

Weak convergence proof (in progress)

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Theorem 1. Let $\nu_t^{(1:N)}$ denote the offspring numbers in an interacting particle system satisfying the standing assumption and such that, for any N sufficiently large, for all finite t , $\mathbb{P}\{\tau_N(t) = \infty\} = 0$. Suppose that there exists a deterministic sequence $(b_N)_{N \geq 1}$ such that $\lim_{N \rightarrow \infty} b_N = 0$ and

$$\frac{1}{(N)_3} \sum_{i=1}^N \mathbb{E}_t\{(\nu_t^{(i)})_3\} \leq b_N \frac{1}{(N)_2} \sum_{i=1}^N \mathbb{E}_t\{(\nu_t^{(i)})_2\} \quad (1)$$

for all N , uniformly in $t \geq 1$. Then the rescaled genealogical process $(G_{\tau_N(t)}^{(n,N)})_{t \geq 0}$ converges weakly to Kingman's n -coalescent as $N \rightarrow \infty$.

Proof. Define $p_t := \max_{\xi \in E} \{1 - p_{\xi\xi}(t)\} = 1 - p_{\Delta\Delta}(t)$, where Δ denotes the trivial partition of $\{1, \dots, n\}$ into singletons. For a proof that the maximum is attained at $\xi = \Delta$, see Lemma 1. Following Möhle (1999), we now construct the two-dimensional Markov process $(Z_t, S_t)_{t \in \mathbb{N}}$ with transition probabilities

$$\mathbb{P}(Z_t = j, S_t = \eta \mid Z_{t-1} = i, S_{t-1} = \xi) = \begin{cases} 1 - p_t & \text{if } j = i \text{ and } \xi = \eta \\ p_{\xi\xi}(t) + p_t - 1 & \text{if } j = i + 1 \text{ and } \xi = \eta \\ p_{\xi\eta}(t) & \text{if } j = i + 1 \text{ and } \xi \neq \eta \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

The construction is such that the marginal (S_t) has the same distribution as the genealogical process of interest, and (Z_t) has jumps at all the times (S_t) does plus some extra jumps. (The definition of p_t ensures that the probability in the second case is non-negative, attaining the value zero when $\xi = \Delta$.)

Denote by $0 = T_0^{(N)} < T_1^{(N)} < \dots$ the jump times of the rescaled process $(Z_{\tau_N(t)})_{t \geq 0}$, and $\omega_i^{(N)} := T_i^{(N)} - T_{i-1}^{(N)}$ the corresponding holding times ($i \in \mathbb{N}$).

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□

Lemma 1. $\max_{\xi \in E} (1 - p_{\xi\xi}(t)) = 1 - p_{\Delta\Delta}(t)$.

Proof. Consider any $\xi \in E$ consisting of k blocks ($1 \leq k \leq n - 1$), and any $\xi' \in E$ consisting of $k + 1$ blocks. From the definition of $p_{\xi\eta}(t)$ (Koskela et al., 2018, Equation (1)),

$$p_{\xi\xi}(t) = \frac{1}{(N)_k} \sum_{\substack{i_1, \dots, i_k \\ \text{all distinct}}} \nu_t^{(i_1)} \dots \nu_t^{(i_k)}. \quad (3)$$

Similarly,

$$\begin{aligned} p_{\xi'\xi'}(t) &= \frac{1}{(N)_{k+1}} \sum_{\substack{i_1, \dots, i_{k+1} \\ \text{all distinct}}} \nu_t^{(i_1)} \dots \nu_t^{(i_k)} \nu_t^{(i_{k+1})} \\ &= \frac{1}{(N)_k(N-k)} \sum_{\substack{i_1, \dots, i_k \\ \text{all distinct}}} \left\{ \nu_t^{(i_1)} \dots \nu_t^{(i_k)} \sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^N \nu_t^{(i_{k+1})} \right\}. \end{aligned} \quad (4)$$

Discarding the zero summands,

$$p_{\xi'\xi'}(t) = \frac{1}{(N)_k(N-k)} \sum_{\substack{i_1, \dots, i_k \\ \text{all distinct:} \\ \nu_t^{(i_1)}, \dots, \nu_t^{(i_k)} > 0}} \left\{ \nu_t^{(i_1)} \dots \nu_t^{(i_k)} \sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^N \nu_t^{(i_{k+1})} \right\}. \quad (5)$$

The inner sum is

$$\sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^N \nu_t^{(i_{k+1})} = \left\{ \sum_{i=1}^N \nu_t^{(i)} - \sum_{i \in \{i_1, \dots, i_k\}} \nu_t^{(i)} \right\} \leq N - k \quad (6)$$

since $\nu_t^{(i_1)}, \dots, \nu_t^{(i_k)}$ are all at least 1. Hence

$$p_{\xi'\xi'}(t) \leq \frac{N-k}{(N)_k(N-k)} \sum_{\substack{i_1, \dots, i_k \\ \text{all distinct:} \\ \nu_t^{(i_1)}, \dots, \nu_t^{(i_k)} > 0}} \nu_t^{(i_1)} \dots \nu_t^{(i_k)} = p_{\xi\xi}(t). \quad (7)$$

Thus $p_{\xi\xi}(t)$ is decreasing in the number of blocks of ξ , and is therefore minimised by taking $\xi = \Delta$, which achieves the maximum n blocks. This choice in turn maximises $1 - p_{\xi\xi}(t)$, as required. \square

Lemma 2. For any $0 < t < \infty$,

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\prod_{r=1}^{\tau_N(t)} (1 - p_r) \right] = e^{-\alpha_n t} \quad (8)$$

where $\alpha_n := n(n-1)/2$.

Proof. The strategy is to find upper and lower bounds on $\mathbb{E} \left[\prod_{r=1}^{\tau_N(t)} (1 - p_r) \right]$, both of which converge to $e^{-\alpha_n t}$.

Lower Bound

From Brown et al. (2020, Equation (14)), taking $\xi = \Delta$, we have

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \alpha_n(1 + O(N^{-1})) [c_N(t) + B_n D_N(t)] \quad (9)$$

where $B_n > 0$ and the $O(N^{-1})$ term does not depend on t . In particular,

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \frac{N^{n-2}}{(N-2)_{n-2}} \alpha_n c_N(t) - \frac{N^{n-3}}{(N-3)_{n-3}} B_n D_N(t). \quad (10)$$

Since $D_N(t) \leq c_N(t)$, a sufficient condition for the bound to be positive is

$$E_t := \left\{ c_N(t) < \frac{(N-3)_{n-3}}{N^{n-3}} \left(\alpha_n \left(1 + \frac{2}{N-2} \right) + B_n \right)^{-1} \right\}. \quad (11)$$

Hence, by a multinomial expansion,

$$\begin{aligned} \prod_{r=1}^{\tau_N(t)} (1 - p_r) &\geq \prod_{r=1}^{\tau_N(t)} \{1 - \alpha_n(1 + O(N^{-1})) [c_N(r) + B_n D_N(r)]\} \times \prod_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r} \\ &= \left(1 + \sum_{k=1}^{\tau_N(t)} \sum_{\substack{r_1 < \dots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^k \{-\alpha_n(1 + O(N^{-1})) [c_N(r_j) + B_n D_N(r_j)]\} \right) \times \prod_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r} \\ &= \prod_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r} + \sum_{k=1}^{\tau_N(t)} \{-\alpha_n(1 + O(N^{-1}))\}^k \left(\prod_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r} \right) \sum_{\substack{r_1 < \dots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^k \{c_N(r_j) + B_n D_N(r_j)\}. \end{aligned} \quad (12)$$

Taking expectations,

$$\begin{aligned}
\mathbb{E} \left[\prod_{r=1}^{\tau_N(t)} (1 - p_r) \right] &\geq \mathbb{E} \left[\prod_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r} \right] \\
&+ \mathbb{E} \left[\sum_{k=1}^{\infty} \{ -\alpha_n(1 + O(N^{-1})) \}^k \mathbb{1}_{\{k \leq \tau_N(t)\}} \mathbb{1}_{\cap_{r=1}^{\tau_N(t)} E_r} \sum_{r_1 < \dots < r_k}^{\tau_N(t)} \prod_{j=1}^k \{ c_N(r_j) + B_n D_N(r_j) \} \right] \\
&= \mathbb{P} \left[\bigcap_{r=1}^{\tau_N(t)} E_r \right] \\
&+ \sum_{k=1}^{\infty} \{ -\alpha_n(1 + O(N^{-1})) \}^k \mathbb{E} \left[\sum_{r_1 < \dots < r_k}^{\tau_N(t)} \prod_{j=1}^k \{ c_N(r_j) + B_n D_N(r_j) \} \middle| k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \\
&\times \mathbb{P} \left[k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right]. \tag{13}
\end{aligned}$$

We want to show that the conditional expectation on the right converges to $t^k/k!$, for reasons that will become clear. The strategy is to upper and lower bound this expectation by quantities that converge to $t^k/k!$.

First the lower bound. Assume that $k \leq \tau_N(t)$, ensuring that the sum is non-empty. From Koskela et al. (2018, Equation (8)),

$$\begin{aligned}
\sum_{r_1 < \dots < r_k}^{\tau_N(t)} \prod_{j=1}^k \{ c_N(r_j) + B_n D_N(r_j) \} &\geq \sum_{r_1 < \dots < r_k}^{\tau_N(t)} \prod_{j=1}^k c_N(r_j) \\
&\geq \frac{1}{k!} \left(\sum_{s=1}^{\tau_N(t)} c_N(s) \right)^k - \frac{1}{k!} \binom{k}{2} \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \left(\sum_{s=1}^{\tau_N(t)} c_N(s) \right)^{k-2} \\
&\geq \frac{1}{k!} t^k - \frac{1}{k!} \binom{k}{2} (t+1)^{k-2} \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \tag{14}
\end{aligned}$$

by the definition of τ_N . Then

$$\begin{aligned}
&\mathbb{E} \left[\sum_{r_1 < \dots < r_k}^{\tau_N(t)} \prod_{j=1}^k \{ c_N(r_j) + B_n D_N(r_j) \} \middle| k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \\
&\geq \frac{1}{k!} t^k - \frac{1}{k!} \binom{k}{2} (t+1)^{k-2} \mathbb{E} \left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \middle| k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \\
&= \frac{1}{k!} t^k - \frac{1}{k!} \binom{k}{2} (t+1)^{k-2} \mathbb{E} \left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \mathbb{1}_{\{k \leq \tau_N(t)\}} \mathbb{1}_{\{\cap_{r=1}^{\tau_N(t)} E_r\}} \right] \left(\mathbb{P} \left[k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \right)^{-1} \\
&\geq \frac{1}{k!} t^k - \frac{1}{k!} \binom{k}{2} (t+1)^{k-2} \mathbb{E} \left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right] \left(\mathbb{P} \left[k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \right)^{-1} \longrightarrow \frac{1}{k!} t^k \tag{15}
\end{aligned}$$

as $N \rightarrow \infty$ using Brown et al. (2020, Equation (5)) and Lemma 3.

Now for the upper bound. We start with a multinomial expansion and some manipulations of the sums:

$$\begin{aligned}
\sum_{\substack{r_1 < \dots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^k \{c_N(r_j) + B_n D_N(r_j)\} &= \frac{1}{k!} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \prod_{j=1}^k \{c_N(r_j) + B_n D_N(r_j)\} \\
&= \frac{1}{k!} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \sum_{\mathcal{I} \subseteq \{1, \dots, k\}} (B_n)^{k-|\mathcal{I}|} \left\{ \prod_{i \in \mathcal{I}} c_N(r_i) \right\} \left\{ \prod_{j \notin \mathcal{I}} D_N(r_j) \right\} \\
&= \frac{1}{k!} \sum_{\mathcal{I} \subseteq \{1, \dots, k\}} (B_n)^{k-|\mathcal{I}|} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i \in \mathcal{I}} c_N(r_i) \right\} \left\{ \prod_{j \notin \mathcal{I}} D_N(r_j) \right\} \\
&= \frac{1}{k!} \sum_{I=0}^k \binom{k}{I} (B_n)^{k-I} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^I c_N(r_i) \right\} \left\{ \prod_{j=I+1}^k D_N(r_j) \right\} \\
&= \frac{1}{k!} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^k c_N(r_i) \right\} \\
&\quad + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (B_n)^{k-I} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^I c_N(r_i) \right\} \left\{ \prod_{j=I+1}^k D_N(r_j) \right\}. \tag{16}
\end{aligned}$$

Then, using that $D_N(s) \leq c_N(s)$ for all s (Koskela et al., 2018, p.9), along with the definition of τ_N ,

$$\begin{aligned}
&\frac{1}{k!} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^k c_N(r_i) \right\} + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (B_n)^{k-I} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^I c_N(r_i) \right\} \left\{ \prod_{j=I+1}^k D_N(r_j) \right\} \\
&\leq \frac{1}{k!} \left(\sum_{r=1}^{\tau_N(t)} c_N(r) \right)^k + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (B_n)^{k-I} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^{k-1} c_N(r_i) \right\} D_N(r_k) \\
&\leq \frac{1}{k!} \{t + c_N(\tau_N(t))\}^k + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (B_n)^{k-I} \left\{ \sum_{\substack{r_1 \neq \dots \neq r_{k-1} \\ \text{all distinct}}}^{\tau_N(t)} \prod_{i=1}^{k-1} c_N(r_i) \right\} \left(\sum_{r_k=1}^{\tau_N(t)} D_N(r_k) \right) \\
&\leq \frac{1}{k!} \{t + c_N(\tau_N(t))\}^k + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (B_n)^{k-I} \left(\sum_{r=1}^{\tau_N(t)} c_N(r) \right)^{k-1} \left(\sum_{r=1}^{\tau_N(t)} D_N(r) \right) \\
&\leq \frac{1}{k!} \{t + c_N(\tau_N(t))\}^k + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (B_n)^{k-I} (t+1)^{k-1} \left(\sum_{r=1}^{\tau_N(t)} D_N(r) \right). \tag{17}
\end{aligned}$$

Taking expectations,

$$\begin{aligned}
& \mathbb{E} \left[\sum_{r_1 < \dots < r_k}^{\tau_N(t)} \prod_{j=1}^k \{c_N(r_j) + B_n D_N(r_j)\} \middle| k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \\
& \leq \frac{1}{k!} \mathbb{E} \left[\{t + c_N(\tau_N(t))\}^k \middle| k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \\
& \quad + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (B_n)^{k-I} (t+1)^{k-1} \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} D_N(r) \middle| k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \\
& = \frac{1}{k!} \mathbb{E} \left[\{t + c_N(\tau_N(t))\}^k \mathbb{1}_{\{k \leq \tau_N(t)\}} \mathbb{1}_{\{\bigcap E_r\}} \right] \left(\mathbb{P} \left[k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \right)^{-1} \\
& \quad + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (B_n)^{k-I} (t+1)^{k-1} \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} D_N(r) \mathbb{1}_{\{k \leq \tau_N(t)\}} \mathbb{1}_{\{\bigcap E_r\}} \right] \left(\mathbb{P} \left[k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \right)^{-1} \\
& \leq \left(\frac{1}{k!} \mathbb{E} \left[\{t + c_N(\tau_N(t))\}^k \right] + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (B_n)^{k-I} (t+1)^{k-1} \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} D_N(r) \right] \right) \left(\mathbb{P} \left[k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \right)^{-1} \\
& \longrightarrow \frac{1}{k!} t^k. \tag{18}
\end{aligned}$$

The limit follows from Lemma 3 and Brown et al. (2020, Equations (3),(4)) along with the fact that, since $c_N(s) \in [0, 1]$ for all s , $\mathbb{E}[c_N(s)^k] \leq \mathbb{E}[c_N(s)]$ for all $k \geq 1$, and the expansion

$$\mathbb{E} \left[\{t + c_N(\tau_N(t))\}^k \right] = \sum_{i=0}^k \binom{k}{i} t^i \mathbb{E} [c_N(\tau_N(t))^{k-i}] \longrightarrow t^k. \tag{19}$$

Combining the upper and lower limits, we conclude that

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{r_1 < \dots < r_k}^{\tau_N(t)} \prod_{j=1}^k \{c_N(r_j) + B_n D_N(r_j)\} \middle| k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] = \frac{1}{k!} t^k \tag{20}$$

and thus

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \mathbb{E} \left[\prod_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r} + \sum_{k=1}^{\tau_N(t)} \{-\alpha_n(1 + O(N^{-1}))\}^k \left(\prod_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r} \right) \sum_{\substack{r_1 < \dots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^k \{c_N(r_j) + B_n D_N(r_j)\} \right] \\
&= \lim_{N \rightarrow \infty} \mathbb{P} \left[\bigcap_{r=1}^{\tau_N(t)} E_r \right] \\
&\quad + \lim_{N \rightarrow \infty} \sum_{k=1}^{\infty} \{-\alpha_n(1 + O(N^{-1}))\}^k \mathbb{E} \left[\mathbb{1}_{\{k \leq \tau_N(t)\}} \left(\prod_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r} \right) \sum_{\substack{r_1 < \dots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^k \{c_N(r_j) + B_n D_N(r_j)\} \right] \\
&= \lim_{N \rightarrow \infty} \mathbb{P} \left[\bigcap_{r=1}^{\tau_N(t)} E_r \right] \\
&\quad + \sum_{k=1}^{\infty} (-\alpha_n)^k \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \dots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^k \{c_N(r_j) + B_n D_N(r_j)\} \middle| k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \times \lim_{N \rightarrow \infty} \mathbb{P} \left[k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \\
&= 1 + \sum_{k=1}^{\infty} (-\alpha_n)^k \frac{t^k}{k!} \times 1 = e^{-\alpha_n t} \tag{21}
\end{aligned}$$

as $N \rightarrow \infty$, where the last line follows from (20) and Lemma 3.

Upper Bound

From Koskela et al. (2018, Lemma 1 Case 1), taking $\xi = \Delta$, we have

$$1 - p_t = p_{\Delta\Delta}(t) \leq 1 - \alpha_n(1 + O(N^{-1})) [c_N(t) - B'_n D_N(t)] \tag{22}$$

where again the $O(N^{-1})$ term does not depend on t . A multinomial expansion as before yields

$$\prod_{r=1}^{\tau_N(t)} (1 - p_r) \leq 1 + \sum_{k=1}^{\tau_N(t)} \{-\alpha_n(1 + O(N^{-1}))\}^k \sum_{\substack{r_1 < \dots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^k \{c_N(r_j) - B'_n D_N(r_j)\}. \tag{23}$$

Analogously to (16), assuming $k \leq \tau_N(t)$ we can write

$$\begin{aligned}
\sum_{\substack{r_1 < \dots < r_k \\ =1}}^{\tau_N(t)} \prod_{j=1}^k \{c_N(r_j) - B'_n D_N(r_j)\} &= \frac{1}{k!} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^k c_N(r_i) \right\} \\
&\quad + \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (-B'_n)^{k-I} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}} \left\{ \prod_{i=1}^I c_N(r_i) \right\} \left\{ \prod_{j=I+1}^k D_N(r_j) \right\}.
\end{aligned} \tag{24}$$

We start by dealing with the second term:

$$\begin{aligned}
& \frac{1}{k!} \sum_{I=0}^{k-1} \binom{k}{I} (-B'_n)^{k-I} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}} \left\{ \prod_{i=1}^I c_N(r_i) \right\} \left\{ \prod_{j=I+1}^k D_N(r_j) \right\} \\
&= \frac{1}{k!} \sum_{\substack{I=0: \\ k-I \text{ even}}}^{k-1} \binom{k}{I} (B'_n)^{k-I} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}} \left\{ \prod_{i=1}^I c_N(r_i) \right\} \left\{ \prod_{j=I+1}^k D_N(r_j) \right\} \\
&\quad - \frac{1}{k!} \sum_{\substack{I=0: \\ k-I \text{ odd}}}^{k-1} \binom{k}{I} (B'_n)^{k-I} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}} \left\{ \prod_{i=1}^I c_N(r_i) \right\} \left\{ \prod_{j=I+1}^k D_N(r_j) \right\}. \tag{25}
\end{aligned}$$

This is lower bounded by

$$0 - \frac{1}{k!} \sum_{\substack{I=0 \\ k-I \text{ odd}}}^{k-1} \binom{k}{I} (B'_n)^{k-I} (t+1)^{k-1} \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \quad (26)$$

using that $c_N(r), D_N(r) \geq 0$ for all r to bound the even terms below, and arguments from (17) to bound the odd terms above. The same arguments lead to the upper bound

$$\frac{1}{k!} \sum_{\substack{I=0 \\ k-I \text{ even}}}^{k-1} \binom{k}{I} (B'_n)^{k-I} (t+1)^{k-1} \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) - 0. \quad (27)$$

By Brown et al. (2020, Equation (4)), both bounds have vanishing expectation as $N \rightarrow \infty$. We are left with the first term in (24), which is upper bounded by

$$\frac{1}{k!} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^k c_N(r_i) \right\} \leq \frac{1}{k!} \left(\sum_{s=1}^{\tau_N(t)} c_N(s) \right)^k \leq \frac{1}{k!} \{t + c_N(\tau_N(t))\}^k \quad (28)$$

the expectation of which converges to $t^k/k!$ as in (19). We use Koskela et al. (2018, Equation (8)) to construct a lower bound:

$$\begin{aligned} \frac{1}{k!} \sum_{\substack{r_1 \neq \dots \neq r_k \\ \text{all distinct}}}^{\tau_N(t)} \left\{ \prod_{i=1}^k c_N(r_i) \right\} &\geq \frac{1}{k!} \left(\sum_{s=1}^{\tau_N(t)} c_N(s) \right)^k - \frac{1}{k!} \binom{k}{2} \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \left(\sum_{s=1}^{\tau_N(t)} c_N(s) \right)^{k-2} \\ &\geq \frac{1}{k!} t^k - \frac{1}{k!} \binom{k}{2} (t+1)^{k-2} \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \end{aligned} \quad (29)$$

The expectation of this bound also converges to $t^k/k!$, using Brown et al. (2020, Equation (5)). We can now conclude that

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{r_1 < \dots < r_k}^{\tau_N(t)} \prod_{j=1}^k \{c_N(r_j) - B'_n D_N(r_j)\} \right] = \frac{1}{k!} t^k \quad (30)$$

and thus, by calculations analogous to (21),

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[1 + \sum_{k=1}^{\tau_N(t)} \{-\alpha_n(1 + O(N^{-1}))\}^k \sum_{r_1 < \dots < r_k}^{\tau_N(t)} \prod_{j=1}^k \{c_N(r_j) - B'_n D_N(r_j)\} \right] = 1 + \sum_{k=1}^{\infty} (-\alpha_n)^k \frac{1}{k!} t^k = e^{-\alpha_n t} \quad (31)$$

as $N \rightarrow \infty$.

We now have upper and lower bounds on $\lim_{N \rightarrow \infty} \mathbb{E} \left[\prod_{r=1}^{\tau_N(t)} (1 - p_r) \right]$, both of which are equal to $e^{-\alpha_n t}$, and the result follows. \square

Lemma 3. For all $k \in \mathbb{N}$, for all $t > 0$,

$$\lim_{N \rightarrow \infty} \mathbb{P} \left[k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] = 1 \quad \text{and} \quad \lim_{N \rightarrow \infty} \mathbb{P} \left[\bigcap_{r=1}^{\tau_N(t)} E_r \right] = 1. \quad (32)$$

Proof. We first construct a constant C_1 such that

$$\mathbb{P} \left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) < C_1\} \right] \leq \mathbb{P} \left[k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \quad (33)$$

and

$$\lim_{N \rightarrow \infty} \mathbb{P} \left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) < C_1\} \right] = 1. \quad (34)$$

Any $C_1 \leq \frac{(N-3)_{n-3}}{N^{n-3}} \left(\alpha_n \left(1 + \frac{2}{N-2} \right) + B_n \right)^{-1}$ will be sufficient for $\bigcap_{r=1}^{\tau_N(t)} E_r$. Furthermore, we can write

$$\{\tau_N(t) \geq k\} = \left\{ \min \left\{ s \geq 1 : \sum_{r=1}^s c_N(r) \geq t \right\} \geq k \right\} = \left\{ \sum_{r=1}^{k-1} c_N(r) < t \right\} \supseteq \bigcap_{r=1}^{k-1} \left\{ c_N(r) < \frac{t}{k} \right\}. \quad (35)$$

A suitable choice to satisfy (33) would thus be

$$C_1 = \min \left\{ \frac{(N-3)_{n-3}}{N^{n-3}} \left(\alpha_n \left(1 + \frac{2}{N-2} \right) + B_n \right)^{-1}, \frac{t}{k} \right\}. \quad (36)$$

Now we show that this choice of C_1 satisfies (34).

$$\begin{aligned} \mathbb{P} \left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) < C_1\} \right] &= 1 - \mathbb{P} \left[\bigcup_{r=1}^{\tau_N(t)} \{c_N(r) \geq C_1\} \right] = 1 - \mathbb{E} [\mathbb{1}_{\bigcup_{r=1}^{\tau_N(t)} \{c_N(r) \geq C_1\}}] \geq 1 - \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} \mathbb{1}_{\{c_N(r) \geq C_1\}} \right] \\ &= 1 - \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} \mathbb{E} [\mathbb{1}_{\{c_N(r) \geq C_1\}} \mid \mathcal{F}_{r-1}] \right] = 1 - \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} \mathbb{P}[\{c_N(r) \geq C_1\} \mid \mathcal{F}_{r-1}] \right] \end{aligned} \quad (37)$$

where the inequality holds by considering the two possible values of $\mathbb{1}_{\bigcup_{r=1}^{\tau_N(t)} \{c_N(r) \geq C_1\}}$, and the second line follows from Lemma 4 with $f(r) = \mathbb{1}_{\{c_N(r) \geq C_1\}}$. Using the generalised Markov inequality,

$$\mathbb{P}[\{c_N(r) \geq C_1\} \mid \mathcal{F}_{r-1}] \leq C_1^{-2} \mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}]. \quad (38)$$

Now, using Lemma 4 again, this time with $f(r) = c_N(r)^2$,

$$\begin{aligned} \mathbb{P} \left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) < C_1\} \right] &\geq 1 - \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} C_1^{-2} \mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}] \right] \\ &= 1 - C_1^{-2} \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}] \right] \\ &= 1 - C_1^{-2} \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} c_N(r)^2 \right] \\ &\xrightarrow{N \rightarrow \infty} 1 - \min\{(\alpha_n + B_n)^2, k^2/t^2\} \times 0 = 1. \end{aligned} \quad (39)$$

Clearly $\mathbb{P} \left[k \leq \tau_N(t), \bigcap_{r=1}^{\tau_N(t)} E_r \right] \leq \mathbb{P} \left[\bigcap_{r=1}^{\tau_N(t)} E_r \right]$, so the second part of (32) follows as an immediate corollary. \square

The following Lemma is taken from Koskela et al. (2018, Lemma 2), where the function is set to $f(t) = c_N(t)$, but the authors remark that the result holds for other choices of function.

Lemma 4. *Let (\mathcal{F}_t) be the filtration generated by the offspring counts $\nu_t^{(1:N)}$ at each generation t . Let $f(t)$ be any \mathcal{F}_t -measurable function. Then*

$$\mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} f(r) \right] = \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}] \right]. \quad (40)$$

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