Weak convergence proof v.3

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1 Useful definitions/properties

This section is a placeholder to provide definitions etc. for cross-referencing. In the end all this stuff will presumably be scattered throughout the introduction or whatever. —SB

Let $\nu_t^{(i)}$ be the number of offspring in generation t of particle i ($t \in \mathbb{N}$, $i = 1, \dots, N$). Let $(\mathcal{F}_t)$ be the reverse-time filtration generated by the offspring counts. Define the following:

$$c_N(t) := \frac{1}{(N)_2} \sum_{i=1}^{N} (\nu_t^{(i)})_2 \tag{1}$$

$$D_N(t) := \frac{1}{N(N)_2} \sum_{i=1}^{N} (\nu_t^{(i)})_2 \left\{ \nu_t^{(i)} + \frac{1}{N} \sum_{j \neq i} (\nu_t^{(j)})^2 \right\} \tag{2}$$

$$\tau_N(t) := \inf \left\{ s \geq 1 : \sum_{r=1}^{s} c_N(r) \geq t \right\} \tag{3}$$

We have the following properties: for all $t \in \mathbb{N}$, $t' > 0$,

$$c_N(t), D_N(t) \in [0, 1] \tag{4}$$

$$D_N(t) \leq c_N(t) \tag{5}$$

$$c_N(t)^2 \leq c_N(t) \tag{6}$$

$$t' \leq \sum_{r=1}^{\tau_N(t')} c_N(r) \leq t' + 1. \tag{7}$$

Proof. For (4): $c_N(t)$ and $D_N(t)$ are clearly non-negative. Both are maximised when one of the offspring counts is equal to N and the rest are zero, in which case $c_N(t) = D_N(t) = 1$.

For (5): as outlined in Koskela et al. (2018, p.9),

$$\begin{aligned} D_N(t) &:= \frac{1}{(N)_2} \sum_{i=1}^{N} (\nu_t^{(i)})_2 \frac{1}{N} \left\{ \nu_t^{(i)} + \frac{1}{N} \sum_{j \neq i}^{N} (\nu_t^{(j)})^2 \right\} \leq \frac{1}{(N)_2} \sum_{i=1}^{N} (\nu_t^{(i)})_2 \frac{1}{N} \left\{ \nu_t^{(i)} + \frac{1}{N} \sum_{j \neq i}^{N} N \nu_t^{(j)} \right\} \\ &= \frac{1}{(N)_2} \sum_{i=1}^{N} (\nu_t^{(i)})_2 \frac{1}{N} \left\{ \sum_{j=1}^{N} \nu_t^{(j)} \right\} \leq \frac{1}{(N)_2} \sum_{i=1}^{N} (\nu_t^{(i)})_2 = c_N(t). \end{aligned}$$

For (6): this is immediate given (4).

For (7): this follows directly from the definition of τ_N in (3). ■

Let $\mathcal{P}_n$ be the space of partitions of $\{1, \dots, n\}$, and denote by Δ the partition of singletons $\{\{1\}, \dots, \{n\}\}$. Let $p_{\xi\eta}(t)$ denote the transition probabilities of the genealogical process ($t \in \mathbb{N}$, $\xi, \eta \in \mathcal{P}_n$). Then, following Koskela et al. (2018) but keeping the terms in N explicit, we have the following upper bound which holds when N is sufficiently large (namely when $N \geq n^2(n-3) + 1$):

$$p_{\Delta\Delta}(t) \leq 1 - \alpha_n \frac{N^{n-2}}{(N-2)_{n-2}} \left[c_N(t) - B'_n D_N(t) \right] \tag{8}$$

where $\alpha_n := \binom{n}{2}$ and B'_n...? A corresponding lower bound (which holds whenever $N \geq 3$), is found by following Brown et al. (2021, (3.14)) but keeping the terms in N explicit:

$$p_{\Delta\Delta}(t) \geq 1 - \alpha_n \frac{N^{n-2}}{(N-2)_{n-2}} \left[c_N(t) + B_n D_N(t)\right] \tag{9}$$

where $B_n =$...? It would make more sense to introduce these bounds in the other order, so that B_n appears before B'_n. State these bounds for general $p_{\xi\xi}$? —SB

2 The theorem

We start by defining a suitable metric space. Let $\mathcal{P}_n$ be the space of partitions of $\{1, \ldots, n\}$. Denote by $\mathcal{X}$ the set of all functions mapping $[0, \infty)$ to $\mathcal{P}_n$ that are right-continuous with left limits. (Our rescaled genealogical process $(\mathcal{G}^{(n,N)}_{\tau_N(t)})_{t\geq 0}$ and our encoding of the n-coalescent are piecewise-constant functions mapping time $t \in [0, \infty)$ to partitions, and thus live in the space $\mathcal{X}$.) Finally, equip the space $\mathcal{P}_n$ with the zero-one metric,

$$\rho(\xi, \eta) = 1 - \delta_{\xi\eta} := \begin{cases} 0 & \text{if } \xi = \eta \\ 1 & \text{otherwise} \end{cases}$$

for any $\xi, \eta \in \mathcal{P}_n$.

Theorem 1. *Let $\nu_t^{(1:N)}$ denote the offspring numbers in an interacting particle system satisfying the standing assumption and such that, for any N sufficiently large, for all finite t, $\mathbb{P}[\tau_N(t) = \infty] = 0$. Suppose that there exists a deterministic sequence $(b_N)_{N\in\mathbb{N}}$ such that $\lim_{N\to\infty} b_N = 0$ and*

$$\frac{1}{(N)_3} \sum_{i=1}^{N} \mathbb{E}_t\{(\nu_t^{(i)})_3\} \leq b_N \frac{1}{(N)_2} \sum_{i=1}^{N} \mathbb{E}_t\{(\nu_t^{(i)})_2\} \tag{10}$$

for all N, uniformly in $t \geq 1$. Then the rescaled genealogical process $(G^{(n,N)}_{\tau_N(t)})_{t\geq 0}$ converges weakly in $(\mathcal{X}, \rho)$ to Kingman's n-coalescent as $N \to \infty$.

As shown in Brown et al. (2021), the following conditions are all consequences of (10): for all $t > s > 0$,

$$\mathbb{E}\left[c_N(\tau_N(t))\right] \to 0 \tag{11}$$

$$\mathbb{E}\left[\sum_{r=\tau_N(s)+1}^{\tau_N(t)} c_N(r)^2\right] \to 0 \tag{12}$$

$$\mathbb{E}\left[\sum_{r=\tau_N(s)+1}^{\tau_N(t)} D_N(r)\right] \to 0 \tag{13}$$

as $N \to \infty$.

Proof of Theorem 1. The structure of the proof follows Möhle (1999), albeit with considerable technical complication due to the dependence between generations in our model. Is this the main/only source of complication? —SB Since we already have convergence of the finite-dimensional distributions (Brown et al. 2021, Theorem 1), strengthening this to weak convergence requires relative compactness of the sequence of processes $\{(G^{(n,N)}_{\tau_N(t)})_{t\geq 0}\}_{N\in\mathbb{N}}$.

Ethier and Kurtz (2009, Chapter 3, Corollary 7.4) provides a necessary and sufficient condition for relative compactness: $\mathcal{P}_n$ is finite and therefore complete and separable, and the sample paths of $(G^{(n,N)}_{\tau_N(t)})_{t\geq 0}$ live in $\mathcal{X}$, so the conditions of the corollary are satisfied. The corollary states that the sequence of processes $\{(G^{(n,N)}_{\tau_N(t)})_{t\geq 0}\}_{N\in\mathbb{N}}$ is relatively compact if and only if the following two conditions hold:

1. For every $\varepsilon > 0$, $t \geq 0$ there exists a compact set $\Gamma \subseteq \mathcal{P}_n$ such that

$$\liminf_{N\to\infty} \mathbb{P}[G^{(n,N)}_{\tau_N(t)} \in \Gamma] \geq 1 - \varepsilon$$

2. For every $\varepsilon > 0$, $t > 0$ there exists $\delta > 0$ such that

$$\liminf_{N\to\infty} \mathbb{P}[\omega(G^{(n,N)}_{\tau_N(\cdot)}, \delta, t) < \varepsilon] \geq 1 - \varepsilon$$

where ω is the modulus of continuity:

$$\omega(G^{(n,N)}_{\tau_N(\cdot)}, \delta, t) := \inf \max_{i\in[K]} \sup_{u,v\in[T_{i-1},T_i)} \rho\left(G^{(n,N)}_{\tau_N(u)}, G^{(n,N)}_{\tau_N(v)}\right)$$

with the infimum taken over all partitions of the form $0 = T_0 < T_1 < \cdots < T_{K-1} < t \leq T_K$ such that $\min_{i\in[K]}(T_i - T_{i-1}) > \delta$. Clarify that such a partition with any K is valid, i.e. K is not fixed. —SB

In our case, Condition 1 is satisfied automatically with $\Gamma = \mathcal{P}_n$, since $\mathcal{P}_n$ is finite and hence compact. Intuitively, Condition 2 ensures that the jumps of the process are well-separated. In our case where ρ is the zero-one metric, we see that $\rho(G^{(n,N)}_{\tau_N(u)}, G^{(n,N)}_{\tau_N(v)})$ is equal to 1 if there is a jump between times u and v, and 0 otherwise. Taking the supremum and maximum then indicates whether there is a jump inside any of the intervals of the given partition; this can only be equal to zero if all of the jumps up to time t occur exactly at the times $T_0, \ldots, T_K$. The infimum over all allowed partitions, then, can only be equal to zero if no two jumps occur less than δ (unscaled) time apart, because of the restriction we placed on these partitions.

The proof is concentrated on proving Condition 2. To do this, we use a coupling with another process that contains all of the jumps of the genealogical process, with the addition of some extra jumps. This process is constructed in such a way that it can be shown to satisfy Condition 2, and hence so does the genealogical process.

Define $p_t := \max_{\xi\in\mathcal{P}_n}\{1-p_{\xi\xi}(t)\} = 1-p_{\Delta\Delta}(t)$, where Δ denotes the trivial partition of singletons $\{\{1\},\ldots,\{n\}\}$. For a proof that the maximum is attained at $\xi = \Delta$, see Lemma 1. Following Möhle (1999), we now construct the two-dimensional conditionally on $\mathcal{F}$? —SB Markov process $(Z_t, S_t)_{t\in\mathbb{N}_0}$ on $\mathbb{N}_0 \times \mathcal{P}_n$ with transition probabilities

$$\mathbb{P}[Z_t = j, S_t = \eta \mid Z_{t-1} = i, S_{t-1} = \xi] = \begin{cases} 1 - p_t & \text{if } j = i \text{ and } \xi = \eta \\ p_{\xi\xi}(t) + p_t - 1 & \text{if } j = i+1 \text{ and } \xi = \eta \\ p_{\xi\eta}(t) & \text{if } j = i+1 \text{ and } \xi \neq \eta \\ 0 & \text{otherwise} \end{cases}$$

and initial state $Z_0 = 0$, $S_0 = \Delta$. The construction is such that the marginal (S_t) has the same distribution as the genealogical process of interest, and (Z_t) has jumps at all the times (S_t) does plus some extra jumps. (The definition of p_t ensures that the probability in the second case is non-negative, attaining the value zero when $\xi = \Delta$.) And the transition probabilities (jump times) of Z do not depend on the current state. —SB

Denote by $0 = T^{(N)}_0 < T^{(N)}_1 < \ldots$ the jump times of the rescaled process $(Z_{\tau_N(t)})_{t\geq 0}$, and by $\varpi^{(N)}_i := T^{(N)}_i - T^{(N)}_{i-1}$ the corresponding holding times.

Suppose that for some $t > 0$, there exists $m \in \mathbb{N}$ and $\delta > 0$ such that $\varpi^{(N)}_i > \delta$ for all $i \in \{1,\ldots,m\}$, and $T^{(N)}_m \geq t$. Then $K_N := \min\{i : T^{(N)}_i \geq t\}$ is well-defined with $1 \leq K_N \leq m$, and $T^{(N)}_1, \ldots, T^{(N)}_{K_N}$ form a partition of the form required for Condition 2. Indeed $(Z_{\tau_N(\cdot)})$ is constant on every interval $[T^{(N)}_{i-1}, T^{(N)}_i)$ by construction, so $\omega((Z_{\tau_N(\cdot)}), \delta, t) = 0$. We therefore have that for each $m \in \mathbb{N}$ and $\delta > 0$,

$$\mathbb{P}[\omega((Z_{\tau_N(\cdot)}), \delta, t) < \varepsilon] \geq \mathbb{P}[T^{(N)}_m \geq t, \varpi^{(N)}_i > \delta \, \forall i \in \{1,\ldots,m\}].$$

Thus a sufficient condition for Condition 2 is: for any $\varepsilon > 0$, $t > 0$, there exist $m \in \mathbb{N}$, $\delta > 0$ such that

$$\liminf_{N\to\infty} \mathbb{P}[T^{(N)}_m \geq t, \varpi^{(N)}_i > \delta \, \forall i \in \{1,\ldots,m\}] \geq 1 - \varepsilon. \tag{14}$$

Since $T^{(N)}_m = \varpi^{(N)}_1 + \cdots + \varpi^{(N)}_m$, there is a positive correlation between $T^{(N)}_m$ and each of the $\varpi^{(N)}_i$, so

$$\begin{aligned} \mathbb{P}[T^{(N)}_m \geq{}& t, \varpi^{(N)}_i > \delta \, \forall i \in \{1,\ldots,m\}] \\ &= \mathbb{P}[T^{(N)}_m \geq t \mid \varpi^{(N)}_i > \delta \, \forall i \in \{1,\ldots,m\}] \, \mathbb{P}[\varpi^{(N)}_i > \delta \, \forall i \in \{1,\ldots,m\}] \\ &\geq \mathbb{P}[T^{(N)}_m \geq t] \, \mathbb{P}[\varpi^{(N)}_i > \delta \, \forall i \in \{1,\ldots,m\}]. \end{aligned}$$

Due to Lemma 2, the limiting distributions of $\varpi^{(N)}_i$ are i.i.d. $\mathrm{Exp}(\alpha_n)$, so

$$\liminf_{N\to\infty} \mathbb{P}[\varpi^{(N)}_i > \delta \, \forall i \in \{1,\ldots,m\}] = (e^{-\alpha_n\delta})^m$$

and

$$\liminf_{N\to\infty} \mathbb{P}[T_m^{(N)} \geq t] = \liminf_{N\to\infty} \mathbb{P}[\varpi_1^{(N)} + \cdots + \varpi_m^{(N)} \geq t] = e^{-\alpha_n \delta} \sum_{i=0}^{m-1} \frac{(\alpha_n t)^i}{i!}.$$

using the series expansion for the Erlang cumulative distribution function. citation? —SB Hence

$$\liminf_{N\to\infty} \mathbb{P}[T_m^{(N)} \geq t, \varpi_i^{(N)} > \delta \, \forall i \in \{1, \dots, m\}] \geq (e^{-\alpha_n \delta})^{m+1} \sum_{i=0}^{m-1} \frac{(\alpha_n t)^i}{i!},$$

which can be made $\geq 1 - \varepsilon$ by taking m sufficiently large and δ sufficiently small. Since this argument applies for any ε and t, (14) and hence Condition 2 is satisfied, and the proof is complete. ■

Lemma 1. $\max_{\xi \in \mathcal{P}_n}(1 - p_{\xi\xi}(t)) = 1 - p_{\Delta\Delta}(t)$.

Proof. Consider any $\xi \in E$ consisting of k blocks ($1 \leq k \leq n-1$), and any $\xi' \in E$ consisting of $k+1$ blocks. From the definition of $p_{\xi\eta}(t)$ (Koskela et al. 2018, Equation (1)),

$$p_{\xi\xi}(t) = \frac{1}{(N)_k} \sum_{\substack{i_1,\dots,i_k \\ \text{all distinct}}} \nu_t^{(i_1)} \cdots \nu_t^{(i_k)}.$$

Similarly,

$$\begin{aligned} p_{\xi'\xi'}(t) &= \frac{1}{(N)_{k+1}} \sum_{\substack{i_1,\dots,i_k,i_{k+1} \\ \text{all distinct}}} \nu_t^{(i_1)} \cdots \nu_t^{(i_k)} \nu_t^{(i_{k+1})} \\ &= \frac{1}{(N)_k (N-k)} \sum_{\substack{i_1,\dots,i_k \\ \text{all distinct}}} \left\{ \nu_t^{(i_1)} \cdots \nu_t^{(i_k)} \sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^{N} \nu_t^{(i_{k+1})} \right\}. \end{aligned}$$

Discarding the zero summands,

$$p_{\xi'\xi'}(t) = \frac{1}{(N)_k (N-k)} \sum_{\substack{i_1,\dots,i_k \\ \text{all distinct:} \\ \nu_t^{(i_1)},\dots,\nu_t^{(i_k)} > 0}} \left\{ \nu_t^{(i_1)} \cdots \nu_t^{(i_k)} \sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^{N} \nu_t^{(i_{k+1})} \right\}.$$

The inner sum is

$$\sum_{\substack{i_{k+1}=1 \\ \text{also distinct}}}^{N} \nu_t^{(i_{k+1})} = \left\{ \sum_{i=1}^{N} \nu_t^{(i)} - \sum_{i \in \{i_1,\dots,i_k\}} \nu_t^{(i)} \right\} \leq N - k$$

since $\nu_t^{(i_1)}, \dots, \nu_t^{(i_k)}$ are all at least 1. Hence

$$p_{\xi'\xi'}(t) \leq \frac{N-k}{(N)_k (N-k)} \sum_{\substack{i_1,\dots,i_k \\ \text{all distinct:} \\ \nu_t^{(i_1)},\dots,\nu_t^{(i_k)} > 0}} \nu_t^{(i_1)} \cdots \nu_t^{(i_k)} = p_{\xi\xi}(t).$$

Thus $p_{\xi\xi}(t)$ is decreasing in the number of blocks of ξ, and is therefore minimised by taking $\xi = \Delta$, which achieves the maximum n blocks. This choice in turn maximises $1 - p_{\xi\xi}(t)$, as required. ■

Lemma 2. *The finite-dimensional distributions of* $\varpi_1^{(N)}, \varpi_2^{(N)}, \dots$ *converge as* $N \to \infty$ *to those of* $\varpi_1, \varpi_2, \dots$*, where the* ϖ_i *are independent* $\mathrm{Exp}(\alpha_n)$ *distributed random variables.*

Proof. There is a continuous bijection between the jump times $T_1^{(N)}, T_2^{(N)}, \dots$ and the holding times $\varpi_1^{(N)}, \varpi_2^{(N)}, \dots$, so convergence of the holding times to $\varpi_1, \varpi_2, \dots$ is equivalent to convergence of the jump times to $T_1, T_2, \dots$, where $T_i := \varpi_1 + \cdots + \varpi_i$. We will work with the jump times, following the structure of Möhle (1999, Lemma 3.2).

The idea is to prove by induction that, for any $k \in \mathbb{N}$ and $t_1, \dots, t_k > 0$,

$$\lim_{N\to\infty} \mathbb{P}[T_1^{(N)} \le t_1, \dots, T_k^{(N)} \le t_k] = \mathbb{P}[T_1 \le t_1, \dots, T_k \le t_k]. \tag{15}$$

Take the basis case $k = 1$. Then

$$\mathbb{P}[T_1 \le t] = \mathbb{P}[\varpi_1 \le t] = 1 - e^{-\alpha_n t}$$

and $T_1^{(N)} > t$ if and only if Z has no jumps up to time t: Expectation appears by tower property to remove (implicit) conditioning in transition probabilities? —SB

$$\mathbb{P}[T_1^{(N)} > t] = \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1 - p_r)\right].$$

Lemma 3 shows that this probability converges to $e^{-\alpha_n t}$ as required.

For the induction step, assume that (15) holds for some k. We have the following decomposition:

$$\mathbb{P}[T_1^{(N)} \le t_1, \dots, T_{k+1}^{(N)} \le t_{k+1}] = \mathbb{P}[T_1^{(N)} \le t_1, \dots, T_k^{(N)} \le t_k] - \mathbb{P}[T_1^{(N)} \le t_1, \dots, T_k^{(N)} \le t_k, T_{k+1}^{(N)} > t_{k+1}].$$

The first term on the RHS converges to $\mathbb{P}[T_1 \le t_1, \dots, T_k \le t_k]$ by the induction hypothesis, and it remains to show that

$$\lim_{N\to\infty} \mathbb{P}[T_1^{(N)} \le t_1, \dots, T_k^{(N)} \le t_k, T_{k+1}^{(N)} > t_{k+1}] = \mathbb{P}[T_1 \le t_1, \dots, T_k \le t_k, T_{k+1} > t_{k+1}].$$

As shown in Möhle (1999), the RHS

$$\mathbb{P}[T_1 \le t_1, \dots, T_k \le t_k, T_{k+1} > t_{k+1}] = \alpha_n^k e^{-\alpha_n t} \sum_{\substack{i_1 \le \cdots \le i_{k-1} \\ \in \{0,\dots,k\}: \\ i_j \ge j \forall j}} \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}.$$

The event on the LHS can be written (Möhle 1999)

$$\mathbb{P}[T_1^{(N)} \le t_1, \dots, T_k^{(N)} \le t_k, T_{k+1}^{(N)} > t_{k+1}] = \mathbb{E}\left[\sum_{\substack{r_1 < \cdots < r_k: \\ r_i \le \tau_N(t_i) \forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right) \left(\prod_{\substack{r=1 \\ \notin \{r_1,\dots,r_k\}}}^{\tau_N(t)} (1 - p_r)\right)\right],$$

that is, there are jumps at some times $r_1, \dots, r_k$ and identity transitions at all other times. Due to Lemmata 4 and 5, this probability converges to the correct limit. This completes the induction. ∎

3 Main components of proof

Lemma 3 (Basis step)**.** *Assume* (10) *holds. For any* $0 < t < \infty$,

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1 - p_r)\right] = e^{-\alpha_n t}$$

where $\alpha_n := n(n-1)/2$.

Proof. We start by showing that $\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1 - p_r)\right] \le e^{-\alpha_n t}$.
From (8) we have for each r

$$1 - p_r = p_{\Delta\Delta}(r) \le 1 - \alpha_n \frac{N^{n-2}}{(N-2)_{n-2}} \left[c_N(r) - B_n' D_N(r)\right]. \tag{16}$$

When $N \geq 3$, a sufficient condition to ensure the bound in (16) is non-negative is that the event

$$E_N^1(r) := \left\{ c_N(r) \leq \frac{(N-2)_{n-2}}{\alpha_n N^{n-2}} \right\} \tag{17}$$

occurs. We will also need to control the sign of $c_N(r) - B_n' D_N(r)$, for which we define the event

$$E_N^2(r) := \{ c_N(r) \geq B_n' D_N(r) \}, \tag{18}$$

and we define $E_N^1 := \bigcap_{r=1}^{\tau_N(t)} E_N^1(r)$ and $E_N^2 := \bigcap_{r=1}^{\tau_N(t)} E_N^2(r)$. Then

$$1 - p_r = p_{\Delta\Delta}(r) \leq 1 - \alpha_n (1 + O(N^{-1})) \left[c_N(r) - B_n' D_N(r) \right] \mathbb{1}_{E_N^1 \cap E_N^2}.$$

Applying a multinomial expansion and then separating the positive and negative terms,

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1 - p_r) &\leq 1 + \sum_{l=1}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} \left[c_N(s_j) - B_n' D_N(s_j) \right] \mathbb{1}_{E_N^1 \cap E_N^2} \\
&= 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} \left[c_N(s_j) - B_n' D_N(s_j) \right] \mathbb{1}_{E_N^1 \cap E_N^2} \\
&\quad - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} \left[c_N(s_j) - B_n' D_N(s_j) \right] \mathbb{1}_{E_N^1 \cap E_N^2}.
\end{aligned} \tag{19}$$

This is further bounded by applying Lemma 9 and then both bounds of Lemma 7(b):

$$\begin{aligned}
\prod_{r=1}^{\tau_N(t)} (1 - p_r) \leq 1 + \Bigg\{ & \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \\
& - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left[\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1 + B_n')^l \right] \Bigg\} \mathbb{1}_{E_N^1 \cap E_N^2} \\
\leq 1 + \Bigg\{ & \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left\{ t^l + c_N(\tau_N(t)) (t+1)^l \right\} \\
& - \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \left[t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \binom{l}{2} (t+1)^{l-2} \right] \\
& - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1 + B_n')^l \Bigg\} \mathbb{1}_{E_N^1 \cap E_N^2}.
\end{aligned}$$

Collecting some terms,

$$\begin{aligned}\prod_{r=1}^{\tau_N(t)}(1-p_r) &\le 1+\sum_{l=1}^{\tau_N(t)}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_{E_N^1\cap E_N^2}+c_N(\tau_N(t))\sum_{\substack{l=2\\ \text{even}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}(t+1)^l\\
&\quad+\left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right)\sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\binom{l}{2}(t+1)^{l-2}\\
&\quad+\left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)\sum_{\substack{l=1\\ \text{odd}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}(t+1)^{l-1}(1+B_n')^l\\
&\le 1+\sum_{l=1}^{\infty}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_{\{\tau_N(t)\ge l\}}\mathbb{1}_{E_N^1\cap E_N^2}+c_N(\tau_N(t))\exp[\alpha_n(1+O(N^{-1}))(t+1)]\\
&\quad+\left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right)\frac{1}{2}\alpha_n^2\exp[\alpha_n(1+O(N^{-1}))(t+1)]\\
&\quad+\left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')]. \qquad (20)\end{aligned}$$

Now, taking the expectation and limit, then applying (11)–(13), and Lemmata 11, 12 and 14 to deal with the indicators,

$$\begin{aligned}\lim_{N\to\infty}\mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right] &\le 1+\sum_{l=1}^{\infty}(-\alpha_n)^l\frac{1}{l!}t^l\lim_{N\to\infty}\mathbb{P}\left[\{\tau_N(t)\ge l\}\cap E_N^1\cap E_N^2\right]+\lim_{N\to\infty}\mathbb{E}\left[c_N(\tau_N(t))\right]\exp[\alpha_n(t+1)]\\
&\quad+\lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right]\frac{1}{2}\alpha_n^2\exp[\alpha_n(t+1)]\\
&\quad+\lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)}D_N(s)\right]\exp[\alpha_n(t+1)(1+B_n')]\\
&=1+\sum_{l=1}^{\infty}(-\alpha_n)^l\frac{1}{l!}t^l=e^{-\alpha_n t}. \qquad (21)\end{aligned}$$

Passing the limit and expectation inside the infinite sum is justified by dominated convergence and Fubini.

It remains to show the corresponding lower bound $\lim_{N\to\infty}\mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right]\ge e^{-\alpha_n t}$.
From (9) we have

$$1-p_t=p_{\Delta\Delta}(t)\ge 1-\frac{N^{n-2}}{(N-2)_{n-2}}\alpha_n[c_N(t)+B_nD_N(t)]$$

where $B_n>0$. Due to (5), a sufficient condition for this bound to be non-negative is

$$E_N^3(r):=\left\{c_N(r)\le\frac{(N-2)_{n-2}}{N^{n-2}}\alpha_n^{-1}(1+B_n)^{-1}\right\}, \qquad (22)$$

and we again define $E_N^3:=\bigcap_{r=1}^{\tau_N(t)}E_N^3(r)$. We now apply a multinomial expansion to the product, and split into positive and negative terms:

$$\prod_{r=1}^{\tau_N(t)}(1-p_r) \geq \left\{1+\sum_{l=1}^{\tau_N(t)}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}[c_N(s_j)+B_nD_N(s_j)]\right\}\mathbb{1}_{E_N^3}$$
$$=\left\{1+\sum_{\substack{l=2\\\text{even}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}[c_N(s_j)+B_nD_N(s_j)]\right.$$
$$\left.-\sum_{\substack{l=1\\\text{odd}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}[c_N(s_j)+B_nD_N(s_j)]\right\}\mathbb{1}_{E_N^3}$$

This is further bounded by applying Lemma 8 and both bounds in Lemma 7(b):

$$\prod_{r=1}^{\tau_N(t)}(1-p_r) \geq \left\{1+\sum_{\substack{l=2\\\text{even}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}c_N(s_j)\right.$$
$$\left.-\sum_{\substack{l=1\\\text{odd}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\left[\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}c_N(s_j)+\left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)(t+1)^{l-1}(1+B_n)^l\right]\right\}\mathbb{1}_{E_N^3}$$
$$\geq\left\{1+\sum_{\substack{l=2\\\text{even}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\left[t^l-\left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right)\binom{l}{2}(t+1)^{l-2}\right]\right.$$
$$\left.-\sum_{\substack{l=1\\\text{odd}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\left[t^l+c_N(\tau_N(t))(t+1)^l+\left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)(t+1)^{l-1}(1+B_n)^l\right]\right\}\mathbb{1}_{E_N^3}.$$

Collecting terms,

$$\prod_{r=1}^{\tau_N(t)}(1-p_r) \geq \sum_{l=0}^{\tau_N(t)}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_{E_N^3}-\left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right)\sum_{\substack{l=2\\\text{even}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\binom{l}{2}(t+1)^{l-2}$$
$$-c_N(\tau_N(t))\sum_{\substack{l=1\\\text{odd}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}(t+1)^l$$
$$-\left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)\sum_{\substack{l=1\\\text{odd}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}(t+1)^{l-1}(1+B_n)^l$$
$$\geq\sum_{l=0}^{\infty}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_{E_N^3}\mathbb{1}_{\{\tau_N(t)\geq l\}}-\left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right)\frac{1}{2}\alpha_n^2\exp[\alpha_n(1+O(N^{-1}))(t+1)]$$
$$-c_N(\tau_N(t))\exp[\alpha_n(1+O(N^{-1}))(t+1)]$$
$$-\left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n)]. \tag{23}$$

Now, taking the expectation and limit, and applying (11)–(13) to show that all but the first sum vanish, and

Lemmata 12 and 11 to show that $\lim_{N\to\infty} \mathbb{P}[\{\tau_N(t) \geq l\} \cap E_N^3] = 1$,

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1-p_r)\right] \geq{}& \sum_{l=0}^{\infty} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l \lim_{N\to\infty} \mathbb{P}\left[\{\tau_N(t) \geq l\} \cap E_N^3\right] \\
&- \lim_{N\to\infty} \mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right] \frac{1}{2}\alpha_n^2 \exp[\alpha_n(t+1)] \\
&- \lim_{N\to\infty} \mathbb{E}\left[c_N(\tau_N(t))\right] \exp[\alpha_n(t+1)] \\
&- \lim_{N\to\infty} \mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} D_N(s)\right] \exp[\alpha_n(t+1)(1+B_n)] \\
={}& \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l = e^{-\alpha_n t}.
\end{aligned} \tag{24}$$

Again, passing the limit and expectation inside the infinite sum is justified by dominated convergence and Fubini. Combining the upper and lower bounds in (21) and (24) respectively concludes the proof. ■

Lemma 4 (Induction step upper bound)**.** *Assume* (10) *holds. Fix* $k \in \mathbb{N}$, $i_0 := 0$, $i_k := k$. *For any sequence of times* $0 = t_0 \leq t_1 \leq \cdots \leq t_k \leq t$,

$$\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right) \left(\prod_{\substack{r=1 \\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\right] \leq \alpha_n^k e^{-\alpha_n t} \sum_{\substack{i_1 \leq \cdots \leq i_{k-1} \\ \in\{0,\ldots,k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}.$$

Proof. We use the bound on $(1-p_r)$ from (16) and apply a multinomial expansion, defining as in (17) and (18) respectively the sequences of events E_N^1 and E_N^2 which ensure the bounds are non-negative:

$$\begin{aligned}
\prod_{\substack{r=1 \\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)} (1-p_r) \leq{}& \prod_{\substack{r=1 \\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)} \left\{1 - \alpha_n(1+O(N^{-1}))[c_N(r) - B_n' D_N(r)] \mathbb{1}_{E_N^1 \cap E_N^2}\right\} \\
={}& 1 + \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} \sum_{\substack{s_1 \neq \cdots \neq s_l \\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - B_n' D_N(s_j)] \mathbb{1}_{E_N^1 \cap E_N^2} \\
={}& 1 + \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - B_n' D_N(s_j)] \mathbb{1}_{E_N^1 \cap E_N^2} \\
&- \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} \sum_{\substack{s_1 \neq \cdots \neq s_l \\ \exists i,i': s_i = r_{i'}}} \prod_{j=1}^{l} [c_N(s_j) - B_n' D_N(s_j)] \mathbb{1}_{E_N^1 \cap E_N^2}.
\end{aligned} \tag{25}$$

The penultimate line above is exactly the expansion we had in the basis step (19), except for the limit on l, and as such following the same arguments gives a bound analogous to that in (20):

$$
\begin{aligned}
1 + \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - B_n' D_N(s_j)] \mathbb{1}_{E_N^1 \cap E_N^2} & \\
\leq 1 + \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_{E_N^1 \cap E_N^2} &+ c_N(\tau_N(t)) \exp[\alpha_n (1+O(N^{-1}))(t+1)] \\
+ \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n (1+O(N^{-1}))(t+1)] & \\
+ \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n (1+O(N^{-1}))(t+1)(1+B_n')]. &
\end{aligned}
$$

For the last line of (25),

$$
\begin{aligned}
- \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} \sum_{\substack{s_1 \neq \cdots \neq s_l \\ \exists i,i': s_i = r_{i'}}} \prod_{j=1}^{l} \{c_N(s_j) - B_n' D_N(s_j)\} \mathbb{1}_{E_N^1 \cap E_N^2} & \\
\leq \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \sum_{\substack{s_1 \neq \cdots \neq s_l \\ \exists i,i': s_i = r_{i'}}} \prod_{j=1}^{l} \{c_N(s_j) + B_n' D_N(s_j)\} & \\
\leq \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \sum_{\substack{s_1 \neq \cdots \neq s_l \\ \exists i,i': s_i = r_{i'}}} (1+B_n')^l \prod_{j=1}^{l} c_N(s_j) & \\
\leq \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1+O(N^{-1})) \frac{1}{(l-1)!} \sum_{s_1 \in \{r_1,\ldots,r_k\}} \sum_{s_2 \neq \cdots \neq s_l}^{\tau_N(t)} (1+B_n')^l \prod_{j=1}^{l} c_N(s_j) & \\
= \sum_{s \in \{r_1,\ldots,r_k\}} c_N(s) \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1+O(N^{-1})) \frac{1}{(l-1)!} (1+B_n')^l \sum_{s_1 \neq \cdots \neq s_{l-1}}^{\tau_N(t)} \prod_{j=1}^{l-1} c_N(s_j) & \\
\leq \sum_{j=1}^{k} c_N(r_j) \sum_{l=1}^{\tau_N(t)-k} \alpha_n^l (1+O(N^{-1})) \frac{1}{(l-1)!} (1+B_n')^l (t+1)^{l-1} & \\
\leq \left(\sum_{j=1}^{k} c_N(r_j) \right) \alpha_n (1+B_n') \exp[\alpha_n (1+O(N^{-1}))(1+B_n')(t+1)], &
\end{aligned}
$$

where the penultimate inequality uses Lemma 7(a). Putting these together, we have

$$
\begin{aligned}
\prod_{\substack{r=1 \\ \notin \{r_1,\ldots,r_k\}}}^{\tau_N(t)} (1-p_r) \leq 1 + \sum_{l=1}^{\tau_N(t)-k} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_{E_N^1 \cap E_N^2} &+ c_N(\tau_N(t)) \exp[\alpha_n (1+O(N^{-1}))(t+1)] \\
+ \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n (1+O(N^{-1}))(t+1)] & \\
+ \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n (1+O(N^{-1}))(t+1)(1+B_n')] & \\
+ \left(\sum_{j=1}^{k} c_N(r_j) \right) \alpha_n (1+B_n') \exp[\alpha_n (1+O(N^{-1}))(1+B_n')(t+1)]. & \qquad (26)
\end{aligned}
$$

Meanwhile, using the bound on p_r from (9) then applying a modification of Lemma 8 where the sum is over ordered indices rather than distinct indices,

$$\begin{aligned}\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\prod_{i=1}^{k}p_{r_i} &\leq \alpha_n^k(1+O(N^{-1}))\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\prod_{i=1}^{k}[c_N(r_i)+B_nD_N(r_i)]\\ &\leq \alpha_n^k(1+O(N^{-1}))\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\prod_{i=1}^{k}c_N(r_i)+\left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)\alpha_n^k(1+O(N^{-1}))(t+1)^{k-1}(1+B_n)^k.\end{aligned}\tag{27}$$

A more liberal (but simpler) bound can be arrived at thus:

$$\begin{aligned}\prod_{i=1}^{k}p_{r_i} &\leq \alpha_n^k(1+O(N^{-1}))\prod_{i=1}^{k}[c_N(r_i)+B_nD_N(r_i)]\\ &\leq \alpha_n^k(1+O(N^{-1}))\prod_{i=1}^{k}c_N(r_i)(1+B_n)\\ &\leq \alpha_n^k(1+O(N^{-1}))(1+B_n)^k\prod_{i=1}^{k}c_N(r_i)\end{aligned}$$

which, using Lemma 7(a), also leads to the deterministic bound

$$\begin{aligned}\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\prod_{i=1}^{k}p_{r_i} &\leq \alpha_n^k(1+O(N^{-1}))(1+B_n)^k\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\prod_{i=1}^{k}c_N(r_i)\\ &\leq \alpha_n^k(1+O(N^{-1}))(1+B_n)^k\frac{1}{k!}\sum_{r_1\neq\cdots\neq r_k}^{\tau_N(t)}\prod_{i=1}^{k}c_N(r_i)\\ &\leq \alpha_n^k(1+O(N^{-1}))(1+B_n)^k\frac{1}{k!}(t+1)^k.\end{aligned}\tag{28}$$

Combining (26) with the other product, the expression inside the expectation in Lemma 4 is bounded above by

$$\begin{aligned}&\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\left(\prod_{i=1}^{k}p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)}(1-p_r)\right)\\ &\leq\left\{1+\sum_{l=1}^{\tau_N(t)-k}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_{E_N^1\cap E_N^2}\right\}\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\prod_{i=1}^{k}p_{r_i}\\ &\quad+\left\{c_N(\tau_N(t))\exp[\alpha_n(1+O(N^{-1}))(t+1)]+\left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right)\frac{1}{2}\alpha_n^2\exp[\alpha_n(1+O(N^{-1}))(t+1)]\right.\\ &\qquad\left.+\left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')]\right\}\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\prod_{i=1}^{k}p_{r_i}\\ &\quad+\exp[\alpha_n(1+O(N^{-1}))(1+B_n')(t+1)]\alpha_n(1+B_n')\sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}}\sum_{j=1}^{k}c_N(r_j)\prod_{i=1}^{k}p_{r_i}.\end{aligned}$$

Applying the various bounds (27)–(28), we have

$$
\begin{aligned}
&\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^k p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\\
&\qquad \le \alpha_n^k(1+O(N^{-1}))\left\{1+\sum_{l=1}^{\tau_N(t)-k}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_{E_N^1\cap E_N^2}\right\}\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\prod_{i=1}^k c_N(r_i)\\
&\qquad\quad+\left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)\alpha_n^k(1+O(N^{-1}))(t+1)^{k-1}(1+B_n)^k\sum_{l=0}^{\tau_N(t)}(\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\\
&\qquad\quad+\left\{c_N(\tau_N(t))\exp[\alpha_n(1+O(N^{-1}))(t+1)]+\left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right)\frac{1}{2}\alpha_n^2\exp[\alpha_n(1+O(N^{-1}))(t+1)]\right.\\
&\qquad\qquad\left.+\left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B'_n)]\right\}\alpha_n^k(1+O(N^{-1}))(1+B_n)^k\frac{1}{k!}(t+1)^k\\
&\qquad\quad+\exp[\alpha_n(1+B'_n)(t+1)]\alpha_n(1+B'_n)\alpha_n^k(1+O(N^{-1}))(1+B_n)^k\\
&\qquad\qquad\qquad\qquad\times\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\sum_{j=1}^k c_N(r_j)\prod_{i=1}^k c_N(r_i).
\end{aligned}
$$

Upon taking the expectation and limit, we have

$$
\begin{aligned}
&\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^k p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\right]\\
&\qquad\le \alpha_n^k\lim_{N\to\infty}\mathbb{E}\left[\left(1+\sum_{l=1}^{\tau_N(t)-k}(-\alpha_n)^l\frac{1}{l!}t^l\mathbb{1}_{E_N^1\cap E_N^2}\right)\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\prod_{i=1}^k c_N(r_i)\right]\\
&\qquad\quad+\lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} D_N(s)\right]\alpha_n^k(t+1)^{k-1}(1+B_n)^k\exp[\alpha_n t]\\
&\qquad\quad+\left\{\lim_{N\to\infty}\mathbb{E}\left[c_N(\tau_N(t))\right]\exp[\alpha_n(t+1)]+\lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right]\frac{1}{2}\alpha_n^2\exp[\alpha_n(t+1)]\right.\\
&\qquad\qquad\left.+\lim_{N\to\infty}\mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} D_N(s)\right]\exp[\alpha_n(t+1)(1+B'_n)]\right\}\alpha_n^k(1+B_n)^k\frac{1}{k!}(t+1)^k\\
&\qquad\quad+\exp[\alpha_n(1+B'_n)(t+1)]\alpha_n^{k+1}(1+B'_n)(1+B_n)^k\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\sum_{j=1}^k c_N(r_j)\prod_{i=1}^k c_N(r_i)\right]. \qquad (29)
\end{aligned}
$$

The middle terms vanish due to (11)–(13) and the expression becomes

$$\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\right] \le \alpha_n^k \lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^{k} c_N(r_i)\right]$$
$$+ \alpha_n^k \sum_{l=1}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l \lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_{\{\tau_N(t)\ge k+l\}} \mathbb{1}_{E_N^1\cap E_N^2} \sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^{k} c_N(r_i)\right]$$
$$+ \exp[\alpha_n(1+B_n')(t+1)]\alpha_n^{k+1}(1+B_n')(1+B_n)^k \lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \sum_{j=1}^{k} c_N(r_j) \prod_{i=1}^{k} c_N(r_i)\right], \tag{30}$$

where passing the limit and expectation inside the infinite sum is justified by dominated convergence and Fubini; see Lemma 16. To simplify the last line,

$$\begin{aligned}
\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \sum_{j=1}^{k} c_N(r_j) \prod_{i=1}^{k} c_N(r_i) &\le \frac{1}{k!} \sum_{r_1\ne\cdots\ne r_k}^{\tau_N(t)} \sum_{j=1}^{k} c_N(r_j) \prod_{i=1}^{k} c_N(r_i) \\
&= \frac{1}{k!} \sum_{r_1\ne\cdots\ne r_k}^{\tau_N(t)} \sum_{j=1}^{k} c_N(r_j)^2 \prod_{i\ne j} c_N(r_i) \\
&\le \frac{1}{k!} \sum_{j=1}^{k} \sum_{s=1}^{\tau_N(t)} c_N(s)^2 \sum_{r_1\ne\cdots\ne r_{k-1}}^{\tau_N(t)} \prod_{i=1}^{k-1} c_N(r_i) \\
&\le \frac{1}{(k-1)!} \sum_{s=1}^{\tau_N(t)} c_N(s)^2 (t+1)^{k-1},
\end{aligned}$$

using Lemma 7(a) for the final inequality. Hence

$$\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \sum_{s\in\{r_1,\ldots,r_k\}} c_N(s) \prod_{i=1}^{k} c_N(r_i)\right] \le \frac{1}{(k-1)!}(t+1)^{k-1} \lim_{N\to\infty} \mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right] = 0$$

by (12). By Lemmata 12, 11 and 14, $\lim_{N\to\infty} \mathbb{P}[\{\tau_N(t)\ge k+l\}\cap E_N^1\cap E_N^2] = 1$, so we can apply Lemma 6 to the remaining expectations in (30), yielding

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\right] &\le \alpha_n^k \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^{k} \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} \\
&= \alpha_n^k e^{-\alpha_n t} \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^{k} \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}
\end{aligned}$$

as required. ∎

Lemma 5 (Induction step lower bound)**.** *Assume* (10) *holds. Fix* $k \in \mathbb{N}$, $i_0 := 0$, $i_k := k$. *For any sequence of times* $0 = t_0 \le t_1 \le \cdots \le t_k \le t$,

$$\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\right] \ge \alpha_n^k e^{-\alpha_n t} \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^{k} \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}.$$

Proof. Firstly,

$$\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\ldots,r_k\}}}^{\tau_N(t)} (1-p_r)\right) \ge \sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right)\left(\prod_{r=1}^{\tau_N(t)} (1-p_r)\right). \tag{31}$$

Now the second product does not depend on $r_1,\ldots,r_k$, and we can use the lower bound from (23):

$$\begin{aligned}\prod_{r=1}^{\tau_N(t)}(1-p_r) \ge{}& \sum_{l=0}^{\tau_N(t)}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_{E_N^3} - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right)\frac{1}{2}\alpha_n^2\exp[\alpha_n(1+O(N^{-1}))(t+1)]\\ &- c_N(\tau_N(t))\exp[\alpha_n(1+O(N^{-1}))(t+1)]\\ &- \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n)]\end{aligned} \tag{32}$$

where E_N^3 is defined as in (22). We will also need an upper bound on this product, which is formed from (20) with a further deterministic bound:

$$\begin{aligned}\prod_{r=1}^{\tau_N(t)}(1-p_r) \le{}& \sum_{l=0}^{\tau_N(t)}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_{\{\tau_N(t)\ge l\}}\mathbb{1}_{E_N^1\cap E_N^2} + c_N(\tau_N(t))\exp[\alpha_n(1+O(N^{-1}))(t+1)]\\ &+ \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right)\frac{1}{2}\alpha_n^2\exp[\alpha_n(1+O(N^{-1}))(t+1)]\\ &+ \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')]\\ \le{}& \exp[\alpha_n(1+O(N^{-1}))t] + \exp[\alpha_n(1+O(N^{-1}))(t+1)]\\ &+ \frac{1}{2}\alpha_n^2(t+1)\exp[\alpha_n(1+O(N^{-1}))(t+1)] + (t+1)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')]\\ \le{}& \left(2+\frac{\alpha_n^2(t+1)}{2}\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)] + (t+1)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')].\end{aligned} \tag{33}$$

Now let us consider the remaining sum-product on the RHS of (31). We use the same bound on p_r as in (16):

$$p_r = 1 - p_{\Delta\Delta}(r) \ge \alpha_n(1+O(N^{-1}))\left[c_N(r) - B_n' D_N(r)\right] \tag{34}$$

where the $O(N^{-1})$ term does not depend on r. When N is large enough for the factor of $(1+O(N^{-1}))$ to be non-negative, the condition that the bound in (34) is non-negative holds on the event E_N^2 that was defined in (18). Then

$$\prod_{i=1}^{k} p_{r_i} \ge \alpha_n^k(1+O(N^{-1}))\prod_{i=1}^{k}\left[c_N(r_i) - B_n' D_N(r_i)\right]\mathbb{1}_{E_N^2}.$$

Applying a modification of Lemma 9 where the sum is over ordered indices rather than distinct indices,

$$\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right) \ge \alpha_n^k(1+O(N^{-1})) \sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^{k} [c_N(r_i) - B_n' D_N(r_i)]\, \mathbb{1}_{E_N^2}$$

$$\ge \alpha_n^k(1+O(N^{-1})) \left\{ \sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^{k} c_N(r_i)\mathbb{1}_{E_N^2} - \frac{1}{k!}\left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)(t+1)^{k-1}(1+B_n')^k \right\}.$$

The above expression is already split into positive and negative terms; a lower bound on (31) can be formed by multiplying the positive terms by the lower bound (32) and the negative terms by the upper bound (33). Thus

$$\begin{aligned}
&\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\\
&\quad\ge \alpha_n^k(1+O(N^{-1})) \sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^{k} c_N(r_i)\mathbb{1}_{E_N^2}\Bigg\{ \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_{E_N^3}\\
&\qquad - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right)\frac{1}{2}\alpha_n^2\exp[\alpha_n(1+O(N^{-1}))(t+1)]\\
&\qquad - c_N(\tau_N(t))\exp[\alpha_n(1+O(N^{-1}))(t+1)]\\
&\qquad - \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n)]\Bigg\}\\
&\quad - \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)\alpha_n^k(1+O(N^{-1}))\frac{1}{k!}(t+1)^{k-1}(1+B_n')^k\Bigg\{\\
&\qquad \left(2+\frac{\alpha_n^2(t+1)}{2}\right)\exp[\alpha_n(1+O(N^{-1}))(t+1)]\\
&\qquad + (t+1)\exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')]\Bigg\}.
\end{aligned}$$

Due to (11)–(13), all but the first line on the RHS of the above have vanishing expectation, leaving

$$\begin{aligned}
&\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \left(\prod_{i=1}^{k} p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)} (1-p_r)\right)\right]\\
&\quad\ge \lim_{N\to\infty}\mathbb{E}\left[\alpha_n^k(1+O(N^{-1}))\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^{k} c_N(r_i)\mathbb{1}_{E_N^2}\sum_{l=0}^{\tau_N(t)}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}t^l\mathbb{1}_{E_N^3}\right]\\
&\quad= \alpha_n^k\sum_{l=0}^{\infty}(-\alpha_n)^l\frac{1}{l!}t^l\lim_{N\to\infty}\mathbb{E}\left[\mathbb{1}_{\{\tau_N(t)\ge l\}}\mathbb{1}_{E_N^2\cap E_N^3}\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\prod_{i=1}^{k} c_N(r_i)\right].
\end{aligned}\tag{35}$$

Passing the limit and expectation inside the infinite sum is justified by dominated convergence and Fubini; see Lemma 16. Lemmata 11 and 14 establish that $\lim_{N\to\infty}\mathbb{P}[E_N^2\cap E_N^3] = 1$ and Lemma 12 deals with the other indicator. We can therefore apply Lemma 6 to conclude that

$$\lim_{N\to\infty}\mathbb{E}\left[\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\left(\prod_{i=1}^{k}p_{r_i}\right)\left(\prod_{\substack{r=1\\ \notin\{r_1,\dots,r_k\}}}^{\tau_N(t)}(1-p_r)\right)\right]\ge\alpha_n^k\sum_{l=0}^{\infty}(-\alpha_n)^l\frac{1}{l!}t^l\sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}}\prod_{j=1}^{k}\frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}$$

$$=\alpha_n^k e^{-\alpha_n t}\sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}}\prod_{j=1}^{k}\frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}$$

as required. ■

Lemma 6. *Assume* (10) *holds. Fix* $k\in\mathbb{N}$, $i_0:=0$, $i_k:=k$. *Let* E_N *be a sequence of events such that* $\lim_{N\to\infty}\mathbb{P}[E_N]=1$. *Then for any sequence of times* $0=t_0\le t_1\le\cdots\le t_k\le t$,

$$\lim_{N\to\infty}\mathbb{E}\left[\mathbb{1}_{E_N}\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\prod_{i=1}^{k}c_N(r_i)\right]=\sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}}\prod_{j=1}^{k}\frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}.$$

Proof. As pointed out by Möhle (1999, p. 460), the sum-product on the left hand side can be expanded as

$$\sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}}\prod_{i=1}^{k}c_N(r_i)=\sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}}\prod_{j=1}^{k}\frac{1}{(i_j-i_{j-1})!}\sum_{\substack{r_{i_{j-1}+1}\ne\cdots\ne r_{i_j}\\ =\tau_N(t_{j-1})+1}}^{\tau_N(t_j)}\prod_{i=i_{j-1}+1}^{i_j}c_N(r_i).$$

By a modification of the upper bound in Lemma 7(b) where the lower limit of the sum is a general time rather than 1,

$$\sum_{\substack{r_{i_{j-1}+1}\ne\cdots\ne r_{i_j}\\ =\tau_N(t_{j-1})+1}}^{\tau_N(t_j)}\prod_{i=i_{j-1}+1}^{i_j}c_N(r_i)\le(t_j-t_{j-1})^{i_j-i_{j-1}}+c_N(\tau_N(t_j))(t_j-t_{j-1}+1)^{i_j-i_{j-1}}$$

Now, taking the product on the outside,

$$\prod_{j=1}^{k} \frac{1}{(i_j - i_{j-1})!} \sum_{\substack{r_{i_{j-1}+1} \neq \cdots \neq r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) \le \prod_{j=1}^{k} \left\{ \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} + c_N(\tau_N(t_j)) \frac{(t_j - t_{j-1}+1)^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right\}$$

$$\le \prod_{j=1}^{k} \left\{ \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} + c_N(\tau_N(t_j))(t_j - t_{j-1}+1)^{i_j - i_{j-1}} \right\}$$

$$= \sum_{\mathcal{I} \subseteq [k]} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} c_N(\tau_N(t_j))(t_j - t_{j-1}+1)^{i_j - i_{j-1}} \right)$$

$$= \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}$$

$$+ \sum_{\mathcal{I} \subset [k]} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} c_N(\tau_N(t_j))(t_j - t_{j-1}+1)^{i_j - i_{j-1}} \right)$$

$$\le \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}$$

$$+ \sum_{\mathcal{I} \subset [k]} \left(\prod_{j \in \mathcal{I}} t^{i_j - i_{j-1}} \right) \left(\prod_{j \notin \mathcal{I}} c_N(\tau_N(t_j))(t+1)^{i_j - i_{j-1}} \right)$$

$$\le \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} + \sum_{\mathcal{I} \subset [k]} c_N(\tau_N(t_{j^\star(\mathcal{I})})) \prod_{j=1}^{k} (t+1)^{i_j - i_{j-1}}$$

$$= \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} + \sum_{\mathcal{I} \subset [k]} c_N(\tau_N(t_{j^\star(\mathcal{I})}))(t+1)^k$$

where, say, $j^\star(\mathcal{I}) := \min\{j \notin \mathcal{I}\}$. Now we are in a position to evaluate the limit in Lemma 6:

$$\lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_{E_N} \sum_{\substack{r_1 < \cdots < r_k: \\ r_i \le \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i) \right] \le \lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1 < \cdots < r_k: \\ r_i \le \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i) \right]$$

$$\le \sum_{\substack{i_1 \le \cdots \le i_{k-1} \\ \in \{0,\dots,k\}: \\ i_j \ge j \forall j}} \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} + \sum_{\substack{i_1 \le \cdots \le i_{k-1} \\ \in \{0,\dots,k\}: \\ i_j \ge j \forall j}} \sum_{\mathcal{I} \subset [k]} \lim_{N\to\infty} \mathbb{E}\left[c_N(\tau_N(t_{j^\star(\mathcal{I})})) \right] (t+1)^k$$

$$= \sum_{\substack{i_1 \le \cdots \le i_{k-1} \\ \in \{0,\dots,k\}: \\ i_j \ge j \forall j}} \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}$$

using (11)

For the corresponding lower bound, by a modification of the lower bound in Lemma 7(b) where the lower limit of the sum is a general time rather than 1,

$$\sum_{\substack{r_{i_{j-1}+1} \neq \cdots \neq r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) \ge (t_j - t_{j-1})^{i_j - i_{j-1}} - \binom{i_j - i_{j-1}}{2} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1}+1)^{i_j - i_{j-1} - 2}$$

$$\ge (t_j - t_{j-1})^{i_j - i_{j-1}} - (i_j - i_{j-1})! \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1}+1)^{i_j - i_{j-1} - 2}.$$

Define the events

$$E_N^4(j) = \left\{ \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) \le \frac{1}{(i_j - i_{j-1})!} \left(\frac{t_j - t_{j-1}}{t_j - t_{j-1} + 1} \right)^{i_j - i_{j-1}} \right\},$$

which is sufficient to ensure the j^{th} term in the following product is non-negative, and define $E_N^4 := \bigcap_{j=1}^k E_N^4(j)$. (If $t_j = t_{j-1}$ then $E_N^4(j)$ has probability one automatically; otherwise the constant on the right is strictly positive and so satisfies the conditions of Lemma 13.) Now, taking a product over j,

$$\begin{aligned}
&\prod_{j=1}^k \frac{1}{(i_j - i_{j-1})!} \sum_{\substack{r_{i_{j-1}+1} \neq \cdots \neq r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) \\
&\ge \prod_{j=1}^k \left\{ \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} - \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{i_j - i_{j-1} - 2} \right\} \mathbb{1}_{E_N^4} \\
&= \sum_{\mathcal{I} \subseteq [k]} (-1)^{k - |\mathcal{I}|} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{i_j - i_{j-1} - 2} \right) \mathbb{1}_{E_N^4} \\
&= \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E_N^4} \\
&\quad + \sum_{\mathcal{I} \subset [k]} (-1)^{k - |\mathcal{I}|} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{i_j - i_{j-1} - 2} \right) \mathbb{1}_{E_N^4} \\
&\ge \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E_N^4} \\
&\quad - \sum_{\mathcal{I} \subset [k]} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t_j - t_{j-1} + 1)^{i_j - i_{j-1} - 2} \right) \\
&\ge \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E_N^4} \\
&\quad - \sum_{\mathcal{I} \subset [k]} \left(\prod_{j \in \mathcal{I}} t^{i_j - i_{j-1}} \right) \left(\prod_{j \notin \mathcal{I}} \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right) (t+1)^{i_j - i_{j-1} - 2} \right) \\
&\ge \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E_N^4} \\
&\quad - \sum_{\mathcal{I} \subset [k]} \left(\sum_{s=\tau_N(t_{j^\star(\mathcal{I})-1})+1}^{\tau_N(t_{j^\star(\mathcal{I})})} c_N(s)^2 \right) \left(\prod_{j \in \mathcal{I}} t^{i_j - i_{j-1}} \right) \left(\prod_{j \notin \mathcal{I}} (t+1)^{i_j - i_{j-1} - 1} \right) \\
&\ge \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E_N^4} - \sum_{\mathcal{I} \subset [k]} \left(\sum_{s=\tau_N(t_{j^\star(\mathcal{I})-1})+1}^{\tau_N(t_{j^\star(\mathcal{I})})} c_N(s)^2 \right) \prod_{j=1}^k (t+1)^{i_j - i_{j-1}} \\
&= \prod_{j=1}^k \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \mathbb{1}_{E_N^4} - \sum_{\mathcal{I} \subset [k]} \left(\sum_{s=\tau_N(t_{j^\star(\mathcal{I})-1})+1}^{\tau_N(t_{j^\star(\mathcal{I})})} c_N(s)^2 \right) (t+1)^k,
\end{aligned}$$

where again we have arbitrarily set $j^\star(\mathcal{I}) := \min\{j \notin \mathcal{I}\}$. We can now evaluate the limit:

$$
\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_{E_N} \sum_{\substack{r_1<\cdots<r_k:\\ r_i\le\tau_N(t_i)\forall i}} \prod_{i=1}^{k} c_N(r_i)\right] &\ge \lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_{E_N\cap E_N^4} \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^{k} \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}\right] \\
&\quad - \lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_{E_N} \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \sum_{\mathcal{I}\subset[k]} \left(\sum_{s=\tau_N(t_{j^\star(\mathcal{I})-1})+1}^{\tau_N(t_{j^\star(\mathcal{I})})} c_N(s)^2\right)(t+1)^k\right] \\
&\ge \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^{k} \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} \lim_{N\to\infty} \mathbb{E}[\mathbb{1}_{E_N\cap E_N^4}] \\
&\quad - \lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \sum_{\mathcal{I}\subset[k]} \left(\sum_{s=\tau_N(t_{j^\star(\mathcal{I})-1})+1}^{\tau_N(t_{j^\star(\mathcal{I})})} c_N(s)^2\right)(t+1)^k\right] \\
&= \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^{k} \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} \lim_{N\to\infty} \mathbb{P}[E_N\cap E_N^4] \\
&\quad - \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \sum_{\mathcal{I}\subset[k]} \lim_{N\to\infty} \mathbb{E}\left[\sum_{s=\tau_N(t_{j^\star(\mathcal{I})-1})+1}^{\tau_N(t_{j^\star(\mathcal{I})})} c_N(s)^2\right](t+1)^k \\
&= \sum_{\substack{i_1\le\cdots\le i_{k-1}\\ \in\{0,\dots,k\}:\\ i_j\ge j\forall j}} \prod_{j=1}^{k} \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}
\end{aligned}
$$

where for the last equality we use (12) to show that the second sum vanishes and Lemma 13 to show that $\lim_{N\to\infty}\mathbb{P}[E_N\cap E_N^4]=1$. We have shown that the upper and lower bounds coincide, so the result follows. ∎

4 Bounds on sum-products

Lemma 7. *Fix $t>0$, $l\in\mathbb{N}$.*

(a)
$$\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \le (t+1)^l$$

(b)
$$t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2\right)\binom{l}{2}(t+1)^{l-2} \le \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \le t^l + c_N(\tau_N(t))(t+1)^l$$

Proof. **(a)** It is a true fact that
$$\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \le \left(\sum_{s=0}^{\tau_N(t)} c_N(s)\right)^l,$$

as can be seen by considering the multinomial expansion of the RHS. Applying (7),

$$\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \leq (t+1)^l. \tag{36}$$

(b) As pointed out in Koskela et al. (2018, Equation (8)),

$$\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \geq \left(\sum_{s=0}^{\tau_N(t)} c_N(s)\right)^l - \binom{l}{2}\left(\sum_{s=0}^{\tau_N(t)} c_N(s)^2\right)\left(\sum_{s=0}^{\tau_N(t)} c_N(s)\right)^{l-2}. \tag{37}$$

Applying (7) on the RHS of (37) yields the lower bound.

For the upper bound we have

$$\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) \leq \left(\sum_{s=0}^{\tau_N(t)} c_N(s)\right)^l \leq \left(\sum_{s=0}^{\tau_N(t)-1} c_N(s) + c_N(\tau_N(t))\right)^l \leq \left[t + c_N(\tau_N(t))\right]^l,$$

using the definition of τ_N. A binomial expansion yields

$$\left[t + c_N(\tau_N(t))\right]^l = t^l + \sum_{i=0}^{l-1}\binom{l}{i} t^i c_N(\tau_N(t))^{l-i} = t^l + c_N(\tau_N(t))\sum_{i=0}^{l-1}\binom{l}{i} t^i c_N(\tau_N(t))^{l-1-i},$$

then by (4),

$$\sum_{i=0}^{l-1}\binom{l}{i} t^i c_N(\tau_N(t))^{l-1-i} \leq \sum_{i=0}^{l-1}\binom{l}{i} t^i \leq (t+1)^l.$$

Putting this together yields the upper bound. ■

Lemma 8. *Fix $t>0$, $l\in\mathbb{N}$. Let B be a positive constant which may depend on n.*

$$\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} \left[c_N(s_j) + BD_N(s_j)\right] \leq \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) + \left(\sum_{s=1}^{\tau_N(t)} D_N(s)\right)(t+1)^{l-1}(1+B)^l.$$

Proof. We start with a binomial expansion:

$$\begin{aligned}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} \left[c_N(s_j) + BD_N(s_j)\right] &= \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \sum_{\mathcal{I}\subseteq[l]} B^{l-|\mathcal{I}|}\left(\prod_{i\in\mathcal{I}} c_N(s_i)\right)\left(\prod_{j\notin\mathcal{I}} D_N(s_j)\right)\\ &= \sum_{\mathcal{I}\subseteq[l]} B^{l-|\mathcal{I}|} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i\in\mathcal{I}} c_N(s_i)\right)\left(\prod_{j\notin\mathcal{I}} D_N(s_j)\right) \end{aligned}\tag{38}$$

where $[l] := \{1,\dots,l\}$. Since the sum is over all permutations of $s_1,\dots,s_l$, we may arbitrarily choose an ordering for $\{1,\dots,l\}$ such that $\mathcal{I} = \{1,\dots,|\mathcal{I}|\}$:

$$\sum_{\mathcal{I}\subseteq[l]} B^{l-|\mathcal{I}|} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i\in\mathcal{I}} c_N(s_i)\right)\left(\prod_{j\notin\mathcal{I}} D_N(s_j)\right) = \sum_{I=0}^{l}\binom{l}{I} B^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right)\left(\prod_{j=I+1}^{l} D_N(s_j)\right).$$

Separating the term $I=l$,

$$\begin{aligned}&\sum_{I=0}^{l}\binom{l}{I} B^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right)\left(\prod_{j=I+1}^{l} D_N(s_j)\right)\\ &\quad= \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) + \sum_{I=0}^{l-1}\binom{l}{I} B^{l-I} \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i)\right)\left(\prod_{j=I+1}^{l} D_N(s_j)\right). \end{aligned}\tag{39}$$

In the second term on the RHS, there is always at least one D_N term, so using (5) we can write

$$\begin{aligned}
\sum_{I=0}^{l-1}\binom{l}{I}B^{l-I}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\left(\prod_{i=1}^{I}c_N(s_i)\right)\left(\prod_{j=I+1}^{l}D_N(s_j)\right) &\leq \sum_{I=0}^{l-1}\binom{l}{I}B^{l-I}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\left(\prod_{i=1}^{l-1}c_N(s_i)\right)D_N(s_l) \\
&\leq \sum_{I=0}^{l-1}\binom{l}{I}B^{l-I}\left(\sum_{s_1\neq\cdots\neq s_{l-1}}^{\tau_N(t)}\prod_{i=1}^{l-1}c_N(s_i)\right)\sum_{s_l=1}^{\tau_N(t)}D_N(s_l) \\
&\leq \sum_{I=0}^{l-1}\binom{l}{I}B^{l-I}(t+1)^{l-1}\sum_{s=1}^{\tau_N(t)}D_N(s) \qquad (40)
\end{aligned}$$

using (36). Finally, by the Binomial Theorem,

$$\sum_{I=0}^{l-1}\binom{l}{I}B^{l-I}(t+1)^{l-1}\sum_{s=1}^{\tau_N(t)}D_N(s) \leq \left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)(t+1)^{l-1}(1+B)^l, \qquad (41)$$

which, together with (39), concludes the proof. ■

Lemma 9. *Fix $t>0$, $l\in\mathbb{N}$. Let B be a positive constant which may depend on n.*

$$\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}\left[c_N(s_j)-BD_N(s_j)\right] \geq \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}c_N(s_j) - \left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)(t+1)^{l-1}(1+B)^l.$$

Proof. A binomial expansion and subsequent manipulation as in (38)–(39) gives

$$\begin{aligned}
\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}\left[c_N(s_j)-BD_N(s_j)\right] &= \sum_{\mathcal{I}\subseteq[l]}(-B)^{l-|\mathcal{I}|}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\left(\prod_{i\in\mathcal{I}}c_N(s_i)\right)\left(\prod_{j\notin\mathcal{I}}D_N(s_j)\right) \\
&= \sum_{I=0}^{l}\binom{l}{I}(-B)^{l-I}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\left(\prod_{i=1}^{I}c_N(s_i)\right)\left(\prod_{j=I+1}^{l}D_N(s_j)\right) \\
&= \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}c_N(s_j) + \sum_{I=0}^{l-1}\binom{l}{I}(-B)^{l-I}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\left(\prod_{i=1}^{I}c_N(s_i)\right)\left(\prod_{j=I+1}^{l}D_N(s_j)\right) \\
&\geq \sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}c_N(s_j) - \sum_{I=0}^{l-1}\binom{l}{I}B^{l-I}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\left(\prod_{i=1}^{I}c_N(s_i)\right)\left(\prod_{j=I+1}^{l}D_N(s_j)\right)
\end{aligned}$$

where the last inequality just multiplies some positive terms by -1. Then (40)–(41) can be applied directly (noting that an upper bound on negative terms gives a lower bound overall):

$$-\sum_{I=0}^{l-1}\binom{l}{I}B^{l-I}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\left(\prod_{i=1}^{I}c_N(s_i)\right)\left(\prod_{j=I+1}^{l}D_N(s_j)\right) \geq -\left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)(t+1)^{l-1}(1+B)^l$$

which concludes the proof. ■

5 Indicators

Lemma 10. *Let $(A_N),(B_N)$ be sequences of events. If $\lim_{N\to\infty}\mathbb{P}[A_N]=1$ and $\lim_{N\to\infty}\mathbb{P}[B_N]=1$ then $\lim_{N\to\infty}\mathbb{P}[A_N\cap B_N]=1$.*

The above might be so obvious as to go unstated, but it is very important because it means we don't have to deal with intersections of dependent events! Here is a little proof just to be sure: —SB

Proof.

$$\begin{aligned}
&\lim_{N\to\infty} \mathbb{P}[A_N] = 1 \textbf{ and } \lim_{N\to\infty} \mathbb{P}[B_N] = 1 \\
\Leftrightarrow &\lim_{N\to\infty} \mathbb{P}[A_N^c] = 0 \textbf{ and } \lim_{N\to\infty} \mathbb{P}[B_N^c] = 0 \\
\Rightarrow &\lim_{N\to\infty} \{\mathbb{P}[A_N^c] + \mathbb{P}[B_N^c]\} = 0 \\
\Rightarrow &\lim_{N\to\infty} \mathbb{P}[A_N^c \cup B_N^c] = 0 \\
\Leftrightarrow &\lim_{N\to\infty} \mathbb{P}[A_N \cap B_N] = 1.
\end{aligned}$$

The only part of this argument that I find potentially controversial is going from the third to the fourth line, which is an application of the sandwich theorem (since $0 \leq \mathbb{P}[A_N^c \cup B_N^c] \leq \mathbb{P}[A_N^c] + \mathbb{P}[B_N^c]$). ■

Lemma 11. *Assume* (12) *holds. Let $K > 0$ be a constant which may depend on n, N but not on r, such that $K^{-2} = O(1)$ as $N \to \infty$. Define the events $E_N(r) := \{c_N(r) < K\}$ and denote $E_N := \bigcap_{r=1}^{\tau_N(t)} E_N(r)$. Then $\lim_{N\to\infty} \mathbb{P}[E_N] = 1$.*

Proof.

$$\begin{aligned}
\mathbb{P}[E_N] = 1 - \mathbb{P}[E_N^c] = 1 - \mathbb{P}\left[\bigcup_{r=1}^{\tau_N(t)} E_N^c(r)\right] &= 1 - \mathbb{E}\left[\mathbb{1}_{\bigcup E_N^c(r)}\right] \geq 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{1}_{E_N^c(r)}\right] \\
= 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}\left[\mathbb{1}_{E_N^c(r)} \mid \mathcal{F}_{r-1}\right]\right] &= 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{P}\left[E_N^c(r) \mid \mathcal{F}_{r-1}\right]\right] \qquad (42)
\end{aligned}$$

where for the second line we apply Lemma 15 with $f(r) = \mathbb{1}_{E_N^c(r)}$. By the generalised Markov inequality,

$$\mathbb{P}[E_N^c(r) \mid \mathcal{F}_{r-1}] = \mathbb{P}[c_N(r) \geq K \mid \mathcal{F}_{r-1}] \leq K^{-2}\mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}].$$

Substituting this into (42) and applying Lemma 15 again, this time with $f(r) = c_N(r)^2$,

$$\mathbb{P}[E_N] \geq 1 - K^{-2}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}]\right] = 1 - K^{-2}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} c_N(r)^2\right].$$

Applying (12), the limit is

$$\lim_{N\to\infty} \mathbb{P}[E_N] = 1 - O(1) \times 0 = 1$$

as required. ■

Lemma 12. *Fix $t > 0$. For any $l \in \mathbb{N}$, $\lim_{N\to\infty} \mathbb{P}[\tau_N(t) \geq l] = 1$.*

Proof. We can replace the event $\{\tau_N(t) \geq l\}$ with an event of the form of E_N in Lemma 11:

$$\{\tau_N(t) \geq l\} = \left\{\min\left\{s \geq 1 : \sum_{r=1}^{s} c_N(r) \geq t\right\} \geq l\right\} = \left\{\sum_{r=1}^{l-1} c_N(r) < t\right\} \supseteq \bigcap_{r=1}^{l-1}\left\{c_N(r) < \frac{t}{l}\right\} \supseteq \bigcap_{r=1}^{\tau_N(t)}\left\{c_N(r) < \frac{t}{l}\right\}.$$

Hence

$$\lim_{N\to\infty} \mathbb{P}[\tau_N(t) \geq l] \geq \lim_{N\to\infty} \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)}\left\{c_N(r) < \frac{t}{l}\right\}\right] = 1$$

by applying Lemma 11 with $K = t/l$. ■

Lemma 13. *Assume* (12) *holds. Fix $k \in \mathbb{N}$, a sequence of times $0 = t_0 \le t_1 \le \cdots \le t_k \le t$, and let $K_1, \dots, K_k$ be strictly positive constants. Define the events*

$$E_N := \bigcap_{j=1}^{k} \left\{ \sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \le K_j \right\}.$$

Then $\lim_{N\to\infty} \mathbb{P}[E_N] = 1$.

Proof.

$$\mathbb{P}[E_N] = 1 - \mathbb{P}[E_N^c] = 1 - \mathbb{P}\left[\bigcup_{j=1}^{k} \left\{ \sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 > K_j \right\} \right] \ge 1 - \sum_{j=1}^{k} \mathbb{P}\left[\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \ge K_j \right].$$

Applying Markov's inequality,

$$\mathbb{P}[E_N] \ge 1 - \sum_{j=1}^{k} K_j^{-1} \mathbb{E}\left[\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \right] \underset{N\to\infty}{\longrightarrow} 1 - \sum_{j=1}^{k} O(1) \times 0 = 1$$

by (12). ■

Lemma 14. *Assume* (13) *holds. Fix $t > 0$. Let K be a constant not depending on N, r, but which may depend on n.*

$$\lim_{N\to\infty} \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) \ge K D_N(r)\} \right] = 1.$$

Proof.

$$\begin{aligned}
\mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) \ge K D_N(r)\} \right] &\ge \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) > K D_N(r)\} \right] \\
&= 1 - \mathbb{P}\left[\bigcup_{r=1}^{\tau_N(t)} \{c_N(r) \le K D_N(r)\} \right] \\
&= 1 - \mathbb{E}\left[\mathbb{1}_{\bigcup\{c_N(r) \le K D_N(r)\}} \right] \\
&\ge 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{1}_{\{c_N(r) \le K D_N(r)\}} \right] \\
&= 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{P}[c_N(r) \le K D_N(r) \mid \mathcal{F}_{r-1}] \right] \qquad (43)
\end{aligned}$$

where the final inequality is an application of Lemma 15 with $f(r) = \mathbb{1}_{\{c_N(r) \le K D_N(r)\}}$.

Fix $0 < \varepsilon < K^{-1}/2$ and assume $N > \max\{\varepsilon^{-1}, (K^{-1} - 2\varepsilon)^{-1}\}$. For each r, i define the event $A_i(r) := \{\nu_r^{(i)} \le N\varepsilon\}$. Conditional on $\mathcal{F}_{r-1}$, we have

$$D_N(r) = \frac{1}{N(N)_2} \sum_{i=1}^{N} (\nu_r^{(j)})_2 \left[\nu_r^{(i)} + \frac{1}{N} \sum_{j \ne i} (\nu_r^{(i)})^2 \right] \mathbb{1}_{A_i^c(r)} + \frac{1}{N(N)_2} \sum_{i=1}^{N} (\nu_r^{(i)})_2 \left[\nu_r^{(i)} + \frac{1}{N} \sum_{j \ne i} (\nu_r^{(j)})^2 \right] \mathbb{1}_{A_i(r)}.$$

For the first term,

$$\frac{1}{N(N)_2} \sum_{i=1}^{N} (\nu_r^{(i)})_2 \left[\nu_r^{(i)} + \frac{1}{N} \sum_{j \ne i} (\nu_r^{(j)})^2 \right] \mathbb{1}_{A_i^c(r)} \le \sum_{i=1}^{N} \mathbb{1}_{A_i^c(r)}.$$

For the second term,

$$\begin{aligned}
\frac{1}{N(N)_2}\sum_{i=1}^{N}(\nu_r^{(i)})_2\left[\nu_r^{(i)}+\frac{1}{N}\sum_{j\neq i}(\nu_r^{(j)})^2\right]\mathbb{1}_{A_i(r)} &\leq \frac{1}{N(N)_2}\sum_{i=1}^{N}(\nu_r^{(i)})_2\nu_r^{(i)}\mathbb{1}_{A_i(r)}+\frac{1}{N^2(N)_2}\sum_{i=1}^{N}(\nu_r^{(i)})_2\sum_{j=1}^{N}(\nu_r^{(j)})^2\mathbb{1}_{A_i(r)} \\
&\leq \frac{1}{N}c_N(r)N\varepsilon+\frac{1}{N^2(N)_2}\sum_{i=1}^{N}(\nu_r^{(i)})_2\sum_{j=1}^{N}(\nu_r^{(j)})_2\mathbb{1}_{A_i(r)} \\
&\quad +\frac{1}{N^2(N)_2}\sum_{i=1}^{N}(\nu_r^{(i)})_2\sum_{j=1}^{N}(\nu_r^{(j)})\mathbb{1}_{A_i(r)} \\
&\leq \varepsilon c_N(r)+\frac{1}{N^2}\sum_{i=1}^{N}\nu_r^{(i)}N\varepsilon c_N(r)+\frac{1}{N^2}c_N(r)N \\
&= c_N(r)\left(2\varepsilon+\frac{1}{N}\right).
\end{aligned}$$

Altogether we have

$$D_N(r)\leq c_N(r)\left(2\varepsilon+\frac{1}{N}\right)+\sum_{i=1}^{N}\mathbb{1}_{A_i^c(r)}.$$

Hence, still conditional on $\mathcal{F}_{r-1}$,

$$\begin{aligned}
\{c_N(r)\leq KD_N(r)\} &\subseteq \left\{c_N(r)\leq Kc_N(r)(2\varepsilon+N^{-1})+K\sum_{i=1}^{N}\mathbb{1}_{A_i^c(r)}\right\} \\
&= \left\{K^{-1}-2\varepsilon-\frac{1}{N}\leq\sum_{i=1}^{N}\frac{\mathbb{1}_{A_i^c(r)}}{c_N(r)}\right\}
\end{aligned}$$

where the ratio $\mathbb{1}_{A_i^c(r)}/c_N(r)$ is well-defined because

$$A_i^c(r)\Rightarrow c_N(r):=\frac{1}{(N)_2}\sum_{j=1}^{N}(\nu_r^{(j)})_2\geq\frac{1}{(N)_2}(\nu_r^{(i)})_2\geq\frac{\varepsilon(N\varepsilon-1)}{N-1}\geq\varepsilon\left(\varepsilon-\frac{1}{N}\right)>0.$$

Hence by Markov's inequality (the conditions on ε, N ensuring the constant is always strictly positive),

$$\begin{aligned}
\mathbb{P}\left[c_N(r)\leq KD_N(r)\mid\mathcal{F}_{r-1}\right] &\leq \mathbb{P}\left[\sum_{i=1}^{N}\mathbb{1}_{A_i^c(r)}\geq\left(K^{-1}-2\varepsilon-\frac{1}{N}\right)\varepsilon\left(\varepsilon-\frac{1}{N}\right)\middle|\mathcal{F}_{r-1}\right] \\
&\leq \frac{1}{\left(K^{-1}-2\varepsilon-\frac{1}{N}\right)\varepsilon\left(\varepsilon-\frac{1}{N}\right)}\mathbb{E}\left[\sum_{i=1}^{N}\mathbb{1}_{A_i^c(r)}\middle|\mathcal{F}_{r-1}\right] \\
&\leq \frac{1}{\left(K^{-1}-2\varepsilon-\frac{1}{N}\right)\varepsilon\left(\varepsilon-\frac{1}{N}\right)}\mathbb{E}\left[\sum_{i=1}^{N}\frac{(\nu_r^{(i)})_3}{(N\varepsilon)_3}\middle|\mathcal{F}_{r-1}\right] \\
&\leq \frac{1}{\left(K^{-1}-2\varepsilon-\frac{1}{N}\right)\varepsilon\left(\varepsilon-\frac{1}{N}\right)}\mathbb{E}\left[\frac{N(N)_2}{(N\varepsilon)_3}D_N(r)\middle|\mathcal{F}_{r-1}\right].
\end{aligned}$$

Applying Lemma 15 once more, with $f(r)=D_N(r)$,

$$\begin{aligned}
\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)}\mathbb{P}[c_N(r)\leq KD_N(r)\mid\mathcal{F}_{r-1}]\right] &\leq \frac{1}{\left(K^{-1}-2\varepsilon-\frac{1}{N}\right)\varepsilon\left(\varepsilon-\frac{1}{N}\right)}\frac{N(N)_2}{(N\varepsilon)_3}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)}\mathbb{E}[D_N(r)\mid\mathcal{F}_{r-1}]\right] \\
&= \frac{1}{\left(K^{-1}-2\varepsilon-\frac{1}{N}\right)\varepsilon\left(\varepsilon-\frac{1}{N}\right)}\frac{N(N)_2}{(N\varepsilon)_3}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)}D_N(r)\right] \\
&\xrightarrow[N\to\infty]{} \frac{1}{(K^{-1}-2\varepsilon)\varepsilon^5}\times 0=0
\end{aligned}$$

due to (13). Substituting this back into (43) concludes the proof. ■

Other useful results

The following Lemma is taken from Koskela et al. (2018, Lemma 2), where the function is set to $f(r) = c_N(r)$, but the authors remark that the result holds for other choices of function.

Lemma 15. *Fix $t > 0$. Let $(\mathcal{F}_r)$ be the backwards-in-time filtration generated by the offspring counts $\nu_r^{(1:N)}$ at each generation r, and let $f(r)$ be any deterministic function of $\nu_r^{(1:N)}$ that is non-negative and bounded. In particular, for all r there exists $B < \infty$ such that $0 \leq f(r) \leq B$. Then*

$$\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} f(r)\right] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right].$$

Proof. Define

$$M_s := \sum_{r=1}^{s} \{f(r) - \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\}.$$

It is easy to establish that (M_s) is a martingale with respect to $(\mathcal{F}_s)$, and $M_0 = 0$. Now fix $K \geq 1$ and note that $\tau_N(t) \wedge K$ is a bounded $\mathcal{F}_t$-stopping time. Hence we can apply the optional stopping theorem:

$$\mathbb{E}[M_{\tau_N(t)\wedge K}] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} \{f(r) - \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\}\right] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} f(r)\right] - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right] = 0.$$

Since this holds for all $K \geq 1$,

$$\lim_{K\to\infty} \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} f(r)\right] = \lim_{K\to\infty} \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right].$$

The monotone convergence theorem allows the limit to pass inside the expectation on each side (since increasing K can only increase each sum, by possibly adding non-negative terms). Hence

$$\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} f(r)\right] = \mathbb{E}\left[\lim_{K\to\infty} \sum_{r=1}^{\tau_N(t)\wedge K} f(r)\right] = \mathbb{E}\left[\lim_{K\to\infty} \sum_{r=1}^{\tau_N(t)\wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right]$$

which concludes the proof. ∎

There are a few instances where Fubini's Theorem and the Dominated Convergence Theorem are needed in order to pass a limit and expectation through an infinite sum. Now we verify that the conditions of these theorems indeed hold. This result, analogous to that in Koskela et al. (2018, Appendix), is used once in Lemma 4 at (29) and once in Lemma 5 at (35).

Lemma 16. *For any fixed $t > 0$,*

$$\mathbb{E}\left[\sum_{l=0}^{\infty} \left|(-\alpha_n)^l (1 + O(N^{-1}))\frac{1}{l!}t^l \sum_{\substack{r_1<\cdots<r_k: \\ r_i\leq\tau_N(t_i)\forall i}} \prod_{i=1}^{k} c_N(r_i)\right|\right] < \infty.$$

Proof.

$$\begin{aligned}\mathbb{E}\left[\sum_{l=0}^{\infty} \left|(-\alpha_n)^l (1 + O(N^{-1}))\frac{1}{l!}t^l \sum_{\substack{r_1<\cdots<r_k: \\ r_i\leq\tau_N(t_i)\forall i}} \prod_{i=1}^{k} c_N(r_i)\right|\right] &\leq \mathbb{E}\left[\sum_{l=0}^{\infty} \alpha_n^l (1 + O(N^{-1}))\frac{1}{l!}t^l (t+1)^k\right] \\ &= \mathbb{E}[\exp\{\alpha_n t(1 + O(N^{-1}))\}(t+1)^k] = \exp\{\alpha_n t(1 + O(N^{-1}))\}(t+1)^k < \infty.\end{aligned}$$

∎

6 Dependency graph

Missing links since this graph was updated:

- Lemma 7(a) is used three times in Lemma 4, but not anywhere else.
- Lemma 7 in the current dependency graph is really referring to Lemma 7(b)
- Lemma 16 is used in Lemmata 5 and 4.
- Lemma 10 is used in Lemmata 3, 4, 5 and 6

—SB

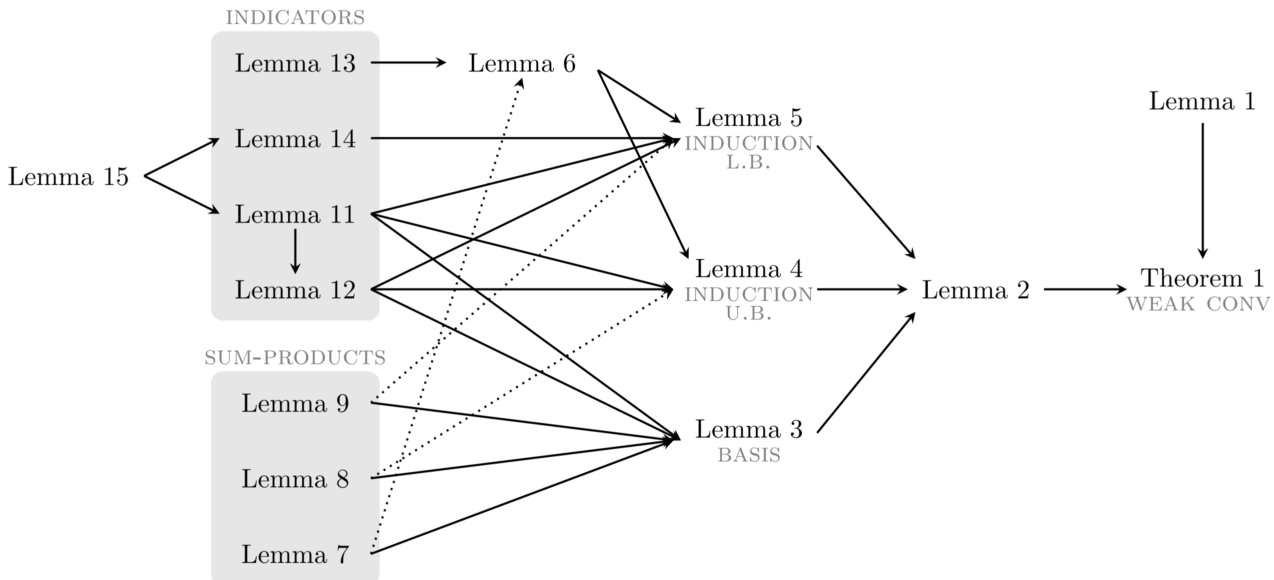

Figure 1: Graph showing dependencies between the lemmata used to prove weak convergence. Dotted arrows indicate dependence via a slight modification of the preceding lemma.

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