Weak convergence proof v.2 (neater) (in progress)

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Lemma 1.

$$t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right)\binom{l}{2}(t+1)^{l-2} \leq \sum_{s_1\neq\cdots\neq s_l}\prod_{j=1}^{l} c_N(s_j) \leq t^l + c_N(\tau_N(t))(t+1)^l. \tag{1}$$

Proof. As pointed out in Koskela et al. (2018, Equation (8)),

$$\sum_{s_1\neq\cdots\neq s_l}\prod_{j=1}^{l} c_N(s_j) \geq \left(\sum_{s=0}^{\tau_N(t)} c_N(s)\right)^l - \binom{l}{2}\left(\sum_{s=0}^{\tau_N(t)} c_N(s)^2\right)\left(\sum_{s=0}^{\tau_N(t)} c_N(s)\right)^{l-2}. \tag{2}$$

By definition of τ_N,

$$t \leq \sum_{s=0}^{\tau_N(t)} c_N(s) \leq t+1. \tag{3}$$

Substituting these bounds into the RHS of (2) yields the lower bound.

It is a true fact that

$$\sum_{s_1\neq\cdots\neq s_l}\prod_{j=1}^{l} c_N(s_j) \leq \left(\sum_{s=0}^{\tau_N(t)} c_N(s)\right)^l, \tag{4}$$

as can be seen by considering the multinomial expansion of the RHS. This is further bounded by

$$\left(\sum_{s=0}^{\tau_N(t)} c_N(s)\right)^l \leq \left(\sum_{s=0}^{\tau_N(t)-1} c_N(s) + c_N(\tau_N(t))\right)^l \leq [t + c_N(\tau_N(t))]^l, \tag{5}$$

again using the definition of τ_N. A binomial expansion yields

$$[t + c_N(\tau_N(t))]^l = t^l + \sum_{i=0}^{l-1}\binom{l}{i}t^i c_N(\tau_N(t))^{l-i} = t^l + c_N(\tau_N(t))\sum_{i=0}^{l-1}\binom{l}{i}t^i c_N(\tau_N(t))^{l-1-i}, \tag{6}$$

then since $c_N(s) \leq 1$ for all s,

$$\sum_{i=0}^{l-1}\binom{l}{i}t^i c_N(\tau_N(t))^{l-1-i} \leq \sum_{i=0}^{l-1}\binom{l}{i}t^i \leq (t+1)^l. \tag{7}$$

Putting this together yields the upper bound. $\square$

Lemma 2. *Let B be a positive constant which may depend on n.*

$$\sum_{s_1\neq\cdots\neq s_l}\prod_{j=1}^{l}[c_N(s_j) + BD_N(s_j)] \leq \sum_{s_1\neq\cdots\neq s_l}\prod_{j=1}^{l} c_N(s_j) + \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)(t+1)^{l-1}(1+B)^l. \tag{8}$$

Proof. We start with a binomial expansion:

$$\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) + BD_N(s_j)] = \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \sum_{\mathcal{I} \subseteq [l]} B^{l-|\mathcal{I}|} \left(\prod_{i \in \mathcal{I}} c_N(s_i) \right) \left(\prod_{j \notin \mathcal{I}} D_N(s_j) \right)$$

$$= \sum_{\mathcal{I} \subseteq [l]} B^{l-|\mathcal{I}|} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i \in \mathcal{I}} c_N(s_i) \right) \left(\prod_{j \notin \mathcal{I}} D_N(s_j) \right) \qquad (9)$$

where $[l] := \{1, \ldots, l\}$. Since the sum is over all permutations of $r_1, \ldots, r_l$, we may arbitrarily choose an ordering for $\{1, \ldots, l\}$ such that $\mathcal{I} = \{1, \ldots, |\mathcal{I}|\}$:

$$\sum_{\mathcal{I} \subseteq [l]} B^{l-|\mathcal{I}|} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i \in \mathcal{I}} c_N(s_i) \right) \left(\prod_{j \notin \mathcal{I}} D_N(s_j) \right) = \sum_{I=0}^{l} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right). \qquad (10)$$

Separating the term $I = l$,

$$\sum_{I=0}^{l} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right) = \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j)$$

$$+ \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right). \qquad (11)$$

In the second line, there is always at least one D_N term, and $c_N(s) \leq D_N(s)$ for all s, so we can write

$$\sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right) \leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) D_N(s_l)$$

$$\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \left(\sum_{s_1 \neq \cdots \neq s_{l-1}}^{\tau_N(t)} \prod_{i=1}^{l-1} c_N(s_i) \right) \sum_{s_l=1}^{\tau_N(t)} D_N(s_l)$$

$$\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} (t+1)^{l-1} \sum_{s=1}^{\tau_N(t)} D_N(s) \qquad (12)$$

using (4) and (3). Finally, by the Binomial Theorem,

$$\sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} (t+1)^{l-1} \sum_{s=1}^{\tau_N(t)} D_N(s) \leq \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1+B)^l, \qquad (13)$$

which, together with (11), concludes the proof. $\square$

Lemma 3. *Let B be a positive constant which may depend on n.*

$$\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - BD_N(s_j)] \geq \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1+B)^l. \qquad (14)$$

Proof. A binomial expansion and subsequent manipulation as in (9)–(11) gives

$$\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}[c_N(s_j)-BD_N(s_j)]=\sum_{\mathcal{I}\subseteq[l]}(-B)^{l-|\mathcal{I}|}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\left(\prod_{i\in\mathcal{I}}c_N(s_i)\right)\left(\prod_{j\notin\mathcal{I}}D_N(s_j)\right)$$

$$=\sum_{I=0}^{l}\binom{l}{I}(-B)^{l-I}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\left(\prod_{i=1}^{I}c_N(s_i)\right)\left(\prod_{j=I+1}^{l}D_N(s_j)\right)$$

$$=\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}c_N(s_j)+\sum_{I=0}^{l-1}\binom{l}{I}(-B)^{l-I}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\left(\prod_{i=1}^{I}c_N(s_i)\right)\left(\prod_{j=I+1}^{l}D_N(s_j)\right)$$

$$\geq\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}c_N(s_j)-\sum_{I=0}^{l-1}\binom{l}{I}B^{l-I}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\left(\prod_{i=1}^{I}c_N(s_i)\right)\left(\prod_{j=I+1}^{l}D_N(s_j)\right)$$

$$(15)$$

where the last inequality just multiplies some positive terms by -1. Then (12)–(13) can be applied directly (noting that an upper bound on negative terms gives a lower bound overall):

$$-\sum_{I=0}^{l-1}\binom{l}{I}B^{l-I}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\left(\prod_{i=1}^{I}c_N(s_i)\right)\left(\prod_{j=I+1}^{l}D_N(s_j)\right)\geq\left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)(t+1)^{l-1}(1+B)^l \qquad (16)$$

which concludes the proof. $\qquad\square$

Lemma 4 (Basis step). *For any* $0<t<\infty$,

$$\lim_{N\to\infty}\mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right]=e^{-\alpha_n t} \qquad (17)$$

where $\alpha_n:=n(n-1)/2$.

Proof. We start by showing that $\lim_{N\to\infty}\mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right]\leq e^{-\alpha_n t}$.

From Koskela et al. (2018, Lemma 1 Case 1), taking $\xi=\Delta$, we have

$$1-p_t=p_{\Delta\Delta}(t)\leq 1-\alpha_n(1+O(N^{-1}))\left[c_N(t)-B_n'D_N(t)\right] \qquad (18)$$

where the $O(N^{-1})$ term does not depend on t. Applying a multinomial expansion and then separating the positive and negative terms,

$$\prod_{r=1}^{\tau_N(t)}(1-p_r)\leq 1+\sum_{l=1}^{\tau_N(t)}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}[c_N(s_j)-B_n'D_N(s_j)]$$

$$=1+\sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}[c_N(s_j)-B_n'D_N(s_j)]$$

$$-\sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}[c_N(s_j)-B_n'D_N(s_j)]. \qquad (19)$$

This is further bounded by applying Lemma 3 and then both bounds of Lemma 1:

$$
\prod_{r=1}^{\tau_N(t)} (1 - p_r) \leq 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j)
$$

$$
- \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left\{ \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1 + B_n')^l \right\}
$$

$$
\leq 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left\{ t^l + c_N(\tau_N(t))(t+1)^l \right\}
$$

$$
- \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left\{ t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \binom{l}{2} (t+1)^{l-2} - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1 + B_n')^l \right\}.
$$

$$(20)$$

A bit of tidying up and we have

$$
\prod_{r=1}^{\tau_N(t)} (1 - p_r) \leq \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l + c_N(\tau_N(t)) \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} (t+1)^l
$$

$$
+ \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \binom{l}{2} (t+1)^{l-2}
$$

$$
+ \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} (t+1)^{l-1} (1 + B_n')^l
$$

$$
\leq \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l + c_N(\tau_N(t)) \exp[\alpha_n (1 + O(N^{-1}))(t+1)]
$$

$$
+ \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n (1 + O(N^{-1}))(t+1)]
$$

$$
+ \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n (1 + O(N^{-1}))(t+1)(1 + B_n')].
$$

$$(21)$$

Now, taking the expectation and limit, and applying Brown et al. (2021, Equations (3.3)–(3.5)) and Lemma ?? (Lemma 2 in the messy weakconv note; not written up here yet),

$$
\lim_{N \to \infty} \mathbb{E} \left[\prod_{r=1}^{\tau_N(t)} (1 - p_r) \right] \leq \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l \lim_{N \to \infty} \mathbb{P}[\tau_N(t) \geq l] + \lim_{N \to \infty} \mathbb{E} \left[c_N(\tau_N(t)) \right] \exp[\alpha_n(t+1)]
$$

$$
+ \lim_{N \to \infty} \mathbb{E} \left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right] \frac{1}{2} \alpha_n^2 \exp[\alpha_n(t+1)]
$$

$$
+ \lim_{N \to \infty} \mathbb{E} \left[\sum_{s=1}^{\tau_N(t)} D_N(s) \right] \exp[\alpha_n(t+1)(1 + B_n')]
$$

$$
= \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l = e^{-\alpha_n t}.
$$

$$(22)$$

It remains to show that $\lim_{N \to \infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right] \geq e^{-\alpha_n t}$.

From Brown et al. (2021, Equation (3.14)), taking $\xi = \Delta$, we have

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \alpha_n(1 + O(N^{-1}))\left[c_N(t) + B_n D_N(t)\right] \tag{23}$$

where $B_n > 0$ and the $O(N^{-1})$ term does not depend on t. In particular,

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \frac{N^{n-2}}{(N-2)_{n-2}}\alpha_n c_N(t) - \frac{N^{n-3}}{(N-3)_{n-3}}B_n D_N(t). \tag{24}$$

Since $D_N(s) \leq c_N(s)$ for all s (Koskela et al., 2018, p.9), a sufficient condition for this bound to be non-negative is

$$E_r := \left\{ c_N(r) \leq \frac{(N-3)_{n-3}}{N^{n-3}}\left(\alpha_n\left(1 + \frac{2}{N-2}\right) + B_n\right)^{-1} \right\}, \tag{25}$$

and we define $E := \bigcap_{r=1}^{\tau_N(t)} E_r$. We now apply a multinomial expansion to the product, and split into positive and negative terms:

$$\prod_{r=1}^{\tau_N(t)}(1-p_r) \geq \left\{ 1 + \sum_{l=1}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1}))\frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) + B_n D_N(s_j)] \right\} \mathbb{1}_E$$

$$= \left\{ 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1}))\frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) + B_n D_N(s_j)] \right.$$

$$\left. - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1}))\frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) + B_n D_N(s_j)] \right\} \mathbb{1}_E \tag{26}$$

This is further bounded by applying Lemma 2 and both bounds in Lemma 1:

$$\prod_{r=1}^{\tau_N(t)}(1-p_r) \geq \left\{ 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1}))\frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \right.$$

$$\left. - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1}))\frac{1}{l!} \left[\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) + \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)(t+1)^{l-1}(1+B_n)^l \right] \right\} \mathbb{1}_E$$

$$\geq \left\{ 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1}))\frac{1}{l!} \left[t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right)\binom{l}{2}(t+1)^{l-2} \right] \right.$$

$$\left. - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1}))\frac{1}{l!} \left[t^l + c_N(\tau_N(t))(t+1)^l + \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)(t+1)^{l-1}(1+B_n)^l \right] \right\} \mathbb{1}_E. \tag{27}$$

Tidying things up,

$$\prod_{r=1}^{\tau_N(t)} (1 - p_r) \geq \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_E - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \binom{l}{2} (t+1)^{l-2}$$

$$- c_N(\tau_N(t)) \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} (t+1)^l$$

$$- \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} (t+1)^{l-1} (1 + B_n)^l$$

$$\geq \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_E - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n (1 + O(N^{-1}))(t+1)]$$

$$- c_N(\tau_N(t)) \exp[\alpha_n (1 + O(N^{-1}))(t+1)]$$

$$- \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n (1 + O(N^{-1}))(t+1)(1 + B_n)]. \tag{28}$$

Now, taking the expectation and limit, and applying Brown et al. (2021, Equations (3.3)–(3.5)) and Lemma ?? (Lemma 2 in the messy weakconv note; not written up here yet),

$$\lim_{N \to \infty} \mathbb{E} \left[\prod_{r=1}^{\tau_N(t)} (1 - p_r) \right] \geq \sum_{l=0}^{\infty} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \lim_{N \to \infty} \mathbb{P}\left[\{\tau_N(t) \geq l\} \cap E \right]$$

$$- \lim_{N \to \infty} \mathbb{E} \left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right] \frac{1}{2} \alpha_n^2 \exp[\alpha_n(t+1)]$$

$$- \lim_{N \to \infty} \mathbb{E}\left[c_N(\tau_N(t)) \right] \exp[\alpha_n(t+1)]$$

$$- \lim_{N \to \infty} \mathbb{E} \left[\sum_{s=1}^{\tau_N(t)} D_N(s) \right] \exp[\alpha_n(t+1)(1 + B_n)]$$

$$= \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l = e^{-\alpha_n t}. \tag{29}$$

Combining the upper and lower bounds in (22) and (29) respectively concludes the proof. $\qquad \square$

Lemma 5 (Induction step upper bound). *Fix $k \in \mathbb{N}$, $i_0 := 0$, $i_k := k$. For any sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_k \leq t$,*

$$\lim_{N \to \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \cdots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^{k} p_{r_i} \right) \left(\prod_{\substack{r=1 \\ \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) \right) \right] \leq \alpha_n^k \sum_{l=0}^{\infty} \frac{1}{l!} (-\alpha_n)^l t^l \sum_{\substack{i_1 \leq \cdots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}. \tag{30}$$

Lemma 6 (Induction step lower bound)**.** *Fix $k \in \mathbb{N}$, $i_0 := 0$, $i_k := k$. For any sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_k \leq t$,*

$$\lim_{N \to \infty} \mathbb{E}\left[\sum_{\substack{r_1 < \cdots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^{k} p_{r_i} \right) \left(\prod_{\substack{r=1 \\ \notin \{r_1, \ldots, r_k\}}}^{\tau_N(t)} (1 - p_r) \right) \right] \geq \alpha_n^k \sum_{l=0}^{\infty} \frac{1}{l!}(-\alpha_n)^l t^l \sum_{\substack{i_1 \leq \cdots \leq i_{k-1} \\ \in \{0, \ldots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}.$$

$$(31)$$

Lemma 7. *For any $t_{j-1}, t_j, i_{j-1}, i_j$, define the event*

$$E_j^\star = \left\{ \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) \leq \frac{(t_j - t_{j-1})^{i_j - i_{j-1} - k+1}}{(i_j - i_{j-1})!} \right\}, \tag{32}$$

and let $E^\star := \bigcap_{j=1}^{k} E_j^\star$. Let E be any event independent of $r_1, \ldots, r_k$ such that $\lim_{N \to \infty} \mathbb{P}[E \cap E^\star] = 1$. Fix $l, k \in \mathbb{N}$, $i_0 := 0$, $i_k := k$. Then for any sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_k \leq t$,

$$\lim_{N \to \infty} \mathbb{E}\left[\mathbb{1}_E \sum_{\substack{r_1 < \cdots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i) \right] = \sum_{\substack{i_1 \leq \cdots \leq i_{k-1} \\ \in \{0, \ldots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}. \tag{33}$$

References

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