That conditioning argument...

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THIS DOCUMENT IS OBSOLETE: SEE “corollary proof details .tex” WHERE THIS CONTENT IS REPRODUCED AND UPDATED.

Note: time is labelled in reverse throughout this document.

Let’s take the conditioning set to be $\mathcal{H}_t = (\mathbf{X}_t, \mathbf{X}_{t-1}, \mathbf{w}_t, \mathbf{w}_{t-1})$. This set works as a separatrix between $\mathbf{a}_t$ and $\mathcal{F}_{t-1}$ as desired. It is different from the one used in (Koskela et al 2019) because we include $\mathbf{w}_t$ directly rather than its explicit expression in terms of $\mathbf{a}_{t+1}, \mathbf{X}_{t+1}, \mathbf{X}_t$ to simplify the notation.

In each case, the choice of parent depends on two factors: the conditional probability of choosing that parent under the given resampling scheme, and the conditional “probability” of a particle moving from that parent’s position at time t to the offspring’s position at time $t-1$. We see the contribution of these two factors in the following.

Multinomial case

Under multinomial resampling, the parental indices $\mathbf{a}_t$ are conditionally independent given $\mathcal{H}_t$. The conditional law of each index is

$$\mathbb{P}[a_t^{(i)} = a_i \mid \mathcal{H}_t] \propto w_t^{(i)} q_{t-1}(X_t^{(a_i)}, X_{t-1}^{(i)}).$$

Since the indices are all independent, their joint conditional law is

$$\mathbb{P}[\mathbf{a}_t = \mathbf{a} \mid \mathcal{H}_t] \propto \prod_{i=1}^{N} w_t^{(i)} q_{t-1}(X_t^{(a_i)}, X_{t-1}^{(i)}).$$

Conditional SMC case

In conditional SMC with multinomial resampling, the parental indices are still independent, but we have to treat $i = 1$ as a special case. We are assuming wlog that the immortal particle is labelled 1 in each generation. We have the following conditional law:

$$\begin{aligned}
&\mathbb{P}[a_t^{(1)} = a_1 \mid \mathcal{H}_t] = \mathbb{I}\{a_1 = 1\} \\
&\mathbb{P}[a_t^{(i)} = a_i \mid \mathcal{H}_t] \propto w_t^{(i)} q_{t-1}(X_t^{(a_i)}, X_{t-1}^{(i)}) \qquad\qquad i = 2, \dots, N.
\end{aligned}$$

The joint conditional law is therefore

$$\mathbb{P}[\mathbf{a}_t = \mathbf{a} \mid \mathcal{H}_t] \propto \mathbb{I}\{a_1 = 1\} \prod_{i=2}^{N} w_t^{(i)} q_{t-1}(X_t^{(a_i)}, X_{t-1}^{(i)}).$$

Stochastic rounding case

If we resample using a stochastic rounding, we lose the independence between parental indices. The set of valid assignments is much smaller, because each family size can vary by no more than one from its expected value.

Defining the family sizes $v_t^{(i)} := |\{j : a_t^{(j)} = i\}|$ as functions of $\mathbf{a}_t$, we have the constraint $v_t^{(i)} \in \{\lfloor Nw_t^{(i)} \rfloor, \lfloor Nw_t^{(i)} \rfloor + 1\}$. This is encoded in the law of $\mathbf{a}_t$ via indicator functions. The joint conditional law is

$$\begin{aligned}\mathbb{P}[\mathbf{a}_t = \mathbf{a} \mid \mathcal{H}_t] \propto \prod_{i=1}^{N} \Big[& \mathbb{I}\{v_t^{(i)} = \lfloor Nw_t^{(i)} \rfloor\}(1 - Nw_t^{(i)} + \lfloor Nw_t^{(i)} \rfloor) \\ & + \mathbb{I}\{v_t^{(i)} = \lfloor Nw_t^{(i)} \rfloor + 1\}(Nw_t^{(i)} - \lfloor Nw_t^{(i)} \rfloor)\Big]\, q_{t-1}(X_t^{(a_i)}, X_{t-1}^{(i)}).\end{aligned}$$

The dependence between parental indices is implicit in the references to $v_t^{(i)}$, since it is a function of the whole vector $\mathbf{a}_t$.