Lemma 1.

$$p_{\xi\xi}(t) = 1 - \binom{|\xi|}{2}(1 + O(N^{-1}))c_N(t).$$

Proof. We will show that the claimed expression is both a lower bound and an upper bound for $p_{\xi\xi}(t)$, beginning with the latter. By definition,

$$\begin{aligned}
p_{\xi\xi}(t) &= \frac{1}{(N)_{|\xi|}} \sum_{\substack{i_1 \neq \ldots \neq i_{|\xi|} = 1 \\ \text{all distinct}}}^{N} \nu_t^{(i_1)} \ldots \nu_t^{(i_{|\xi|})} \\
&= \frac{1}{(N)_{|\xi|}} \left[\left(\sum_{i=1}^{N} \nu_t^{(i)} \right)^{|\xi|} - \binom{|\xi|}{2} \sum_{i=1}^{N} (\nu_t^{(i)})^2 \left(\sum_{j=1}^{N} \nu_t^{(j)} \right)^{|\xi|-2} \right] \\
&= \frac{1}{(N)_{|\xi|}} \left[N^{|\xi|} - N^{|\xi|-2} \binom{|\xi|}{2} \sum_{i=1}^{N} (\nu_t^{(i)})^2 \right] \\
&= \frac{N^{|\xi|}}{(N)_{|\xi|}} \left[1 - \binom{|\xi|}{2} \frac{1}{N^2} \sum_{i=1}^{N} (\nu_t^{(i)})^2 \right].
\end{aligned}$$

Now

$$\frac{1}{N^2} \sum_{i=1}^{N} (\nu_t^{(i)})^2 = \frac{1}{N^2} \left(N + \sum_{i=1}^{N} (\nu_t^{(i)})_2 \right) = \frac{1}{N} + \frac{1 + O(N^{-1})}{\cancel{(N)}_2} c_N(t),$$

typo

so that

$$p_{\xi\xi}(t) = \frac{N^{|\xi}}{(N)_{|\xi|}} \left[1 - \binom{|\xi|}{2} \frac{1}{N} - \binom{|\xi|}{2} (1 + O(N^{-1})) c_N(t) \right]. \tag{1}$$

Next,

$$\begin{aligned}
\frac{N^{|\xi|}}{(N)_{|\xi|}} &= \left[\left(1 - \frac{1}{N} \right) \cdots \left(1 - \frac{|\xi| - 1}{N} \right) \right]^{-1} \\
&= \left[1 - \left(\sum_{i=1}^{|\xi|-1} i \right) \frac{1}{N} + \left(\sum_{i=1}^{|\xi|-1} \sum_{j \neq i}^{|\xi|-1} ij \right) \frac{1}{N^2} + O(N^{-3}) \right]^{-1},
\end{aligned}$$

and we have that

$$\sum_{i=1}^{|\xi|-1} i = \binom{|\xi|}{2},$$

Presumably a known result

$$\sum_{i=1}^{|\xi|-1} \sum_{j \neq i}^{|\xi|-1} ij = \sum_{i=1}^{|\xi|-1} i \left[\binom{|\xi|}{2} - i \right] = \binom{|\xi|}{2}^2 - \sum_{i=1}^{|\xi|-1} i^2 = \binom{|\xi|}{2}^2 - \binom{|\xi|}{2} \frac{2|\xi| - 1}{3},$$

so that

$$\frac{N^{|\xi|}}{(N)_{|\xi|}} = \left[1 - \binom{|\xi|}{2} \frac{1}{N} + \left\{ \binom{|\xi|}{2}^2 - \binom{|\xi|}{2} \frac{2|\xi| - 1}{3} \right\} \frac{1}{N^2} + O(N^{-3}) \right]^{-1}.$$

A Taylor expansion gives

$$\frac{1}{1 + x} = 1 - x + x^2 + O(x^3),$$

so that

$$\frac{N^{|\xi|}}{(N)_{|\xi|}} = 1 + \binom{|\xi|}{2}\frac{1}{N} - \left\{\binom{|\xi|}{2}^2 - \binom{|\xi|}{2}\frac{2|\xi|-1}{3}\right\}\frac{1}{N^2} + \binom{|\xi|}{2}^2\frac{1}{N^2} + O(N^{-3})$$
$$= 1 + \binom{|\xi|}{2}\frac{1}{N} + \binom{|\xi|}{2}\frac{2|\xi|-1}{3}\frac{1}{N^2} + O(N^{-3}).$$

Substituting back into (1), we obtain

when I ksi I = 2

$$p_{\xi\xi}(t)$$
$$= \left[1 + \binom{|\xi|}{2}\frac{1}{N} + \binom{|\xi|}{2}\frac{2|\xi|-1}{3}\frac{1}{N^2} + O(N^{-3})\right]\left[1 - \binom{|\xi|}{2}\frac{1}{N} - \binom{|\xi|}{2}(1+O(N^{-1}))c_N(t)\right]$$
$$= 1 - \left\{\binom{|\xi|}{2}^2 - \binom{|\xi|}{2}\frac{2|\xi|-1}{3}\right\}\frac{1}{N^2} + O(N^{-3}) - \binom{|\xi|}{2}(1+O(N^{-1}))c_N(t)$$

expand and factorise

when I ksi I = 2

$$= 1 - \binom{|\xi|}{2}\frac{(3|\xi|-1)(|\xi|-2)}{6}\frac{1}{N^2} + O(N^{-3}) - \binom{|\xi|}{2}(1+O(N^{-1}))c_N(t).$$

It is clear the second order term is always non-positive, and so we have the asymptotic bound

true when I ksi I > 2

$$p_{\xi\xi}(t) \le 1 - \binom{|\xi|}{2}(1+O(N^{-1}))c_N(t)$$

as soon as $|\xi| \ge 3$. For $|\xi| = 2$, we need to check the signs of higher order terms. We have

N^2 / (N)_2 =

$$\frac{N^2}{N(N-1)} = \frac{1}{1-1/N} = 1 + \frac{1}{N} + \frac{1}{N^2} + \frac{1}{N^3} + \ldots,$$

so that

$$p_{\xi\xi}(t) = \left[1 + \frac{1}{N} + \frac{1}{N^2} + \frac{1}{N^3} + \ldots\right]\left[1 - \frac{1}{N} - (1+O(N^{-1}))c_N(t)\right]$$
$$= 1 - (1+O(N^{-1}))c_N(t).$$

All the corrections cancel, and hence the upper bound holds for $|\xi| = 2$, and in fact is an asymptotic equality in that case.

in case I ksi I = 1, we know p_ksiksi =1 as we have fully coalesced.

For a lower bound, let $\kappa_i := \#\{j : b_j = i\}$ denote the multiplicity of mergers of size i, with the slight abuse of terminology in that κ_1 counts non-merger events. In particular, we have that $\kappa_1 + 2\kappa_2 + \ldots |\xi|\kappa_{|\xi|} = |\xi|$. Now

$$p_{\xi\xi}(t) = 1 - \frac{1}{(N)_{|\xi|}}\sum_{k=1}^{|\xi|-1}\sum_{\substack{b_1 \ge \ldots \ge b_k = 1 \\ b_1 + \ldots + b_k = |\xi|}}^{|\xi|}\frac{|\xi|!}{\prod_{j=1}^{|\xi|}(j!)^{\kappa_j}\kappa_j!}\sum_{\substack{i_1 \ne \ldots \ne i_k = 1 \\ \text{all distinct}}}^{N}(\nu_t^{(i_1)})_{b_1}\ldots(\nu_t^{(i_k)})_{b_k},$$

because the right hand side subtracts the probabilities of all possible merger events. See [Fu, 2006, eq (11)] for the combinatorial factor. The omitted $k = |\xi|$ summand would correspond to the probability of an identity transition. The non-increasing ordering of $(b_1, \ldots, b_k)$ is arbitrary, but without loss of generality: choosing any ordering of the same set of merger sizes would give the same result.

since it's not an identity transition

Because $b_1 \ge 2$ and the summands are all non-negative, we can separate out one pair-merger, replace falling factorials with exponents, and write

$$p_{\xi\xi}(t) \ge 1$$
$$- \frac{1}{(N)_{|\xi|}}\sum_{i=1}^{N}(\nu_t^{(i)})_2\sum_{k=1}^{|\xi|-2}\sum_{\substack{b_1 \ge \ldots \ge b_k = 1 \\ b_1 + \ldots + b_k = |\xi|}}^{|\xi|-2}\frac{|\xi|!}{\prod_{j=1}^{|\xi|}(j!)^{\kappa_j}\kappa_j!}\sum_{\substack{i_2 \ne \ldots \ne i_k \ne i \\ \text{all distinct}}}^{N}(\nu_t^{(i)})^{b_1-2}(\nu_t^{(i_2)})^{b_2}\ldots(\nu_t^{(i_k)})^{b_k}.$$

Now, for each $k \in \{1, \ldots, |\xi| - 2\}$ we have

$$\frac{|\xi|!}{\prod_{j=1}^{|\xi|} (j!)^{\kappa_j} \kappa_j!} = \binom{|\xi|}{b_1, \ldots, b_k} \prod_{j=1}^{|\xi|} \frac{1}{\kappa_j!} = \binom{|\xi|}{2} \frac{\binom{|\xi|-2}{b_1-2, b_2, \ldots, b_k}}{\binom{b_1}{2}} \prod_{j=1}^{|\xi|} \frac{1}{\kappa_j!}$$

unfamiliar notation:
multinomial coefficient

$$\leq \binom{|\xi|}{2} \binom{|\xi| - 2}{b_1 - 2, b_2, \ldots, b_k},$$

which gives

$$p_{\xi\xi}(t) \geq 1$$
$$- \frac{\binom{|\xi|}{2}}{(N)_{|\xi|}} \sum_{i=1}^{N} (\nu_t^{(i)})_2 \sum_{k=1}^{|\xi|-2} \sum_{\substack{b_1 \geq \ldots \geq b_k = 1 \\ b_1 + \ldots + b_k = |\xi|}}^{|\xi|-2} \binom{|\xi| - 2}{b_1 - 2, b_2, \ldots, b_k} \sum_{\substack{i_2 \neq \ldots \neq i_k \neq i \\ \text{all distinct}}}^{N} (\nu_t^{(i)})^{b_1 - 2} (\nu_t^{(i_2)})^{b_2} \ldots (\nu_t^{(i_k)})^{b_k}.$$

We also have the following identity, which is due to the fact that the left hand side is just an obtuse way to write a multinomial expansion of the right hand side:

$$\sum_{k=1}^{|\xi|-2} \sum_{\substack{b_1 \geq \ldots \geq b_k = 1 \\ b_1 + \ldots + b_k = |\xi|}}^{|\xi|-2} \binom{|\xi| - 2}{b_1 - 2, b_2, \ldots, b_k} \sum_{\substack{i_2 \neq \ldots \neq i_k \neq i \\ \text{all distinct}}}^{N} (\nu_t^{(i)})^{b_1 - 2} (\nu_t^{(i_2)})^{b_2} \ldots (\nu_t^{(i_k)})^{b_k}$$
$$= \left(\sum_{j \neq i}^{N} \nu_t^{(j)} \right)^{|\xi|-2} \leq N^{|\xi|-2}.$$

Thus

$$p_{\xi\xi}(t) \geq 1 - \frac{\binom{|\xi|}{2}}{(N)_{|\xi|}} \sum_{i=1}^{N} (\nu_t^{(i)})_2 N^{|\xi|-2} = 1 - \binom{|\xi|}{2} \frac{1 + O(N^{-1})}{(N)_2} c_N(t),$$

as required. □

References

Yun-Xin Fu. Exact coalescent for the Wright-Fisher model. *Theor. Popln Biol.*, 69:385–394, 2006.