

Resampling in Sequential Monte Carlo

Suzie Brown

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Outline

1. Introduction to sequential Monte Carlo
2. How to resample
3. Properties of resampling schemes
4. Link with genealogies

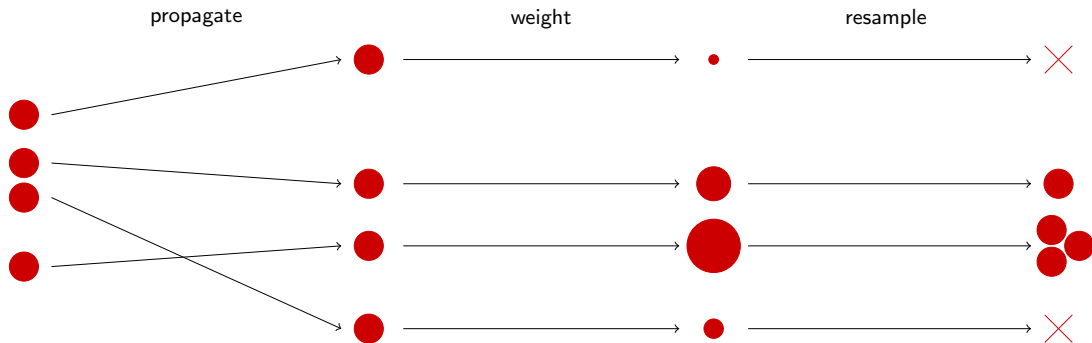
Sequential Monte Carlo

Motivation

- ▶ Want to approximate a sequence of target measures $(\eta_t)_{t \in \mathbb{N}}$
- ▶ Use a system of N particles with dynamics ‘mimicking’ the target
- ▶ A ‘particle’ consists of a position and a weight: $(x_t^{(i)}, w_t^{(i)}) = (x_i, w_i)$
- ▶ Approximate the measures η_t by (random) empirical measures η_t^N consisting of atoms at the particle positions

Sequential Monte Carlo

Illustration



Resampling

Motivation

- ▶ Resampling is necessary to prevent *weight degeneracy*
- ▶ But resampling causes *ancestral degeneracy*

Resampling

Motivation

- ▶ Resampling is necessary to prevent *weight degeneracy*
- ▶ But resampling causes *ancestral degeneracy*
- ▶ Strategy: resample in a way that minimises 'unnecessary coalescences'

Resampling

Definition

We will take valid resampling schemes to be those satisfying

- ▶ The total number of particles N remains fixed
- ▶ The particles after resampling are equally weighted
- ▶ The scheme is unbiased: the expected number of offspring of particle i is equal to Nw_i for each i

Multinomial Resampling¹

Definition

Parental indices $a_i \in \{1, \dots, N\}$:

$$(a_i \mid w_{1:N}) \stackrel{iid}{\sim} \text{Categorical}(N, w_{1:N})$$

Offspring numbers $v_i \in \{0, \dots, N\}$ such that $\sum v_i = N$:

$$(v_{1:N} \mid w_{1:N}) \sim \text{Multinomial}(1 : N, w_{1:N})$$

¹Efron & Tibshirani (1994) 'An introduction to the bootstrap'

Multinomial Resampling

Inversion Sampling

Draw uniform random variables

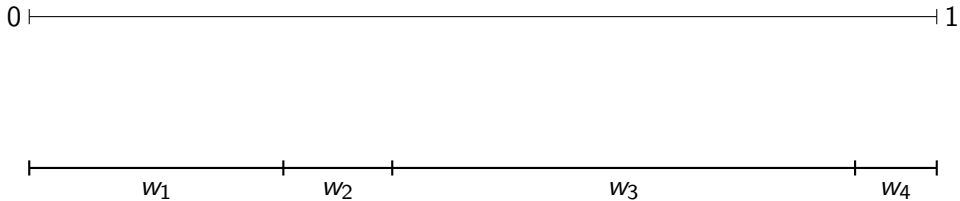
$$U_i \stackrel{iid}{\sim} \text{Uniform}(0, 1); \quad i = 1, \dots, N$$

and determine the parental indices by inversion

$$a_i = \inf \left\{ k : \sum_{j=1}^k w_j \geq U_i \right\}$$

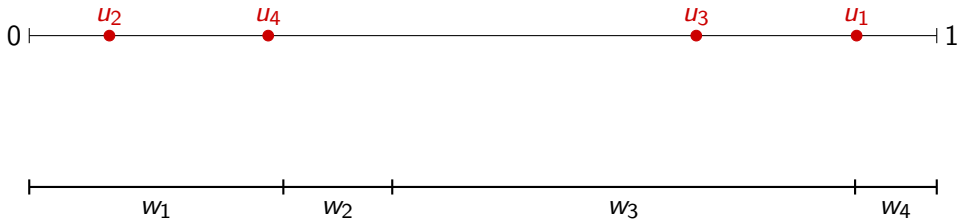
Multinomial Resampling

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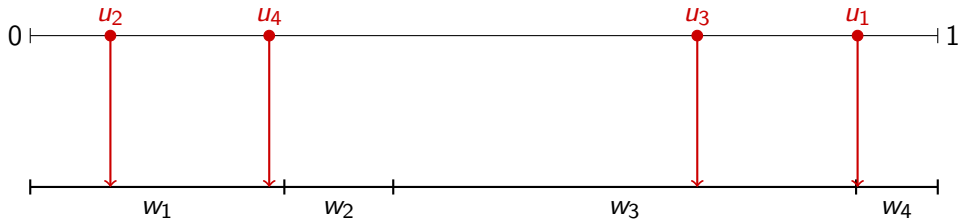
Multinomial Resampling

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Multinomial Resampling

Inversion Sampling



Residual Resampling^{2,3}

Definition

1. Deterministically assign $\lfloor Nw_i \rfloor$ offspring to particle i ; $i=1, \dots, N$
2. There are $R := N - \sum_{i=1}^N \lfloor Nw_i \rfloor$ offspring still to be assigned
3. Assign these randomly according to the residual weights $r_i := \frac{1}{R}(Nw_i - \lfloor Nw_i \rfloor)$

²Liu & Chen (1998) 'Sequential Monte Carlo methods for dynamic systems'

³Whitley (1994) 'A genetic algorithm tutorial'

Residual Resampling

Definition


If residuals are assigned using multinomial resampling, offspring counts are distributed

$$v_{1:N} \stackrel{d}{=} \lfloor Nw_{1:N} \rfloor + \text{Multinomial}(R, r_{1:N})$$


Residual Resampling

Illustration

$w_1 = 0.28$ 

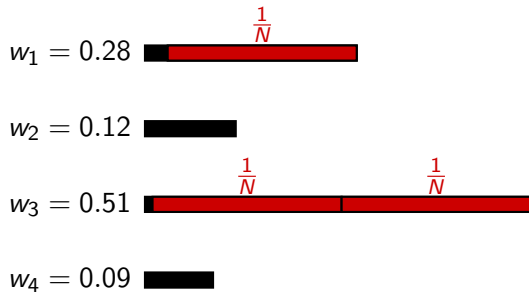
$w_2 = 0.12$ 

$w_3 = 0.51$ 

$w_4 = 0.09$ 

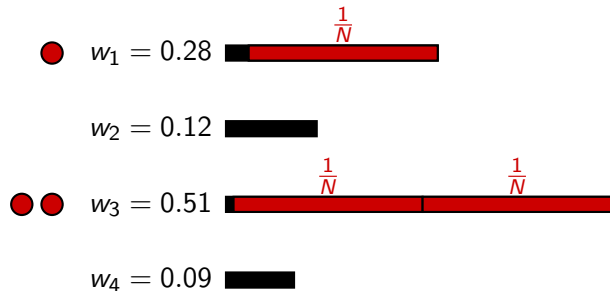
Residual Resampling

Illustration



Residual Resampling

Illustration



Residual Resampling

Illustration

● $r_1 \propto 0.03$ ■


$r_2 \propto 0.12$ ■■■■

●● $r_3 \propto 0.01$ ■


$r_4 \propto 0.09$ ■■■■

Residual Resampling

Illustration

● $r_1 = 0.12$ 

$r_2 = 0.48$ 

●● $r_3 = 0.04$ 

$r_4 = 0.36$ 

Stratified Resampling⁴

Definition

Draw uniformly from each stratified interval

$$U_i \sim \text{Uniform} \left(\frac{i-1}{N}, \frac{i}{N} \right); \quad i = 1, \dots, N$$

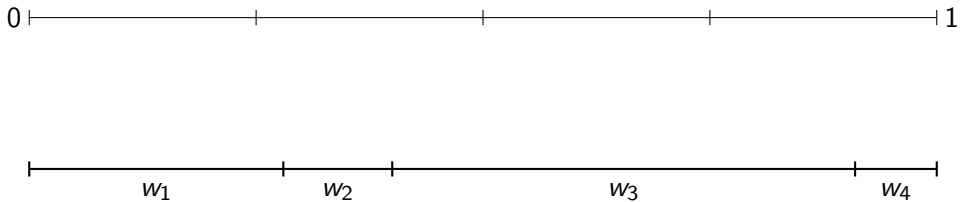
and determine the parental indices by inversion

$$a_i = \inf \left\{ k : \sum_{j=1}^k w_j \geq U_i \right\}$$

⁴Kitagawa (1996) 'Monte Carlo filter and smoother for non-Gaussian nonlinear state space models'

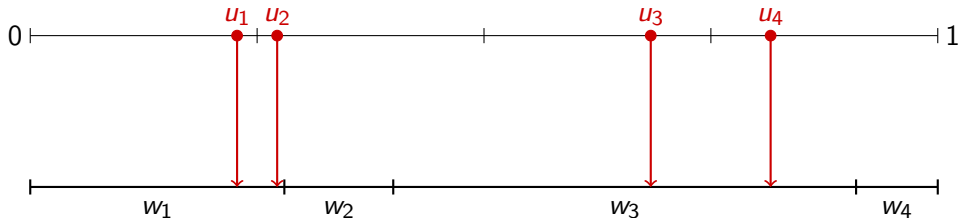
Stratified Resampling

Inversion Sampling



Stratified Resampling

Inversion Sampling



Systematic Resampling^{5,6}

Definition

Draw uniformly from $[0, \frac{1}{N}]$, and add multiples of $\frac{1}{N}$

$$U_1 \sim \text{Uniform} \left(0, \frac{1}{N} \right)$$

$$U_i = U_1 + \frac{i-1}{N}; \quad i = 2, \dots, N$$

and determine the parental indices by inversion

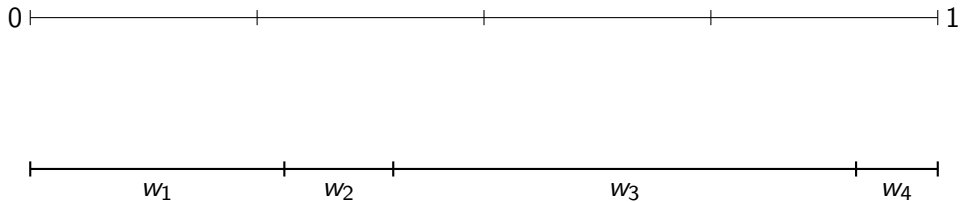
$$a_i = \inf \left\{ k : \sum_{j=1}^k w_j \geq U_i \right\}$$

⁵Carpenter, Clifford & Fearnhead (1999) 'Improved particle filter for nonlinear problems'

⁶Whitley (1994) 'A genetic algorithm tutorial'

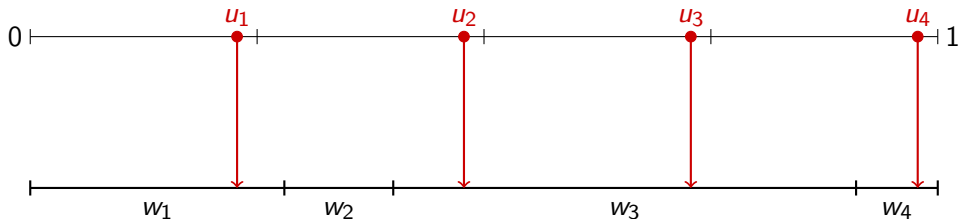
Systematic Resampling

Inversion Sampling



Systematic Resampling

Inversion Sampling



Properties of Resampling Schemes

Support of Offspring Counts

Suppose $w_i \in [\frac{k}{N}, \frac{k+1}{N}]$.

What are the possible values for v_i ?

Multinomial: $v_i \in \{0, \dots, N\}$

Residual: $v_i \in \{k, \dots, k + R\}$

Stratified: $v_i \in \{k - 1, k, k + 1, k + 2\}$

Systematic: $v_i \in \{k, k + 1\}$

Properties of Resampling Schemes

One-Step Variance

Consider variance of our estimator, conditional on the previous step:

$$\text{Var} \left[\frac{1}{N} \sum_{i=1}^N \varphi(X_t^{(i)}) \mid \mathcal{G}_{t-1} \right]$$

⁷Douc, Cappé & Moulines (2005) 'Comparison of resampling schemes for particle filtering'

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In this sense we have⁷

$$\text{Var}[\text{stratified}] \leq \text{Var}[\text{multinomial}]$$

$$\text{Var}[\text{residual-stratified}] \leq \text{Var}[\text{residual-multinomial}] \leq \text{Var}[\text{multinomial}]$$

⁷Douc, Cappé & Moulines (2005) 'Comparison of resampling schemes for particle filtering'

Properties of Resampling Schemes

Permutation Invariance

Stratified and systematic resampling are sensitive to the ordering of the particles.

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Example

$$N = 6$$

$$w_{1:N} = \frac{1}{12}(3, 3, 2, 2, 1, 1)$$

Is it possible to sample offspring counts $v_i = (1, 1, 1, 1, 1, 1)$?

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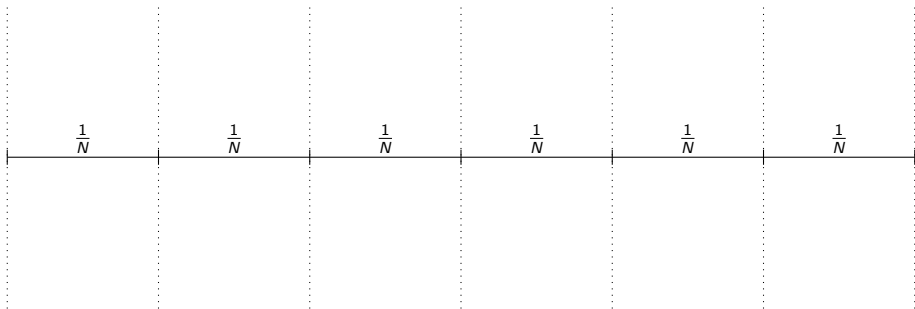
$$w_{1:N} = \frac{1}{12}(3, 3, 2, 2, 1, 1)$$

Is it possible to sample offspring counts $v_i = (1, 1, 1, 1, 1, 1)$?

Answer: it depends on the ordering!

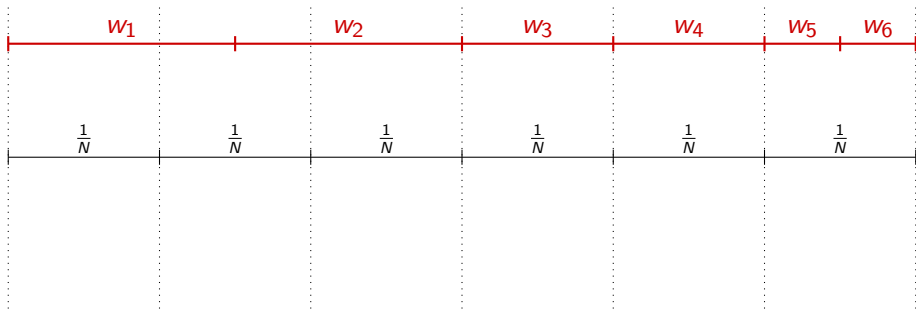
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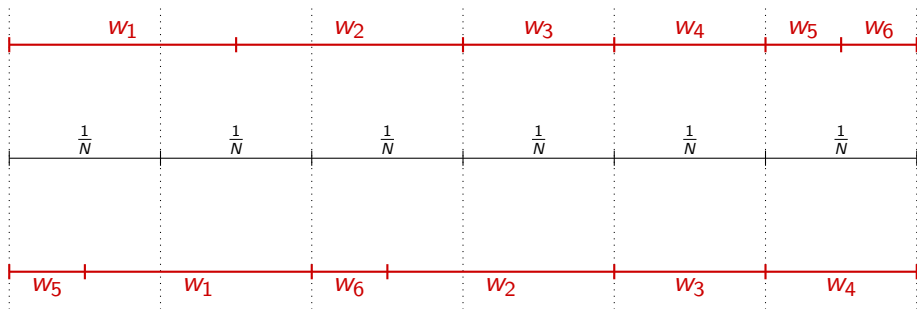
Properties of Resampling Schemes

Permutation Invariance



Properties of Resampling Schemes

Permutation Invariance



Properties of Resampling Schemes

Degeneracy under Equal Weights

- ▶ Suppose all of the weights are multiples of $\frac{1}{N}$.
- ▶ Then residual, stratified and systematic resampling all yield purely deterministic assignments of offspring.
- ▶ In particular, if $w_{1:N} = \frac{1}{N}(1, \dots, 1)$, these schemes do not resample at all (assigning exactly one offspring to each particle).
- ▶ Under reasonable conditions, this situation has zero measure.

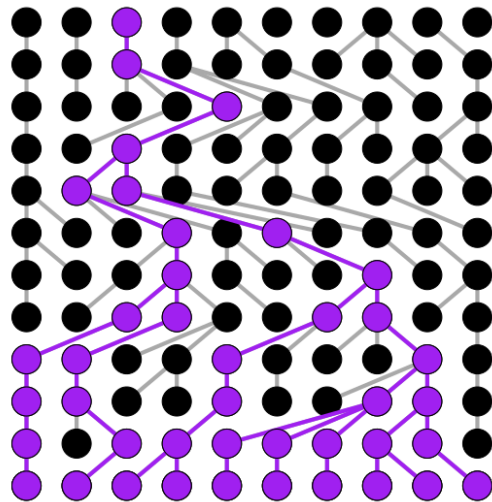
Properties of Resampling Schemes

Summary

	$\sup v_i - Nw_i $	low variance	invariant under permutations	degenerate if $w_{1:N} \propto (1, \dots, 1)$
multinomial	N	\times	\checkmark	\times
residual	R	\checkmark	\checkmark	\checkmark
stratified	2	\checkmark	\times	\checkmark
systematic	1	?	\times	\checkmark

Resampling and Genealogies

- ▶ Resampling creates a genealogy (family tree) of particles
- ▶ Properties of the genealogy affect performance of the SMC algorithm
- ▶ Different resampling schemes give different forms of genealogies
- ▶ Basic quantity for analysing genealogies is the pair coalescence probability



Coalescence Probability

Definition

The probability that a randomly chosen pair of particles at generation t share a common ancestor at generation $(t - 1)$

$$c_N = \frac{1}{N(N-1)} \sum_{i=1}^N v_i(v_i - 1)$$

Coalescence Probability

Example

Consider the case where we have only two particles ($N = 2$)

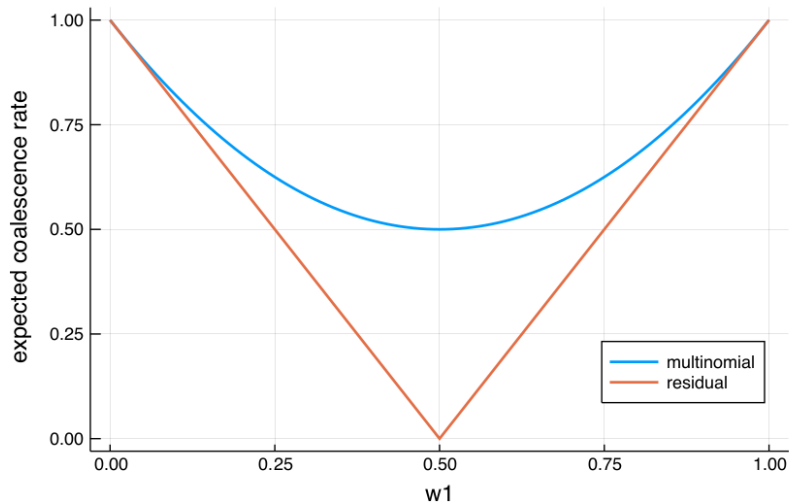
$$c_2 = \frac{1}{2} [v_1(v_1 - 1) + v_2(v_2 - 1)]$$

The expectation of c_2 conditional on knowing the weights (w_1, w_2) is

$$\begin{aligned} c_2 &= \frac{1}{2} \mathbb{E}[v_1(v_1 - 1) \mid w_{1:2}] + \frac{1}{2} \mathbb{E}[v_2(v_2 - 1) \mid w_{1:2}] \\ &= \mathbb{P}[v_1 = 2 \mid w_{1:2}] + \mathbb{P}[v_2 = 2 \mid w_{1:2}] \end{aligned}$$

Coalescence Probability

Example



- ▶ We proved that asymptotically (as $N \rightarrow \infty$) residual resampling dominates multinomial in terms of expected coalescence probability
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- ▶ It just remains to prove it for $N = 4, 5, \dots$

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- ▶ We also proved it in cases $N = 2$ and $N = 3$
- ▶ We conjecture that it holds for all finite N too
- ▶ It just remains to prove it for $N = 4, 5, \dots$
- ▶ We proved that systematic resampling (and some others) dominate multinomial in expected coalescence probability, for all N .

References I