

# Residual resampling in SMC

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THIS DOCUMENT IS OBSOLETE. The calculations in section 1 are flawed because the definition of  $c_N$  used is not correct. Sections 2 and 3 have been rendered irrelevant since we simplified the theorem conditions to involve  $\mathbb{E}_t[(\nu_i)_2]$  and  $\mathbb{E}_t[(\nu_i)_3]$  only.

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The offspring counts are sampled according to:

$$\begin{aligned} v_t^{(i)} &= \lfloor Nw_t^{(i)} \rfloor + X_i \\ X_i &\sim \text{Multinomial}(N - k, (\bar{w}_t^{(1)}, \dots, \bar{w}_t^{(N)})) \end{aligned}$$

where  $k := \sum_{i=1}^N \lfloor Nw_t^{(i)} \rfloor$  is the number of offspring assigned deterministically, and  $\bar{w}_t^{(i)} := \frac{Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor}{N - k}$  are the residual weights. Let us also define the residuals  $r_i := Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor$ .

## 1 Coalescence rate

We shall calculate the expected coalescence rate in the case of residual resampling. The coalescence rate is defined in ? as

$$c_N(t) := \frac{1}{(N)_2} \sum_{i=1}^N \mathbb{E}[(v_t^{(i)})_2].$$

The inner expectation comes out as

$$\begin{aligned} \mathbb{E}[(v_t^{(i)})_2] &= \mathbb{E}[(v_t^{(i)})^2] - \mathbb{E}[v_t^{(i)}] \\ &= \lfloor Nw_t^{(i)} \rfloor^2 + 2\lfloor Nw_t^{(i)} \rfloor r_i + r_i \left(1 - \frac{r_i}{N - k} + r_i\right) - Nw_t^{(i)} \\ &= \lfloor Nw_t^{(i)} \rfloor^2 - \lfloor Nw_t^{(i)} \rfloor + 2\lfloor Nw_t^{(i)} \rfloor r_i + r_i^2 \left(1 - \frac{1}{N - k}\right) \\ &= (Nw_t^{(i)})^2 - \lfloor Nw_t^{(i)} \rfloor - \frac{r_i^2}{N - k} \end{aligned}$$

so we get

$$\begin{aligned} \mathbb{E}[c_N^r(t) | \mathcal{F}_{t-1}] &= \frac{1}{(N)_2} \mathbb{E} \left[ \sum_{i=1}^N \mathbb{E}[(v_t^{(i)})_2] | \mathcal{F}_{t-1} \right] \\ &= \frac{N}{N-1} \sum_{i=1}^N \mathbb{E}[(w_t^{(i)})^2 | \mathcal{F}_{t-1}] - \frac{1}{(N)_2} \sum_{i=1}^N \mathbb{E} \left[ \frac{r_i^2}{N - k} | \mathcal{F}_{t-1} \right] - \frac{1}{(N)_2} \mathbb{E}[k | \mathcal{F}_{t-1}] \\ &= \mathbb{E}[c_N^m(t) | \mathcal{F}_{t-1}] \left(1 + \frac{1}{N-1}\right) - \frac{1}{(N)_2} \mathbb{E} \left[ \frac{\sum_{i=1}^N (Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor)^2}{\sum_{j=1}^N (Nw_t^{(j)} - \lfloor Nw_t^{(j)} \rfloor)} | \mathcal{F}_{t-1} \right] \\ &\quad - \frac{1}{(N)_2} \mathbb{E} \left[ \sum_{i=1}^N \lfloor Nw_t^{(i)} \rfloor | \mathcal{F}_{t-1} \right] \end{aligned}$$

### Sanity check:

When the weights are all equal,  $w_t^{(i)} \equiv 1/N$ , we should have  $\mathbb{E}[c_N^r(t)|\mathcal{F}_{t-1}] = 0$  since each particle will have exactly one offspring so it is impossible for any lineages to coalesce. In this case we have  $\mathbb{E}[c_N^m(t)|\mathcal{F}_{t-1}] = \sum_{i=1}^N \mathbb{E}[(w_t^{(i)})^2|\mathcal{F}_{t-1}] = 1/N$  for multinomial resampling. We also have that  $Nw_t^{(i)} \equiv \lfloor Nw_t^{(i)} \rfloor \equiv 1$  and hence  $r_i = 0$  and  $k = N$ . Thus the RHS comes out as

$$\frac{1}{N} \frac{N}{N-1} - 0 - \frac{1}{(N)_2} N = \frac{1}{N-1} - \frac{1}{N-1} = 0$$

as expected.

## 2 Squared coalescence rate and other awful expressions

First let's write down various moments that will be needed (where  $i \neq j$ ).

$$\begin{aligned} \mathbb{E}[(v_t^{(i)})^2|\mathcal{F}_{t-1}] &= \lfloor Nw_t^{(i)} \rfloor^2 + 2\lfloor Nw_t^{(i)} \rfloor Nw_t^{(i)} + (N)_2(w_t^{(i)})^2 \\ \mathbb{E}[(v_t^{(i)})^3|\mathcal{F}_{t-1}] &= \lfloor Nw_t^{(i)} \rfloor^3 + 3\lfloor Nw_t^{(i)} \rfloor^2 Nw_t^{(i)} + 3\lfloor Nw_t^{(i)} \rfloor (N)_2(w_t^{(i)})^2 + (N)_3(w_t^{(i)})^3 \\ \mathbb{E}[(v_t^{(i)})^4|\mathcal{F}_{t-1}] &= \lfloor Nw_t^{(i)} \rfloor^4 + 4\lfloor Nw_t^{(i)} \rfloor^3 Nw_t^{(i)} + 6\lfloor Nw_t^{(i)} \rfloor^2 (N)_2(w_t^{(i)})^2 + 4\lfloor Nw_t^{(i)} \rfloor (N)_3(w_t^{(i)})^3 + (N)_4(w_t^{(i)})^4 \\ \mathbb{E}[v_t^{(i)} v_t^{(j)}|\mathcal{F}_{t-1}] &= \lfloor Nw_t^{(i)} \rfloor \lfloor Nw_t^{(j)} \rfloor + \lfloor Nw_t^{(i)} \rfloor Nw_t^{(j)} + \lfloor Nw_t^{(j)} \rfloor Nw_t^{(i)} + (N)_2 w_t^{(i)} w_t^{(j)} \\ \mathbb{E}[(v_t^{(i)})^2 v_t^{(j)}|\mathcal{F}_{t-1}] &= \lfloor Nw_t^{(i)} \rfloor^2 \lfloor Nw_t^{(j)} \rfloor + \lfloor Nw_t^{(i)} \rfloor^2 Nw_t^{(j)} + 2\lfloor Nw_t^{(i)} \rfloor \lfloor Nw_t^{(j)} \rfloor Nw_t^{(i)} + 2\lfloor Nw_t^{(i)} \rfloor (N)_2 w_t^{(i)} w_t^{(j)} \\ &\quad + \lfloor Nw_t^{(j)} \rfloor (N)_2(w_t^{(i)})^2 + (N)_3(w_t^{(i)})^2 w_t^{(j)} \\ \mathbb{E}[(v_t^{(i)})^2 (v_t^{(j)})^2|\mathcal{F}_{t-1}] &= \lfloor Nw_t^{(i)} \rfloor^2 \lfloor Nw_t^{(j)} \rfloor^2 + 2\lfloor Nw_t^{(i)} \rfloor^2 \lfloor Nw_t^{(j)} \rfloor Nw_t^{(j)} + 2\lfloor Nw_t^{(i)} \rfloor \lfloor Nw_t^{(j)} \rfloor^2 Nw_t^{(i)} \\ &\quad + \lfloor Nw_t^{(i)} \rfloor^2 (N)_2(w_t^{(j)})^2 + 4\lfloor Nw_t^{(i)} \rfloor \lfloor Nw_t^{(j)} \rfloor (N)_2 w_t^{(i)} w_t^{(j)} + \lfloor Nw_t^{(j)} \rfloor^2 (N)_2(w_t^{(i)})^2 \\ &\quad + 2\lfloor Nw_t^{(i)} \rfloor (N)_3 w_t^{(i)} (w_t^{(j)})^2 + 2\lfloor Nw_t^{(j)} \rfloor (N)_3(w_t^{(i)})^2 w_t^{(j)} + (N)_4(w_t^{(i)})^2 (w_t^{(j)})^2 \end{aligned}$$

For the squared coalescence rate, expanding the falling factorials appropriately, we get

$$\begin{aligned} \mathbb{E}[(c_N^r(t))^2|\mathcal{F}_{t-1}] &= \frac{1}{(N)_2^2} \left( \sum_{i=1}^N \mathbb{E}[(v_t^{(i)})_2^2|\mathcal{F}_{t-1}] + \sum_{i=1}^N \sum_{j \neq i} \mathbb{E}[(v_t^{(i)})_2 (v_t^{(j)})_2|\mathcal{F}_{t-1}] \right) \\ &= \frac{1}{(N)_2^2} \sum_{i=1}^N \left( \mathbb{E}[(v_t^{(i)})^4|\mathcal{F}_{t-1}] - 2\mathbb{E}[(v_t^{(i)})^3|\mathcal{F}_{t-1}] + \mathbb{E}[(v_t^{(i)})^2|\mathcal{F}_{t-1}] \right) \\ &\quad + \frac{1}{(N)_2^2} \sum_{i=1}^N \sum_{j \neq i} \left( \mathbb{E}[(v_t^{(i)})^2 (v_t^{(j)})^2|\mathcal{F}_{t-1}] - \mathbb{E}[(v_t^{(i)})^2 v_t^{(j)}|\mathcal{F}_{t-1}] - \mathbb{E}[v_t^{(i)} (v_t^{(j)})^2|\mathcal{F}_{t-1}] + \mathbb{E}[v_t^{(i)} v_t^{(j)}|\mathcal{F}_{t-1}] \right) \end{aligned}$$

Then we can try plugging in the expressions derived above and find that none of the terms cancel. Maybe if we're clever we can factorise it or something.

## 3 Mega-merger rate

Now the rate of super0binary mergers...

$$\begin{aligned} \mathbb{E}[D_N(t)|\mathcal{F}_{t-1}] &= \frac{1}{N(N)_2} \sum_{i=1}^N \left( \mathbb{E}[(v_t^{(i)})^3|\mathcal{F}_{t-1}] - \mathbb{E}[(v_t^{(i)})^2|\mathcal{F}_{t-1}] \right) \\ &\quad + \frac{1}{N(N)_2} \sum_{i=1}^N \sum_{j \neq i} \left( \mathbb{E}[(v_t^{(i)})^2 (v_t^{(j)})^2|\mathcal{F}_{t-1}] - \mathbb{E}[v_t^{(i)} (v_t^{(j)})^2|\mathcal{F}_{t-1}] \right) \end{aligned}$$