Cheat sheet: standard theorems

Suzie Brown

6 August 2021

1 Moments of Multinomial distribution

Source: Mosimann 1962, p.67

Let $(X_1, \ldots, X_k) \sim \text{Multinomial}(n, (p_1, \ldots, p_k))$. Then

$$\mathbb{E}[(X_1)_{a_1} \cdots (X_k)_{a_k}] = (n)_a p_1^{a_1} \cdots p_k^{a_k}$$

where $a := a_1 + \cdots + a_k$.

2 Bernoulli inequality

Source: Wolfram MathWorld

Let $r \geq 1$ and $x \geq -1$. Then

$$(1 + x)^r \geq 1 + rx.$$

3 Filtered Borel-Cantelli II

Source: Durrett 2019, Theorem 4.3.4

Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a filtration with $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of events such that $B_n \in \mathcal{F}_n$ for all n. Then

$$\{B_n \text{ infinitely often}\} = \left\{ \sum_{n=1}^{\infty} \mathbb{P}[B_n \mid \mathcal{F}_{n-1}] = \infty \right\}.$$

4 Binomial series

Source: Wikipedia

Let $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$. Then

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

and

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

5 Exponentials

Source: Wikipedia

Let $x \in \mathbb{R}$. Then

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

and

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

6 Geometric series

Source: Wikipedia

Let $n \in \mathbb{N}$ and $r \in \mathbb{R}$. Then

$$\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}$$

and, if $|r| < 1$,

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}.$$

7 Optional Stopping Theorem

Source: Wikipedia

Let $(X_t)_{t\in\mathbb{N}}$ be a martingale and τ a stopping time, with respect to some filtration $(\mathcal{F}_t)$. If any of the following conditions holds:

- τ is almost surely bounded;
- $\mathbb{E}[\tau] < \infty$ and $\mathbb{E}[|X_{t+1} - X_t| \mid \mathcal{F}_t]$ is bounded uniformly in $t < \tau$;
- $X_{t\wedge\tau}$ is almost surely bounded;

then $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$ almost surely.

8 Dominated Convergence Theorem

Source: Wikipedia

Let (f_n) be a sequence of measurable functions which converges pointwise to some function f. If there exists a Lebesgue integrable function g such that

$$|f_n(x)| \leq g(x)$$

for all n, x then f is Lebesgue integrable and

$$\lim_{n\to\infty} \int |f_n - f| d\mu = 0.$$

9 Monotone Convergence Theorem

Source: Wikipedia

9.1 for real numbers

Let $(a_n)_{n\in\mathbb{N}}$ be a montone sequence of real numbers, i.e. either $a_{n+1} \leq a_n$ for all n, or $a_{n+1} \geq a_n$ for all n. Then the limit

$$a := \lim_{n\to\infty} a_n$$

exists if and only if (a_n) is bounded.

9.2 for series

Let $(a_{j,k})_{j,k\in\mathbb{N}}$ be a matrix of non-negative real numbers such that $a_{j+1,k} \geq a_{j,k}$ for all j, k. Then

$$\lim_{j\to\infty} \sum_k a_{j,k} = \sum_k \lim_{j\to\infty} a_{j,k}.$$

10 Fubini's theorem

Source: Wikipedia

Let X and Y be σ-finite measure spaces. Let f be a measurable function on the product space $X \times Y$. If

$$\int_{X\times Y} |f(x,y)| d(x,y) < \infty$$

then

$$\int_{X\times Y} f(x,y) d(x,y) = \int_X \int_Y f(x,y) dy\, dx = \int_Y \int_X f(x,y) dx\, dy.$$

References

Durrett, Richard (2019). *Probability: Theory and Examples.* 5th ed. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press. DOI: 10.1017/9781108591034.

Mosimann, James E. (1962). "On the Compound Multinomial Distribution, the Multivariate β-Distribution, and Correlations among Proportions". In: *Biometrika* 49.1/2, pp. 65–82.