Weak convergence proof v.2

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Bounds on sum-products

Lemma 1. *Fix* $t > 0$, $l \in \mathbb{N}$.

$$t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \binom{l}{2} (t+1)^{l-2} \leq \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \leq t^l + c_N(\tau_N(t))(t+1)^l. \tag{1}$$

Proof. As pointed out in Koskela et al. (2018, Equation (8)),

$$\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \geq \left(\sum_{s=0}^{\tau_N(t)} c_N(s) \right)^l - \binom{l}{2} \left(\sum_{s=0}^{\tau_N(t)} c_N(s)^2 \right) \left(\sum_{s=0}^{\tau_N(t)} c_N(s) \right)^{l-2}. \tag{2}$$

By definition of τ_N,

$$t \leq \sum_{s=0}^{\tau_N(t)} c_N(s) \leq t+1. \tag{3}$$

Substituting these bounds into the RHS of (2) yields the lower bound.

It is a true fact that

$$\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \leq \left(\sum_{s=0}^{\tau_N(t)} c_N(s) \right)^l, \tag{4}$$

as can be seen by considering the multinomial expansion of the RHS. This is further bounded by

$$\left(\sum_{s=0}^{\tau_N(t)} c_N(s) \right)^l \leq \left(\sum_{s=0}^{\tau_N(t)-1} c_N(s) + c_N(\tau_N(t)) \right)^l \leq \left[t + c_N(\tau_N(t)) \right]^l, \tag{5}$$

again using the definition of τ_N. A binomial expansion yields

$$\left[t + c_N(\tau_N(t)) \right]^l = t^l + \sum_{i=0}^{l-1} \binom{l}{i} t^i c_N(\tau_N(t))^{l-i} = t^l + c_N(\tau_N(t)) \sum_{i=0}^{l-1} \binom{l}{i} t^i c_N(\tau_N(t))^{l-1-i}, \tag{6}$$

then since $c_N(s) \leq 1$ for all s,

$$\sum_{i=0}^{l-1} \binom{l}{i} t^i c_N(\tau_N(t))^{l-1-i} \leq \sum_{i=0}^{l-1} \binom{l}{i} t^i \leq (t+1)^l. \tag{7}$$

Putting this together yields the upper bound. ∎

Lemma 2. *Fix $t > 0$, $l \in \mathbb{N}$. Let B be a positive constant which may depend on n.*

$$\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) + BD_N(s_j)] \leq \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1}(1+B)^l. \tag{8}$$

Proof. We start with a binomial expansion:

$$\begin{aligned}\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) + BD_N(s_j)] &= \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \sum_{\mathcal{I} \subseteq [l]} B^{l-|\mathcal{I}|} \left(\prod_{i \in \mathcal{I}} c_N(s_i) \right) \left(\prod_{j \notin \mathcal{I}} D_N(s_j) \right) \\ &= \sum_{\mathcal{I} \subseteq [l]} B^{l-|\mathcal{I}|} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i \in \mathcal{I}} c_N(s_i) \right) \left(\prod_{j \notin \mathcal{I}} D_N(s_j) \right) \end{aligned} \tag{9}$$

where $[l] := \{1, \ldots, l\}$. Since the sum is over all permutations of $s_1, \ldots, s_l$, we may arbitrarily choose an ordering for $\{1, \ldots, l\}$ such that $\mathcal{I} = \{1, \ldots, |\mathcal{I}|\}$:

$$\sum_{\mathcal{I} \subseteq [l]} B^{l-|\mathcal{I}|} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i \in \mathcal{I}} c_N(s_i) \right) \left(\prod_{j \notin \mathcal{I}} D_N(s_j) \right) = \sum_{I=0}^{l} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right). \tag{10}$$

Separating the term $I = l$,

$$\begin{aligned}\sum_{I=0}^{l} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right) &= \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \\ &+ \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right). \end{aligned} \tag{11}$$

In the second line, there is always at least one D_N term, and $c_N(s) \geq D_N(s)$ for all s (Koskela et al., 2018, p.9), so we can write

$$\begin{aligned}\sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right) &\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{l-1} c_N(s_i) \right) D_N(s_l) \\ &\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \left(\sum_{s_1 \neq \cdots \neq s_{l-1}}^{\tau_N(t)} \prod_{i=1}^{l-1} c_N(s_i) \right) \sum_{s_l=1}^{\tau_N(t)} D_N(s_l) \\ &\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} (t+1)^{l-1} \sum_{s=1}^{\tau_N(t)} D_N(s) \end{aligned} \tag{12}$$

using (4) and (3). Finally, by the Binomial Theorem,

$$\sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} (t+1)^{l-1} \sum_{s=1}^{\tau_N(t)} D_N(s) \leq \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1}(1+B)^l, \tag{13}$$

which, together with (11), concludes the proof. ■

Lemma 3. *Fix $t > 0$, $l \in \mathbb{N}$. Let B be a positive constant which may depend on n.*

$$\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - BD_N(s_j)] \geq \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1}(1+B)^l. \tag{14}$$

Proof. A binomial expansion and subsequent manipulation as in (9)–(11) gives

$$\begin{aligned}
\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - BD_N(s_j)] &= \sum_{\mathcal{I}\subseteq[l]} (-B)^{l-|\mathcal{I}|} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i\in\mathcal{I}} c_N(s_i)\right) \left(\prod_{j\notin\mathcal{I}} D_N(s_j)\right) \\
&= \sum_{I=0}^{l} \binom{l}{I} (-B)^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right) \left(\prod_{j=I+1}^{l} D_N(s_j)\right) \\
&= \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) + \sum_{I=0}^{l-1} \binom{l}{I} (-B)^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right) \left(\prod_{j=I+1}^{l} D_N(s_j)\right) \\
&\geq \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) - \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right) \left(\prod_{j=I+1}^{l} D_N(s_j)\right)
\end{aligned} \tag{15}$$

where the last inequality just multiplies some positive terms by -1. Then (12)–(13) can be applied directly (noting that an upper bound on negative terms gives a lower bound overall):

$$-\sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right) \left(\prod_{j=I+1}^{l} D_N(s_j)\right) \geq -\left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right) (t+1)^{l-1}(1+B)^l \tag{16}$$

which concludes the proof. ■

Main components of weak convergence

Lemma 4 (Basis step)**.** *For any* $0 < t < \infty$,

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1-p_r)\right] = e^{-\alpha_n t} \tag{17}$$

where $\alpha_n := n(n-1)/2$.

Proof. We start by showing that $\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right] \leq e^{-\alpha_n t}$.
From Koskela et al. (2018, Lemma 1 Case 1), taking $\xi = \Delta$, we have for each r

$$1 - p_r = p_{\Delta\Delta}(r) \leq 1 - \alpha_n(1 + O(N^{-1}))\,[c_N(r) - B_n' D_N(r)] \tag{18}$$

where the $O(N^{-1})$ term does not depend on r. When N is large enough, a sufficient condition to ensure the bound in (18) is non-negative is the event

$$E_r := \left\{c_N(r) \leq \alpha_n^{-1}\right\} \tag{19}$$

and we define $E := \bigcap_{r=1}^{\tau_N(t)} E_r$, so

$$1 - p_r = p_{\Delta\Delta}(r) \leq 1 - \alpha_n(1 + O(N^{-1}))\,[c_N(r) - B_n' D_N(r)]\,\mathbb{1}_E. \tag{20}$$

Applying a multinomial expansion and then separating the positive and negative terms,

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1-p_r) &\leq 1 + \sum_{l=1}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - B_n' D_N(s_j)]\,\mathbb{1}_E \\
&= 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - B_n' D_N(s_j)]\,\mathbb{1}_E \\
&\quad - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - B_n' D_N(s_j)]\,\mathbb{1}_E.
\end{aligned} \tag{21}$$

This is further bounded by applying Lemma 3 and then both bounds of Lemma 1:

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1-p_r) \leq 1 + \Bigg\{ &\sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \\
&- \sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \left[\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1+B_n')^l \right] \Bigg\} \mathbb{1}_E \\
\leq 1 + \Bigg\{ &\sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \left\{ t^l + c_N(\tau_N(t))(t+1)^l \right\} \\
&- \sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \left[t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \binom{l}{2} (t+1)^{l-2} - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1+B_n')^l \right] \Bigg\} \mathbb{1}_E.
\end{aligned} \tag{22}$$

Collecting some terms,

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1-p_r) &\leq 1 + \sum_{l=1}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_E + c_N(\tau_N(t)) \sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} (t+1)^l \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \binom{l}{2} (t+1)^{l-2} \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} (t+1)^{l-1} (1+B_n')^l \\
&\leq 1 + \sum_{l=1}^{\infty} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_{\{\tau_N(t) \geq l\}} \mathbb{1}_E + c_N(\tau_N(t)) \exp[\alpha_n (1+O(N^{-1}))(t+1)] \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n (1+O(N^{-1}))(t+1)] \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n (1+O(N^{-1}))(t+1)(1+B_n')].
\end{aligned} \tag{23}$$

Now, taking the expectation and limit, then applying Brown et al. (2021, Equations (3.3)–(3.5)), and Lemmata 10 and 12 to deal with the indicators,

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1-p_r) \right] &\leq 1 + \sum_{l=1}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l \lim_{N\to\infty} \mathbb{P}\left[\{\tau_N(t) \geq l\} \cap E \right] + \lim_{N\to\infty} \mathbb{E}\left[c_N(\tau_N(t)) \right] \exp[\alpha_n (t+1)] \\
&\quad + \lim_{N\to\infty} \mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right] \frac{1}{2} \alpha_n^2 \exp[\alpha_n (t+1)] \\
&\quad + \lim_{N\to\infty} \mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} D_N(s) \right] \exp[\alpha_n (t+1)(1+B_n')] \\
&= 1 + \sum_{l=1}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l = e^{-\alpha_n t}.
\end{aligned} \tag{24}$$

It remains to show the corresponding lower bound $\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1-p_r) \right] \geq e^{-\alpha_n t}$.

From Brown et al. (2021, Equation (3.14)), taking $\xi = \Delta$, we have

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \alpha_n(1 + O(N^{-1}))\,[c_N(t) + B_n D_N(t)] \tag{25}$$

where $B_n > 0$ and the $O(N^{-1})$ term does not depend on t. In particular,

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \frac{N^{n-2}}{(N-2)_{n-2}}\alpha_n[c_N(t) + B_n D_N(t)]. \tag{26}$$

Since $D_N(s) \leq c_N(s)$ for all s (Koskela et al., 2018, p.9), a sufficient condition for this bound to be non-negative is

$$E_r := \left\{ c_N(r) \leq \frac{(N-2)_{n-2}}{N^{n-2}}\alpha_n^{-1}(1 + B_n)^{-1} \right\}, \tag{27}$$

and we again define $E := \bigcap_{r=1}^{\tau_N(t)} E_r$. We now apply a multinomial expansion to the product, and split into positive and negative terms:

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)}(1-p_r) &\geq \left\{ 1 + \sum_{l=1}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1}))\frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) + B_n D_N(s_j)] \right\} \mathbb{1}_E \\
&= \left\{ 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1}))\frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) + B_n D_N(s_j)] \right. \\
&\qquad \left. - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1}))\frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) + B_n D_N(s_j)] \right\} \mathbb{1}_E
\end{aligned} \tag{28}$$

This is further bounded by applying Lemma 2 and both bounds in Lemma 1:

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)}(1-p_r) &\geq \left\{ 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1}))\frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \right. \\
&\qquad \left. - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1}))\frac{1}{l!} \left[\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1}(1+B_n)^l \right] \right\} \mathbb{1}_E \\
&\geq \left\{ 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1}))\frac{1}{l!} \left[t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \binom{l}{2} (t+1)^{l-2} \right] \right. \\
&\qquad \left. - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1}))\frac{1}{l!} \left[t^l + c_N(\tau_N(t))(t+1)^l + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1}(1+B_n)^l \right] \right\} \mathbb{1}_E.
\end{aligned} \tag{29}$$

Collecting terms,

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)}(1-p_r) \geq \sum_{l=0}^{\tau_N(t)}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_E - \left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right)\sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\binom{l}{2}(t+1)^{l-2} \\
- c_N(\tau_N(t))\sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}(t+1)^l \\
- \left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)\sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}(t+1)^{l-1}(1+B_n)^l \\
\geq \sum_{l=0}^{\infty}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_E\mathbb{1}_{\{\tau_N(t)\geq l\}} - \left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right)\frac{1}{2}\alpha_n^2\exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
- c_N(\tau_N(t))\exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
- \left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n)]. \qquad (30)
\end{aligned}$$

Now, taking the expectation and limit, and applying Brown et al. (2021, Equations (3.3)–(3.5)) to show that all but the first sum vanish, and Lemmata 10 and 9 to show that $\lim_{N\to\infty}\mathbb{P}[\{\tau_N(t)\geq l\}\cap E]=1$,

$$\begin{aligned}
\lim_{N\to\infty}\mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right] \geq \sum_{l=0}^{\infty}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\lim_{N\to\infty}\mathbb{P}\left[\{\tau_N(t)\geq l\}\cap E\right] \\
- \lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right]\frac{1}{2}\alpha_n^2\exp[\alpha_n(t+1)] \\
- \lim_{N\to\infty}\mathbb{E}\left[c_N(\tau_N(t))\right]\exp[\alpha_n(t+1)] \\
- \lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)}D_N(s)\right]\exp[\alpha_n(t+1)(1+B_n)] \\
= \sum_{l=0}^{\infty}(-\alpha_n)^l\frac{1}{l!}t^l = e^{-\alpha_n t}. \qquad (31)
\end{aligned}$$

Combining the upper and lower bounds in (24) and (31) respectively concludes the proof. ■

Lemma 5 (Induction step upper bound). *Fix $k\in\mathbb{N}$, $i_0:=0$, $i_k:=k$. For any sequence of times $0=t_0\leq t_1\leq\cdots\leq t_k\leq t$,*

$$\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\left(\prod_{i=1}^{k}p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)}(1-p_r)\right)\right] \leq \alpha_n^k\sum_{l=0}^{\infty}\frac{1}{l!}(-\alpha_n)^l t^l\sum_{\substack{i_1\leq\cdots\leq i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\geq j\forall j}}\prod_{j=1}^{k}\frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}. \qquad (32)$$

Proof. We use the bound on $(1-p_r)$ from (18) and apply a multinomial expansion, defining as in (19) the event E

which ensures the bound is non-negative:

$$\begin{aligned}
\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r) &\leq \prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} \left\{1-\alpha_n(1+O(N^{-1}))[c_N(r)-B_n'D_N(r)]\mathbb{1}_E\right\} \\
&= 1+\sum_{l=1}^{\tau_N(t)-k}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}\sum_{\substack{s_1\neq\dots\neq s_l\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)}\prod_{j=1}^{l}[c_N(s_j)-B_n'D_N(s_j)]\mathbb{1}_E \\
&= 1+\sum_{l=1}^{\tau_N(t)-k}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}\sum_{s_1\neq\dots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}[c_N(s_j)-B_n'D_N(s_j)]\mathbb{1}_E \\
&\quad -\sum_{l=1}^{\tau_N(t)-k}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}\sum_{\substack{s_1\neq\dots\neq s_l\\ \exists i,i':s_i=r_{i'}}}\prod_{j=1}^{l}[c_N(s_j)-B_n'D_N(s_j)]\mathbb{1}_E. \qquad (33)
\end{aligned}$$

The penultimate line above is exactly the expansion we had in the basis step (21), except for the limit on l, and as such following the same arguments gives a bound like that in (23):

$$\begin{aligned}
1+\sum_{l=1}^{\tau_N(t)-k}&(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}\sum_{s_1\neq\dots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}[c_N(s_j)-B_n'D_N(s_j)]\mathbb{1}_E \\
&\leq 1+\sum_{l=1}^{\tau_N(t)-k}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_E + c_N(\tau_N(t))\exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right)\frac{1}{2}\alpha_n^2\exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')]. \qquad (34)
\end{aligned}$$

For the last line of (33),

$$\begin{aligned}
-\sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1+O(N^{-1}))\frac{1}{l!} &\sum_{\substack{s_1\neq\cdots\neq s_l\\ \exists i,i':s_i=r_{i'}}} \prod_{j=1}^{l}\{c_N(s_j)-B_n' D_N(s_j)\}\mathbb{1}_E \\
&\leq \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1+O(N^{-1}))\frac{1}{l!} \sum_{\substack{s_1\neq\cdots\neq s_l\\ \exists i,i':s_i=r_{i'}}} \prod_{j=1}^{l}\{c_N(s_j)+B_n' D_N(s_j)\} \\
&\leq \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1+O(N^{-1}))\frac{1}{l!} \sum_{\substack{s_1\neq\cdots\neq s_l\\ \exists i,i':s_i=r_{i'}}} (1+B_n')^l \prod_{j=1}^{l} c_N(s_j) \\
&\leq \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1+O(N^{-1}))\frac{1}{(l-1)!} \sum_{s_1\in\{r_1,\dots,r_k\}} \sum_{s_2\neq\cdots\neq s_l}^{\tau_N(t)} (1+B_n')^l \prod_{j=1}^{l} c_N(s_j) \\
&= \sum_{s\in\{r_1,\dots,r_k\}} c_N(s) \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1+O(N^{-1}))\frac{1}{(l-1)!}(1+B_n')^l \sum_{s_1\neq\cdots\neq s_{l-1}}^{\tau_N(t)} \prod_{j=1}^{l-1} c_N(s_j) \\
&\leq \sum_{j=1}^{k} c_N(r_j) \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1+O(N^{-1}))\frac{1}{(l-1)!}(1+B_n')^l (t+1)^{l-1} \\
&\leq \left(\sum_{j=1}^{k} c_N(r_j)\right) \alpha_n (1+B_n') \exp[\alpha_n(1+O(N^{-1}))(1+B_n')(t+1)]. \qquad (35)
\end{aligned}$$

Putting these together, we have

$$\begin{aligned}
\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r) \leq 1 &+ \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1+O(N^{-1}))\frac{1}{l!} t^l \mathbb{1}_E + c_N(\tau_N(t)) \exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
&+ \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right) \frac{1}{2}\alpha_n^2 \exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
&+ \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right) \exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')] \\
&+ \left(\sum_{j=1}^{k} c_N(r_j)\right) \alpha_n (1+B_n') \exp[\alpha_n(1+O(N^{-1}))(1+B_n')(t+1)]. \qquad (36)
\end{aligned}$$

Meanwhile, using the bound on p_r from (25) then applying a modification of Lemma 2,

$$\begin{aligned}
\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}} \prod_{i=1}^{k} p_{r_i} &\leq \alpha_n^k (1+O(N^{-1})) \sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}} \prod_{i=1}^{k} [c_N(r_i)+B_n D_N(r_i)] \\
&\leq \alpha_n^k (1+O(N^{-1})) \sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}} \prod_{i=1}^{k} c_N(r_i) + \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right) \alpha_n^k (1+O(N^{-1}))(t+1)^{k-1}(1+B_n)^k. \qquad (37)
\end{aligned}$$

A more liberal (but simpler) bound can be arrived at thus:

$$\begin{aligned}\prod_{i=1}^{k} p_{r_i} &\leq \alpha_n^k (1+O(N^{-1})) \prod_{i=1}^{k} [c_N(r_i) + B_n D_N(r_i)] \\ &\leq \alpha_n^k (1+O(N^{-1})) \prod_{i=1}^{k} c_N(r_i)(1+B_n) \\ &\leq \alpha_n^k (1+O(N^{-1}))(1+B_n)^k \prod_{i=1}^{k} c_N(r_i) \end{aligned} \tag{38}$$

which also leads to the deterministic bound

$$\begin{aligned}\sum_{\substack{r_1<\cdots<r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} p_{r_i} &\leq \alpha_n^k (1+O(N^{-1}))(1+B_n)^k \sum_{\substack{r_1<\cdots<r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i) \\ &\leq \alpha_n^k (1+O(N^{-1}))(1+B_n)^k \frac{1}{k!} \sum_{r_1 \neq \cdots \neq r_k}^{\tau_N(t)} \prod_{i=1}^{k} c_N(r_i) \\ &\leq \alpha_n^k (1+O(N^{-1}))(1+B_n)^k \frac{1}{k!} (t+1)^k. \end{aligned} \tag{39}$$

Combining (36) with the other product, the expression inside the expectation in (32) is bounded above by

$$\begin{aligned}\sum_{\substack{r_1<\cdots<r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right) \left(\prod_{\substack{r=1 \\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right) \leq \left\{1 + \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_E \right\} \sum_{\substack{r_1<\cdots<r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} p_{r_i} \\ + \left\{ c_N(\tau_N(t)) \exp[\alpha_n(1+O(N^{-1}))(t+1)] + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right) \frac{1}{2}\alpha_n^2 \exp[\alpha_n(1+O(N^{-1}))(t+1)] \right. \\ \left. + \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right) \exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')] \right\} \sum_{\substack{r_1<\cdots<r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} p_{r_i} \\ + \exp[\alpha_n(1+O(N^{-1}))(1+B_n')(t+1)] \alpha_n (1+B_n') \sum_{\substack{r_1<\cdots<r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \sum_{j=1}^{k} c_N(r_j) \prod_{i=1}^{k} p_{r_i}. \end{aligned} \tag{40}$$

Applying the various bounds (37)–(39), we have

$$\begin{aligned}\sum_{\substack{r_1<\cdots<r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right) \left(\prod_{\substack{r=1 \\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right) \leq \alpha_n^k (1+O(N^{-1})) \left\{1 + \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_E \right\} \sum_{\substack{r_1<\cdots<r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i) \\ + \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right) \alpha_n^k (1+O(N^{-1}))(t+1)^{k-1}(1+B_n)^k \sum_{l=0}^{\tau_N(t)} (\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l \\ + \left\{ c_N(\tau_N(t)) \exp[\alpha_n(1+O(N^{-1}))(t+1)] + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right) \frac{1}{2}\alpha_n^2 \exp[\alpha_n(1+O(N^{-1}))(t+1)] \right. \\ \left. + \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right) \exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')] \right\} \alpha_n^k (1+O(N^{-1}))(1+B_n)^k \frac{1}{k!}(t+1)^k \\ + \exp[\alpha_n(1+B_n')(t+1)] \alpha_n (1+B_n') \alpha_n^k (1+O(N^{-1}))(1+B_n)^k \\ \times \sum_{\substack{r_1<\cdots<r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \sum_{j=1}^{k} c_N(r_j) \prod_{i=1}^{k} c_N(r_i). \end{aligned} \tag{41}$$

Upon taking the expectation and limit, we have

$$\begin{aligned}
\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\left(\prod_{i=1}^{k}p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)}(1-p_r)\right)\right] &\le \alpha_n^k\lim_{N\to\infty}\mathbb{E}\left[\left(1+\sum_{l=1}^{\tau_N(t)-k}(-\alpha_n)^l\frac{1}{l!}t^l\mathbb{1}_E\right)\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\prod_{i=1}^{k}c_N(r_i)\right]\\
&+\lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)}D_N(s)\right]\alpha_n^k(t+1)^{k-1}(1+B_n)^k\exp[\alpha_n t]\\
&+\left\{\lim_{N\to\infty}\mathbb{E}\left[c_N(\tau_N(t))\right]\exp[\alpha_n(t+1)]+\lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right]\frac{1}{2}\alpha_n^2\exp[\alpha_n(t+1)]\right.\\
&\qquad\left.+\lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)}D_N(s)\right]\exp[\alpha_n(t+1)(1+B_n')]\right\}\alpha_n^k(1+B_n)^k\frac{1}{k!}(t+1)^k\\
&+\exp[\alpha_n(1+B_n')(t+1)]\alpha_n^{k+1}(1+B_n')(1+B_n)^k\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\sum_{j=1}^{k}c_N(r_j)\prod_{i=1}^{k}c_N(r_i)\right]. \qquad (42)
\end{aligned}$$

The middle terms vanish due to Brown et al. (2021, Equations (3.3)–(3.5)) and the expression becomes

$$\begin{aligned}
\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\left(\prod_{i=1}^{k}p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)}(1-p_r)\right)\right] &\le \alpha_n^k\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\prod_{i=1}^{k}c_N(r_i)\right]\\
&+\alpha_n^k\sum_{l=1}^{\infty}(-\alpha_n)^l\frac{1}{l!}t^l\lim_{N\to\infty}\mathbb{E}\left[\mathbb{1}_{\{\tau_N(t)\ge k+l\}}\mathbb{1}_E\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\prod_{i=1}^{k}c_N(r_i)\right]\\
&+\exp[\alpha_n(1+B_n')(t+1)]\alpha_n^{k+1}(1+B_n')(1+B_n)^k\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\sum_{j=1}^{k}c_N(r_j)\prod_{i=1}^{k}c_N(r_i)\right]. \qquad (43)
\end{aligned}$$

To simplify the last line,

$$\begin{aligned}
\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\sum_{j=1}^{k}c_N(r_j)\prod_{i=1}^{k}c_N(r_i) &\le \frac{1}{k!}\sum_{r_1\ne\cdots\ne r_k}^{\tau_N(t)}\sum_{j=1}^{k}c_N(r_j)\prod_{i=1}^{k}c_N(r_i)\\
&=\frac{1}{k!}\sum_{r_1\ne\cdots\ne r_k}^{\tau_N(t)}\sum_{j=1}^{k}c_N(r_j)^2\prod_{i\ne j}c_N(r_i)\\
&\le\frac{1}{k!}\sum_{j=1}^{k}\sum_{s=1}^{\tau_N(t)}c_N(s)^2\sum_{r_1\ne\cdots\ne r_{k-1}}^{\tau_N(t)}\prod_{i=1}^{k-1}c_N(r_i)\\
&\le\frac{1}{(k-1)!}\sum_{s=1}^{\tau_N(t)}c_N(s)^2(t+1)^{k-1} \qquad (44)
\end{aligned}$$

hence

$$\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\sum_{s\in\{r_1,\dots,r_k\}}c_N(s)\prod_{i=1}^{k}c_N(r_i)\right]\le\frac{1}{(k-1)!}(t+1)^{k-1}\lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right]=0 \qquad (45)$$

by Brown et al. (2021, Equation (3.5)). By Lemmata 10 and 9, $\lim_{N\to\infty} \mathbb{P}[\{\tau_N(t) \geq k+l\} \cap E] = 1$, so we can apply Lemma 7 to the remaining expectations in (43), yielding

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}} \left(\prod_{i=1}^k p_{r_i}\right) \left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\right] &\leq \alpha_n^k \sum_{l=0}^\infty (-\alpha_n)^l \frac{1}{l!} t^l \sum_{\substack{i_1\leq\dots\leq i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\geq j\forall j}} \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} \\
&= \alpha_n^k e^{-\alpha_n t} \sum_{\substack{i_1\leq\dots\leq i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\geq j\forall j}} \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} \qquad (46)
\end{aligned}$$

as required. ∎

Lemma 6 (Induction step lower bound). *Fix $k\in\mathbb{N}$, $i_0 := 0$, $i_k := k$. For any sequence of times $0 = t_0 \leq t_1 \leq \dots \leq t_k \leq t$,*

$$\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\dots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}} \left(\prod_{i=1}^k p_{r_i}\right) \left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\right] \geq \alpha_n^k \sum_{l=0}^\infty \frac{1}{l!}(-\alpha_n)^l t^l \sum_{\substack{i_1\leq\dots\leq i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\geq j\forall j}} \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}. \qquad (47)$$

Proof. Firstly,

$$\sum_{\substack{r_1<\dots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}} \left(\prod_{i=1}^k p_{r_i}\right) \left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right) \geq \sum_{\substack{r_1<\dots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}} \left(\prod_{i=1}^k p_{r_i}\right) \left(\prod_{r=1}^{\tau_N(t)} (1-p_r)\right). \qquad (48)$$

Now the second product does not depend on $r_1,\dots,r_k$, and we can use the lower bound from (30):

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1-p_r) \geq \sum_{l=0}^{\tau_N(t)} &(-\alpha_n)^l (1+O(N^{-1}))\frac{1}{l!} t^l \mathbb{1}_E - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right) \frac{1}{2}\alpha_n^2 \exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
&- c_N(\tau_N(t)) \exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
&- \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right) \exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n)] \qquad (49)
\end{aligned}$$

where E is defined as in (27). We will also need an upper bound on this product, which is formed from (23) with a further deterministic bound:

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1-p_r) \leq &\sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1}))\frac{1}{l!} t^l + c_N(\tau_N(t)) \exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
&+ \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right) \frac{1}{2}\alpha_n^2 \exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
&+ \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right) \exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')] \\
\leq &\exp[\alpha_n(1+O(N^{-1}))t] + \exp[\alpha_n(1+O(N^{-1}))(t+1)] \\
&+ \frac{1}{2}\alpha_n^2(t+1)\exp[\alpha_n(1+O(N^{-1}))(t+1)] + (t+1)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')] \\
\leq &\left(2+\frac{\alpha_n^2(t+1)}{2}\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)] + (t+1)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')]. \qquad (50)
\end{aligned}$$

Now let us consider the remaining sum-product on the RHS of (48). We use the same bound on p_r as in (18):

$$p_r = 1 - p_{\Delta\Delta}(r) \geq \alpha_n(1 + O(N^{-1}))\,[c_N(r) - B_n' D_N(r)] \tag{51}$$

where the $O(N^{-1})$ term does not depend on r. When N is large enough for the factor of $(1 + O(N^{-1}))$ to be non-negative, a sufficient condition to ensure the bound in (51) is non-negative is the event

$$E_r' := \{c_N(r) \geq B_n' D_N(r)\} \tag{52}$$

and we define $E' := \bigcap_{r=1}^{\tau_N(t)} E_r'$. Then

$$\prod_{i=1}^{k} p_{r_i} \geq \alpha_n^k(1 + O(N^{-1}))\prod_{i=1}^{k}[c_N(r_i) - B_n' D_N(r_i)]\,\mathbb{1}_{E'}. \tag{53}$$

Applying a modification of Lemma 3,

$$\begin{aligned}\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\left(\prod_{i=1}^{k} p_{r_i}\right) &\geq \alpha_n^k(1 + O(N^{-1}))\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\prod_{i=1}^{k}[c_N(r_i) - B_n' D_N(r_i)]\,\mathbb{1}_{E'}\\ &\geq \alpha_n^k(1 + O(N^{-1}))\left\{\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\prod_{i=1}^{k} c_N(r_i)\mathbb{1}_{E'} - \frac{1}{k!}\left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)(t+1)^{k-1}(1+B_n')^k\right\}.\end{aligned} \tag{54}$$

The above expression is already split into positive and negative terms; a lower bound on (48) can be formed by multiplying the positive terms by the lower bound (49) and the negative terms by the upper bound (50). Thus

$$\begin{aligned}\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\left(\prod_{i=1}^{k} p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)}(1-p_r)\right) \geq \alpha_n^k(1 + O(N^{-1}))&\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\prod_{i=1}^{k} c_N(r_i)\mathbb{1}_{E'}\Bigg\{\\ &\sum_{l=0}^{\tau_N(t)}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\,\mathbb{1}_E\\ &-\left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right)\frac{1}{2}\alpha_n^2\exp[\alpha_n(1+O(N^{-1}))(t+1)]\\ &-c_N(\tau_N(t))\exp[\alpha_n(1+O(N^{-1}))(t+1)]\\ &-\left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n)]\Bigg\}\\ -\left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)\alpha_n^k(1+O(N^{-1}))\frac{1}{k!}(t+1)^{k-1}(1+B_n')^k&\Bigg\{\\ &\left(2+\frac{\alpha_n^2(t+1)}{2}\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)]\\ &+(t+1)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')]\Bigg\}.\end{aligned} \tag{55}$$

Due to Brown et al. (2021, Equations (3.3)–(3.5)), all but the first two lines in the above have vanishing expectation, leaving

$$\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\right]$$

$$\geq \lim_{N\to\infty} \mathbb{E}\left[\alpha_n^k(1+O(N^{-1}))\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}} \prod_{i=1}^{k} c_N(r_i)\mathbb{1}_{E'}\sum_{l=0}^{\tau_N(t)}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_E\right]$$

$$= \alpha_n^k\sum_{l=0}^{\infty}(-\alpha_n)^l\frac{1}{l!}t^l\lim_{N\to\infty}\mathbb{E}\left[\mathbb{1}_{\{\tau_N(t)\geq l\}}\mathbb{1}_{E\cap E'}\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\prod_{i=1}^{k}c_N(r_i)\right] \tag{56}$$

Lemmata 9 and 12 establish that $\lim_{N\to\infty}\mathbb{P}[E\cap E']=1$ and Lemma 10 deals with the other indicator. We can therefore apply Lemma 7 to conclude that

$$\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\right] \geq \alpha_n^k\sum_{l=0}^{\infty}(-\alpha_n)^l\frac{1}{l!}t^l \sum_{\substack{i_1\leq\cdots\leq i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\geq j\forall j}}\prod_{j=1}^{k}\frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}$$

$$= \alpha_n^k e^{-\alpha_n t}\sum_{\substack{i_1\leq\cdots\leq i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\geq j\forall j}}\prod_{j=1}^{k}\frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} \tag{57}$$

as required. ■

Lemma 7. *Fix $k\in\mathbb{N}$, $i_0:=0$, $i_k:=k$. Let E be any event independent of $r_1,\ldots,r_k$ such that $\lim_{N\to\infty}\mathbb{P}[E]=1$. Then for any sequence of times $0=t_0\leq t_1\leq\cdots\leq t_k\leq t$,*

$$\lim_{N\to\infty}\mathbb{E}\left[\mathbb{1}_E\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\prod_{i=1}^{k}c_N(r_i)\right] = \sum_{\substack{i_1\leq\cdots\leq i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\geq j\forall j}}\prod_{j=1}^{k}\frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}. \tag{58}$$

Proof. As pointed out by Möhle (1999, p. 460), the sum-product on the left hand side can be expanded as

$$\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\prod_{i=1}^{k}c_N(r_i) = \sum_{\substack{i_1\leq\cdots\leq i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\geq j\forall j}}\prod_{j=1}^{k}\sum_{\substack{r_{i_{j-1}+1}<\cdots<r_{i_j}\\ =\tau_N(t_{j-1})+1}}^{\tau_N(t_j)}\prod_{i=i_{j-1}+1}^{i_j}c_N(r_i). \tag{59}$$

Moreover,

$$\sum_{\substack{r_{i_{j-1}+1}<\cdots<r_{i_j}\\ =\tau_N(t_{j-1})+1}}^{\tau_N(t_j)}\prod_{i=i_{j-1}+1}^{i_j}c_N(r_i) = \frac{1}{(i_j-i_{j-1})!}\sum_{\substack{r_{i_{j-1}+1}\neq\cdots\neq r_{i_j}\\ =\tau_N(t_{j-1})+1}}^{\tau_N(t_j)}\prod_{i=i_{j-1}+1}^{i_j}c_N(r_i). \tag{60}$$

By a modification of the upper bound in Lemma 1,

$$\sum_{\substack{r_{i_{j-1}+1}\neq\cdots\neq r_{i_j}\\ =\tau_N(t_{j-1})+1}}^{\tau_N(t_j)}\prod_{i=i_{j-1}+1}^{i_j}c_N(r_i) \leq (t_j-t_{j-1})^{i_j-i_{j-1}}+c_N(\tau_N(t_j))(t_j-t_{j-1}+1)^{i_j-i_{j-1}}$$

$$\leq (t_j-t_{j-1})^{i_j-i_{j-1}}+c_N(\tau_N(t_j))(t_j-t_{j-1}+1)^{k}. \tag{61}$$

Now, taking the product on the outside,

$$\begin{aligned}
\prod_{j=1}^{k} \sum_{\substack{r_{i_{j-1}+1}<\cdots<r_{i_j} \\ =\tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) &\leq \prod_{j=1}^{k} \left\{ \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} + c_N(\tau_N(t_j)) \frac{(1 + t_j - t_{j-1})^k}{(i_j - i_{j-1})!} \right\} \\
&\leq \prod_{j=1}^{k} \left\{ \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} + c_N(\tau_N(t_j))(1 + t_j - t_{j-1})^k \right\} \\
&= \sum_{\mathcal{I} \subseteq [k]} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} c_N(\tau_N(t_j))(1 + t_j - t_{j-1})^k \right) \\
&= \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \\
&\quad + \sum_{\mathcal{I} \subset [k]} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} c_N(\tau_N(t_j))(1 + t_j - t_{j-1})^k \right) \\
&\leq \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \\
&\quad + \sum_{\mathcal{I} \subset [k]} c_N(\tau_N(t_{j^*})) \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) (1 + t_j - t_{j-1})^{k^2} \qquad (62)
\end{aligned}$$

where, say, $j^* := \min\{j \notin \mathcal{I}\}$. Now we are in a position to evaluate the limit in (58):

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_E \sum_{\substack{r_1<\cdots<r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i) \right] &\leq \lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i) \right] \\
\leq \sum_{\substack{i_1 \leq \cdots \leq i_{k-1} \\ \in \{0,\dots,k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \\
+ \sum_{\substack{i_1 \leq \cdots \leq i_{k-1} \\ \in \{0,\dots,k\}: \\ i_j \geq j \forall j}} \sum_{\mathcal{I} \subset [k]} \lim_{N\to\infty} \mathbb{E}\left[c_N(\tau_N(t_{j^*})) \right] \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) (1 + t_j - t_{j-1})^{k^2} \\
= \sum_{\substack{i_1 \leq \cdots \leq i_{k-1} \\ \in \{0,\dots,k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \qquad (63)
\end{aligned}$$

using Brown et al. (2021, Equation (3.3)).

For the corresponding lower bound, by a modification of the lower bound in Lemma 1,

$$\begin{aligned}
\sum_{\substack{r_{i_{j-1}+1}\neq\cdots\neq r_{i_j} \\ =\tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) &\geq (t_j - t_{j-1})^{i_j - i_{j-1}} - \binom{i_j - i_{j-1}}{2} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{i_j - i_{j-1} - 2} \\
&\geq (t_j - t_{j-1})^{i_j - i_{j-1}} - (i_j - i_{j-1})! \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{k-1}. \qquad (64)
\end{aligned}$$

Define the event

$$E_j^\star = \left\{ \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) \leq \frac{(t_j - t_{j-1})^{i_j - i_{j-1} - k+1}}{(i_j - i_{j-1})!} \right\}, \tag{65}$$

(If $t_j = t_{j-1}$ then $E_j^\star$ has probability 1 even without the limit; otherwise this is in the right form for Lemma 11.) which is sufficient to ensure the j^{th} term in the following product is non-negative, and define $E^\star := \bigcap_{j=1}^k E_j^\star$. Now, taking a product over j,

$$\begin{aligned}
\prod_{j=1}^k \sum_{\substack{r_{i_{j-1}+1} < \cdots < r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) &\geq \prod_{j=1}^k \left\{ \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} - \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{k-1} \right\} \mathbb{1}_{E^\star} \\
&= \sum_{\mathcal{I} \subseteq [k]} (-1)^{k - |\mathcal{I}|} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{k-1} \right) \mathbb{1}_{E^\star} \\
&= \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E^\star} \\
&\quad + \sum_{\mathcal{I} \subset [k]} (-1)^{k - |\mathcal{I}|} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{k-1} \right) \mathbb{1}_{E^\star} \\
&\geq \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E^\star} \\
&\quad - \sum_{\mathcal{I} \subset [k]} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{k-1} \right) \\
&\geq \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E^\star} \\
&\quad - \sum_{\mathcal{I} \subset [k]} \left(\sum_{s=\tau_N(t_{j^*-1})+1}^{\tau_N(t_{j^*})} c_N(s)^2 \right) \left(\prod_{j \in \mathcal{I}} (t_j - t_{j-1})^k \right) \left(\prod_{j \notin \mathcal{I}} (t_j - t_{j-1} + 1)^k \right) \\
&\geq \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E^\star} - \sum_{\mathcal{I} \subset [k]} \left(\sum_{s=\tau_N(t_{j^*-1})+1}^{\tau_N(t_{j^*})} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{k^2},
\end{aligned} \tag{66}$$

where again we have arbitrarily set $j^\star := \min\{j \notin \mathcal{I}\}$. We can now evaluate the limit:

$$
\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_E \sum_{\substack{r_1<\dots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^k c_N(r_i)\right] &\ge \lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_{E\cap E^\star} \sum_{\substack{i_1\le\dots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}\right] \\
&\quad - \lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_E \sum_{\substack{i_1\le\dots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \sum_{\mathcal{I}\subset[k]} \left(\sum_{s=\tau_N(t_{j^*-1})+1}^{\tau_N(t_{j^*})} c_N(s)^2\right)(t_j-t_{j-1}+1)^{k^2}\right] \\
&\ge \sum_{\substack{i_1\le\dots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} \lim_{N\to\infty} \mathbb{E}[\mathbb{1}_{E\cap E^\star}] \\
&\quad - \lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{i_1\le\dots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \sum_{\mathcal{I}\subset[k]} \left(\sum_{s=\tau_N(t_{j^*-1})+1}^{\tau_N(t_{j^*})} c_N(s)^2\right)(t_j-t_{j-1}+1)^{k^2}\right] \\
&= \sum_{\substack{i_1\le\dots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} \lim_{N\to\infty} \mathbb{P}[E\cap E^\star] \\
&\quad - \sum_{\substack{i_1\le\dots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \sum_{\mathcal{I}\subset[k]} \lim_{N\to\infty} \mathbb{E}\left[\sum_{s=\tau_N(t_{j^*-1})+1}^{\tau_N(t_{j^*})} c_N(s)^2\right](t_j-t_{j-1}+1)^{k^2} \\
&= \sum_{\substack{i_1\le\dots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} \qquad (67)
\end{aligned}
$$

where for the last equality we use Brown et al. (2021, Equation (3.5)) to show that the second sum vanishes and Lemma 11 to show that $\lim_{N\to\infty}\mathbb{P}[E\cap E^\star]=1$. We have shown that the upper and lower bounds coincide, so the result follows. ■

Indicators

Lemma 8. *Let A, B be events. If $\lim \mathbb{P}[A] = 1$ and $\lim \mathbb{P}[B] = 1$ then $\lim \mathbb{P}[A\cap B] = 1$.*

The above might be so obvious as to go unstated, but it is very important because it means we don't have to deal with intersections of dependent events!

Lemma 9. *Let $K > 0$ be a constant which may depend on n, N but not on r, such that $K^{-2} = O(1)$ as $N\to\infty$. Define the events $E_r := \{c_N(r) < K\}$ and denote $E := \bigcap_{r=1}^{\tau_N(t)} E_r$. Then $\lim_{N\to\infty}\mathbb{P}[E] = 1$.*

Proof.

$$\begin{aligned}\mathbb{P}[E] = 1 - \mathbb{P}[E^c] = 1 - \mathbb{P}\left[\bigcup_{r=1}^{\tau_N(t)} E_r^c\right] &= 1 - \mathbb{E}\left[\mathbb{1}_{\bigcup E_r^c}\right] \geq 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r^c}\right] \\ &= 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}\left[\mathbb{1}_{E_r^c} \mid \mathcal{F}_{r-1}\right]\right] = 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{P}\left[E_r^c \mid \mathcal{F}_{r-1}\right]\right] \end{aligned} \tag{68}$$

where for the second line we apply Lemma 13 with $f(r) = \mathbb{1}_{E_r^c}$. By the generalised Markov inequality,

$$\mathbb{P}[E_r^c \mid \mathcal{F}_{r-1}] = \mathbb{P}[c_N(r) \geq K \mid \mathcal{F}_{r-1}] \leq K^{-2}\mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}]. \tag{69}$$

Substituting this into (68) and applying Lemma 13 again, this time with $f(r) = c_N(r)^2$,

$$\mathbb{P}[E] \geq 1 - K^{-2}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}]\right] = 1 - K^{-2}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} c_N(r)^2\right]. \tag{70}$$

Applying Brown et al. (2021, Equation (3.5)), the limit is

$$\lim_{N\to\infty} \mathbb{P}[E] = 1 - O(1) \times 0 = 1 \tag{71}$$

as required. ∎

> **Lemma 10.** *Fix $t > 0$. For any $l \in \mathbb{N}$, $\lim_{N\to\infty} \mathbb{P}[\tau_N(t) \geq l] = 1$.*

Proof. We can replace the event $\{\tau_N(t) \geq l\}$ with an event of the form of E in Lemma 9:

$$\{\tau_N(t) \geq l\} = \left\{\min\left\{s \geq 1 : \sum_{r=1}^{s} c_N(r) \geq t\right\} \geq l\right\} = \left\{\sum_{r=1}^{l-1} c_N(r) < t\right\} \supseteq \bigcap_{r=1}^{l-1} \left\{c_N(r) < \frac{t}{l}\right\} \supseteq \bigcap_{r=1}^{\tau_N(t)} \left\{c_N(r) < \frac{t}{l}\right\}. \tag{72}$$

Hence

$$\lim_{N\to\infty} \mathbb{P}[\tau_N(t) \geq l] \geq \lim_{N\to\infty} \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \left\{c_N(r) < \frac{t}{l}\right\}\right] = 1 \tag{73}$$

by applying Lemma 9 with $K = t/l$. ∎

> **Lemma 11.** *Fix $k \in \mathbb{N}$, a sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_k \leq t$, and let $K_1, \ldots, K_k$ be strictly positive constants such that for each j, $K_j^{-1} = O(1)$ as $N \to \infty$. Define the event*
>
> $$E^\star := \bigcap_{j=1}^{k} \left\{\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \leq K_j\right\}. \tag{74}$$
>
> *Then $\lim_{N\to\infty} \mathbb{P}[E^\star] = 1$.*

Proof.

$$\mathbb{P}[E^\star] = 1 - \mathbb{P}[(E^\star)^c] = 1 - \mathbb{P}\left[\bigcup_{j=1}^{k} \left\{\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 > K_j\right\}\right] \geq 1 - \sum_{j=1}^{k} \mathbb{P}\left[\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \geq K_j\right]. \tag{75}$$

Applying Markov's inequality,

$$\mathbb{P}[E^\star] \geq 1 - \sum_{j=1}^{k} K_j^{-1}\mathbb{E}\left[\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2\right] \xrightarrow[N\to\infty]{} 1 - \sum_{j=1}^{k} O(1) \times 0 = 1 \tag{76}$$

by Brown et al. (2021, Equation (3.5)). ∎

Lemma 12. *Fix $t > 0$. Let K be a constant not depending on N, r, but which may depend on n.*

$$\lim_{N\to\infty} \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) \geq KD_N(r)\}\right] = 1. \tag{77}$$

Proof.

$$\begin{aligned}
\mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) \geq KD_N(r)\}\right] &\geq \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) > KD_N(r)\}\right] \\
&= 1 - \mathbb{P}\left[\bigcup_{r=1}^{\tau_N(t)} \{c_N(r) \leq KD_N(r)\}\right] \\
&= 1 - \mathbb{E}\left[\mathbb{1}_{\bigcup\{c_N(r)\leq KD_N(r)\}}\right] \\
&\geq 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{1}_{\{c_N(r)\leq KD_N(r)\}}\right] \\
&= 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{P}[c_N(r) \leq KD_N(r) \mid \mathcal{F}_{r-1}]\right]
\end{aligned} \tag{78}$$

where the final inequality is an application of Lemma 13 with $f(r) = \mathbb{1}_{\{c_N(r)\leq KD_N(r)\}}$.

Fix $0 < \varepsilon < K^{-1}/2$ and assume $N > \max\{\varepsilon^{-1}, (K^{-1} - 2\varepsilon)^{-1}\}$. For each r, i define the event $A_i(r) := \{\nu_r^{(i)} \leq N\varepsilon\}$. Conditional on $\mathcal{F}_{r-1}$, we have

$$D_N(r) = \frac{1}{N(N)_2}\sum_{i=1}^N (\nu_r^{(j)})_2 \left[\nu_r^{(i)} + \frac{1}{N}\sum_{j\neq i}(\nu_r^{(i)})^2\right]\mathbb{1}_{A_i^c(r)} + \frac{1}{N(N)_2}\sum_{i=1}^N (\nu_r^{(i)})_2 \left[\nu_r^{(i)} + \frac{1}{N}\sum_{j\neq i}(\nu_r^{(j)})^2\right]\mathbb{1}_{A_i(r)}. \tag{79}$$

For the first term,

$$\frac{1}{N(N)_2}\sum_{i=1}^N (\nu_r^{(i)})_2 \left[\nu_r^{(i)} + \frac{1}{N}\sum_{j\neq i}(\nu_r^{(j)})^2\right]\mathbb{1}_{A_i^c(r)} \leq \sum_{i=1}^N \mathbb{1}_{A_i^c(r)}. \tag{80}$$

For the second term,

$$\begin{aligned}
\frac{1}{N(N)_2}\sum_{i=1}^N (\nu_r^{(i)})_2 \left[\nu_r^{(i)} + \frac{1}{N}\sum_{j\neq i}(\nu_r^{(j)})^2\right]\mathbb{1}_{A_i(r)} &\leq \frac{1}{N(N)_2}\sum_{i=1}^N (\nu_r^{(i)})_2 \nu_r^{(i)}\mathbb{1}_{A_i(r)} + \frac{1}{N^2(N)_2}\sum_{i=1}^N (\nu_r^{(i)})_2 \sum_{j=1}^N (\nu_r^{(j)})^2 \mathbb{1}_{A_i(r)} \\
&\leq \frac{1}{N} c_N(r) N\varepsilon + \frac{1}{N^2(N)_2}\sum_{i=1}^N (\nu_r^{(i)})_2 \sum_{j=1}^N (\nu_r^{(j)})_2 \mathbb{1}_{A_i(r)} \\
&\quad + \frac{1}{N^2(N)_2}\sum_{i=1}^N (\nu_r^{(i)})_2 \sum_{j=1}^N (\nu_r^{(j)}) \mathbb{1}_{A_i(r)} \\
&\leq \varepsilon c_N(r) + \frac{1}{N^2}\sum_{i=1}^N \nu_r^{(i)} N\varepsilon c_N(r) + \frac{1}{N^2} c_N(r) N \\
&= c_N(r)\left(2\varepsilon + \frac{1}{N}\right).
\end{aligned} \tag{81}$$

Altogether we have

$$D_N(r) \leq c_N(r)\left(2\varepsilon + \frac{1}{N}\right) + \sum_{i=1}^N \mathbb{1}_{A_i^c(r)}. \tag{82}$$

Hence, still conditional on $\mathcal{F}_{r-1}$,

$$\begin{aligned}\{c_N(r) \le K D_N(r)\} &\subseteq \left\{ c_N(r) \le K c_N(r)(2\varepsilon + N^{-1}) + K \sum_{i=1}^{N} \mathbb{1}_{A_i^c(r)} \right\} \\ &= \left\{ K^{-1} - 2\varepsilon - \frac{1}{N} \le \sum_{i=1}^{N} \frac{\mathbb{1}_{A_i^c(r)}}{c_N(r)} \right\} \end{aligned} \tag{83}$$

where the ratio $\mathbb{1}_{A_i^c(r)}/c_N(r)$ is well-defined because

$$A_i^c(r) \Rightarrow c_N(r) := \frac{1}{(N)_2} \sum_{j=1}^{N} (\nu_r^{(j)})_2 \ge \frac{1}{(N)_2} (\nu_r^{(i)})_2 \ge \frac{\varepsilon(N\varepsilon - 1)}{N-1} \ge \varepsilon \left(\varepsilon - \frac{1}{N} \right) > 0. \tag{84}$$

Hence by Markov's inequality (the conditions on ε, N ensuring the constant is always strictly positive),

$$\begin{aligned}\mathbb{P}\left[c_N(r) \le K D_N(r) \mid \mathcal{F}_{r-1}\right] &\le \mathbb{P}\left[\sum_{i=1}^{N} \mathbb{1}_{A_i^c(r)} \ge \left(K^{-1} - 2\varepsilon - \frac{1}{N} \right) \varepsilon \left(\varepsilon - \frac{1}{N} \right) \middle| \mathcal{F}_{r-1} \right] \\ &\le \frac{1}{\left(K^{-1} - 2\varepsilon - \frac{1}{N}\right) \varepsilon \left(\varepsilon - \frac{1}{N}\right)} \mathbb{E}\left[\sum_{i=1}^{N} \mathbb{1}_{A_i^c(r)} \middle| \mathcal{F}_{r-1} \right] \\ &\le \frac{1}{\left(K^{-1} - 2\varepsilon - \frac{1}{N}\right) \varepsilon \left(\varepsilon - \frac{1}{N}\right)} \mathbb{E}\left[\sum_{i=1}^{N} \frac{(\nu_r^{(i)})_3}{(N\varepsilon)_3} \middle| \mathcal{F}_{r-1} \right] \\ &\le \frac{1}{\left(K^{-1} - 2\varepsilon - \frac{1}{N}\right) \varepsilon \left(\varepsilon - \frac{1}{N}\right)} \mathbb{E}\left[\frac{N(N)_2}{(N\varepsilon)_3} D_N(r) \middle| \mathcal{F}_{r-1} \right]. \end{aligned} \tag{85}$$

Applying Lemma 13 once more, with $f(r) = D_N(r)$,

$$\begin{aligned}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{P}[c_N(r) \le K D_N(r) \mid \mathcal{F}_{r-1}] \right] &\le \frac{1}{\left(K^{-1} - 2\varepsilon - \frac{1}{N}\right) \varepsilon \left(\varepsilon - \frac{1}{N}\right)} \frac{N(N)_2}{(N\varepsilon)_3} \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[D_N(r) \mid \mathcal{F}_{r-1}] \right] \\ &= \frac{1}{\left(K^{-1} - 2\varepsilon - \frac{1}{N}\right) \varepsilon \left(\varepsilon - \frac{1}{N}\right)} \frac{N(N)_2}{(N\varepsilon)_3} \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} D_N(r) \right] \\ &\xrightarrow[N\to\infty]{} \frac{1}{(K^{-1} - 2\varepsilon)\varepsilon^5} \times 0 = 0. \end{aligned} \tag{86}$$

Substituting this back into (78) concludes the proof. ■

Other useful results

The following Lemma is taken from Koskela et al. (2018, Lemma 2), where the function is set to $f(r) = c_N(r)$, but the authors remark that the result holds for other choices of function.

Lemma 13. *Fix $t > 0$. Let $(\mathcal{F}_r)$ be the backwards-in-time filtration generated by the offspring counts $\nu_r^{(1:N)}$ at each generation r, and let $f(r)$ be any deterministic function of $\nu_r^{(1:N)}$ that is non-negative and bounded. In particular, for all r there exists $B < \infty$ such that $0 \le f(r) \le B$. Then*

$$\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} f(r) \right] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}] \right]. \tag{87}$$

Proof. Define

$$M_s := \sum_{r=1}^{s} \left\{ f(r) - \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}] \right\}. \tag{88}$$

It is easy to establish that (M_s) is a martingale with respect to $(\mathcal{F}_s)$, and $M_0 = 0$. Now fix $K \geq 1$ and note that $\tau_N(t) \wedge K$ is a bounded $\mathcal{F}_t$-stopping time. Hence we can apply the optional stopping theorem:

$$\mathbb{E}[M_{\tau_N(t)\wedge K}] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} \{f(r) - \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\}\right] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} f(r)\right] - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right] = 0. \quad (89)$$

Since this holds for all $K \geq 1$,

$$\lim_{K\to\infty} \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} f(r)\right] = \lim_{K\to\infty} \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right]. \quad (90)$$

The monotone convergence theorem allows the limit to pass inside the expectation on each side (since increasing K can only increase each sum, by possibly adding non-negative terms). Hence

$$\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} f(r)\right] = \mathbb{E}\left[\lim_{K\to\infty} \sum_{r=1}^{\tau_N(t)\wedge K} f(r)\right] = \mathbb{E}\left[\lim_{K\to\infty} \sum_{r=1}^{\tau_N(t)\wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right] \quad (91)$$

which concludes the proof. ■

Dependency graph

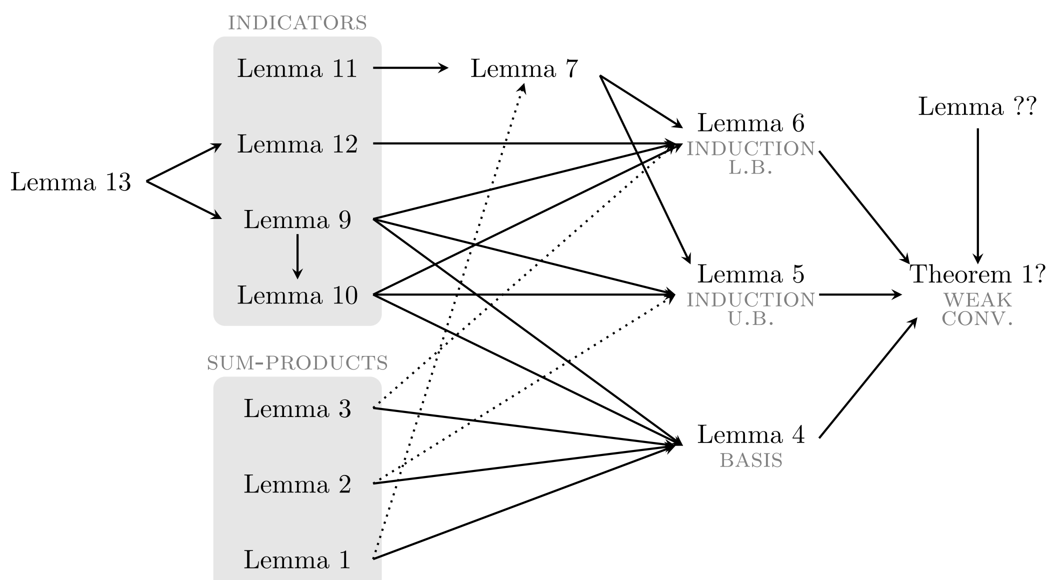

Figure 1: Graph showing dependencies between the lemmata used to prove weak convergence. Dotted arrows indicate dependence via a slight modification of the preceding lemma.

References

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