Residual Resampling v2.0 (in progress)

Suzie Brown

10 December 2020

$$r_i := Nw_i - \lfloor Nw_i \rfloor \tag{1}$$

$$R := \sum_{j=1}^{N} r_j = N - \sum_{j=1}^{N} \lfloor Nw_j \rfloor \tag{2}$$

$$\tag{3}$$

Case $R = 0$

$$\sum_{i=1}^{N} \mathbb{E}[(\nu_i)_2 \mid \mathcal{H}_t] = \sum_{i=1}^{N} (\lfloor Nw_i \rfloor)_2 \geq 2|\{\lfloor Nw_i \rfloor \geq 2\}| \tag{4}$$

and

$$\sum_{i=1}^{N} \mathbb{E}[(\nu_i)_3 \mid \mathcal{H}_t] = \sum_{i=1}^{N} (\lfloor Nw_i \rfloor)_3 \leq (a^2)_3 |\{\lfloor Nw_i \rfloor \geq 2\}| \leq a^6 |\{\lfloor Nw_i \rfloor \geq 2\}| \tag{5}$$

so

$$b_N := \frac{1}{N-2} \frac{a^6}{2} \tag{6}$$

will suffice. **NB:** b_N must be deterministic, so we can't actually choose it based on which R case we fall into. We can just set it to its maximum between this case and the next one.

Case $R \neq 0$

$$\nu_i \overset{d}{=} \lfloor Nw_i \rfloor + \mathrm{Bin}(R, r_i/R) \tag{7}$$

$$\mathbb{E}[\nu_i \mid \mathcal{H}_t] = \lfloor Nw_i \rfloor + r_i = Nw_i \tag{8}$$

$$\begin{aligned} \mathbb{E}[(\nu_i)_2 \mid \mathcal{H}_t] &= (\lfloor Nw_i \rfloor)_2 + 2\lfloor Nw_i \rfloor r_i + \frac{R-1}{R} r_i^2 = \lfloor Nw_i \rfloor^2 - \lfloor Nw_i \rfloor + 2\lfloor Nw_i \rfloor r_i + r_i^2 - \frac{r_i^2}{R} \\ &= \{\lfloor Nw_i \rfloor + r_i\}^2 - \lfloor Nw_i \rfloor - \frac{r_i^2}{R} = N^2 w_i^2 - \lfloor Nw_i \rfloor - \frac{r_i^2}{R} \geq N^2 w_i^2 - Nw_i \end{aligned} \tag{9}$$

$$\begin{aligned} \mathbb{E}[(\nu_i)_3 \mid \mathcal{H}_t] &= (\lfloor Nw_i \rfloor)_3 + 3(\lfloor Nw_i \rfloor)_2 r_i + 3\lfloor Nw_i \rfloor \frac{R-1}{R} r_i^2 + \frac{(R-1)(R-2)}{R^2} r_i^3 \\ &\leq (\lfloor Nw_i \rfloor)_3 + 3(\lfloor Nw_i \rfloor)_2 r_i + 3\lfloor Nw_i \rfloor r_i^2 + r_i^3 \\ &\leq \lfloor Nw_i \rfloor^3 + 3\lfloor Nw_i \rfloor^2 r_i + 3\lfloor Nw_i \rfloor r_i^2 + r_i^3 = \{\lfloor Nw_i \rfloor + r_i\}^3 = N^3 w_i^3 \leq a^6 \end{aligned} \tag{10}$$

$(X)_1$	$=$	X
$(X)_2$	$=$	$X^2 - X$
$(X)_3$	$=$	$X^3 - 3X^2 + 2X$
X	$=$	$(X)_1$
X^2	$=$	$(X)_2 + (X)_1$
X^3	$=$	$(X)_3 + 3(X)_2 + (X)_1$

Table 1: Conversions between standard and factorial powers

$(X+Y)_1$	$=$	$X + Y$
$(X+Y)_2$	$=$	$(X)_2 + 2XY + (Y)_2$
$(X+Y)_3$	$=$	$(X)_3 + 3(X)_2Y + 3X(Y)_2 + (Y)_3$

Table 2: Expansions of mixed factorial powers

$\mathbb{E}[(X)_1]$	$=$	np
$\mathbb{E}[(X)_2]$	$=$	$n(n-1)p^2$
$\mathbb{E}[(X)_3]$	$=$	$n(n-1)(n-2)p^3$
$\mathbb{E}[X]$	$=$	np
$\mathbb{E}[X^2]$	$=$	$np(1+(n-1)p)$
$\mathbb{E}[X^3]$	$=$	$np(1+(n-1)p(3+(n-2)p))$

Table 3: Moments of the Binomial distirbution; $X \sim \mathrm{Bin}(n, p)$