Comparing expected coalescence rates for multinomial & residual resampling

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Case $N = 2$

We can calculate the expected coalescence rates explicitly. With only $N = 2$ particles, the coalescence rate becomes

$$\mathbb{E}[c_N(t)|w_t^{(1:3)}] = \frac{1}{(N)_2}\sum_{i=1}^{N}\mathbb{E}\left[(v_t^{(i)})_2|w_t^{(1:3)}\right] = \mathbb{P}[v_t^{(1)} = 0] + \mathbb{P}[v_t^{(1)} = 2]$$

For residual resampling,

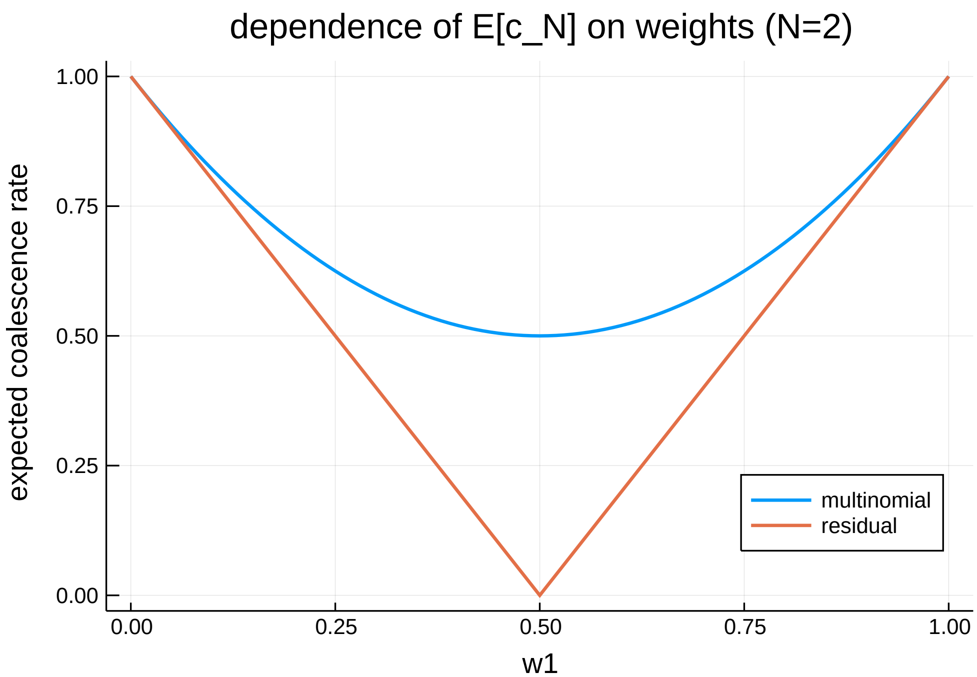
$$\mathbb{E}[c_2^r(t)|w_t^{(1:3)}] = \mathbb{I}\{w_t^{(1)} \geq 1/2\}(2w_t^{(1)} - 1) + \mathbb{I}\{w_t^{(1)} < 1/2\}(2w_t^{(2)} - 1)$$

And for multinomial resampling,

$$\begin{aligned}\mathbb{E}[c_2^m(t)|w_t^{(1:3)}] &= (w_t^{(1)})^2 + (w_t^{(2)})^2 \\ &= \mathbb{I}\{w_t^{(1)} \geq 1/2\}((w_t^{(1)})^2 + (w_t^{(2)})^2) + \mathbb{I}\{w_t^{(1)} < 1/2\}((w_t^{(1)})^2 + (w_t^{(2)})^2) \\ &\geq \mathbb{I}\{w_t^{(1)} \geq 1/2\}(w_t^{(1)})^2 + \mathbb{I}\{w_t^{(1)} < 1/2\}(w_t^{(2)})^2\end{aligned}$$

Then since $(w_t^{(i)} - 1)^2 = (w_t^{(i)})^2 - 2w_t^{(i)} + 1 \geq 0$, we have that $(w_t^{(i)})^2 \geq 2w_t^{(i)} - 1$ and hence we can conclude

$$\mathbb{E}[c_2^m(t)|w_t^{(1:3)}] \geq \mathbb{E}[c_2^r(t)|w_t^{(1:3)}]. \quad \square$$



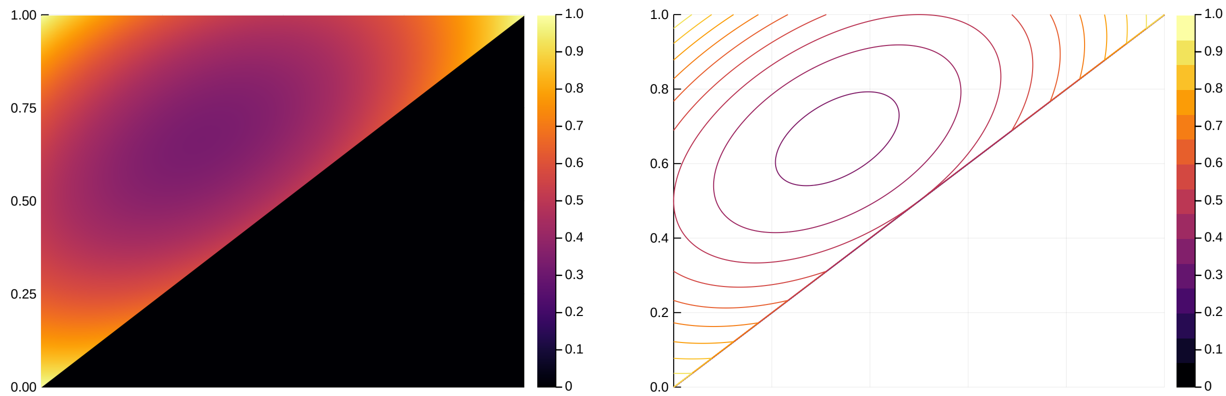


Figure 1: Multinomial resampling when $N = 3$

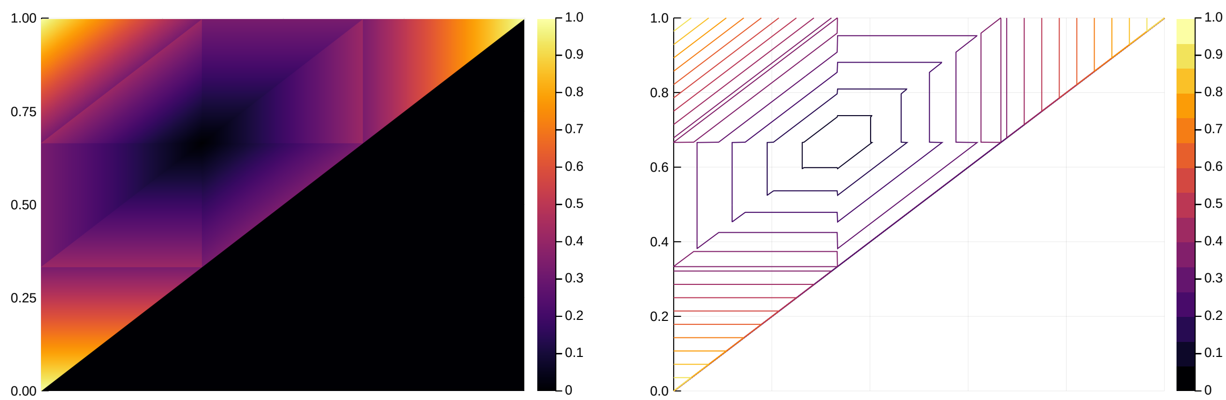


Figure 2: Residual resampling when $N = 3$

Case $N = 3$

Given a weight vector $(w_t^{(1)}, w_t^{(2)}, w_t^{(3)})$, let $w_{(1)} \geq w_{(2)} \geq w_{(3)}$ denote the weights sorted from high to low.

Case	Weights	Offspring counts	Conditional probabilities	$\mathbb{E}[c_2^r(t) \mid w_t^{(1:3)}]$
(A)	$w_{(1)} = 1$	$(3,0,0)$	1	1
(B)	$2/3 < w_{(1)} < 1$	$(3,0,0)$	$3w_{(1)} - 2$	$12w_{(1)} - 6$
		$(2,1,0)$	$3w_{(2)}$	
		$(2,0,1)$	$3w_{(3)}$	
(C)	$w_{(1)} = 2/3$	$(2,1,0)$	$3w_{(2)}$	2
		$(2,0,1)$	$3w_{(3)}$	
(D1)	$1/3 < w_{(1)} < 2/3$ and	$(2,1,0)$	$3w_{(1)} - 1$	$2 - 6w_{(3)}$
	$1/3 \leq w_{(2)} < 2/3$	$(1,2,0)$	$3w_{(2)} - 1$	
		$(1,1,1)$	$3w_{(3)}$	
(D2)	$1/3 < w_{(1)} < 2/3$ and	$(3,0,0)$	$(3/2)^2(w_{(1)} - 1/3)^2$	$(3/2)(3w_{(1)} - 1)(w_{(1)} + 1)$
	$w_{(2)} < 1/3$	$(2,1,0)$	$(3/2)^2 2(w_{(1)} - 1/3)w_{(2)}$	
		$(2,0,1)$	$(3/2)^2 2(w_{(1)} - 1/3)w_{(3)}$	
		$(1,2,0)$	$(3/2)^2 w_{(2)}^2$	
		$(1,0,2)$	$(3/2)^2 w_{(3)}^2$	
		$(1,1,1)$	$(3/2)^2 2w_{(2)}w_{(3)}$	
(E)	$w_{(1)} = 1/3$	$(1,1,1)$	1	0