

Weak convergence proof v.2 (neater) (in progress)

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Bounds on sum-products

Lemma 1.

$$t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \binom{l}{2} (t+1)^{l-2} \leq \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l c_N(s_j) \leq t^l + c_N(\tau_N(t))(t+1)^l. \quad (1)$$

Proof. As pointed out in Koskela et al. (2018, Equation (8)),

$$\sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l c_N(s_j) \geq \left(\sum_{s=0}^{\tau_N(t)} c_N(s) \right)^l - \binom{l}{2} \left(\sum_{s=0}^{\tau_N(t)} c_N(s)^2 \right) \left(\sum_{s=0}^{\tau_N(t)} c_N(s) \right)^{l-2}. \quad (2)$$

By definition of τ_N ,

$$t \leq \sum_{s=0}^{\tau_N(t)} c_N(s) \leq t+1. \quad (3)$$

Substituting these bounds into the RHS of (2) yields the lower bound.

It is a true fact that

$$\sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l c_N(s_j) \leq \left(\sum_{s=0}^{\tau_N(t)} c_N(s) \right)^l, \quad (4)$$

as can be seen by considering the multinomial expansion of the RHS. This is further bounded by

$$\left(\sum_{s=0}^{\tau_N(t)} c_N(s) \right)^l \leq \left(\sum_{s=0}^{\tau_N(t)-1} c_N(s) + c_N(\tau_N(t)) \right)^l \leq [t + c_N(\tau_N(t))]^l, \quad (5)$$

again using the definition of τ_N . A binomial expansion yields

$$[t + c_N(\tau_N(t))]^l = t^l + \sum_{i=0}^{l-1} \binom{l}{i} t^i c_N(\tau_N(t))^{l-i} = t^l + c_N(\tau_N(t)) \sum_{i=0}^{l-1} \binom{l}{i} t^i c_N(\tau_N(t))^{l-1-i}, \quad (6)$$

then since $c_N(s) \leq 1$ for all s ,

$$\sum_{i=0}^{l-1} \binom{l}{i} t^i c_N(\tau_N(t))^{l-1-i} \leq \sum_{i=0}^{l-1} \binom{l}{i} t^i \leq (t+1)^l. \quad (7)$$

Putting this together yields the upper bound. ■

Lemma 2. Let B be a positive constant which may depend on n .

$$\sum_{s_1 \neq \dots \neq s_l} \prod_{j=1}^l [c_N(s_j) + BD_N(s_j)] \leq \sum_{s_1 \neq \dots \neq s_l} \prod_{j=1}^l c_N(s_j) + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1+B)^l. \quad (8)$$

Proof. We start with a binomial expansion:

$$\begin{aligned} \sum_{s_1 \neq \dots \neq s_l} \prod_{j=1}^l [c_N(s_j) + BD_N(s_j)] &= \sum_{s_1 \neq \dots \neq s_l} \sum_{\mathcal{I} \subseteq [l]} B^{l-|\mathcal{I}|} \left(\prod_{i \in \mathcal{I}} c_N(s_i) \right) \left(\prod_{j \notin \mathcal{I}} D_N(s_j) \right) \\ &= \sum_{\mathcal{I} \subseteq [l]} B^{l-|\mathcal{I}|} \sum_{s_1 \neq \dots \neq s_l} \left(\prod_{i \in \mathcal{I}} c_N(s_i) \right) \left(\prod_{j \notin \mathcal{I}} D_N(s_j) \right) \end{aligned} \quad (9)$$

where $[l] := \{1, \dots, l\}$. Since the sum is over all permutations of r_1, \dots, r_l , we may arbitrarily choose an ordering for $\{1, \dots, l\}$ such that $\mathcal{I} = \{1, \dots, |\mathcal{I}|\}$:

$$\sum_{\mathcal{I} \subseteq [l]} B^{l-|\mathcal{I}|} \sum_{s_1 \neq \dots \neq s_l} \left(\prod_{i \in \mathcal{I}} c_N(s_i) \right) \left(\prod_{j \notin \mathcal{I}} D_N(s_j) \right) = \sum_{I=0}^l \binom{l}{I} B^{l-I} \sum_{s_1 \neq \dots \neq s_l} \left(\prod_{i=1}^I c_N(s_i) \right) \left(\prod_{j=I+1}^l D_N(s_j) \right). \quad (10)$$

Separating the term $I = l$,

$$\begin{aligned} \sum_{I=0}^l \binom{l}{I} B^{l-I} \sum_{s_1 \neq \dots \neq s_l} \left(\prod_{i=1}^I c_N(s_i) \right) \left(\prod_{j=I+1}^l D_N(s_j) \right) &= \sum_{s_1 \neq \dots \neq s_l} \prod_{j=1}^l c_N(s_j) \\ &\quad + \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \dots \neq s_l} \left(\prod_{i=1}^I c_N(s_i) \right) \left(\prod_{j=I+1}^l D_N(s_j) \right). \end{aligned} \quad (11)$$

In the second line, there is always at least one D_N term, and $c_N(s) \leq D_N(s)$ for all s , so we can write

$$\begin{aligned} \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \dots \neq s_l} \left(\prod_{i=1}^I c_N(s_i) \right) \left(\prod_{j=I+1}^l D_N(s_j) \right) &\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \dots \neq s_l} \left(\prod_{i=1}^{l-1} c_N(s_i) \right) D_N(s_l) \\ &\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \left(\sum_{s_1 \neq \dots \neq s_{l-1}} \prod_{i=1}^{l-1} c_N(s_i) \right) \sum_{s_l=1}^{\tau_N(t)} D_N(s_l) \\ &\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} (t+1)^{l-1} \sum_{s=1}^{\tau_N(t)} D_N(s) \end{aligned} \quad (12)$$

using (4) and (3). Finally, by the Binomial Theorem,

$$\sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} (t+1)^{l-1} \sum_{s=1}^{\tau_N(t)} D_N(s) \leq \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1+B)^l, \quad (13)$$

which, together with (11), concludes the proof. ■

Lemma 3. Let B be a positive constant which may depend on n .

$$\sum_{s_1 \neq \dots \neq s_l} \prod_{j=1}^l [c_N(s_j) - BD_N(s_j)] \geq \sum_{s_1 \neq \dots \neq s_l} \prod_{j=1}^l c_N(s_j) - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1+B)^l. \quad (14)$$

Proof. A binomial expansion and subsequent manipulation as in (9)–(11) gives

$$\begin{aligned}
\sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l [c_N(s_j) - BD_N(s_j)] &= \sum_{\mathcal{I} \subseteq [l]} (-B)^{l-|\mathcal{I}|} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \left(\prod_{i \in \mathcal{I}} c_N(s_i) \right) \left(\prod_{j \notin \mathcal{I}} D_N(s_j) \right) \\
&= \sum_{I=0}^l \binom{l}{I} (-B)^{l-I} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^I c_N(s_i) \right) \left(\prod_{j=I+1}^l D_N(s_j) \right) \\
&= \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l c_N(s_j) + \sum_{I=0}^{l-1} \binom{l}{I} (-B)^{l-I} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^I c_N(s_i) \right) \left(\prod_{j=I+1}^l D_N(s_j) \right) \\
&\geq \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l c_N(s_j) - \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^I c_N(s_i) \right) \left(\prod_{j=I+1}^l D_N(s_j) \right)
\end{aligned} \tag{15}$$

where the last inequality just multiplies some positive terms by -1 . Then (12)–(13) can be applied directly (noting that an upper bound on negative terms gives a lower bound overall):

$$-\sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^I c_N(s_i) \right) \left(\prod_{j=I+1}^l D_N(s_j) \right) \geq \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1+B)^l \tag{16}$$

which concludes the proof. ■

Main components of weak convergence

Lemma 4 (Basis step). *For any $0 < t < \infty$,*

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\prod_{r=1}^{\tau_N(t)} (1 - p_r) \right] = e^{-\alpha_n t} \tag{17}$$

where $\alpha_n := n(n-1)/2$.

Proof. We start by showing that $\lim_{N \rightarrow \infty} \mathbb{E} \left[\prod_{r=1}^{\tau_N(t)} (1 - p_r) \right] \leq e^{-\alpha_n t}$.

From Koskela et al. (2018, Lemma 1 Case 1), taking $\xi = \Delta$, we have for each r

$$1 - p_r = p_{\Delta\Delta}(r) \leq 1 - \alpha_n(1 + O(N^{-1})) [c_N(r) - B'_n D_N(r)] \tag{18}$$

where the $O(N^{-1})$ term does not depend on r . Applying a multinomial expansion and then separating the positive and negative terms,

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1 - p_r) &\leq 1 + \sum_{l=1}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l [c_N(s_j) - B'_n D_N(s_j)] \\
&= 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l [c_N(s_j) - B'_n D_N(s_j)] \\
&\quad - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l [c_N(s_j) - B'_n D_N(s_j)].
\end{aligned} \tag{19}$$

This is further bounded by applying Lemma 3 and then both bounds of Lemma 1:

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1 - p_r) &\leq 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l c_N(s_j) \\
&\quad - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left\{ \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l c_N(s_j) - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1 + B'_n)^l \right\} \\
&\leq 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \{t^l + c_N(\tau_N(t))(t+1)^l\} \\
&\quad - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left\{ t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \binom{l}{2} (t+1)^{l-2} - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1 + B'_n)^l \right\}.
\end{aligned} \tag{20}$$

Collecting some terms,

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1 - p_r) &\leq \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l + c_N(\tau_N(t)) \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} (t+1)^l \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \binom{l}{2} (t+1)^{l-2} \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} (t+1)^{l-1} (1 + B'_n)^l \\
&\leq \sum_{l=0}^{\infty} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_{\{\tau_N(t) \geq l\}} + c_N(\tau_N(t)) \exp[\alpha_n(1 + O(N^{-1}))(t+1)] \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n(1 + O(N^{-1}))(t+1)] \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n(1 + O(N^{-1}))(t+1)(1 + B'_n)].
\end{aligned} \tag{21}$$

Now, taking the expectation and limit, then applying Brown et al. (2021, Equations (3.3)–(3.5)) and Lemma 10,

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathbb{E} \left[\prod_{r=1}^{\tau_N(t)} (1 - p_r) \right] &\leq \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l \lim_{N \rightarrow \infty} \mathbb{P}[\tau_N(t) \geq l] + \lim_{N \rightarrow \infty} \mathbb{E}[c_N(\tau_N(t))] \exp[\alpha_n(t+1)] \\
&\quad + \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right] \frac{1}{2} \alpha_n^2 \exp[\alpha_n(t+1)] \\
&\quad + \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{s=1}^{\tau_N(t)} D_N(s) \right] \exp[\alpha_n(t+1)(1 + B'_n)] \\
&= \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l = e^{-\alpha_n t}.
\end{aligned} \tag{22}$$

It remains to show the corresponding lower bound $\lim_{N \rightarrow \infty} \mathbb{E} \left[\prod_{r=1}^{\tau_N(t)} (1 - p_r) \right] \geq e^{-\alpha_n t}$.

From Brown et al. (2021, Equation (3.14)), taking $\xi = \Delta$, we have

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \alpha_n(1 + O(N^{-1})) [c_N(t) + B_n D_N(t)] \tag{23}$$

where $B_n > 0$ and the $O(N^{-1})$ term does not depend on t . In particular,

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \frac{N^{n-2}}{(N-2)_{n-2}} \alpha_n c_N(t) - \frac{N^{n-3}}{(N-3)_{n-3}} \alpha_n B_n D_N(t). \quad (24)$$

Since $D_N(s) \leq c_N(s)$ for all s (Koskela et al., 2018, p.9), a sufficient condition for this bound to be non-negative is

$$E_r := \left\{ c_N(r) \leq \frac{(N-3)_{n-3}}{N^{n-3}} \frac{1}{\alpha_n} \left(\frac{N}{N-2} + B_n \right)^{-1} \right\}, \quad (25)$$

and we define $E := \bigcap_{r=1}^{\tau_N(t)} E_r$. We now apply a multinomial expansion to the product, and split into positive and negative terms:

$$\begin{aligned} \prod_{r=1}^{\tau_N(t)} (1 - p_r) &\geq \left\{ 1 + \sum_{l=1}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l [c_N(s_j) + B_n D_N(s_j)] \right\} \mathbb{1}_E \\ &= \left\{ 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l [c_N(s_j) + B_n D_N(s_j)] \right. \\ &\quad \left. - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l [c_N(s_j) + B_n D_N(s_j)] \right\} \mathbb{1}_E \end{aligned} \quad (26)$$

This is further bounded by applying Lemma 2 and both bounds in Lemma 1:

$$\begin{aligned} \prod_{r=1}^{\tau_N(t)} (1 - p_r) &\geq \left\{ 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l c_N(s_j) \right. \\ &\quad \left. - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left[\sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l c_N(s_j) + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1 + B_n)^l \right] \right\} \mathbb{1}_E \\ &\geq \left\{ 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left[t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \binom{l}{2} (t+1)^{l-2} \right] \right. \\ &\quad \left. - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left[t^l + c_N(\tau_N(t)) (t+1)^l + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1 + B_n)^l \right] \right\} \mathbb{1}_E. \end{aligned} \quad (27)$$

Collecting terms,

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1 - p_r) &\geq \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_E - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \binom{l}{2} (t+1)^{l-2} \\
&\quad - c_N(\tau_N(t)) \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} (t+1)^l \\
&\quad - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} (t+1)^{l-1} (1 + B_n)^l \\
&\geq \sum_{l=0}^{\infty} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_E \mathbb{1}_{\{\tau_N(t) \geq l\}} - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n(1 + O(N^{-1}))(t+1)] \\
&\quad - c_N(\tau_N(t)) \exp[\alpha_n(1 + O(N^{-1}))(t+1)] \\
&\quad - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n(1 + O(N^{-1}))(t+1)(1 + B_n)]. \tag{28}
\end{aligned}$$

Now, taking the expectation and limit, and applying Brown et al. (2021, Equations (3.3)–(3.5)) to show that all but the first sum vanish, and Lemmata 10 and 9 to show that $\lim_{N \rightarrow \infty} \mathbb{P}[\{\tau_N(t) \geq l\} \cap E] = 1$,

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathbb{E} \left[\prod_{r=1}^{\tau_N(t)} (1 - p_r) \right] &\geq \sum_{l=0}^{\infty} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \lim_{N \rightarrow \infty} \mathbb{P}[\{\tau_N(t) \geq l\} \cap E] \\
&\quad - \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right] \frac{1}{2} \alpha_n^2 \exp[\alpha_n(t+1)] \\
&\quad - \lim_{N \rightarrow \infty} \mathbb{E}[c_N(\tau_N(t))] \exp[\alpha_n(t+1)] \\
&\quad - \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{s=1}^{\tau_N(t)} D_N(s) \right] \exp[\alpha_n(t+1)(1 + B_n)] \\
&= \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l = e^{-\alpha_n t}. \tag{29}
\end{aligned}$$

Combining the upper and lower bounds in (22) and (29) respectively concludes the proof. ■

Lemma 5 (Induction step upper bound). *Fix $k \in \mathbb{N}$, $i_0 := 0$, $i_k := k$. For any sequence of times $0 = t_0 \leq t_1 \leq \dots \leq t_k \leq t$,*

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^k p_{r_i} \right) \left(\prod_{\substack{r=1 \\ r \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) \right) \right] \leq \alpha_n^k \sum_{l=0}^{\infty} \frac{1}{l!} (-\alpha_n)^l t^l \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}. \tag{30}$$

Proof. We use the bound on $(1 - p_r)$ from (18) and apply a multinomial expansion:

$$\begin{aligned}
\prod_{\substack{r=1 \\ \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) &\leq \prod_{\substack{r=1 \\ \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} \{1 - \alpha_n(1 + O(N^{-1}))[c_N(r) - B'_n D_N(r)]\} \\
&= 1 + \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{\substack{s_1 \neq \dots \neq s_l \\ \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} \prod_{j=1}^l [c_N(s_j) - B'_n D_N(s_j)] \\
&= 1 + \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l [c_N(s_j) - B'_n D_N(s_j)] \\
&\quad - \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{\substack{s_1 \neq \dots \neq s_l \\ \exists i, i' : s_i = r_{i'}}}^{\tau_N(t)} \prod_{j=1}^l [c_N(s_j) - B'_n D_N(s_j)]. \tag{31}
\end{aligned}$$

The penultimate line above is exactly the expansion we had in the basis step (19), except for the limit on l , and as such following the same arguments gives a bound like that in (21):

$$\begin{aligned}
1 + \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \dots \neq s_l}^{\tau_N(t)} \prod_{j=1}^l [c_N(s_j) - B'_n D_N(s_j)] \\
\leq \sum_{l=0}^{\tau_N(t)-k} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l + c_N(\tau_N(t)) \exp[\alpha_n(1 + O(N^{-1}))(t + 1)] \\
\quad + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n(1 + O(N^{-1}))(t + 1)] \\
\quad + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n(1 + O(N^{-1}))(t + 1)(1 + B'_n)]. \tag{32}
\end{aligned}$$

For the last line of (31),

$$\begin{aligned}
& - \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{\substack{s_1 \neq \dots \neq s_l \\ \exists i, i': s_i = r_{i'}}} \prod_{j=1}^l \{c_N(s_j) - B'_n D_N(s_j)\} \\
& \leq \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{\substack{s_1 \neq \dots \neq s_l \\ \exists i, i': s_i = r_{i'}}} \prod_{j=1}^l \{c_N(s_j) + B'_n D_N(s_j)\} \\
& \leq \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{\substack{s_1 \neq \dots \neq s_l \\ \exists i, i': s_i = r_{i'}}} (1 + B'_n)^l \prod_{j=1}^l c_N(s_j) \\
& \leq \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1 + O(N^{-1})) \frac{1}{(l-1)!} \sum_{s_1 \in \{r_1, \dots, r_k\}} \sum_{s_2 \neq \dots \neq s_l}^{\tau_N(t)} (1 + B'_n)^l \prod_{j=1}^l c_N(s_j) \\
& = \sum_{s \in \{r_1, \dots, r_k\}} c_N(s) \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1 + O(N^{-1})) \frac{1}{(l-1)!} (1 + B'_n)^l \sum_{s_1 \neq \dots \neq s_{l-1}}^{\tau_N(t)} \prod_{j=1}^{l-1} c_N(s_j) \\
& \leq \sum_{j=1}^k c_N(r_j) \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1 + O(N^{-1})) \frac{1}{(l-1)!} (1 + B'_n)^l (t+1)^{l-1} \\
& \leq \left(\sum_{j=1}^k c_N(r_j) \right) \alpha_n (1 + B'_n) \exp[\alpha_n (1 + O(N^{-1})) (1 + B'_n) (t+1)]. \tag{33}
\end{aligned}$$

Putting these together, we have

$$\begin{aligned}
\prod_{\substack{r=1 \\ r \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) & \leq \sum_{l=0}^{\tau_N(t)-k} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l + c_N(\tau_N(t)) \exp[\alpha_n (1 + O(N^{-1})) (t+1)] \\
& + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n (1 + O(N^{-1})) (t+1)] \\
& + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n (1 + O(N^{-1})) (t+1) (1 + B'_n)] \\
& + \left(\sum_{j=1}^k c_N(r_j) \right) \alpha_n (1 + B'_n) \exp[\alpha_n (1 + O(N^{-1})) (1 + B'_n) (t+1)]. \tag{34}
\end{aligned}$$

Meanwhile, using the bound on p_r from (23) then applying a modification of Lemma 2,

$$\begin{aligned}
\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k p_{r_i} & \leq \alpha_n^k (1 + O(N^{-1})) \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k [c_N(r_i) + B_n D_N(r_i)] \\
& \leq \alpha_n^k (1 + O(N^{-1})) \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \alpha_n^k (1 + O(N^{-1})) (t+1)^{k-1} (1 + B_n)^k. \tag{35}
\end{aligned}$$

A more liberal bound can be arrived at thus:

$$\begin{aligned}
\prod_{i=1}^k p_{r_i} &\leq \alpha_n^k (1 + O(N^{-1})) \prod_{i=1}^k [c_N(r_i) + B_n D_N(r_i)] \\
&\leq \alpha_n^k (1 + O(N^{-1})) \prod_{i=1}^k c_N(r_i) (1 + B_n) \\
&\leq \alpha_n^k (1 + O(N^{-1})) (1 + B_n)^k \prod_{i=1}^k c_N(r_i)
\end{aligned} \tag{36}$$

which also leads to the deterministic bound

$$\begin{aligned}
\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k p_{r_i} &\leq \alpha_n^k (1 + O(N^{-1})) (1 + B_n)^k \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) \\
&\leq \alpha_n^k (1 + O(N^{-1})) (1 + B_n)^k \frac{1}{k!} \sum_{r_1 \neq \dots \neq r_k}^{\tau_N(t)} \prod_{i=1}^k c_N(r_i) \\
&\leq \alpha_n^k (1 + O(N^{-1})) (1 + B_n)^k \frac{1}{k!} (t+1)^k.
\end{aligned} \tag{37}$$

Combining (34) with the other product, the expression inside the expectation in (30) is bounded above by

$$\begin{aligned}
&\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^k p_{r_i} \right) \left(\prod_{\substack{r=1 \\ r \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) \right) \leq \sum_{l=0}^{\tau_N(t)-k} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k p_{r_i} \\
&\quad + \left\{ c_N(\tau_N(t)) \exp[\alpha_n (1 + O(N^{-1}))(t+1)] + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n (1 + O(N^{-1}))(t+1)] \right. \\
&\quad \left. + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n (1 + O(N^{-1}))(t+1)(1 + B'_n)] \right\} \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k p_{r_i} \\
&\quad + \exp[\alpha_n (1 + O(N^{-1}))(1 + B'_n)(t+1)] \alpha_n (1 + B'_n) \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \sum_{j=1}^k c_N(r_j) \prod_{i=1}^k p_{r_i}.
\end{aligned} \tag{38}$$

Applying the various bounds (35)–(37), we have

$$\begin{aligned}
&\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^k p_{r_i} \right) \left(\prod_{\substack{r=1 \\ r \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) \right) \leq \alpha_n^k (1 + O(N^{-1})) \sum_{l=0}^{\tau_N(t)-k} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \alpha_n^k (1 + O(N^{-1})) (t+1)^{k-1} (1 + B_n)^k \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \\
&\quad + \left\{ c_N(\tau_N(t)) \exp[\alpha_n (1 + O(N^{-1}))(t+1)] + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n (1 + O(N^{-1}))(t+1)] \right. \\
&\quad \left. + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n (1 + O(N^{-1}))(t+1)(1 + B'_n)] \right\} \alpha_n^k (1 + O(N^{-1})) (1 + B_n)^k \frac{1}{k!} (t+1)^k \\
&\quad + \exp[\alpha_n (1 + B'_n)(t+1)] \alpha_n (1 + B'_n) \alpha_n^k (1 + O(N^{-1})) (1 + B_n)^k \\
&\quad \times \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \sum_{j=1}^k c_N(r_j) \prod_{i=1}^k c_N(r_i).
\end{aligned} \tag{39}$$

Upon taking the expectation and limit, we have

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^k p_{r_i} \right) \left(\prod_{\substack{r=1 \\ r \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) \right) \right] &\leq \alpha_n^k \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{l=0}^{\tau_N(t)-k} (-\alpha_n)^l \frac{1}{l!} t^l \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) \right] \\
&+ \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{s=1}^{\tau_N(t)} D_N(s) \right] \alpha_n^k (t+1)^{k-1} (1+B_n)^k \exp[\alpha_n t] \\
&+ \left\{ \lim_{N \rightarrow \infty} \mathbb{E} [c_N(\tau_N(t))] \exp[\alpha_n(t+1)] + \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right] \frac{1}{2} \alpha_n^2 \exp[\alpha_n(t+1)] \right. \\
&\quad \left. + \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{s=1}^{\tau_N(t)} D_N(s) \right] \exp[\alpha_n(t+1)(1+B'_n)] \right\} \alpha_n^k (1+B_n)^k \frac{1}{k!} (t+1)^k \\
&+ \exp[\alpha_n(1+B'_n)(t+1)] \alpha_n^{k+1} (1+B'_n)(1+B_n)^k \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \sum_{j=1}^k c_N(r_j) \prod_{i=1}^k c_N(r_i) \right]. \quad (40)
\end{aligned}$$

The middle terms vanish due to Brown et al. (2021, Equations (3.3)–(3.5)) and the expression becomes

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^k p_{r_i} \right) \left(\prod_{\substack{r=1 \\ r \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) \right) \right] &\leq \alpha_n^k \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l \lim_{N \rightarrow \infty} \mathbb{E} \left[\mathbb{1}_{\{\tau_N(t) \geq k+l\}} \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) \right] \\
&+ \exp[\alpha_n(1+B'_n)(t+1)] \alpha_n^{k+1} (1+B'_n)(1+B_n)^k \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \sum_{j=1}^k c_N(r_j) \prod_{i=1}^k c_N(r_i) \right]. \quad (41)
\end{aligned}$$

To simplify the second line,

$$\begin{aligned}
\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \sum_{j=1}^k c_N(r_j) \prod_{i=1}^k c_N(r_i) &\leq \frac{1}{k!} \sum_{r_1 \neq \dots \neq r_k}^{\tau_N(t)} \sum_{j=1}^k c_N(r_j) \prod_{i=1}^k c_N(r_i) \\
&= \frac{1}{k!} \sum_{r_1 \neq \dots \neq r_k}^{\tau_N(t)} \sum_{j=1}^k c_N(r_j)^2 \prod_{i \neq j} c_N(r_i) \\
&\leq \frac{1}{k!} \sum_{j=1}^k \sum_{s=1}^{\tau_N(t)} c_N(s)^2 \sum_{r_1 \neq \dots \neq r_{k-1}}^{\tau_N(t)} \prod_{i=1}^{k-1} c_N(r_i) \\
&\leq \frac{1}{(k-1)!} \sum_{s=1}^{\tau_N(t)} c_N(s)^2 (t+1)^{k-1} \quad (42)
\end{aligned}$$

hence

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \sum_{s \in \{r_1, \dots, r_k\}} c_N(s) \prod_{i=1}^k c_N(r_i) \right] \leq \frac{1}{(k-1)!} (t+1)^{k-1} \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right] = 0 \quad (43)$$

by Brown et al. (2021, Equation (3.5)). By Lemma 10, $\lim_{N \rightarrow \infty} \mathbb{P}[\tau_N(t) \geq k+l] = 1$, so we can apply Lemma 7 to

the remaining expectation in (41), yielding

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^k p_{r_i} \right) \left(\prod_{\substack{r=1 \\ r \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) \right) \right] &\leq \alpha_n^k \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l \lim_{N \rightarrow \infty} \mathbb{E} \left[\mathbb{1}_{\{\tau_N(t) \geq l\}} \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) \right] \\
&= \alpha_n^k \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \\
&= \alpha_n^k e^{-\alpha_n t} \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \tag{44}
\end{aligned}$$

as required. ■

Lemma 6 (Induction step lower bound). *Fix $k \in \mathbb{N}$, $i_0 := 0$, $i_k := k$. For any sequence of times $0 = t_0 \leq t_1 \leq \dots \leq t_k \leq t$,*

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^k p_{r_i} \right) \left(\prod_{\substack{r=1 \\ r \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) \right) \right] \geq \alpha_n^k \sum_{l=0}^{\infty} \frac{1}{l!} (-\alpha_n)^l t^l \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}. \tag{45}$$

Proof. Firstly,

$$\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^k p_{r_i} \right) \left(\prod_{\substack{r=1 \\ r \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) \right) \geq \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^k p_{r_i} \right) \left(\prod_{r=1}^{\tau_N(t)} (1 - p_r) \right). \tag{46}$$

Now the second product does not depend on r_1, \dots, r_k , and we can use the lower bound from (28):

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1 - p_r) &\geq \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_E - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n(1 + O(N^{-1}))(t + 1)] \\
&\quad - c_N(\tau_N(t)) \exp[\alpha_n(1 + O(N^{-1}))(t + 1)] \\
&\quad - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n(1 + O(N^{-1}))(t + 1)(1 + B_n)] \tag{47}
\end{aligned}$$

where E is defined as in (25). We will also need an upper bound on this product, which is formed from (21) with

a further deterministic bound:

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1 - p_r) &\leq \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l + c_N(\tau_N(t)) \exp[\alpha_n(1 + O(N^{-1}))(t+1)] \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n(1 + O(N^{-1}))(t+1)] \\
&\quad + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n(1 + O(N^{-1}))(t+1)(1 + B'_n)] \\
&\leq \exp[\alpha_n(1 + O(N^{-1}))t] + \exp[\alpha_n(1 + O(N^{-1}))(t+1)] \\
&\quad + \frac{1}{2} \alpha_n^2 (t+1) \exp[\alpha_n(1 + O(N^{-1}))(t+1)] + (t+1) \exp[\alpha_n(1 + O(N^{-1}))(t+1)(1 + B'_n)] \\
&\leq \left(2 + \frac{\alpha_n^2(t+1)}{2} \right) \exp[\alpha_n(1 + O(N^{-1}))(t+1)] + (t+1) \exp[\alpha_n(1 + O(N^{-1}))(t+1)(1 + B'_n)]. \quad (48)
\end{aligned}$$

Now let us consider the remaining sum-product. We use the same bound on p_r as in (18):

$$p_r = 1 - p_{\Delta\Delta}(r) \geq \alpha_n(1 + O(N^{-1})) [c_N(r) - B'_n D_N(r)]. \quad (49)$$

A sufficient condition to ensure this bound is non-negative is the event

$$E'_r := \{c_N(r) \geq B'_n D_N(r)\} \quad (50)$$

and we define $E' := \bigcap_{r=1}^{\tau_N(t)} E'_r$. Applying a modification of Lemma 3,

$$\begin{aligned}
\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^k p_{r_i} \right) &\geq \alpha_n^k (1 + O(N^{-1})) \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k [c_N(r_i) - B'_n D_N(r_i)] \mathbb{1}_{E'} \\
&\geq \alpha_n^k (1 + O(N^{-1})) \left\{ \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) \mathbb{1}_{E'} - \frac{1}{k!} \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{k-1} (1 + B'_n)^k \right\}. \quad (51)
\end{aligned}$$

The above expression is already split into even and odd terms; a lower bound on (46) can be formed by multiplying

the even terms by the lower bound (47) and the odd terms by the upper bound (48). Thus

$$\begin{aligned}
\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^k p_{r_i} \right) \left(\prod_{\substack{r=1 \\ r \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) \right) &\geq \alpha_n^k (1 + O(N^{-1})) \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) \mathbb{1}_{E'} \left\{ \right. \\
&\quad \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_E \\
&\quad - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n (1 + O(N^{-1})) (t+1)] \\
&\quad - c_N(\tau_N(t)) \exp[\alpha_n (1 + O(N^{-1})) (t+1)] \\
&\quad \left. - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n (1 + O(N^{-1})) (t+1)(1 + B_n)] \right\} \\
&\quad - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \alpha_n^k (1 + O(N^{-1})) \frac{1}{k!} (t+1)^{k-1} (1 + B'_n)^k \left\{ \right. \\
&\quad \left(2 + \frac{\alpha_n^2 (t+1)}{2} \right) \exp[\alpha_n (1 + O(N^{-1})) (t+1)] \\
&\quad \left. + (t+1) \exp[\alpha_n (1 + O(N^{-1})) (t+1)(1 + B'_n)] \right\}. \tag{52}
\end{aligned}$$

Due to Brown et al. (2021, Equations (3.3)–(3.5)), all of the negative terms have vanishing expectation, leaving

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^k p_{r_i} \right) \left(\prod_{\substack{r=1 \\ r \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) \right) \right] \\
\geq \lim_{N \rightarrow \infty} \mathbb{E} \left[\alpha_n^k (1 + O(N^{-1})) \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) \mathbb{1}_{E'} \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_E \right] \\
= \alpha_n^k \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l \lim_{N \rightarrow \infty} \mathbb{E} \left[\mathbb{1}_{\{\tau_N(t) \geq l\}} \mathbb{1}_{E \cap E'} \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) \right] \tag{53}
\end{aligned}$$

Lemmata 9 and 12 establish that $\lim_{N \rightarrow \infty} \mathbb{P}[E \cap E'] = 1$ and Lemma 10 deals with the other indicator. We can therefore apply Lemma 7 to conclude that

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^k p_{r_i} \right) \left(\prod_{\substack{r=1 \\ r \notin \{r_1, \dots, r_k\}}}^{\tau_N(t)} (1 - p_r) \right) \right] &\geq \alpha_n^k \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \\
&= \alpha_n^k e^{-\alpha_n t} \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \tag{54}
\end{aligned}$$

as required. ■

Lemma 7. Fix $l, k \in \mathbb{N}$, $i_0 := 0$, $i_k := k$. Let E be any event independent of r_1, \dots, r_k such that $\lim_{N \rightarrow \infty} \mathbb{P}[E] = 1$. Then for any sequence of times $0 = t_0 \leq t_1 \leq \dots \leq t_k \leq t$,

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\mathbb{1}_E \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) \right] = \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}. \quad (55)$$

Proof. As pointed out by Möhle (1999, p. 460), the sum-product on the left hand side can be expanded as

$$\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) = \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \sum_{\substack{r_{i_{j-1}+1} < \dots < r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i). \quad (56)$$

Moreover,

$$\sum_{\substack{r_{i_{j-1}+1} < \dots < r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) = \frac{1}{(i_j - i_{j-1})!} \sum_{\substack{r_{i_{j-1}+1} \neq \dots \neq r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i). \quad (57)$$

An argument akin to (4) gives us an upper bound:

$$\begin{aligned} \sum_{\substack{r_{i_{j-1}+1} \neq \dots \neq r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) &\leq \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s) \right)^{i_j - i_{j-1}} \leq [t_j - t_{j-1} + c_N(\tau_N(t_j))]^{i_j - i_{j-1}} \\ &= \sum_{l=0}^{i_j - i_{j-1}} \binom{i_j - i_{j-1}}{l} (t_j - t_{j-1})^l [c_N(\tau_N(t_j))]^{i_j - i_{j-1} - l} \\ &= (t_j - t_{j-1})^{i_j - i_{j-1}} \\ &\quad + c_N(\tau_N(t_j)) \sum_{l=0}^{i_j - i_{j-1} - 1} \binom{i_j - i_{j-1}}{l} (t_j - t_{j-1})^l [c_N(\tau_N(t_j))]^{i_j - i_{j-1} - 1 - l} \\ &\leq (t_j - t_{j-1})^{i_j - i_{j-1}} + c_N(\tau_N(t_j))(t_j - t_{j-1} + 1)^k, \end{aligned} \quad (58)$$

using in the last line that $c_N \leq 1$ and $0 \leq i_j - i_{j-1} \leq k$. Now, taking the product on the outside,

$$\begin{aligned}
\prod_{j=1}^k \sum_{\substack{r_{i_{j-1}+1} < \dots < r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) &\leq \prod_{j=1}^k \left\{ \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} + c_N(\tau_N(t_j)) \frac{(1 + t_j - t_{j-1})^k}{(i_j - i_{j-1})!} \right\} \\
&\leq \prod_{j=1}^k \left\{ \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} + c_N(\tau_N(t_j)) (1 + t_j - t_{j-1})^k \right\} \\
&= \sum_{\mathcal{I} \subseteq [k]} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} c_N(\tau_N(t_j)) (1 + t_j - t_{j-1})^k \right) \\
&= \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \\
&\quad + \sum_{\mathcal{I} \subseteq [k]} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} c_N(\tau_N(t_j)) (1 + t_j - t_{j-1})^k \right) \\
&\leq \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \\
&\quad + \sum_{\mathcal{I} \subseteq [k]} c_N(\tau_N(t_{j^*})) \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) (1 + t_j - t_{j-1})^{k^2} \tag{59}
\end{aligned}$$

where, say, $j^* := \min\{j \notin \mathcal{I}\}$. Now we are in a position to evaluate the limit in (55):

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathbb{E} \left[\mathbb{1}_E \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) \right] &\leq \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) \right] \\
&\leq \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \\
&\quad + \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \sum_{\mathcal{I} \subseteq [k]} \lim_{N \rightarrow \infty} \mathbb{E} [c_N(\tau_N(t_{j^*}))] \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) (1 + t_j - t_{j-1})^{k^2} \\
&= \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \tag{60}
\end{aligned}$$

using Brown et al. (2021, Equation (3.3)). For the corresponding lower bound, by a slight modification of (2),

$$\begin{aligned}
\sum_{\substack{r_{i_{j-1}+1} \neq \dots \neq r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) &\geq \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s) \right)^{i_j-i_{j-1}} \\
&\quad - \binom{i_j-i_{j-1}}{2} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s) \right)^{i_j-i_{j-1}-2} \\
&\geq (t_j - t_{j-1})^{i_j-i_{j-1}} - \binom{i_j-i_{j-1}}{2} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{i_j-i_{j-1}-2} \\
&\geq (t_j - t_{j-1})^{i_j-i_{j-1}} - (i_j - i_{j-1})! \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{k-1}. \quad (61)
\end{aligned}$$

Define the event

$$E_j^* = \left\{ \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) \leq \frac{(t_j - t_{j-1})^{i_j-i_{j-1}-k+1}}{(i_j - i_{j-1})!} \right\}, \quad (62)$$

which is sufficient to ensure the j^{th} term in the following is non-negative, and let $E^* := \bigcap_{j=1}^k E_j^*$. Now, taking a product over j as in (59),

$$\begin{aligned}
\prod_{j=1}^k \sum_{\substack{r_{i_{j-1}+1} < \dots < r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) &\geq \prod_{j=1}^k \left\{ \frac{(t_j - t_{j-1})^{i_j-i_{j-1}}}{(i_j - i_{j-1})!} - \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{k-1} \right\} \mathbb{1}_{E^*} \\
&= \sum_{\mathcal{I} \subseteq [k]} (-1)^{k-|\mathcal{I}|} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j-i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{k-1} \right) \mathbb{1}_{E^*} \\
&= \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j-i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E^*} \\
&\quad + \sum_{\mathcal{I} \subseteq [k]} (-1)^{k-|\mathcal{I}|} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j-i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{k-1} \right) \\
&\geq \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j-i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E^*} \\
&\quad - \sum_{\substack{\mathcal{I} \subseteq [k]: \\ k-|\mathcal{I}| \text{ odd}}} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j-i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{k-1} \right) \\
&\geq \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j-i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E^*} \\
&\quad - \sum_{\mathcal{I} \subseteq [k]} \left(\sum_{s=\tau_N(t_{j^*-1})+1}^{\tau_N(t_{j^*})} c_N(s)^2 \right) \left(\prod_{j \in \mathcal{I}} (t_j - t_{j-1})^k \right) \left(\prod_{j \notin \mathcal{I}} (t_j - t_{j-1} + 1)^k \right) \\
&\geq \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j-i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E^*} - \sum_{\mathcal{I} \subseteq [k]} \left(\sum_{s=\tau_N(t_{j^*-1})+1}^{\tau_N(t_{j^*})} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{k^2}. \quad (63)
\end{aligned}$$

We can now evaluate the limit:

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \mathbb{E} \left[\mathbb{1}_E \sum_{\substack{r_1 < \dots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^k c_N(r_i) \right] \geq \lim_{N \rightarrow \infty} \mathbb{E} \left[\mathbb{1}_{E \cap E^*} \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right] \\
& \quad - \lim_{N \rightarrow \infty} \mathbb{E} \left[\mathbb{1}_E \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \sum_{\mathcal{I} \subset [k]} \left(\sum_{s=\tau_N(t_{j^*-1})+1}^{\tau_N(t_{j^*})} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{k^2} \right] \\
& \geq \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \lim_{N \rightarrow \infty} \mathbb{E}[\mathbb{1}_{E \cap E^*}] \\
& \quad - \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \sum_{\mathcal{I} \subset [k]} \left(\sum_{s=\tau_N(t_{j^*-1})+1}^{\tau_N(t_{j^*})} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{k^2} \right] \\
& = \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \lim_{N \rightarrow \infty} \mathbb{P}[E \cap E^*] \\
& \quad - \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \sum_{\mathcal{I} \subset [k]} \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{s=\tau_N(t_{j^*-1})+1}^{\tau_N(t_{j^*})} c_N(s)^2 \right] (t_j - t_{j-1} + 1)^{k^2} \\
& = \sum_{\substack{i_1 \leq \dots \leq i_{k-1} \\ \in \{0, \dots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \tag{64}
\end{aligned}$$

where for the last equality we use Brown et al. (2021, Equation (3.5)) to show that the second sum vanishes and Lemma 11 to show that $\lim_{N \rightarrow \infty} \mathbb{P}[E \cap E^*] = 1$. We have shown that the upper and lower bounds coincide, so the result follows. \blacksquare

Indicators

Lemma 8. *Let K be a constant which may depend on n, N but not on r , such that $K^{-2} = O(1)$ as $N \rightarrow \infty$. Define the events $E_r := \{c_N(r) < K\}$ and denote $E := \bigcap_{r=1}^{\tau_N(t)} E_r$. Then $\lim_{N \rightarrow \infty} \mathbb{P}[E] = 1$.*

Proof.

$$\begin{aligned}
\mathbb{P}[E] &= 1 - \mathbb{P}[E^c] = 1 - \mathbb{P} \left[\bigcup_{r=1}^{\tau_N(t)} E_r^c \right] = 1 - \mathbb{E} [\mathbb{1}_{\bigcup_{r=1}^{\tau_N(t)} E_r^c}] \geq 1 - \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r^c} \right] \\
&= 1 - \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} \mathbb{E} [\mathbb{1}_{E_r^c} \mid \mathcal{F}_{r-1}] \right] = 1 - \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} \mathbb{P}[E_r^c \mid \mathcal{F}_{r-1}] \right] \tag{65}
\end{aligned}$$

where for the second line we apply Lemma 13 with $f(r) = 1_{E_r^c}$. By the generalised Markov inequality,

$$\mathbb{P}[E_r^c \mid \mathcal{F}_{r-1}] = \mathbb{P}[c_N(r) \geq K \mid \mathcal{F}_{r-1}] \leq \mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}] K^{-2}. \quad (66)$$

Substituting this into (66) and applying Lemma 13 again, this time with $f(r) = c_N(r)^2$,

$$\mathbb{P}[E] \geq 1 - K^{-2} \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}] \right] = 1 - K^{-2} \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} c_N(r)^2 \right]. \quad (67)$$

Applying Brown et al. (2021, Equation (3.5)), the limit is

$$\lim_{N \rightarrow \infty} \mathbb{P}[E] = 1 - O(1) \times 0 = 1 \quad (68)$$

as required. ■

Lemma 9. Fix $t > 0$. For any $l \in \mathbb{R}$, $\lim_{N \rightarrow \infty} \mathbb{P}[\tau_N(t) \geq l] = 1$.

Proof.

$$\{\tau_N(t) \geq l\} = \left\{ \min \left\{ s \geq 1 : \sum_{r=1}^s c_N(r) \geq t \right\} \geq l \right\} = \left\{ \sum_{r=1}^{l-1} c_N(r) < t \right\} \supseteq \bigcap_{r=1}^{l-1} \left\{ c_N(r) < \frac{t}{l} \right\} \supseteq \bigcap_{r=1}^{\tau_N(t)} \left\{ c_N(r) < \frac{t}{l} \right\}. \quad (69)$$

Hence

$$\lim_{N \rightarrow \infty} \mathbb{P}[\tau_N(t) \geq l] \geq \lim_{N \rightarrow \infty} \mathbb{P} \left[\bigcap_{r=1}^{\tau_N(t)} \left\{ c_N(r) < \frac{t}{l} \right\} \right] \quad (70)$$

and the result follows by applying Lemma 9 with $K = t/l$. ■

Lemma 10. Define the event

$$E^* := \bigcap_{j=1}^k \left\{ \sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \leq \frac{(t_j - t_{j-1})^{i_j - i_{j-1} - k + 1}}{(i_j - i_{j-1})!} \right\}. \quad (71)$$

Then $\lim_{N \rightarrow \infty} \mathbb{P}[E^*] = 1$.

Proof.

$$\begin{aligned} E^* &\supseteq \left\{ \sum_{j=1}^k \sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \leq \sum_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1} - k + 1}}{(i_j - i_{j-1})!} \right\} \\ &= \left\{ \sum_{s=1}^{\tau_N(t)} c_N(s)^2 \leq \sum_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1} - k + 1}}{(i_j - i_{j-1})!} \right\} \\ &\supseteq \left\{ \sum_{s=1}^{\tau_N(t)} c_N(s)^2 \leq \frac{1}{k!} \sum_{j=1}^k (t_j - t_{j-1})^{i_j - i_{j-1} - k + 1} \right\}. \end{aligned} \quad (72)$$

To simplify the RHS further, consider the possible values of $(i_j - i_{j-1} - k + 1) \in \{-k + 1, \dots, 1\}$: This simplification isn't necessary for the result, but it makes the expressions less cumbersome later on.

Case $(i_j - i_{j-1} - k + 1) < 0$:

$$\sum_{j=1}^k (t_j - t_{j-1})^{i_j - i_{j-1} - k + 1} \geq \sum_{j=1}^k t^{i_j - i_{j-1} - k + 1} \geq \sum_{j=1}^k t^{-k + 1} = kt^{-k + 1}. \quad (73)$$

Case $(i_j - i_{j-1} - k + 1) = 0$:

$$\sum_{j=1}^k (t_j - t_{j-1})^{i_j - i_{j-1} - k + 1} = \sum_{j=1}^k 1 = k. \quad (74)$$

Case $(i_j - i_{j-1} - k + 1) = 1$:

$$\sum_{j=1}^k (t_j - t_{j-1})^{i_j - i_{j-1} - k + 1} = \sum_{j=1}^k (t_j - t_{j-1}) = t_k - t_0 = t. \quad (75)$$

Altogether

$$\sum_{j=1}^k (t_j - t_{j-1})^{i_j - i_{j-1} - k + 1} \geq \min\{kt^{-k+1}, k, t\} = \min\{kt^{-k+1}, t\} = t \min\{kt^{-k}, 1\}, \quad (76)$$

so

$$E^\star \supseteq \left\{ \sum_{s=1}^{\tau_N(t)} c_N(s)^2 < \frac{t}{k!} \min\{kt^{-k}, 1\} \right\}. \quad (77)$$

Using Markov's inequality,

$$\begin{aligned} \mathbb{P}[E^\star] &\geq \mathbb{P} \left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 < \frac{t}{k!} \min\{kt^{-k}, 1\} \right] = 1 - \mathbb{P} \left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \geq \frac{t}{k!} \min\{kt^{-k}, 1\} \right] \\ &\geq 1 - \frac{k!}{t} \max\{1, k^{-1}t^k\} \mathbb{E} \left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right], \end{aligned} \quad (78)$$

and by Brown et al. (2021, Equation (3.5))

$$\lim_{N \rightarrow \infty} \mathbb{P}[E^\star] = 1 - O(1) \times 0 = 1 \quad (79)$$

as required. ■

Lemma 11. *Let K be a constant not depending on N, r , but which may depend on n .*

$$\lim_{N \rightarrow \infty} \mathbb{P} \left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) \geq KD_N(r)\} \right] = 1. \quad (80)$$

Proof.

$$\begin{aligned} \mathbb{P} \left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) \geq KD_N(r)\} \right] &\geq \mathbb{P} \left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) > KD_N(r)\} \right] \\ &= 1 - \mathbb{P} \left[\bigcup_{r=1}^{\tau_N(t)} \{c_N(r) \leq KD_N(r)\} \right] \\ &= 1 - \mathbb{E} [\mathbb{1}_{\bigcup_{r=1}^{\tau_N(t)} \{c_N(r) \leq KD_N(r)\}}] \\ &\geq 1 - \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} \mathbb{1}_{\{c_N(r) \leq KD_N(r)\}} \right] \\ &= 1 - \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} \mathbb{P}[c_N(r) \leq KD_N(r) \mid \mathcal{F}_{r-1}] \right] \end{aligned} \quad (81)$$

where the final inequality is an application of Lemma 13 with $f(r) = \mathbb{1}_{\{c_N(r) \leq K D_N(r)\}}$.

Fix $0 < \varepsilon < K^{-1}/2$ and let $N > \max\{\varepsilon^{-1}, ((\binom{n-2}{2} - 2\varepsilon)^{-1})\}$. For each r, i define the event $A_i(r) := \{\nu_r^{(i)} \leq N\varepsilon\}$. Conditional on \mathcal{F}_{r-1} , we have

$$D_N(r) = \frac{1}{N(N)_2} \sum_{i=1}^N (\nu_r^{(j)})_2 \left[\nu_r^{(i)} + \frac{1}{N} \sum_{j \neq i} (\nu_r^{(j)})^2 \right] \mathbb{1}_{A_i(r)^c} + \frac{1}{N(N)_2} \sum_{i=1}^N (\nu_r^{(i)})_2 \left[\nu_r^{(i)} + \frac{1}{N} \sum_{j \neq i} (\nu_r^{(j)})^2 \right] \mathbb{1}_{A_i(r)}. \quad (82)$$

For the first term,

$$\frac{1}{N(N)_2} \sum_{i=1}^N (\nu_r^{(i)})_2 \left[\nu_r^{(i)} + \frac{1}{N} \sum_{j \neq i} (\nu_r^{(j)})^2 \right] \mathbb{1}_{A_i(r)^c} \leq \sum_{i=1}^N \mathbb{1}_{A_i(r)^c}. \quad (83)$$

For the second term,

$$\begin{aligned} \frac{1}{N(N)_2} \sum_{i=1}^N (\nu_r^{(i)})_2 \left[\nu_r^{(i)} + \frac{1}{N} \sum_{j \neq i} (\nu_r^{(j)})^2 \right] \mathbb{1}_{A_i(r)} &\leq \frac{1}{N(N)_2} \sum_{i=1}^N (\nu_r^{(i)})_2 \nu_r^{(i)} \mathbb{1}_{A_i(r)} + \frac{1}{N^2(N)_2} \sum_{i=1}^N (\nu_r^{(i)})_2 \sum_{j=1}^N (\nu_r^{(j)})^2 \mathbb{1}_{A_i(r)} \\ &\leq \frac{1}{N} c_N(r) N\varepsilon + \frac{1}{N^2(N)_2} \sum_{i=1}^N (\nu_r^{(i)})_2 \sum_{j=1}^N (\nu_r^{(j)})^2 \mathbb{1}_{A_i(r)} \\ &\quad + \frac{1}{N^2(N)_2} \sum_{i=1}^N (\nu_r^{(i)})_2 \sum_{j=1}^N (\nu_r^{(j)}) \mathbb{1}_{A_i(r)} \\ &\leq \varepsilon c_N(r) + \frac{1}{N^2} \sum_{i=1}^N \nu_r^{(i)} N\varepsilon c_N(r) + \frac{1}{N^2} c_N(r) N \\ &= c_N(r) \left(2\varepsilon + \frac{1}{N} \right). \end{aligned} \quad (84)$$

Hence, conditional on \mathcal{F}_{r-1} ,

$$\begin{aligned} \{c_N(r) \geq K D_N(r)\} &\supseteq \left\{ c_N(r) \leq K c_N(r) (2\varepsilon + N^{-1}) + K \sum_{i=1}^N \mathbb{1}_{A_i(r)^c} \right\} \\ &= \left\{ K^{-1} - 2\varepsilon - \frac{1}{N} \leq \sum_{i=1}^N \mathbb{1}_{A_i(r)^c} c_N(r)^{-1} \right\} \end{aligned} \quad (85)$$

where the ratio $\mathbb{1}_{A_i(r)^c}/c_N(r)$ is well-defined because

$$A_i(r)^c \Rightarrow c_N(r) := \frac{1}{(N)_2} \sum_{j=1}^N (\nu_r^{(j)})_2 \geq \frac{1}{(N)_2} (\nu_r^{(i)})_2 \geq \frac{\varepsilon(N\varepsilon - 1)}{N - 1} \geq \varepsilon \left(\varepsilon - \frac{1}{N} \right) > 0. \quad (86)$$

Hence by Markov's inequality (the conditions on ε, N ensuring the constant is always strictly positive),

$$\begin{aligned} \mathbb{P}[c_N(r) \leq K D_N(r) \mid \mathcal{F}_{r-1}] &\leq \mathbb{P} \left[\sum_{i=1}^N \mathbb{1}_{A_i(r)^c} \geq \left(K^{-1} - 2\varepsilon - \frac{1}{N} \right) \varepsilon \left(\varepsilon - \frac{1}{N} \right) \middle| \mathcal{F}_{r-1} \right] \\ &\leq \frac{1}{\left(K^{-1} - 2\varepsilon - \frac{1}{N} \right) \varepsilon \left(\varepsilon - \frac{1}{N} \right)} \mathbb{E} \left[\sum_{i=1}^N \mathbb{1}_{A_i(r)^c} \middle| \mathcal{F}_{r-1} \right] \\ &\leq \frac{1}{\left(K^{-1} - 2\varepsilon - \frac{1}{N} \right) \varepsilon \left(\varepsilon - \frac{1}{N} \right)} \mathbb{E} \left[\sum_{i=1}^N \frac{(\nu_r^{(i)})_3}{(N\varepsilon)_3} \middle| \mathcal{F}_{r-1} \right] \\ &\leq \frac{1}{\left(K^{-1} - 2\varepsilon - \frac{1}{N} \right) \varepsilon \left(\varepsilon - \frac{1}{N} \right)} \mathbb{E} \left[\frac{N(N)_2}{(N\varepsilon)_3} D_N(r) \middle| \mathcal{F}_{r-1} \right]. \end{aligned} \quad (87)$$

Applying Lemma 13 once more, with $f(r) = D_N(r)$,

$$\begin{aligned} \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} \mathbb{P}[c_N(r) \leq K D_N(r) \mid \mathcal{F}_{r-1}] \right] &\leq \frac{1}{(K^{-1} - 2\varepsilon - \frac{1}{N}) \varepsilon (\varepsilon - \frac{1}{N})} \frac{N(N)_2}{(N\varepsilon)_3} \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[D_N(r) \mid \mathcal{F}_{r-1}] \right] \\ &= \frac{1}{(K^{-1} - 2\varepsilon - \frac{1}{N}) \varepsilon (\varepsilon - \frac{1}{N})} \frac{N(N)_2}{(N\varepsilon)_3} \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} D_N(r) \right] \\ &\xrightarrow{N \rightarrow \infty} \frac{1}{(K^{-1} - 2\varepsilon) \varepsilon^5} \times 0 = 0. \end{aligned} \quad (88)$$

Substituting this back into (82) concludes the proof. \blacksquare

Other useful results

The following Lemma is taken from Koskela et al. (2018, Lemma 2), where the function is set to $f(t) = c_N(t)$, but the authors remark that the result holds for other choices of function.

Lemma 12. *Let (\mathcal{F}_t) be the backwards-in-time filtration generated by the offspring counts $\nu_t^{(1:N)}$ at each generation t , and let $f(t)$ be any deterministic function of $\nu_t^{(1:N)}$ that is non-negative and bounded. In particular, for all t there exists $B < \infty$ such that $0 \leq f(t) \leq B$. Then*

$$\mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} f(r) \right] = \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}] \right]. \quad (89)$$

Proof. Define

$$M_s := \sum_{r=1}^s \{f(r) - \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\}. \quad (90)$$

It is easy to establish that (M_s) is a martingale with respect to (\mathcal{F}_s) , and $M_0 = 0$. Now fix $K \geq 1$ and note that $\tau_N(t) \wedge K$ is a bounded \mathcal{F}_t -stopping time. Hence we can apply the optional stopping theorem:

$$\mathbb{E}[M_{\tau_N(t) \wedge K}] = \mathbb{E} \left[\sum_{r=1}^{\tau_N(t) \wedge K} \{f(r) - \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\} \right] = \mathbb{E} \left[\sum_{r=1}^{\tau_N(t) \wedge K} f(r) \right] - \mathbb{E} \left[\sum_{r=1}^{\tau_N(t) \wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}] \right] = 0. \quad (91)$$

Since this holds for all $K \geq 1$,

$$\lim_{K \rightarrow \infty} \mathbb{E} \left[\sum_{r=1}^{\tau_N(t) \wedge K} f(r) \right] = \lim_{K \rightarrow \infty} \mathbb{E} \left[\sum_{r=1}^{\tau_N(t) \wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}] \right]. \quad (92)$$

The monotone convergence theorem allows the limit to pass inside the expectation on each side (since increasing K can only increase each sum, by possibly adding non-negative terms). Hence

$$\mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} f(r) \right] = \mathbb{E} \left[\lim_{K \rightarrow \infty} \sum_{r=1}^{\tau_N(t) \wedge K} f(r) \right] = \mathbb{E} \left[\lim_{K \rightarrow \infty} \sum_{r=1}^{\tau_N(t) \wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}] \right] = \mathbb{E} \left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}] \right] \quad (93)$$

which concludes the proof. \blacksquare

References

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