Recall the quantities

$$c_N(t) := \frac{1}{(N)_2} \sum_{i=1}^{N} (\nu_t^{(i)})_2,$$
$$D_N(t) := \frac{1}{N(N)_2} \sum_{i=1}^{N} (\nu_t^{(i)})_2 \left(\nu_t^{(i)} + \frac{1}{N} \sum_{j \neq i}^{N} (\nu_t^{(j)})^2 \right);$$

the assumptions

$$\mathbb{E}[c_N(t)] \to 0, \tag{1}$$
$$\mathbb{E}\left[\sum_{r=\tau_N(s)+1}^{\tau_N(t)} D_N(r) \right] \to 0, \tag{2}$$
$$\mathbb{E}\left[\sum_{r=\tau_N(s)+1}^{\tau_N(t)} c_N(r)^2 \right] \to 0; \tag{3}$$

and the reverse-time filtration $\mathcal{F}_t := \sigma(\boldsymbol{\nu}_s; 1 \leq s \leq t)$.

Lemma 1. *We have*

$$c_N(t)^2 \leq \frac{N}{N-1} D_N(t),$$

so that (2) $\Rightarrow$ (3).

Proof.

$$\begin{aligned}
c_N(t)^2 &= \frac{1}{N(N-1)(N)_2} \sum_{i=1}^{N} (\nu_t^{(i)})_2 \left(\nu_t^{(i)}(\nu_t^{(i)} - 1) + \sum_{j \neq i}^{N} (\nu_t^{(j)})_2 \right) \\
&= \frac{1}{N(N)_2} \sum_{i=1}^{N} (\nu_t^{(i)})_2 \left(\frac{\nu_t^{(i)}(\nu_t^{(i)} - 1)}{N-1} + \frac{1}{N-1} \sum_{j \neq i}^{N} (\nu_t^{(j)})_2 \right) \\
&\leq \frac{1}{N(N)_2} \sum_{i=1}^{N} (\nu_t^{(i)})_2 \left(\nu_t^{(i)} + \frac{1}{N-1} \sum_{j \neq i}^{N} (\nu_t^{(j)})_2 \right) \\
&\leq \frac{1}{N(N)_2} \sum_{i=1}^{N} (\nu_t^{(i)})_2 \left(\nu_t^{(i)} + \frac{N/(N-1)}{N} \sum_{j \neq i}^{N} (\nu_t^{(j)})^2 \right) \\
&\leq \frac{N/(N-1)}{N(N)_2} \sum_{i=1}^{N} (\nu_t^{(i)})_2 \left(\nu_t^{(i)} + \frac{1}{N} \sum_{j \neq i}^{N} (\nu_t^{(j)})^2 \right) = \frac{N}{N-1} D_N(t).
\end{aligned}$$

□

We now introduce a new assumption: for some deterministic sequence $b_N \to 0$, we have

$$\frac{1}{(N)_3} \sum_{i=1}^{N} \mathbb{E}[(\nu_t^{(i)})_3 | \mathcal{F}_{t-1}] \leq b_N \mathbb{E}[c_N(t) | \mathcal{F}_{t-1}] \tag{4}$$

uniformly in $t \geq 1$.

Lemma 2. (4) $\Rightarrow$ (1) *and* (4) $\Rightarrow$ (2).

Proof. We prove the two implications separately, starting with the former. Following the proof of [Möhle and Sagitov, 2003, Lemma 5.5], we fix $\varepsilon > 0$ and define the event $A_i := \{\nu_t^{(i)} \leq N\varepsilon\}$. Now

$$\begin{aligned}
\mathbb{E}[c_N(t)|\mathcal{F}_{t-1}] &= \frac{1}{(N)_2}\sum_{i=1}^{N}\mathbb{E}[(\nu_t^{(i)})_2|\mathcal{F}_{t-1}] \\
&= \frac{1}{(N)_2}\sum_{i=1}^{N}\Big\{\mathbb{E}[(\nu_t^{(i)})_2\mathbb{1}_{A_i}|\mathcal{F}_{t-1}] + \mathbb{E}[(\nu_t^{(i)})_2\mathbb{1}_{A_i^c}|\mathcal{F}_{t-1}]\Big\} \\
&\leq \frac{\varepsilon}{N-1}\sum_{i=1}^{N}\mathbb{E}[\nu_t^{(i)}\mathbb{1}_{A_i}|\mathcal{F}_{t-1}] + \sum_{i=1}^{N}\mathbb{E}[\mathbb{1}_{A_i^c}|\mathcal{F}_{t-1}] \\
&\leq \{1+O(N^{-1})\}\varepsilon + \sum_{i=1}^{N}\mathbb{P}(\nu_t^{(i)} > N\varepsilon|\mathcal{F}_{t-1}). \qquad (5)
\end{aligned}$$

For $N \geq 3/\varepsilon$, Markov's inequality yields

$$\begin{aligned}
\sum_{i=1}^{N}\mathbb{P}(\nu_t^{(i)} > N\varepsilon|\mathcal{F}_{t-1}) &\leq \frac{1}{(N\varepsilon)_3}\sum_{i=1}^{N}\mathbb{E}[(\nu_t^{(i)})_3|\mathcal{F}_{t-1}] = \frac{\{1+O(N^{-1})\}}{\varepsilon^3(N)_3}\sum_{i=1}^{N}\mathbb{E}[(\nu_t^{(i)})_3|\mathcal{F}_{t-1}] \\
&\leq \{1+O(N^{-1})\}\frac{b_N}{\varepsilon^3}\mathbb{E}[c_N(t)|\mathcal{F}_{t-1}]. \qquad (6)
\end{aligned}$$

Substituting (6) into (5) and using $c_N(t) \leq 1$ results in

$$\mathbb{E}[c_N(t)|\mathcal{F}_{t-1}] \leq \{1+O(N^{-1})\}\left(\varepsilon + \frac{b_N}{\varepsilon^3}\right) \to \varepsilon$$

because $b_N \to 0$. Since $\varepsilon > 0$ was arbitrary, we have

$$\mathbb{E}[c_N(t)] = \mathbb{E}[\mathbb{E}[c_N(t)|\mathcal{F}_{t-1}]] \to 0$$

as $N \to \infty$.

We will show (4) $\Rightarrow$ (2) in two parts, the first of which is

$$\begin{aligned}
\frac{1}{N(N)_2}\sum_{i=1}^{N}\mathbb{E}[(\nu_t^{(i)})_2\nu_t^{(i)}|\mathcal{F}_{t-1}] &= \frac{1}{N(N)_2}\sum_{i=1}^{N}\mathbb{E}[(\nu_t^{(i)})_3 + 2(\nu_t^{(i)})_2|\mathcal{F}_{t-1}] \\
&\leq \frac{1}{(N)_3}\sum_{i=1}^{N}\mathbb{E}[(\nu_t^{(i)})_3|\mathcal{F}_{t-1}] + \frac{2}{N}\mathbb{E}[c_N(t)|\mathcal{F}_{t-1}] \\
&\leq \left(b_N + \frac{2}{N}\right)\mathbb{E}[c_N(t)|\mathcal{F}_{t-1}]. \qquad (7)
\end{aligned}$$

For the second, note

$$\begin{aligned}
\frac{1}{N^2(N)_2}\sum_{i\neq j}^{N}\mathbb{E}[(\nu_t^{(i)})_2(\nu_t^{(j)})^2|\mathcal{F}_{t-1}] &= \frac{1}{N^2(N)_2}\sum_{i\neq j=1}^{N}\mathbb{E}[(\nu_t^{(i)})_2(\nu_t^{(j)})_2 + (\nu_t^{(i)})_2\nu_t^{(j)}|\mathcal{F}_{t-1}] \\
&\leq \frac{1}{N^2(N)_2}\sum_{i\neq j}^{N}\mathbb{E}[(\nu_t^{(i)})_2(\nu_t^{(j)})_2|\mathcal{F}_{t-1}] + \frac{\mathbb{E}[c_N(t)|\mathcal{F}_{t-1}]}{N}. \qquad (8)
\end{aligned}$$

Now

$$\begin{aligned}\sum_{i\neq j}^{N}\mathbb{E}[(\nu_t^{(i)})_2(\nu_t^{(j)})_2|\mathcal{F}_{t-1}] &= \sum_{i\neq j}^{N}\Big\{\mathbb{E}[(\nu_t^{(i)})_2(\nu_t^{(j)})_2\mathbb{1}_{A_i}|\mathcal{F}_{t-1}]+\mathbb{E}[(\nu_t^{(i)})_2(\nu_t^{(j)})_2\mathbb{1}_{A_i^c}|\mathcal{F}_{t-1}]\Big\}\\ &\leq N\varepsilon\sum_{i\neq j}^{N}\mathbb{E}[\nu_t^{(i)}(\nu_t^{(j)})_2\mathbb{1}_{A_i}|\mathcal{F}_{t-1}]+N^3\sum_{i\neq j}^{N}\mathbb{E}[\nu_t^{(j)}\mathbb{1}_{A_i^c}|\mathcal{F}_{t-1}]\\ &\leq N^2(N)_2\varepsilon\mathbb{E}[c_N(t)|\mathcal{F}_{t-1}]+N^4\sum_{i=1}^{N}\mathbb{P}(\nu_t^{(i)}>N\varepsilon|\mathcal{F}_{t-1}). \qquad (9)\end{aligned}$$

Substituting (6) into (9) yields

$$\sum_{i\neq j}^{N}\mathbb{E}[(\nu_t^{(i)})_2(\nu_t^{(j)})_2|\mathcal{F}_{t-1}]\leq N^4\{1+O(N^{-1})\}\left(\varepsilon+\frac{b_N}{\varepsilon^3}\right)\mathbb{E}[c_N(t)|\mathcal{F}_{t-1}], \qquad (10)$$

and substituting (10) into (8) gives

$$\frac{1}{N^2(N)_2}\sum_{i\neq j}^{N}\mathbb{E}[(\nu_t^{(i)})_2(\nu_t^{(j)})^2|\mathcal{F}_{t-1}]\leq\left(\{1+O(N^{-1})\}\Big[\varepsilon+\frac{b_N}{\varepsilon^3}\Big]+\frac{1}{N}\right)\mathbb{E}[c_N(t)|\mathcal{F}_{t-1}]. \qquad (11)$$

Finally, invoking Lemma 2 from our paper twice, with (7) and (11) in between, gives

$$\begin{aligned}\mathbb{E}\left[\sum_{r=\tau_N(s)+1}^{\tau_N(t)}D_N(r)\right] &= \mathbb{E}\left[\sum_{r=\tau_N(s)+1}^{\tau_N(t)}\mathbb{E}[D_N(r)|\mathcal{F}_{t-1}]\right]\\ &\leq\left(\{1+O(N^{-1})\}\left[\varepsilon+\frac{b_N}{\varepsilon^3}\right]+\frac{3}{N}+b_N\right)\mathbb{E}\left[\sum_{r=\tau_N(s)+1}^{\tau_N(t)}c_N(r)\right]\\ &\leq\left(\{1+O(N^{-1})\}\left[\varepsilon+\frac{b_N}{\varepsilon^3}\right]+\frac{3}{N}+b_N\right)(t-s+1)\to\varepsilon(t-s+1),\end{aligned}$$

and recalling that $\varepsilon>0$ was arbitrary concludes the proof. □

References

M. Möhle and S. Sagitov. Coalescent patterns in exchangeable diploid population models. *J. Math. Biol.*, 47:337–352, 2003.