Resampling in Sequential Monte Carlo

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Outline

1. Sequential Monte Carlo
2. How to resample
3. Properties of resampling schemes
4. Link with genealogies

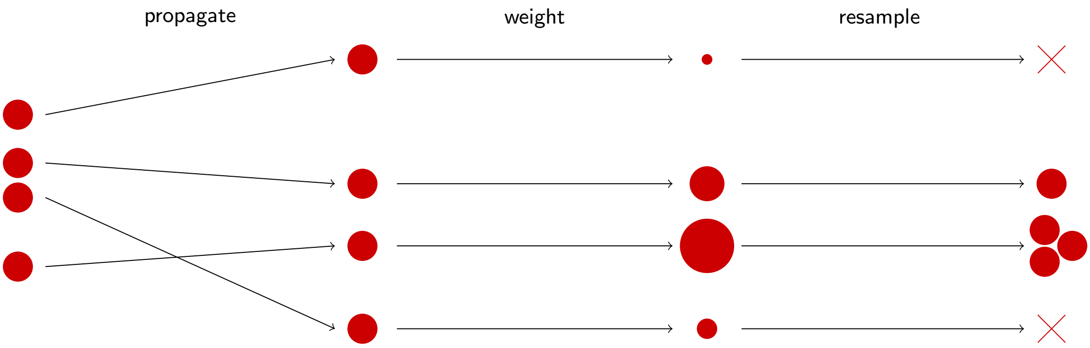
Sequential Monte Carlo

Motivation

- Want to approximate a sequence of target measures $(\eta_n)_{n\in\mathbb{N}}$
- Use a system of N particles with dynamics 'mimicking' the target
- Approximate the measures η_n by (random) empirical measures η_n^N consisting of atoms at the particle positions

Sequential Monte Carlo

Illustration



Resampling

Motivation

- Resampling is necessary to prevent *weight degeneracy*
- But resampling causes *ancestral degeneracy*

Resampling
Motivation

- Resampling is necessary to prevent *weight degeneracy*
- But resampling causes *ancestral degeneracy*
- Strategy: resample in a way that minimises ‘unnecessary coalescences’

Resampling
Definition

We will take valid resampling schemes to be those satisfying

- The total number of particles N remains fixed
- The particles after resampling are equally weighted
- The scheme is unbiased: the expected number of offspring of particle i is equal to Nw_i for each i

Multinomial Resampling[1]

Definition

Parental indices $a_i \in \{1, \dots, N\}$:

$$(a_i \mid w_{1:N}) \overset{iid}{\sim} \text{Categorical}(N, w_{1:N})$$

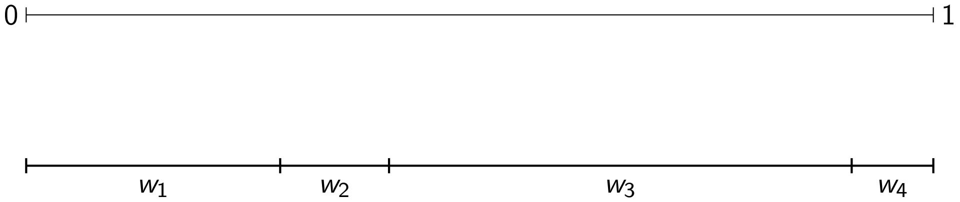
Offspring numbers $v_i \in \{0, \dots, N\}$ such that $\sum v_i = N$:

$$(v_{1:N} \mid w_{1:N}) \sim \text{Multinomial}(1:N, w_{1:N})$$

[1] Efron & Tibshirani (1994) 'An introduction to the bootstrap'

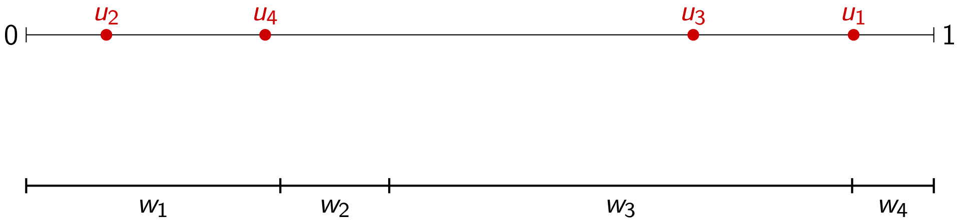
Multinomial Resampling

Inversion Sampling



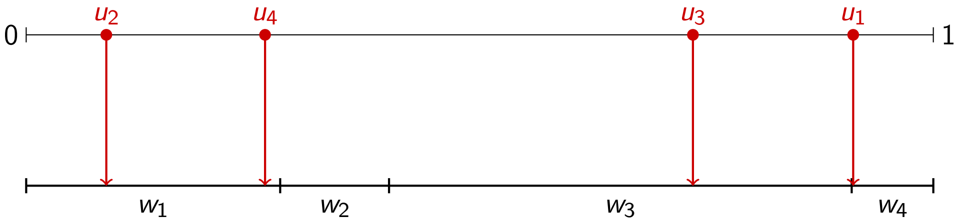
Multinomial Resampling

Inversion Sampling



Multinomial Resampling

Inversion Sampling



Residual Resampling[2,3]

Definition

1. Deterministically assign $\lfloor Nw_i \rfloor$ offspring to particle i; i=1,. . . , N
2. There are $R := N - \sum_{i=1}^{N} \lfloor Nw_i \rfloor$ offspring still to be assigned
3. Assign these randomly according to the residual weights $r_i := \frac{1}{R}(Nw_i - \lfloor Nw_i \rfloor)$

[2] Liu & Chen (1998) 'Sequential Monte Carlo methods for dynamic systems'
[3] Whitley (1994) 'A genetic algorithm tutorial'

Residual Resampling

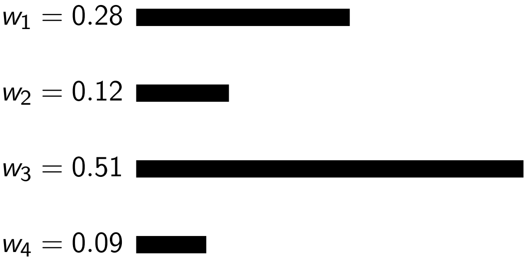
Definition

If residuals are assigned using multinomial resampling, offspring counts are distributed

$$v_{1:N} \stackrel{d}{=} \lfloor N w_{1:N} \rfloor + \text{Multinomial}(R, r_{1:N})$$

Residual Resampling

Illustration



Residual Resampling

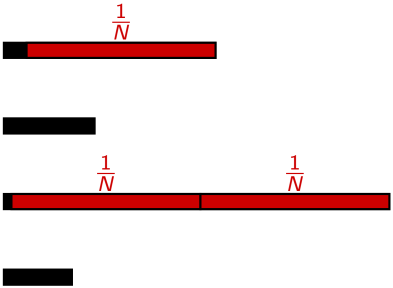
Illustration

$w_1 = 0.28$

$w_2 = 0.12$

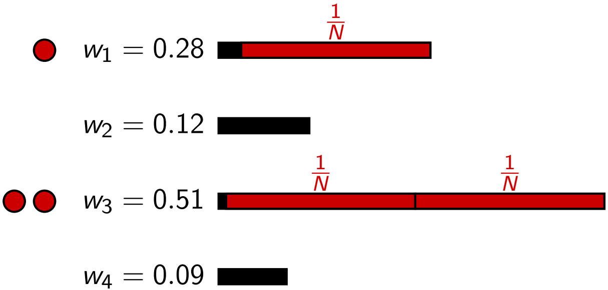
$w_3 = 0.51$

$w_4 = 0.09$



Residual Resampling

Illustration



Residual Resampling

Illustration

$r_1 \propto 0.03$

$r_2 \propto 0.12$

$r_3 \propto 0.01$

$r_4 \propto 0.09$

Residual Resampling

Illustration

$r_1 = 0.12$

$r_2 = 0.48$

$r_3 = 0.04$

$r_4 = 0.36$

Stratified Resampling[4]

Definition

Draw uniformly from each stratified interval

$$U_i \sim \text{Uniform}\left(\frac{i-1}{N}, \frac{i}{N}\right); \qquad i = 1, \dots, N$$

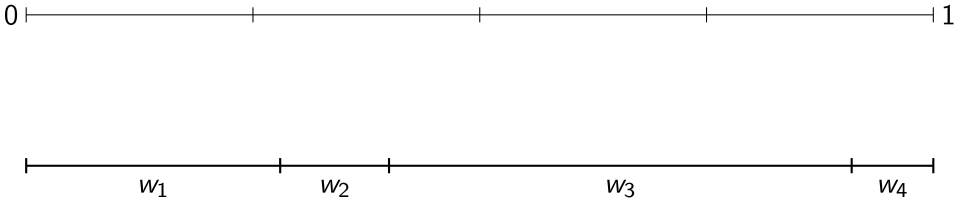
and determine the parental indices by inversion

$$a_i = \inf\left\{k : \sum_{j=1}^{k} w_j \geq U_i\right\}$$

[4] Kitagawa (1996) ‘Monte Carlo filter and smoother for non-Gaussian nonlinear state space models’

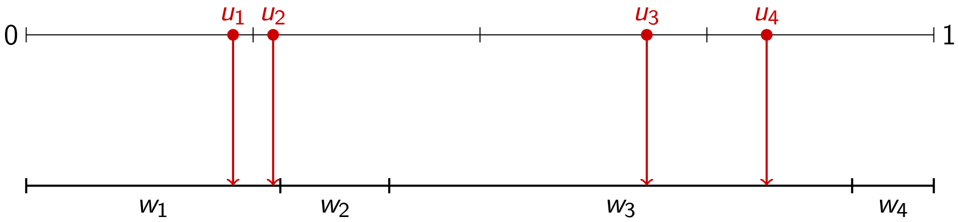
Stratified Resampling

Inversion Sampling



Stratified Resampling

Inversion Sampling



Systematic Resampling[5,6]

Definition

Draw uniformly from $[0, \frac{1}{N}]$, and add multiples of $\frac{1}{N}$

$$U_1 \sim \text{Uniform}\left(0, \frac{1}{N}\right)$$

$$U_i = U_1 + \frac{i-1}{N}; \qquad i = 2, \dots, N$$

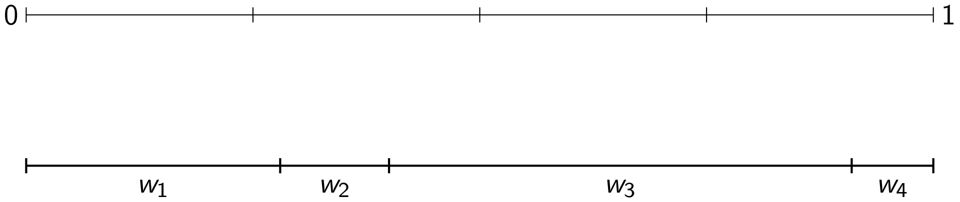
and determine the parental indices by inversion

$$a_i = \inf\left\{k : \sum_{j=1}^{k} w_j \geq U_i\right\}$$

[5] Carpenter, Clifford & Fearnhead (1999) 'Improved particle filter for nonlinear problems'
[6] Whitley (1994) 'A genetic algorithm tutorial'

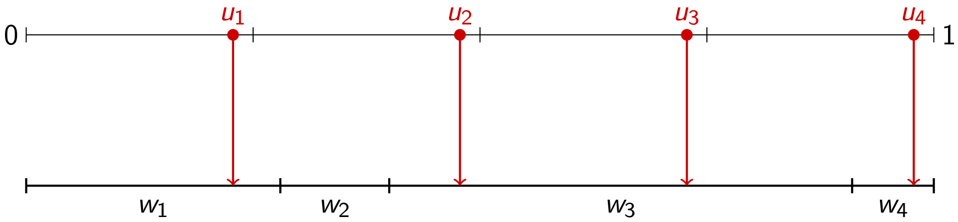
Systematic Resampling

Inversion Sampling



Systematic Resampling

Inversion Sampling



Coalescence Probability

Definition

The probability that a randomly chosen pair of particles at generation t share a common ancestor at generation $(t-1)$

$$c_N = \frac{1}{N(N-1)} \sum_{i=1}^{N} v_i(v_i - 1)$$

Coalescence Probability

Example

Consider the case where we have only two particles ($N = 2$)

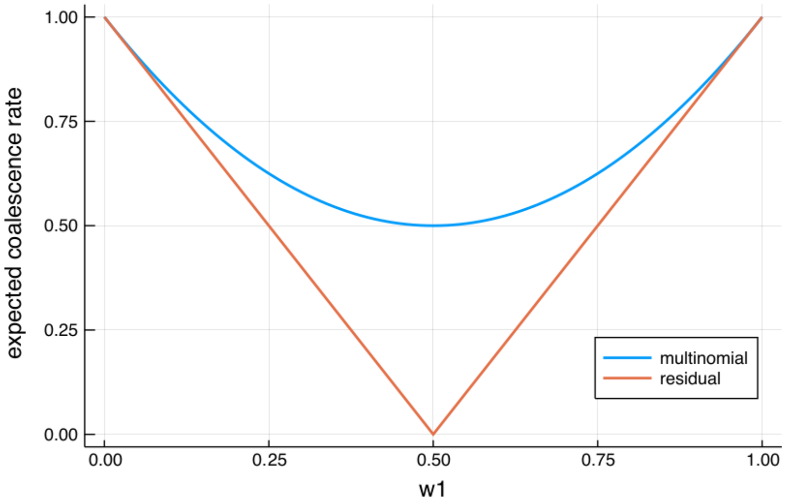
$$c_2 = \frac{1}{2}\left[v_1(v_1 - 1) + v_2(v_2 - 1)\right]$$

The expectation of c_2 conditional on knowing the weights (w_1, w_2) is

$$\begin{aligned} c_2 &= \frac{1}{2}\mathbb{E}[v_1(v_1 - 1) \mid w_{1:2}] + \frac{1}{2}\mathbb{E}[v_2(v_2 - 1) \mid w_{1:2}] \\ &= \mathbb{P}[v_1 = 2 \mid w_{1:2}] + \mathbb{P}[v_2 = 2 \mid w_{1:2}] \end{aligned}$$

Coalescence Probability

Example



References I