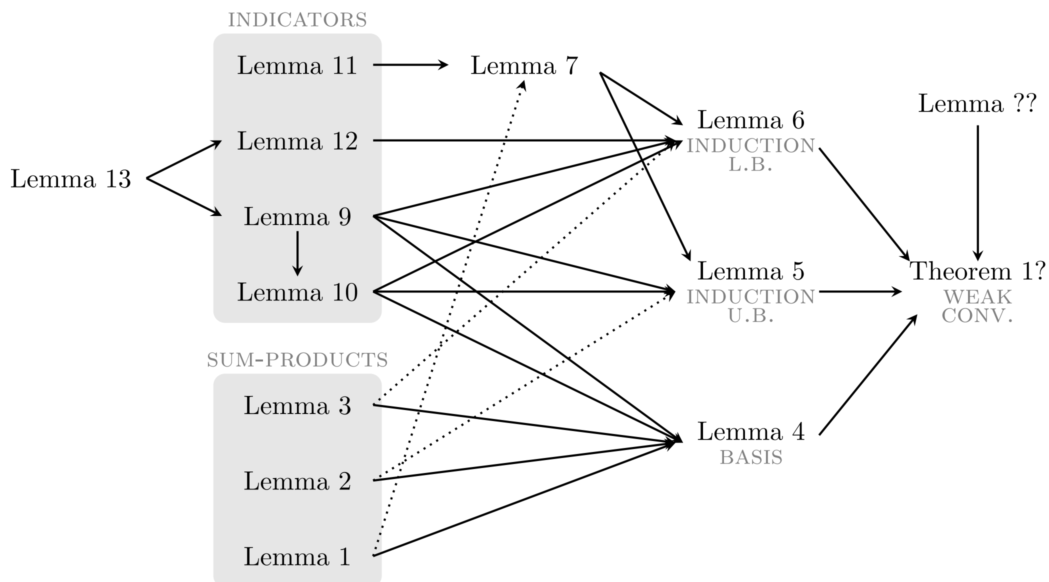

Figure 1: Graph showing dependencies between the lemmata used to prove weak convergence. Dotted arrows indicate dependence via a slight modification of the preceding lemma.

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