Lemma 1.

$$1 - C_{|\xi|}\{1 + O(N^{-1})\}D_N(t) - \binom{|\xi|}{2}\{1 + O(N^{-1})\}c_N(t) \leq p_{\xi\xi}(t),$$

for a constant $C_{|\xi|} > 0$ that does not depend on N.

Proof. Let $\kappa_i := \#\{j : b_j = i\}$ denote the multiplicity of mergers of size i, with the slight abuse of terminology in that κ_1 counts non-merger events. In particular, we have that $\kappa_1 + 2\kappa_2 + \ldots |\xi|\kappa_{|\xi|} = |\xi|$. Now

$$p_{\xi\xi}(t) = 1 - \frac{1}{(N)_{|\xi|}} \sum_{k=1}^{|\xi|-1} \sum_{\substack{b_1 \geq \ldots \geq b_k = 1 \\ b_1 + \ldots + b_k = |\xi|}}^{|\xi|} \frac{|\xi|!}{\prod_{j=1}^{|\xi|}(j!)^{\kappa_j}\kappa_j!} \sum_{\substack{i_1 \neq \ldots \neq i_k = 1 \\ \text{all distinct}}}^{N} (\nu_t^{(i_1)})_{b_1} \ldots (\nu_t^{(i_k)})_{b_k},$$

because the right hand side subtracts the probabilities of all possible merger events. See (**?**, eq (11)) for the combinatorial factor. The omitted $k = |\xi|$ summand would correspond to the probability of an identity transition. The non-increasing ordering of $(b_1, \ldots, b_k)$ is arbitrary, but without loss of generality: choosing any ordering of the same set of merger sizes would give the same result.

Firstly, we separate out the $k = |\xi| - 1$ term, which covers isolated binary mergers, and note that in that case the only possible b-vector is $(2, 1, \ldots, 1)$, for which

$$\frac{|\xi|!}{\prod_{j=1}^{|\xi|}(j!)^{\kappa_j}\kappa_j!} = \frac{|\xi|!}{2!(|\xi| - 2)!} = \binom{|\xi|}{2},$$

and

$$\begin{aligned}
&\sum_{i_1 \neq \ldots \neq i_{|\xi|-1} = 1}^{N} (\nu_t^{(i_1)})_2 \nu_t^{(i_2)} \ldots \nu_t^{(i_{|\xi|-1})} \\
&\leq \sum_{i=1}^{N} (\nu_t^{(i)})_2 \left[(N - \nu_t^{(i)})^{|\xi|-2} - \binom{|\xi| - 2}{2} \sum_{j \neq i}^{N} (\nu_t^{(j)})^2 (N - \nu_t^{(i)})^{|\xi|-4} \right] \\
&\leq N^{|\xi|-2} \sum_{i=1}^{N} (\nu_t^{(i)})_2,
\end{aligned}$$

Thus

$$\begin{aligned}
p_{\xi\xi}(t) \geq 1 &- \binom{|\xi|}{2} \frac{1 + O(N^{-1})}{(N)_2} \sum_{i=1}^{N} (\nu_t^{(i)})_2 \\
&- \frac{1}{(N)_{|\xi|}} \sum_{k=1}^{|\xi|-2} \sum_{\substack{b_1 \geq \ldots \geq b_k = 1 \\ b_1 + \ldots + b_k = |\xi|}}^{|\xi|} \frac{|\xi|!}{\prod_{j=1}^{|\xi|}(j!)^{\kappa_j}\kappa_j!} \sum_{\substack{i_1 \neq \ldots \neq i_k = 1 \\ \text{all distinct}}}^{N} (\nu_t^{(i_1)})_{b_1} \ldots (\nu_t^{(i_k)})_{b_k}.
\end{aligned}$$

For the other summands, we have

$$\frac{|\xi|!}{\prod_{j=1}^{|\xi|}(j!)^{\kappa_j}\kappa_j!} \leq |\xi|!$$

and (similarly to Lemma 1, Case 3 in our paper),

$$\sum_{\substack{i_1 \neq \ldots \neq i_k = 1 \\ \text{all distinct}}}^{N} (\nu_t^{(i_1)})_{b_1} \ldots (\nu_t^{(i_k)})_{b_k} \leq \sum_{i=1}^{N} (\nu_t^{(i)})_2 \left\{ N^{|\xi|-2} - \sum_{\substack{j_1 \neq \ldots \neq j_{|\xi|-2} = 1 \\ \text{all distinct and } \neq i}}^{N} \nu_t^{(j_1)} \ldots \nu_t^{(j_{|\xi|-2})} \right\}$$

$$= \sum_{i=1}^{N} (\nu_t^{(i)})_2 \left\{ N^{|\xi|-2} - (N - \nu_t^{(i)})^{|\xi|-2} + \binom{|\xi|-2}{2} \sum_{j \neq i} (\nu_t^{(j)})^2 \left(\sum_{k \neq i} \nu_t^{(k)} \right)^{|\xi|-4} \right\}$$

$$\leq \sum_{i=1}^{N} (\nu_t^{(i)})_2 \left\{ (|\xi|-2) \nu_t^{(i)} N^{|\xi|-3} + \binom{|\xi|-2}{2} \sum_{j \neq i} (\nu_t^{(j)})^2 N^{|\xi|-4} \right\},$$

where the last step uses $(N-x)^b \geq N^b - bxN^{b-1}$. Overall

$$p_{\xi\xi}(t) \geq 1 - \binom{|\xi|}{2} \frac{1 + O(N^{-1})}{(N)_2} \sum_{i=1}^{N} (\nu_t^{(i)})_2$$

$$- \frac{N^{|\xi|-3}}{(N)_{|\xi|}} \sum_{k=1}^{|\xi|-2} \sum_{\substack{b_1 \geq \ldots \geq b_k = 1 \\ b_1 + \ldots + b_k = |\xi|}}^{|\xi|} |\xi|! \sum_{i=1}^{N} (\nu_t^{(i)})_2 \left\{ (|\xi|-2) \nu_t^{(i)} N^{|\xi|-3} + \binom{|\xi|-2}{2} \sum_{j \neq i} (\nu_t^{(j)})^2 N^{|\xi|-4} \right\}.$$

The summand in the third term depends neither on k nor on $b_1, \ldots, b_k$, and the number of terms in those sums is bounded above by $(|\xi|-2)\gamma_{|\xi|-2}$, where γ_n is the number of integer partitions of n. By (**?**, Section 2), $\gamma_n < Ke^{2\sqrt{2n}}/n$ for a constant $K > 0$ independent of n. Thus

$$p_{\xi\xi}(t) \geq 1 - \binom{|\xi|}{2} \frac{1 + O(N^{-1})}{(N)_2} \sum_{i=1}^{N} (\nu_t^{(i)})_2$$

$$- Ke^{2\sqrt{2(|\xi|-2)}} |\xi|! \binom{|\xi|-2}{2} \frac{N^{|\xi|-3}}{(N)_{|\xi|}} \sum_{i=1}^{N} (\nu_t^{(i)})_2 \left\{ \nu_t^{(i)} + \frac{1}{N} \sum_{j \neq i} (\nu_t^{(j)})^2 \right\}$$

$$= 1 - \binom{|\xi|}{2} \{1 + O(N^{-1})\} c_N(t) - C_{|\xi|} \{1 + O(N^{-1})\} D_N(t)),$$

where $C_{|\xi|} > 0$ depends on $|\xi|$ but not on N. □

In order to use Lemma 1 to remove assumption (6) from (**?**, Theorem 1), it is necessary to rewrite the argument for the lower bound. The upper bound does not use assumption (6). We do this below.

Proof of Theorem 1 without Assumption (6).

$$\chi_d \geq \sum_{s_1 < \ldots < s_\alpha = \tau_N(t_{d-1})+1}^{\tau_N(t_d)\,\star} (\tilde{Q}^\alpha)_{\eta_{d-1}\eta_d} \left\{ \prod_{r=1}^{\alpha} \left(c_N(s_r) - \binom{n-2}{2} \{1 + O(N^{-1})\} D_N(s_r) \right) \right\}$$

$$\times \prod_{\substack{r = \tau_N(t_{d-1})+1 \\ r \neq s_1, \ldots, r \neq s_\alpha}}^{\tau_N(t_d)} \left\{ 1 - C_n \{1 + O(N^{-1})\} D_N(r) \right.$$

$$\left. - \binom{|\eta_{d-1}| - |\{i : s_i < r\}|}{2} \{1 + O(N^{-1})\} c_N(r) \right\}.$$

A multinomial expansion of the product on the last line yields

$$\chi_d \geq \sum_{\beta=0}^{\tau_N(t_d)-\tau_N(t_{d-1})-\alpha} (\tilde{Q}^\alpha)_{\eta_{d-1}\eta_d} \sum_{(\lambda,\mu)\in\Pi_2([\alpha+\beta]):|\lambda|=\alpha} \{1+O(N^{-1})\}^\beta$$
$$\times \sideset{}{^\star}\sum_{s_1<\ldots<s_{\alpha+\beta}=\tau_N(t_{d-1})+1}^{\tau_N(t_d)} \left\{ \prod_{r\in\lambda} \left[c_N(s_r) - \binom{n-2}{2}\{1+O(N^{-1})\}D_N(s_r) \right] \right\}$$
$$\times \prod_{r\in\mu} \left\{ -\binom{|\eta_{d-1}| - |\{i\in\lambda : i<r\}|}{2} c_N(s_r) - C_n D_N(s_r) \right\}.$$

Expanding the product over λ gives

$$\chi_d \geq \sum_{\beta=0}^{\tau_N(t_d)-\tau_N(t_{d-1})-\alpha} (\tilde{Q}^\alpha)_{\eta_{d-1}\eta_d} \sum_{(\lambda,\mu,\pi)\in\Pi_3([\alpha+\beta]):|\mu|=\beta} \binom{n-2}{2}^{|\pi|} (-1)^{|\pi|} \{1+O(N^{-1})\}^{\beta+|\pi|}$$
$$\times \sideset{}{^\star}\sum_{s_1<\ldots<s_{\alpha+\beta}=\tau_N(t_{d-1})+1}^{\tau_N(t_d)} \left\{ \prod_{r\in\lambda} c_N(s_r) \right\} \left\{ \prod_{r\in\pi} D_N(s_r) \right\}$$
$$\times \prod_{r\in\mu} \left\{ -\binom{|\eta_{d-1}| - |\{i\in\lambda\cup\pi : i<r\}|}{2} c_N(s_r) - C_n D_N(s_r) \right\},$$

and expanding the product over μ results in

$$\chi_d \geq \sum_{\beta=0}^{\tau_N(t_d)-\tau_N(t_{d-1})-\alpha} (\tilde{Q}^\alpha)_{\eta_{d-1}\eta_d} \sum_{(\lambda,\mu,\pi,\sigma)\in\Pi_4([\alpha+\beta]):|\mu|+|\sigma|=\beta} C_n^{|\sigma|} \binom{n-2}{2}^{|\pi|} (-1)^{|\pi|+|\sigma|}$$
$$\times \{1+O(N^{-1})\}^{\beta+|\pi|} \left\{ \prod_{r\in\mu} - \binom{|\eta_{d-1}| - |\{i\in\lambda\cup\pi : i<r\}|}{2} \right\}$$
$$\times \sideset{}{^\star}\sum_{s_1<\ldots<s_{\alpha+\beta}=\tau_N(t_{d-1})+1}^{\tau_N(t_d)} \left\{ \prod_{r\in\lambda\cup\mu} c_N(s_r) \right\} \prod_{r\in\pi\cup\sigma} D_N(s_r).$$

Via a further multinomial expansion, the lower bound for the k-step transition probability can be written as

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{d=1}^k \chi_d \right] \geq \lim_{N\to\infty} \mathbb{E}\left[\sum_{\beta_1=0}^\infty \cdots \sum_{\beta_k=0}^\infty \sum_{(\lambda_1,\mu_1,\pi_1,\sigma_1)\in\Pi_4([\alpha_1+\beta_1]):|\mu_1|+|\sigma_1|=\beta_1} \cdots \right.$$
$$\sum_{(\lambda_k,\mu_k,\pi_k,\sigma_k)\in\Pi_4([\alpha_k+\beta_k]):|\mu_k|+|\sigma_k|=\beta_k} C_n^{\sum_{d=1}^k |\sigma_d|} \binom{n-2}{2}^{\sum_{d=1}^k |\pi_d|}$$
$$\times (-1)^{\sum_{d=1}^k |\pi_d|+|\sigma_d|} \{1+O(N^{-1})\}^{|\boldsymbol{\beta}|+\sum_{d=1}^k |\pi_d|}$$
$$\times \left\{ \prod_{d=1}^k (\tilde{Q}^{\alpha_d})_{\eta_{d-1}\eta_d} \prod_{r\in\mu_d} - \binom{|\eta_{d-1}| - |\{i\in\lambda_d\cup\pi_d : i<r\}|}{2} \right\}$$
$$\times \sideset{}{^\star}\sum_{s_1^{(1)}<\ldots<s_{\alpha_1+\beta_1}^{(1)}=\tau_N(t_0)+1}^{\tau_N(t_1)} \cdots \sideset{}{^\star}\sum_{s_1^{(k)}<\ldots<s_{\alpha_k+\beta_k}^{(k)}=\tau_N(t_{k-1})+1}^{\tau_N(t_k)}$$
$$\left. \prod_{d=1}^k \mathbb{1}_{\{\tau_N(t_d)-\tau_N(t_{d-1})\geq\alpha_d+\beta_d\}} \left\{ \prod_{r\in\lambda_d\cup\mu_d} c_N(s_r^{(d)}) \right\} \prod_{r\in\pi_d\cup\sigma_d} D_N(s_r^{(d)}) \right].$$

An argument completely analogous to that in (**?**, Appendix) shows that passing the expectation and the limit through the infinite sums is justified, whereupon the contribution of terms with $\sum_{d=1}^{k} |\pi_d|+|\sigma_d| > 0$ vanishes. To see why, follow the argument used to show that the contribution of multiple merger trajectories vanishes in the corresponding upper bound in **?**. That leaves

$$\begin{aligned}\lim_{N\to\infty} \mathbb{E}\left[\prod_{d=1}^{k} \chi_d\right] \geq \sum_{\beta_1=0}^{\infty} \cdots \sum_{\beta_k=0}^{\infty} \sum_{(\lambda_1,\mu_1)\in\Pi_2([\alpha_1+\beta_1]):|\mu_1|=\beta_1} \cdots \sum_{(\lambda_k,\mu_k)\in\Pi_2([\alpha_k+\beta_k]):|\mu_k|=\beta_k} \\ \left\{\prod_{d=1}^{k} (\tilde{Q}^{\alpha_d})_{\eta_{d-1}\eta_d} \prod_{r\in\mu_d} -\binom{|\eta_{d-1}| - |\{i \in \lambda_d \cup \pi_d : i < r\}|}{2}\right\} \\ \times \lim_{N\to\infty} \mathbb{E}\Bigg[\sum_{s_1^{(1)}<\ldots<s_{\alpha_1+\beta_1}^{(1)}=\tau_N(t_0)+1}^{\tau_N(t_1)\ \star} \cdots \sum_{s_1^{(k)}<\ldots<s_{\alpha_k+\beta_k}^{(k)}=\tau_N(t_{k-1})+1}^{\tau_N(t_k)\ \star} \\ \prod_{d=1}^{k} \mathbb{1}_{\{\tau_N(t_d)-\tau_N(t_{d-1})\geq\alpha_d+\beta_d\}} \left\{\prod_{r\in\lambda_d\cup\mu_d} c_N(s_r^{(d)})\right\}\Bigg]. \end{aligned} \tag{1}$$

Recall (**?**, Eq (11)):

$$\sum_{(\lambda,\mu)\in\Pi_2([\alpha+\beta]):|\mu|=\beta} (\tilde{Q}^{\alpha})_{\eta_{d-1}\eta_d} \prod_{r\in\mu} -\binom{|\eta_{d-1}| - |\{i \in \lambda \cup \pi : i < r\}|}{2} = (Q^{\alpha+\beta})_{\eta_{d-1}\eta_d},$$

and note that applying this k times in (1) yields

$$\begin{aligned}\lim_{N\to\infty} \mathbb{E}\left[\prod_{d=1}^{k} \chi_d\right] \geq \sum_{\beta_1=0}^{\infty} \cdots \sum_{\beta_k=0}^{\infty} \left\{\prod_{d=1}^{k} (Q^{\alpha_d+\beta_d})_{\eta_{d-1}\eta_d}\right\} \\ \times \lim_{N\to\infty} \mathbb{E}\Bigg[\left\{\prod_{d=1}^{k} \mathbb{1}_{\{\tau_N(t_d)-\tau_N(t_{d-1})\geq\alpha_d+\beta_d\}}\right\} \sum_{s_1^{(1)}<\ldots<s_{\alpha_1+\beta_1}^{(1)}=\tau_N(t_0)+1}^{\tau_N(t_1)\ \star} \\ \cdots \sum_{s_1^{(k)}<\ldots<s_{\alpha_k+\beta_k}^{(k)}=\tau_N(t_{k-1})+1}^{\tau_N(t_k)\ \star} \prod_{d=1}^{k} \prod_{r\in\lambda_d\cup\mu_d} c_N(s_r^{(d)})\Bigg]. \end{aligned}$$

We now apply (**?**, Eq (14)) to those terms with a negative sign ($|\boldsymbol{\beta}|$ odd) and (**?**, Eq (15)) to those terms with a positive sign ($|\boldsymbol{\beta}|$ even), and obtain

$$\begin{aligned}\lim_{N\to\infty} \mathbb{E}\left[\prod_{d=1}^{k} \chi_d\right] \geq \sum_{\beta_1=0}^{\infty} \cdots \sum_{\beta_k=0}^{\infty} \left\{\prod_{d=1}^{k} (Q^{\alpha_d+\beta_d})_{\eta_{d-1}\eta_d} \frac{(t_d - t_{d-1})^{\alpha_d+\beta_d}}{(\alpha_d+\beta_d)!}\right\} \\ \times \lim_{N\to\infty} \mathbb{E}\left[\prod_{d=1}^{k} \mathbb{1}_{\{\tau_N(t_d)-\tau_N(t_{d-1})\geq\alpha_d+\beta_d\}}\right]. \end{aligned}$$

An invocation of (**?**, Eq (16)) concludes the proof. □