

Weak convergence proof (in progress)

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September 7, 2020

Theorem 1. Let $\nu_t^{(1:N)}$ denote the offspring numbers in an interacting particle system satisfying the standing assumption and such that, for any N sufficiently large, $\mathbb{P}\{\tau_N(t) = \infty\} = 0$ for all finite t . Suppose that there exists a deterministic sequence $(b_N)_{N \geq 1}$ such that $\lim_{N \rightarrow \infty} b_N = 0$ and

$$\frac{1}{(N)_3} \sum_{i=1}^N \mathbb{E}_t\{(\nu_t^{(i)})_3\} \leq b_N \frac{1}{(N)_2} \sum_{i=1}^N \mathbb{E}_t\{(\nu_t^{(i)})_2\} \quad (1)$$

for all N , uniformly in $t \geq 1$. Then the rescaled genealogical process $(G_{\tau_N(t)}^{(n,N)})_{t \geq 0}$ converges weakly to Kingman's n -coalescent as $N \rightarrow \infty$.

Proof. Define $p_t := \max_{\xi \in E} \{1 - p_{\xi\xi}(t)\} = 1 - p_{\Delta\Delta}(t)$, where Δ denotes the trivial partition of $\{1, \dots, n\}$ into singletons. For a proof that the maximum is attained at $\xi = \Delta$, see Lemma 1. Following Möhle (1999), we now construct the two-dimensional Markov process $(Z_t, S_t)_{t \in \mathbb{N}}$ with transition probabilities

$$\mathbb{P}(Z_t = j, S_t = \eta \mid Z_{t-1} = i, S_{t-1} = \xi) = \begin{cases} 1 - p_t & \text{if } j = i \text{ and } \xi = \eta \\ p_{\xi\xi}(t) + p_t - 1 & \text{if } j = i + 1 \text{ and } \xi = \eta \\ p_{\xi\eta}(t) & \text{if } j = i + 1 \text{ and } \xi \neq \eta \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

The construction is such that the marginal (S_t) has the same distribution as the genealogical process of interest, and (Z_t) has jumps at all the times (S_t) does plus some extra jumps. (The definition of p_t ensures that the probability in the second case is non-negative, attaining the value zero when $\xi = \Delta$.)

Denote by $0 = T_0^{(N)} < T_1^{(N)} < \dots$ the jump times of the rescaled process $(Z_{\tau_N(t)})_{t \geq 0}$, and $\omega_i^{(N)} := T_i^{(N)} - T_{i-1}^{(N)}$ the corresponding holding times ($i \in \mathbb{N}$).

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□

Lemma 1. $\max_{\xi \in E} (1 - p_{\xi\xi}(t)) = 1 - p_{\Delta\Delta}(t)$.

Proof. Consider any $\xi \in E$ consisting of k blocks ($1 \leq k \leq n - 1$), and any $\xi' \in E$ consisting of $k + 1$ blocks. From the definition of $p_{\xi\xi}(t)$ (Koskela et al., 2018, Equation (1)),

$$p_{\xi\xi}(t) = \frac{1}{(N)_k} \sum_{\substack{i_1, \dots, i_k \\ \text{all distinct}}} \nu_t^{(i_1)} \dots \nu_t^{(i_k)}. \quad (3)$$

Similarly,

$$\begin{aligned} p_{\xi'\xi'}(t) &= \frac{1}{(N)_{k+1}} \sum_{\substack{i_1, \dots, i_{k+1} \\ \text{all distinct}}} \nu_t^{(i_1)} \dots \nu_t^{(i_k)} \nu_t^{(i_{k+1})} \\ &= \frac{1}{(N)_k(N-k)} \sum_{\substack{i_1, \dots, i_k \\ \text{all distinct}}} \left\{ \nu_t^{(i_1)} \dots \nu_t^{(i_k)} \sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^N \nu_t^{(i_{k+1})} \right\}. \end{aligned}$$

Discarding some zero summands,

$$p_{\xi'\xi'}(t) = \frac{1}{(N)_k(N-k)} \sum_{\substack{i_1, \dots, i_k \\ \text{all distinct:} \\ \nu_t^{(i_1)}, \dots, \nu_t^{(i_k)} > 0}} \left\{ \nu_t^{(i_1)} \dots \nu_t^{(i_k)} \sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^N \nu_t^{(i_{k+1})} \right\}. \quad (4)$$

The inner sum is

$$\sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^N \nu_t^{(i_{k+1})} = \left\{ \sum_{i=1}^N \nu_t^{(i)} - \sum_{i \in \{i_1, \dots, i_k\}} \nu_t^{(i)} \right\} \leq N - k \quad (5)$$

since $\nu_t^{(i_1)}, \dots, \nu_t^{(i_k)}$ are all at least 1. Hence

$$p_{\xi'\xi'}(t) \leq \frac{N-k}{(N)_k(N-k)} \sum_{\substack{i_1, \dots, i_k \\ \text{all distinct:} \\ \nu_t^{(i_1)}, \dots, \nu_t^{(i_k)} > 0}} \nu_t^{(i_1)} \dots \nu_t^{(i_k)} = p_{\xi\xi}(t). \quad (6)$$

Thus $p_{\xi\xi}(t)$ is decreasing in the number of blocks of ξ , and is therefore minimised by taking $\xi = \Delta$, which achieves the maximum n blocks. This choice in turn maximises $1 - p_{\xi\xi}(t)$, as required. \square

References

- Koskela, J., Jenkins, P. A., Johansen, A. M. and Spanò, D. (2018), ‘Asymptotic genealogies of interacting particle systems with an application to sequential Monte Carlo’, *arXiv preprint arXiv:1804.01811*.
- Möhle, M. (1999), ‘Weak convergence to the coalescent in neutral population models’, *Journal of Applied Probability* **36**(2), 446–460.