Residual resampling with multinomial residuals

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29 April 2021

(A1) The conditional distribution of parental indices $a_t^{(1:N)}$ given offspring counts $\nu_t^{(1:N)}$ is uniform over all assignments such that $|\{j : a_t^{(j)} = i\}| = \nu_t^{(i)}$ for all i.

Corollary 1. *Consider an SMC algorithm using residual resampling with multinomial residuals, such that (A1) is satisfied. Assume that there exists a constant $a \in [1, \infty)$ such that for all x, x', t,*

$$\frac{1}{a} \leq g_t(x, x') \leq a.$$

Assume that $\mathbb{P}[\tau_N(t) = \infty] = 0$ for all finite t. Let $(G_t^{(n,N)})_{t\geq 0}$ denote the genealogy of a random sample of n terminal particles from the output of the algorithm when the total number of particles used is N. Then, for any fixed n, the time-scaled genealogy $(G_{\tau_N(t)}^{(n,N)})_{t\geq 0}$ converges to Kingman's n-coalescent as $N \to \infty$, in the sense of finite-dimensional distributions.

Proof. With residual-multinomial resampling, for each i

$$\nu_t^{(i)} \mid w_t^{(1:N)} \overset{d}{=} \lfloor N w_t^{(i)} \rfloor + X_i$$

where $X_i \sim \text{Binomial}(R, r_i)$. As usual, $R := N - \sum_{i=1}^{N} \lfloor N w_t^{(i)} \rfloor$ and $r_i := (N w_t^{(i)} - \lfloor N w_t^{(i)} \rfloor)/R$. If $R = 0$ then $r_i = 0$ for all i and the following calculations remain correct. —SB We can therefore compute

$$\begin{aligned}
\mathbb{E}[(\nu_t^{(i)})_2 \mid w_t^{(1:N)}] &= \mathbb{E}\left[(\lfloor N w_t^{(i)} \rfloor + X_i)(\lfloor N w_t^{(i)} \rfloor + X_i - 1) \,\middle|\, w_t^{(1:N)} \right] \\
&= (\lfloor N w_t^{(i)} \rfloor)_2 + 2\lfloor N w_t^{(i)} \rfloor E[X_i \mid w_t^{(1:N)}] + \mathbb{E}[(X_i)_2 \mid w_t^{(1:N)}] \\
&= (\lfloor N w_t^{(i)} \rfloor)_2 + 2\lfloor N w_t^{(i)} \rfloor R r_i + (R)_2 r_i^2
\end{aligned}$$

using the moments of the Binomial distribution. We also have

$$
\begin{aligned}
\mathbb{E}[(\nu_t^{(i)})_3 \mid w_t^{(1:N)}] &= \mathbb{E}\left[(\lfloor Nw_t^{(i)}\rfloor + X_i)(\lfloor Nw_t^{(i)}\rfloor + X_i - 1)(\lfloor Nw_t^{(i)}\rfloor + X_i - 2) \,\middle|\, w_t^{(1:N)}\right] \\
&= \lfloor Nw_t^{(i)}\rfloor^3 + \lfloor Nw_t^{(i)}\rfloor^2 \mathbb{E}[3X_i - 3 \mid w_t^{(1:N)}] \\
&\qquad + \lfloor Nw_t^{(i)}\rfloor \mathbb{E}[X_i(X_i-1) + X_i(X_i-2) + (X_i-1)(X_i-2) \mid w_t^{(1:N)}] \\
&\qquad + \mathbb{E}[(X_i)_3 \mid w_t^{(1:N)}] \\
&= \lfloor Nw_t^{(i)}\rfloor^3 - 3\lfloor Nw_t^{(i)}\rfloor^2 + 3\lfloor Nw_t^{(i)}\rfloor^2 \mathbb{E}[X_i \mid w_t^{(1:N)}] \\
&\qquad + \lfloor Nw_t^{(i)}\rfloor \mathbb{E}[3X_i^2 - 6X_i + 2 \mid w_t^{(1:N)}] + E[(X_i)_3 \mid w_t^{(1:N)}] \\
&= \left(\lfloor Nw_t^{(i)}\rfloor^3 - 3\lfloor Nw_t^{(i)}\rfloor^2 + 2\lfloor Nw_t^{(i)}\rfloor\right) + 3\left(\lfloor Nw_t^{(i)}\rfloor^2 - \lfloor Nw_t^{(i)}\rfloor\right)\mathbb{E}[X_i \mid w_t^{(1:N)}] \\
&\qquad + 3\lfloor Nw_t^{(i)}\rfloor \mathbb{E}[(X_i)_2 \mid w_t^{(1:N)}] + \mathbb{E}[(X_i)_3 \mid w_t^{(1:N)}] \\
&= (\lfloor Nw_t^{(i)}\rfloor)_3 + 3(\lfloor Nw_t^{(i)}\rfloor)_2 R r_i + 3\lfloor Nw_t^{(i)}\rfloor (R)_2 r_i^2 + (R)_3 r_i^3 \\
&\leq \left(\lfloor Nw_t^{(i)}\rfloor + Rr_i\right)\left\{(\lfloor Nw_t^{(i)}\rfloor)_2 + 2\lfloor Nw_t^{(i)}\rfloor R r_i + (R)_2 r_i^2\right\} \\
&= Nw_t^{(i)} \mathbb{E}[(\nu_t^{(i)})_2 \mid w_t^{(1:N)}] \\
&\leq a^2 \mathbb{E}[(\nu_t^{(i)})_2 \mid w_t^{(1:N)}],
\end{aligned}
$$

using the almost sure bound $w_t^{(i)} \leq a^2/N$.

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To complete the proof we need to exchange the conditioning on $w_t^{(1:N)}$ for conditioning on $\mathcal{H}_t$ so we can then invoke the D-separation and tower property to get

$$
\frac{1}{(N)_3}\sum_{i=1}^{N} \mathbb{E}_t[(\nu_t^{(i)})_3] \leq b_N \frac{1}{(N)_2}\sum_{i=1}^{N} \mathbb{E}_t[(\nu_t^{(i)})_2]
$$

for some sequence $b_N \to 0$. Perhaps we should expect bounds on q_t to be required, so that ε will also appear in b_N. —SB

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