Coupling argument for residual resampling

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Consider a population of N particles at time t, having fixed weights $(w_t^{(1)}, \dots, w_t^{(N)})$ respectively. These will be the parents, and we consider assigning to them N offspring which will form the next generation. Let $(a_1, \dots, a_N)$ denote the vector of parental indices of each child, and $(v_1, \dots, v_N)$ the resulting offspring counts for each parent. That is, $v_i = \sum_{j=1}^N \mathbb{I}\{a_j = i\}$. We will use a superscript m or r to denote the multinomial or residual schemes respectively.

In the case of multinomial resampling, we have the following:

$$
\begin{aligned}
&(a_1^m, \dots, a_N^m) \sim \text{Categorical}(1:N, w_t^{(1:N)}) \\
&(v_1^m, \dots, v_N^m) \sim \text{Multinomial}(N, w_t^{(1:N)})
\end{aligned}
$$

In residual resampling, the first $k := \sum_{i=1}^N \lfloor N w_t^{(i)} \rfloor$ offspring are assigned deterministically, and the remaining $N-k$ are assigned randomly:

$$
(v_1^r, \dots, v_N^r) \overset{d}{=} \text{Multinomial}(N-k, w_t^{(1:N)}) + \lfloor N w_t^{(1:N)} \rfloor
$$

This distribution can be achieved by doing multinomial resampling as usual, but then overwriting the parental indices of k offspring (to be chosen at random) with the k deterministic assignments (since the offspring are exchangeable and marginals of the Multinomial/Categorical distribution are Multinomial/Categorical).

The probability that a randomly chosen pair of children share the same parent is, with multinomial resampling

$$
p^m = \sum_{i=1}^N (w_t^{(i)})^2
$$

and with residual resampling

$$
\begin{aligned}
p^r &= \sum_{i=1}^N \left[\frac{N-k}{N} w_t^{(i)} \left(\frac{N-k-1}{N} w_t^{(i)} + \frac{\lfloor N w_t^{(i)} \rfloor}{N} \right) + \frac{\lfloor N w_t^{(i)} \rfloor}{N} \left(\frac{N-k}{N} w_t^{(i)} + \frac{k-1}{k} \frac{\lfloor N w_t^{(i)} \rfloor - 1}{N} \right) \right] \\
&= \sum_{i=1}^N \left[\frac{1}{N^2} \left(w_t^{(i)}(N-k) + \lfloor N w_t^{(i)} \rfloor \right)^2 - (N-k)(w_t^{(i)})^2 - \frac{k-1}{k} \lfloor N w_t^{(i)} \rfloor - \frac{1}{k} \lfloor N w_t^{(i)} \rfloor^2 \right] \\
&\le \sum_{i=1}^N \frac{1}{N^2} \left((N-k) w_t^{(i)} + \lfloor N w_t^{(i)} \rfloor \right)^2
\end{aligned}
$$