Weak convergence proof v.2 (neater) (in progress)

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Bounds on sum-products

Lemma 1.

$$t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \binom{l}{2} (t+1)^{l-2} \leq \sum_{s_1 \neq \cdots \neq s_l} \prod_{j=1}^{l} c_N(s_j) \leq t^l + c_N(\tau_N(t))(t+1)^l. \tag{1}$$

Proof. As pointed out in Koskela et al. (2018, Equation (8)),

$$\sum_{s_1 \neq \cdots \neq s_l} \prod_{j=1}^{l} c_N(s_j) \geq \left(\sum_{s=0}^{\tau_N(t)} c_N(s) \right)^l - \binom{l}{2} \left(\sum_{s=0}^{\tau_N(t)} c_N(s)^2 \right) \left(\sum_{s=0}^{\tau_N(t)} c_N(s) \right)^{l-2}. \tag{2}$$

By definition of τ_N,

$$t \leq \sum_{s=0}^{\tau_N(t)} c_N(s) \leq t+1. \tag{3}$$

Substituting these bounds into the RHS of (2) yields the lower bound.

It is a true fact that

$$\sum_{s_1 \neq \cdots \neq s_l} \prod_{j=1}^{l} c_N(s_j) \leq \left(\sum_{s=0}^{\tau_N(t)} c_N(s) \right)^l, \tag{4}$$

as can be seen by considering the multinomial expansion of the RHS. This is further bounded by

$$\left(\sum_{s=0}^{\tau_N(t)} c_N(s) \right)^l \leq \left(\sum_{s=0}^{\tau_N(t)-1} c_N(s) + c_N(\tau_N(t)) \right)^l \leq [t + c_N(\tau_N(t))]^l, \tag{5}$$

again using the definition of τ_N. A binomial expansion yields

$$[t + c_N(\tau_N(t))]^l = t^l + \sum_{i=0}^{l-1} \binom{l}{i} t^i c_N(\tau_N(t))^{l-i} = t^l + c_N(\tau_N(t)) \sum_{i=0}^{l-1} \binom{l}{i} t^i c_N(\tau_N(t))^{l-1-i}, \tag{6}$$

then since $c_N(s) \leq 1$ for all s,

$$\sum_{i=0}^{l-1} \binom{l}{i} t^i c_N(\tau_N(t))^{l-1-i} \leq \sum_{i=0}^{l-1} \binom{l}{i} t^i \leq (t+1)^l. \tag{7}$$

Putting this together yields the upper bound. ∎

Lemma 2. *Let B be a positive constant which may depend on n.*

$$\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) + BD_N(s_j)] \leq \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) + \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right)(t+1)^{l-1}(1+B)^l. \tag{8}$$

Proof. We start with a binomial expansion:

$$\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) + BD_N(s_j)] = \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \sum_{\mathcal{I} \subseteq [l]} B^{l-|\mathcal{I}|} \left(\prod_{i \in \mathcal{I}} c_N(s_i) \right) \left(\prod_{j \notin \mathcal{I}} D_N(s_j) \right)$$

$$= \sum_{\mathcal{I} \subseteq [l]} B^{l-|\mathcal{I}|} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i \in \mathcal{I}} c_N(s_i) \right) \left(\prod_{j \notin \mathcal{I}} D_N(s_j) \right) \tag{9}$$

where $[l] := \{1, \ldots, l\}$. Since the sum is over all permutations of $r_1, \ldots, r_l$, we may arbitrarily choose an ordering for $\{1, \ldots, l\}$ such that $\mathcal{I} = \{1, \ldots, |\mathcal{I}|\}$:

$$\sum_{\mathcal{I} \subseteq [l]} B^{l-|\mathcal{I}|} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i \in \mathcal{I}} c_N(s_i) \right) \left(\prod_{j \notin \mathcal{I}} D_N(s_j) \right) = \sum_{I=0}^{l} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right). \tag{10}$$

Separating the term $I = l$,

$$\sum_{I=0}^{l} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right) = \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j)$$

$$+ \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right). \tag{11}$$

In the second line, there is always at least one D_N term, and $c_N(s) \leq D_N(s)$ for all s, so we can write

$$\sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right) \leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{l-1} c_N(s_i) \right) D_N(s_l)$$

$$\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \left(\sum_{s_1 \neq \cdots \neq s_{l-1}}^{\tau_N(t)} \prod_{i=1}^{l-1} c_N(s_i) \right) \sum_{s_l=1}^{\tau_N(t)} D_N(s_l)$$

$$\leq \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} (t+1)^{l-1} \sum_{s=1}^{\tau_N(t)} D_N(s) \tag{12}$$

using (4) and (3). Finally, by the Binomial Theorem,

$$\sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} (t+1)^{l-1} \sum_{s=1}^{\tau_N(t)} D_N(s) \leq \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right)(t+1)^{l-1}(1+B)^l, \tag{13}$$

which, together with (11), concludes the proof. ∎

Lemma 3. *Let B be a positive constant which may depend on n.*

$$\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - BD_N(s_j)] \geq \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right)(t+1)^{l-1}(1+B)^l. \tag{14}$$

Proof. A binomial expansion and subsequent manipulation as in (9)–(11) gives

$$
\sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - B D_N(s_j)] = \sum_{\mathcal{I} \subseteq [l]} (-B)^{l-|\mathcal{I}|} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i \in \mathcal{I}} c_N(s_i) \right) \left(\prod_{j \notin \mathcal{I}} D_N(s_j) \right)
$$

$$
= \sum_{I=0}^{l} \binom{l}{I} (-B)^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right)
$$

$$
= \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) + \sum_{I=0}^{l-1} \binom{l}{I} (-B)^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right)
$$

$$
\geq \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) - \sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right) \tag{15}
$$

where the last inequality just multiplies some positive terms by -1. Then (12)–(13) can be applied directly (noting that an upper bound on negative terms gives a lower bound overall):

$$
-\sum_{I=0}^{l-1} \binom{l}{I} B^{l-I} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \left(\prod_{i=1}^{I} c_N(s_i) \right) \left(\prod_{j=I+1}^{l} D_N(s_j) \right) \geq \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1} (1+B)^l \tag{16}
$$

which concludes the proof. ∎

Main components of weak convergence

Lemma 4 (Basis step). *For any $0 < t < \infty$,*

$$
\lim_{N \to \infty} \mathbb{E} \left[\prod_{r=1}^{\tau_N(t)} (1 - p_r) \right] = e^{-\alpha_n t} \tag{17}
$$

where $\alpha_n := n(n-1)/2$.

Proof. We start by showing that $\lim_{N \to \infty} \mathbb{E} \left[\prod_{r=1}^{\tau_N(t)} (1 - p_r) \right] \leq e^{-\alpha_n t}$.
From Koskela et al. (2018, Lemma 1 Case 1), taking $\xi = \Delta$, we have

$$
1 - p_t = p_{\Delta\Delta}(t) \leq 1 - \alpha_n (1 + O(N^{-1})) [c_N(t) - B'_n D_N(t)] \tag{18}
$$

where the $O(N^{-1})$ term does not depend on t. Applying a multinomial expansion and then separating the positive and negative terms,

$$
\prod_{r=1}^{\tau_N(t)} (1 - p_r) \leq 1 + \sum_{l=1}^{\tau_N(t)} (-\alpha_n)^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - B'_n D_N(s_j)]
$$

$$
= 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - B'_n D_N(s_j)]
$$

$$
- \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1 + O(N^{-1})) \frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} [c_N(s_j) - B'_n D_N(s_j)]. \tag{19}
$$

This is further bounded by applying Lemma 3 and then both bounds of Lemma 1:

$$
\prod_{r=1}^{\tau_N(t)} (1-p_r) \le 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1}))\frac{1}{l!} \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j)
$$

$$
- \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1}))\frac{1}{l!} \left\{ \sum_{s_1 \neq \cdots \neq s_l}^{\tau_N(t)} \prod_{j=1}^{l} c_N(s_j) - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) (t+1)^{l-1}(1+B_n')^l \right\}
$$

$$
\le 1 + \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1}))\frac{1}{l!} \left\{ t^l + c_N(\tau_N(t))(t+1)^l \right\}
$$

$$
- \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1}))\frac{1}{l!} \left\{ t^l - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right)\binom{l}{2}(t+1)^{l-2} - \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right)(t+1)^{l-1}(1+B_n')^l \right\}.
$$

$$(20)$$

A bit of tidying up and we have

$$
\prod_{r=1}^{\tau_N(t)} (1-p_r) \le \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1}))\frac{1}{l!}t^l + c_N(\tau_N(t)) \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1}))\frac{1}{l!}(t+1)^l
$$

$$
+ \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1}))\frac{1}{l!}\binom{l}{2}(t+1)^{l-2}
$$

$$
+ \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1}))\frac{1}{l!}(t+1)^{l-1}(1+B_n')^l
$$

$$
\le \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1}))\frac{1}{l!}t^l + c_N(\tau_N(t)) \exp[\alpha_n(1+O(N^{-1}))(t+1)]
$$

$$
+ \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2}\alpha_n^2 \exp[\alpha_n(1+O(N^{-1}))(t+1)]
$$

$$
+ \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n')]. \tag{21}
$$

Now, taking the expectation and limit, and applying Brown et al. (2021, Equations (3.3)–(3.5)) and Lemma ?? (Lemma 2 in the messy weakconv note; not written up here yet),

$$
\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1-p_r) \right] \le \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!}t^l \lim_{N\to\infty} \mathbb{P}\left[\tau_N(t) \ge l\right] + \lim_{N\to\infty} \mathbb{E}\left[c_N(\tau_N(t))\right] \exp[\alpha_n(t+1)]
$$

$$
+ \lim_{N\to\infty} \mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right] \frac{1}{2}\alpha_n^2 \exp[\alpha_n(t+1)]
$$

$$
+ \lim_{N\to\infty} \mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} D_N(s) \right] \exp[\alpha_n(t+1)(1+B_n')]
$$

$$
= \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!}t^l = e^{-\alpha_n t}. \tag{22}
$$

It remains to show that $\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)}(1-p_r)\right] \geq e^{-\alpha_n t}$.

From Brown et al. (2021, Equation (3.14)), taking $\xi = \Delta$, we have

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \alpha_n(1 + O(N^{-1}))\left[c_N(t) + B_n D_N(t)\right] \tag{23}$$

where $B_n > 0$ and the $O(N^{-1})$ term does not depend on t. In particular,

$$1 - p_t = p_{\Delta\Delta}(t) \geq 1 - \frac{N^{n-2}}{(N-2)_{n-2}}\alpha_n c_N(t) - \frac{N^{n-3}}{(N-3)_{n-3}}B_n D_N(t). \tag{24}$$

Since $D_N(s) \leq c_N(s)$ for all s (Koskela et al., 2018, p.9), a sufficient condition for this bound to be non-negative is

$$E_r := \left\{ c_N(r) \leq \frac{(N-3)_{n-3}}{N^{n-3}}\left(\alpha_n\left(1 + \frac{2}{N-2}\right) + B_n\right)^{-1}\right\}, \tag{25}$$

and we define $E := \bigcap_{r=1}^{\tau_N(t)} E_r$. We now apply a multinomial expansion to the product, and split into positive and negative terms:

$$\prod_{r=1}^{\tau_N(t)}(1-p_r) \geq \left\{1 + \sum_{l=1}^{\tau_N(t)}(-\alpha_n)^l(1+O(N^{-1}))\frac{1}{l!}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}\left[c_N(s_j) + B_n D_N(s_j)\right]\right\}\mathbb{1}_E$$

$$= \left\{1 + \sum_{\substack{l=2\\\text{even}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}\left[c_N(s_j) + B_n D_N(s_j)\right]\right.$$

$$\left.- \sum_{\substack{l=1\\\text{odd}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}\left[c_N(s_j) + B_n D_N(s_j)\right]\right\}\mathbb{1}_E \tag{26}$$

This is further bounded by applying Lemma 2 and both bounds in Lemma 1:

$$\prod_{r=1}^{\tau_N(t)}(1-p_r) \geq \left\{1 + \sum_{\substack{l=2\\\text{even}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}c_N(s_j)\right.$$

$$\left.- \sum_{\substack{l=1\\\text{odd}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\left[\sum_{s_1\neq\cdots\neq s_l}^{\tau_N(t)}\prod_{j=1}^{l}c_N(s_j) + \left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)(t+1)^{l-1}(1+B_n)^l\right]\right\}\mathbb{1}_E$$

$$\geq \left\{1 + \sum_{\substack{l=2\\\text{even}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\left[t^l - \left(\sum_{s=1}^{\tau_N(t)}c_N(s)^2\right)\binom{l}{2}(t+1)^{l-2}\right]\right.$$

$$\left.- \sum_{\substack{l=1\\\text{odd}}}^{\tau_N(t)}\alpha_n^l(1+O(N^{-1}))\frac{1}{l!}\left[t^l + c_N(\tau_N(t))(t+1)^l + \left(\sum_{s=1}^{\tau_N(t)}D_N(s)\right)(t+1)^{l-1}(1+B_n)^l\right]\right\}\mathbb{1}_E. \tag{27}$$

Tidying things up,

$$\prod_{r=1}^{\tau_N(t)} (1-p_r) \geq \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_E - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \sum_{\substack{l=2 \\ \text{even}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} \binom{l}{2} (t+1)^{l-2}$$

$$- c_N(\tau_N(t)) \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} (t+1)^l$$

$$- \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \sum_{\substack{l=1 \\ \text{odd}}}^{\tau_N(t)} \alpha_n^l (1+O(N^{-1})) \frac{1}{l!} (t+1)^{l-1} (1+B_n)^l$$

$$\geq \sum_{l=0}^{\tau_N(t)} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l \mathbb{1}_E - \left(\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right) \frac{1}{2} \alpha_n^2 \exp[\alpha_n (1+O(N^{-1}))(t+1)]$$

$$- c_N(\tau_N(t)) \exp[\alpha_n(1+O(N^{-1}))(t+1)]$$

$$- \left(\sum_{s=1}^{\tau_N(t)} D_N(s) \right) \exp[\alpha_n(1+O(N^{-1}))(t+1)(1+B_n)]. \tag{28}$$

Now, taking the expectation and limit, and applying Brown et al. (2021, Equations (3.3)–(3.5)) and Lemma ?? (Lemma 2 in the messy weakconv note; not written up here yet),

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{r=1}^{\tau_N(t)} (1-p_r) \right] \geq \sum_{l=0}^{\infty} (-\alpha_n)^l (1+O(N^{-1})) \frac{1}{l!} t^l \lim_{N\to\infty} \mathbb{P}[\{\tau_N(t) \geq l\} \cap E]$$

$$- \lim_{N\to\infty} \mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right] \frac{1}{2} \alpha_n^2 \exp[\alpha_n(t+1)]$$

$$- \lim_{N\to\infty} \mathbb{E}[c_N(\tau_N(t))] \exp[\alpha_n(t+1)]$$

$$- \lim_{N\to\infty} \mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} D_N(s) \right] \exp[\alpha_n(t+1)(1+B_n)]$$

$$= \sum_{l=0}^{\infty} (-\alpha_n)^l \frac{1}{l!} t^l = e^{-\alpha_n t}. \tag{29}$$

Combining the upper and lower bounds in (22) and (29) respectively concludes the proof. ∎

I have proofs for the next three lemmata, I'm just working on a presentation that might be intelligible to someone other than myself.

Lemma 5 (Induction step upper bound). *Fix $k \in \mathbb{N}$, $i_0 := 0$, $i_k := k$. For any sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_k \leq t$,*

$$\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{r_1 < \cdots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^{k} p_{r_i} \right) \left(\prod_{\substack{r=1 \\ \notin \{r_1, \ldots, r_k\}}}^{\tau_N(t)} (1-p_r) \right) \right] \leq \alpha_n^k \sum_{l=0}^{\infty} \frac{1}{l!} (-\alpha_n)^l t^l \sum_{\substack{i_1 \leq \cdots \leq i_{k-1} \\ \in \{0, \ldots, k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}. \tag{30}$$

Lemma 6 (Induction step lower bound). *Fix $k \in \mathbb{N}$, $i_0 := 0$, $i_k := k$. For any sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_k \leq t$,*

$$\lim_{N \to \infty} \mathbb{E}\left[\sum_{\substack{r_1 < \cdots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \left(\prod_{i=1}^{k} p_{r_i} \right) \left(\prod_{\substack{r=1 \\ \notin \{r_1,\ldots,r_k\}}}^{\tau_N(t)} (1 - p_r) \right) \right] \geq \alpha_n^k \sum_{l=0}^{\infty} \frac{1}{l!} (-\alpha_n)^l t^l \sum_{\substack{i_1 \leq \cdots \leq i_{k-1} \\ \in \{0,\ldots,k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}.$$

(31)

Lemma 7. *Fix $l, k \in \mathbb{N}$, $i_0 := 0$, $i_k := k$. Let E be any event independent of $r_1, \ldots, r_k$ such that $\lim_{N \to \infty} \mathbb{P}[E] = 1$. Then for any sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_k \leq t$,*

$$\lim_{N \to \infty} \mathbb{E}\left[\mathbb{1}_E \sum_{\substack{r_1 < \cdots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i) \right] = \sum_{\substack{i_1 \leq \cdots \leq i_{k-1} \\ \in \{0,\ldots,k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}.$$

(32)

Proof. As pointed out by Möhle (1999, p. 460), the sum-product on the left hand side can be expanded as

$$\sum_{\substack{r_1 < \cdots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i) = \sum_{\substack{i_1 \leq \cdots \leq i_{k-1} \\ \in \{0,\ldots,k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^{k} \sum_{\substack{r_{i_{j-1}+1} < \cdots < r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i).$$

(33)

Moreover,

$$\sum_{\substack{r_{i_{j-1}+1} < \cdots < r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) = \frac{1}{(i_j - i_{j-1})!} \sum_{\substack{r_{i_{j-1}+1} \neq \cdots \neq r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i).$$

(34)

An argument akin to (4) gives us an upper bound:

$$\sum_{\substack{r_{i_{j-1}+1} \neq \cdots \neq r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) \leq \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s) \right)^{i_j - i_{j-1}} \leq [t_j - t_{j-1} + c_N(\tau_N(t_j))]^{i_j - i_{j-1}}$$

$$= \sum_{l=0}^{i_j - i_{j-1}} \binom{i_j - i_{j-1}}{l} (t_j - t_{j-1})^l [c_N(\tau_N(t_j))]^{i_j - i_{j-1} - l}$$

$$= (t_j - t_{j-1})^{i_j - i_{j-1}}$$

$$+ c_N(\tau_N(t_j)) \sum_{l=0}^{i_j - i_{j-1} - 1} \binom{i_j - i_{j-1}}{l} (t_j - t_{j-1})^l [c_N(\tau_N(t_j))]^{i_j - i_{j-1} - 1 - l}$$

$$\leq (t_j - t_{j-1})^{i_j - i_{j-1}} + c_N(\tau_N(t_j))(t_j - t_{j-1} + 1)^k,$$

(35)

using in the last line that $c_N \leq 1$ and $0 \leq i_j - i_{j-1} \leq k$. Now, taking the product on the outside,

$$
\prod_{j=1}^{k} \sum_{\substack{r_{i_{j-1}+1} < \cdots < r_{i_j} \\ = \tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) \leq \prod_{j=1}^{k} \left\{ \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} + c_N(\tau_N(t_j)) \frac{(1 + t_j - t_{j-1})^k}{(i_j - i_{j-1})!} \right\}
$$

$$
\leq \prod_{j=1}^{k} \left\{ \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} + c_N(\tau_N(t_j))(1 + t_j - t_{j-1})^k \right\}
$$

$$
= \sum_{\mathcal{I} \subseteq [k]} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} c_N(\tau_N(t_j))(1 + t_j - t_{j-1})^k \right)
$$

$$
= \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}
$$

$$
+ \sum_{\mathcal{I} \subset [k]} \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) \left(\prod_{j \notin \mathcal{I}} c_N(\tau_N(t_j))(1 + t_j - t_{j-1})^k \right)
$$

$$
\leq \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}
$$

$$
+ \sum_{\mathcal{I} \subset [k]} c_N(\tau_N(t_{j^*})) \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) (1 + t_j - t_{j-1})^{k^2} \qquad (36)
$$

where, say, $j^* := \min\{j \notin \mathcal{I}\}$. Now we are in a position to evaluate the limit in (32):

$$
\lim_{N \to \infty} \mathbb{E} \left[\mathbb{1}_E \sum_{\substack{r_1 < \cdots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i) \right] \leq \lim_{N \to \infty} \mathbb{E} \left[\sum_{\substack{r_1 < \cdots < r_k: \\ r_i \leq \tau_N(t_i) \forall i}} \prod_{i=1}^{k} c_N(r_i) \right]
$$

$$
\leq \sum_{\substack{i_1 \leq \cdots \leq i_{k-1} \\ \in \{0,\ldots,k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}
$$

$$
+ \sum_{\substack{i_1 \leq \cdots \leq i_{k-1} \\ \in \{0,\ldots,k\}: \\ i_j \geq j \forall j}} \sum_{\mathcal{I} \subset [k]} \lim_{N \to \infty} \mathbb{E} \left[c_N(\tau_N(t_{j^*})) \right] \left(\prod_{j \in \mathcal{I}} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \right) (1 + t_j - t_{j-1})^{k^2}
$$

$$
= \sum_{\substack{i_1 \leq \cdots \leq i_{k-1} \\ \in \{0,\ldots,k\}: \\ i_j \geq j \forall j}} \prod_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1}}}{(i_j - i_{j-1})!} \qquad (37)
$$

using Brown et al. (2021, Equation (3.3)). For the corresponding lower bound, by a slight modification of (2),

$$
\sum_{\substack{r_{i_{j-1}+1}\neq\cdots\neq r_{i_j}\\=\tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) \geq \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)\right)^{i_j-i_{j-1}}
$$

$$
- \binom{i_j-i_{j-1}}{2}\left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2\right)\left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)\right)^{i_j-i_{j-1}-2}
$$

$$
\geq (t_j-t_{j-1})^{i_j-i_{j-1}} - \binom{i_j-i_{j-1}}{2}\left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2\right)(t_j-t_{j-1}+1)^{i_j-i_{j-1}-2}
$$

$$
\geq (t_j-t_{j-1})^{i_j-i_{j-1}} - (i_j-i_{j-1})!\left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2\right)(t_j-t_{j-1}+1)^{k-1}. \qquad (38)
$$

Define the event

$$
E_j^\star = \left\{\left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2\right) \leq \frac{(t_j-t_{j-1})^{i_j-i_{j-1}-k+1}}{(i_j-i_{j-1})!}\right\}, \qquad (39)
$$

which is sufficient to ensure the jth term in the following is non-negative, and let $E^\star := \bigcap_{j=1}^k E_j^\star$. Now, taking a product over j as in (36),

$$
\prod_{j=1}^k \sum_{\substack{r_{i_{j-1}+1}<\cdots<r_{i_j}\\=\tau_N(t_{j-1})+1}}^{\tau_N(t_j)} \prod_{i=i_{j-1}+1}^{i_j} c_N(r_i) \geq \prod_{j=1}^k \left\{\frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} - \left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2\right)(t_j-t_{j-1}+1)^{k-1}\right\}\mathbb{1}_{E^\star}
$$

$$
= \sum_{\mathcal{I}\subseteq[k]} (-1)^{k-|\mathcal{I}|}\left(\prod_{j\in\mathcal{I}}\frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}\right)\left(\prod_{j\notin\mathcal{I}}\left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2\right)(t_j-t_{j-1}+1)^{k-1}\right)\mathbb{1}_{E^\star}
$$

$$
= \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}\mathbb{1}_{E^\star}
$$

$$
+ \sum_{\mathcal{I}\subset[k]} (-1)^{k-|\mathcal{I}|}\left(\prod_{j\in\mathcal{I}}\frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}\right)\left(\prod_{j\notin\mathcal{I}}\left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2\right)(t_j-t_{j-1}+1)^{k-1}\right)
$$

$$
\geq \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}\mathbb{1}_{E^\star}
$$

$$
- \sum_{\substack{\mathcal{I}\subset[k]:\\k-|\mathcal{I}|\text{ odd}}}\left(\prod_{j\in\mathcal{I}}\frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}\right)\left(\prod_{j\notin\mathcal{I}}\left(\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2\right)(t_j-t_{j-1}+1)^{k-1}\right)
$$

$$
\geq \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}\mathbb{1}_{E^\star}
$$

$$
- \sum_{\mathcal{I}\subset[k]}\left(\sum_{s=\tau_N(t_{j^*-1})+1}^{\tau_N(t_{j^*})} c_N(s)^2\right)\left(\prod_{j\in\mathcal{I}}(t_j-t_{j-1})^k\right)\left(\prod_{j\notin\mathcal{I}}(t_j-t_{j-1}+1)^k\right)
$$

$$
\geq \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}\mathbb{1}_{E^\star} - \sum_{\mathcal{I}\subset[k]}\left(\sum_{s=\tau_N(t_{j^*-1})+1}^{\tau_N(t_{j^*})} c_N(s)^2\right)(t_j-t_{j-1}+1)^{k^2}. \qquad (40)
$$

We can now evaluate the limit:

$$
\lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_E \sum_{\substack{r_1<\cdots<r_k:\\ r_i\leq\tau_N(t_i)\forall i}} \prod_{i=1}^k c_N(r_i)\right] \geq \lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_{E\cap E^\star} \sum_{\substack{i_1\leq\cdots\leq i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\geq j\forall j}} \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!}\right]
$$

$$
-\lim_{N\to\infty} \mathbb{E}\left[\mathbb{1}_E \sum_{\substack{i_1\leq\cdots\leq i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\geq j\forall j}} \sum_{\mathcal{I}\subset[k]} \left(\sum_{s=\tau_N(t_{j^*-1})+1}^{\tau_N(t_{j^*})} c_N(s)^2\right)(t_j-t_{j-1}+1)^{k^2}\right]
$$

$$
\geq \sum_{\substack{i_1\leq\cdots\leq i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\geq j\forall j}} \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} \lim_{N\to\infty} \mathbb{E}[\mathbb{1}_{E\cap E^\star}]
$$

$$
-\lim_{N\to\infty} \mathbb{E}\left[\sum_{\substack{i_1\leq\cdots\leq i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\geq j\forall j}} \sum_{\mathcal{I}\subset[k]} \left(\sum_{s=\tau_N(t_{j^*-1})+1}^{\tau_N(t_{j^*})} c_N(s)^2\right)(t_j-t_{j-1}+1)^{k^2}\right]
$$

$$
= \sum_{\substack{i_1\leq\cdots\leq i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\geq j\forall j}} \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} \lim_{N\to\infty} \mathbb{P}[E\cap E^\star]
$$

$$
- \sum_{\substack{i_1\leq\cdots\leq i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\geq j\forall j}} \sum_{\mathcal{I}\subset[k]} \lim_{N\to\infty} \mathbb{E}\left[\sum_{s=\tau_N(t_{j^*-1})+1}^{\tau_N(t_{j^*})} c_N(s)^2\right](t_j-t_{j-1}+1)^{k^2}
$$

$$
= \sum_{\substack{i_1\leq\cdots\leq i_{k-1}\\ \in\{0,\ldots,k\}:\\ i_j\geq j\forall j}} \prod_{j=1}^k \frac{(t_j-t_{j-1})^{i_j-i_{j-1}}}{(i_j-i_{j-1})!} \tag{41}
$$

where for the last equality we use Brown et al. (2021, Equation (3.5)) to show that the second sum vanishes and Lemma 11 to show that $\lim_{N\to\infty} \mathbb{P}[E\cap E^\star]=1$. We have shown that the upper and lower bounds coincide, so the result follows. ∎

Indicators

Lemma 8. *Let A, B be events.* *Sequences of events, really. Dependence on some incremental variable is implicit, also in the limit notation.* *If $\lim \mathbb{P}[A]=1$ and $\lim \mathbb{P}[B]=1$ then $\lim \mathbb{P}[A\cap B]=1$.*

The above might be so obvious as to go unstated, but it is very important because it means we don't have to deal with intersections of dependent events! Here is a little proof just to be sure:

Proof.

$$
\lim \mathbb{P}[A]=1 \text{ and } \lim \mathbb{P}[B]=1
$$
$$
\Leftrightarrow \lim \mathbb{P}[A^c]=0 \text{ and } \lim \mathbb{P}[B^c]=0
$$
$$
\Rightarrow \lim\{\mathbb{P}[A^c]+\mathbb{P}[B^c]\}=0
$$
$$
\Rightarrow \lim \mathbb{P}[A^c\cup B^c]=0
$$
$$
\Leftrightarrow \lim \mathbb{P}[A\cap B]=1. \tag{42}
$$

The only part of this argument that I find potentially controversial is going from the third to the fourth line, which is an application of the sandwich theorem (since $0 \leq \mathbb{P}[A^c \cup B^c] \leq \mathbb{P}[A^c] + \mathbb{P}[B^c]$). ∎

Lemma 9. *Let K be a constant which may depend on n, N but not on r, such that $K^{-2} = O(1)$ as $N \to \infty$. Define the events $E_r := \{c_N(r) < K\}$ and denote $E := \bigcap_{r=1}^{\tau_N(t)} E_r$. Then $\lim_{N \to \infty} \mathbb{P}[E] = 1$.*

Proof.

$$\mathbb{P}[E] = 1 - \mathbb{P}[E^c] = 1 - \mathbb{P}\left[\bigcup_{r=1}^{\tau_N(t)} E_r^c\right] = 1 - \mathbb{E}\left[\mathbb{1}_{\bigcup E_r^c}\right] \geq 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{1}_{E_r^c}\right]$$

$$= 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}\left[\mathbb{1}_{E_r^c} \mid \mathcal{F}_{r-1}\right]\right] = 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{P}\left[E_r^c \mid \mathcal{F}_{r-1}\right]\right] \tag{43}$$

where for the second line we apply Lemma 13 with $f(r) = \mathbb{1}_{E_r^c}$. By the generalised Markov inequality,

$$\mathbb{P}[E_r^c \mid \mathcal{F}_{r-1}] = \mathbb{P}[c_N(r) \geq K \mid \mathcal{F}_{r-1}] \leq \mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}]K^{-2}. \tag{44}$$

Substituting this into (43) and applying Lemma 13 again, this time with $f(r) = c_N(r)^2$,

$$\mathbb{P}[E] \geq 1 - K^{-2}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[c_N(r)^2 \mid \mathcal{F}_{r-1}]\right] = 1 - K^{-2}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} c_N(r)^2\right]. \tag{45}$$

Applying Brown et al. (2021, Equation (3.5)), the limit is

$$\lim_{N \to \infty} \mathbb{P}[E] = 1 - O(1) \times 0 = 1 \tag{46}$$

as required. ∎

Lemma 10. *Fix $t > 0$. For any $l \in \mathbb{R}$, $\lim_{N \to \infty} \mathbb{P}[\tau_N(t) \geq l] = 1$.*

Proof.

$$\{\tau_N(t) \geq l\} = \left\{\min\left\{s \geq 1 : \sum_{r=1}^{s} c_N(r) \geq t\right\} \geq l\right\} = \left\{\sum_{r=1}^{l-1} c_N(r) < t\right\} \supseteq \bigcap_{r=1}^{l-1}\left\{c_N(r) < \frac{t}{l}\right\} \supseteq \bigcap_{r=1}^{\tau_N(t)}\left\{c_N(r) < \frac{t}{l}\right\}. \tag{47}$$

Hence

$$\lim_{N \to \infty} \mathbb{P}[\tau_N(t) \geq l] \geq \lim_{N \to \infty} \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)}\left\{c_N(r) < \frac{t}{l}\right\}\right] \tag{48}$$

and the result follows by applying Lemma 9 with $K = t/l$. ∎

Lemma 11. *Define the event*

$$E^\star := \bigcap_{j=1}^{k}\left\{\sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \leq \frac{(t_j - t_{j-1})^{i_j - i_{j-1} - k + 1}}{(i_j - i_{j-1})!}\right\}. \tag{49}$$

Then $\lim_{N \to \infty} \mathbb{P}[E^\star] = 1$.

Proof.

$$E^\star \supseteq \left\{ \sum_{j=1}^{k} \sum_{s=\tau_N(t_{j-1})+1}^{\tau_N(t_j)} c_N(s)^2 \le \sum_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1} - k + 1}}{(i_j - i_{j-1})!} \right\}$$

$$= \left\{ \sum_{s=1}^{\tau_N(t)} c_N(s)^2 \le \sum_{j=1}^{k} \frac{(t_j - t_{j-1})^{i_j - i_{j-1} - k + 1}}{(i_j - i_{j-1})!} \right\}$$

$$\supseteq \left\{ \sum_{s=1}^{\tau_N(t)} c_N(s)^2 \le \frac{1}{k!} \sum_{j=1}^{k} (t_j - t_{j-1})^{i_j - i_{j-1} - k + 1} \right\}. \tag{50}$$

To simplify the RHS further, consider the possible values of $(i_j - i_{j-1} - k + 1) \in \{-k+1, \dots, 1\}$: This simplification isn't necessary for the result, but it makes the expressions less cumbersome later on.
Case $(i_j - i_{j-1} - k + 1) < 0$:

$$\sum_{j=1}^{k} (t_j - t_{j-1})^{i_j - i_{j-1} - k + 1} \ge \sum_{j=1}^{k} t^{i_j - i_{j-1} - k + 1} \ge \sum_{j=1}^{k} t^{-k+1} = k t^{-k+1}. \tag{51}$$

Case $(i_j - i_{j-1} - k + 1) = 0$:

$$\sum_{j=1}^{k} (t_j - t_{j-1})^{i_j - i_{j-1} - k + 1} = \sum_{j=1}^{k} 1 = k. \tag{52}$$

Case $(i_j - i_{j-1} - k + 1) = 1$:

$$\sum_{j=1}^{k} (t_j - t_{j-1})^{i_j - i_{j-1} - k + 1} = \sum_{j=1}^{k} (t_j - t_{j-1}) = t_k - t_0 = t. \tag{53}$$

Altogether

$$\sum_{j=1}^{k} (t_j - t_{j-1})^{i_j - i_{j-1} - k + 1} \ge \min\{k t^{-k+1}, k, t\} = \min\{k t^{-k+1}, t\} = t \min\{k t^{-k}, 1\}, \tag{54}$$

so

$$E^\star \supseteq \left\{ \sum_{s=1}^{\tau_N(t)} c_N(s)^2 < \frac{t}{k!} \min\{k t^{-k}, 1\} \right\}. \tag{55}$$

Using Markov's inequality,

$$\mathbb{P}[E^\star] \ge \mathbb{P}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 < \frac{t}{k!} \min\{k t^{-k}, 1\} \right] = 1 - \mathbb{P}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \ge \frac{t}{k!} \min\{k t^{-k}, 1\} \right]$$

$$\ge 1 - \frac{k!}{t} \max\{1, k^{-1} t^k\} \mathbb{E}\left[\sum_{s=1}^{\tau_N(t)} c_N(s)^2 \right], \tag{56}$$

and by Brown et al. (2021, Equation (3.5))

$$\lim_{N \to \infty} \mathbb{P}[E^\star] = 1 - O(1) \times 0 = 1 \tag{57}$$

as required. $\blacksquare$

Lemma 12. *Let K be a constant not depending on N, r, but which may depend on n.*

$$\lim_{N \to \infty} \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) \ge K D_N(r)\} \right] = 1. \tag{58}$$

Proof.

$$\mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) \geq KD_N(r)\}\right] \geq \mathbb{P}\left[\bigcap_{r=1}^{\tau_N(t)} \{c_N(r) > KD_N(r)\}\right]$$

$$= 1 - \mathbb{P}\left[\bigcup_{r=1}^{\tau_N(t)} \{c_N(r) \leq KD_N(r)\}\right]$$

$$= 1 - \mathbb{E}\left[\mathbb{1}_{\bigcup\{c_N(r) \leq KD_N(r)\}}\right]$$

$$\geq 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{1}_{\{c_N(r) \leq KD_N(r)\}}\right]$$

$$= 1 - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{P}[c_N(r) \leq KD_N(r) \mid \mathcal{F}_{r-1}]\right] \tag{59}$$

where the final inequality is an application of Lemma 13 with $f(r) = \mathbb{1}_{\{c_N(r) \leq KD_N(r)\}}$.

Fix $0 < \varepsilon < K^{-1}/2$ and let $N > \max\{\varepsilon^{-1}, (\binom{n-2}{2} - 2\varepsilon)^{-1}\}$. For each r, i define the event $A_i(r) := \{\nu_r^{(i)} \leq N\varepsilon\}$. Conditional on $\mathcal{F}_{r-1}$, we have

$$D_N(r) = \frac{1}{N(N)_2} \sum_{i=1}^{N} (\nu_r^{(j)})_2 \left[\nu_r^{(i)} + \frac{1}{N}\sum_{j \neq i}(\nu_r^{(i)})^2\right] \mathbb{1}_{A_i(r)^c} + \frac{1}{N(N)_2}\sum_{i=1}^{N}(\nu_r^{(i)})_2\left[\nu_r^{(i)} + \frac{1}{N}\sum_{j \neq i}(\nu_r^{(j)})^2\right]\mathbb{1}_{A_i(r)}. \tag{60}$$

For the first term,

$$\frac{1}{N(N)_2}\sum_{i=1}^{N}(\nu_r^{(i)})_2\left[\nu_r^{(i)} + \frac{1}{N}\sum_{j \neq i}(\nu_r^{(j)})^2\right]\mathbb{1}_{A_i(r)^c} \leq \sum_{i=1}^{N}\mathbb{1}_{A_i(r)^c}. \tag{61}$$

For the second term,

$$\frac{1}{N(N)_2}\sum_{i=1}^{N}(\nu_r^{(i)})_2\left[\nu_r^{(i)} + \frac{1}{N}\sum_{j \neq i}(\nu_r^{(j)})^2\right]\mathbb{1}_{A_i(r)} \leq \frac{1}{N(N)_2}\sum_{i=1}^{N}(\nu_r^{(i)})_2\nu_r^{(i)}\mathbb{1}_{A_i(r)} + \frac{1}{N^2(N)_2}\sum_{i=1}^{N}(\nu_r^{(i)})_2\sum_{j=1}^{N}(\nu_r^{(j)})^2\mathbb{1}_{A_i(r)}$$

$$\leq \frac{1}{N}c_N(r)N\varepsilon + \frac{1}{N^2(N)_2}\sum_{i=1}^{N}(\nu_r^{(i)})_2\sum_{j=1}^{N}(\nu_r^{(j)})_2\mathbb{1}_{A_i(r)}$$

$$+ \frac{1}{N^2(N)_2}\sum_{i=1}^{N}(\nu_r^{(i)})_2\sum_{j=1}^{N}(\nu_r^{(j)})\mathbb{1}_{A_i(r)}$$

$$\leq \varepsilon c_N(r) + \frac{1}{N^2}\sum_{i=1}^{N}\nu_r^{(i)}N\varepsilon c_N(r) + \frac{1}{N^2}c_N(r)N$$

$$= c_N(r)\left(2\varepsilon + \frac{1}{N}\right). \tag{62}$$

Hence, conditional on $\mathcal{F}_{r-1}$,

$$\{c_N(r) \geq KD_N(r)\} \supseteq \left\{c_N(r) \leq Kc_N(r)(2\varepsilon + N^{-1}) + K\sum_{i=1}^{N}\mathbb{1}_{A_i(r)^c}\right\}$$

$$= \left\{K^{-1} - 2\varepsilon - \frac{1}{N} \leq \sum_{i=1}^{N}\mathbb{1}_{A_i(r)^c}c_N(r)^{-1}\right\} \tag{63}$$

where the ratio $\mathbb{1}_{A_i(r)^c}/c_N(r)$ is well-defined because

$$A_i(r)^c \Rightarrow c_N(r) := \frac{1}{(N)_2}\sum_{j=1}^{N}(\nu_r^{(j)})_2 \geq \frac{1}{(N)_2}(\nu_r^{(i)})_2 \geq \frac{\varepsilon(N\varepsilon - 1)}{N - 1} \geq \varepsilon\left(\varepsilon - \frac{1}{N}\right) > 0. \tag{64}$$

Hence by Markov's inequality (the conditions on ε, N ensuring the constant is always strictly positive),

$$\mathbb{P}\left[c_N(r) \le K D_N(r) \mid \mathcal{F}_{r-1}\right] \le \mathbb{P}\left[\sum_{i=1}^N \mathbb{1}_{A_i(r)^c} \ge \left(K^{-1} - 2\varepsilon - \frac{1}{N}\right)\varepsilon\left(\varepsilon - \frac{1}{N}\right)\middle| \mathcal{F}_{r-1}\right]$$

$$\le \frac{1}{\left(K^{-1} - 2\varepsilon - \frac{1}{N}\right)\varepsilon\left(\varepsilon - \frac{1}{N}\right)}\mathbb{E}\left[\sum_{i=1}^N \mathbb{1}_{A_i(r)^c}\middle| \mathcal{F}_{r-1}\right]$$

$$\le \frac{1}{\left(K^{-1} - 2\varepsilon - \frac{1}{N}\right)\varepsilon\left(\varepsilon - \frac{1}{N}\right)}\mathbb{E}\left[\sum_{i=1}^N \frac{(\nu_r^{(i)})_3}{(N\varepsilon)_3}\middle| \mathcal{F}_{r-1}\right]$$

$$\le \frac{1}{\left(K^{-1} - 2\varepsilon - \frac{1}{N}\right)\varepsilon\left(\varepsilon - \frac{1}{N}\right)}\mathbb{E}\left[\frac{N(N)_2}{(N\varepsilon)_3} D_N(r)\middle| \mathcal{F}_{r-1}\right]. \tag{65}$$

Applying Lemma 13 once more, with $f(r) = D_N(r)$,

$$\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{P}[c_N(r) \le K D_N(r) \mid \mathcal{F}_{r-1}]\right] \le \frac{1}{\left(K^{-1} - 2\varepsilon - \frac{1}{N}\right)\varepsilon\left(\varepsilon - \frac{1}{N}\right)}\frac{N(N)_2}{(N\varepsilon)_3}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)}\mathbb{E}[D_N(r) \mid \mathcal{F}_{r-1}]\right]$$

$$= \frac{1}{\left(K^{-1} - 2\varepsilon - \frac{1}{N}\right)\varepsilon\left(\varepsilon - \frac{1}{N}\right)}\frac{N(N)_2}{(N\varepsilon)_3}\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} D_N(r)\right]$$

$$\xrightarrow[N\to\infty]{} \frac{1}{(K^{-1} - 2\varepsilon)\varepsilon^5} \times 0 = 0. \tag{66}$$

Substituting this back into (59) concludes the proof. ∎

Other useful results

The following Lemma is taken from Koskela et al. (2018, Lemma 2), where the function is set to $f(t) = c_N(t)$, but the authors remark that the result holds for other choices of function.

Lemma 13. *Let $(\mathcal{F}_t)$ be the backwards-in-time filtration generated by the offspring counts $\nu_t^{(1:N)}$ at each generation t, and let $f(t)$ be any deterministic function of $\nu_t^{(1:N)}$ that is non-negative and bounded. In particular, for all t there exists $B < \infty$ such that $0 \le f(t) \le B$. Then*

$$\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} f(r)\right] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right]. \tag{67}$$

Proof. Define

$$M_s := \sum_{r=1}^{s} \{f(r) - \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\}. \tag{68}$$

It is easy to establish that (M_s) is a martingale with respect to $(\mathcal{F}_s)$, and $M_0 = 0$. Now fix $K \ge 1$ and note that $\tau_N(t) \wedge K$ is a bounded $\mathcal{F}_t$-stopping time. Hence we can apply the optional stopping theorem:

$$\mathbb{E}[M_{\tau_N(t)\wedge K}] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} \{f(r) - \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\}\right] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} f(r)\right] - \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right] = 0. \tag{69}$$

Since this holds for all $K \ge 1$,

$$\lim_{K\to\infty} \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} f(r)\right] = \lim_{K\to\infty} \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)\wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right]. \tag{70}$$

The monotone convergence theorem allows the limit to pass inside the expectation on each side (since increasing K can only increase each sum, by possibly adding non-negative terms). Hence

$$\mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} f(r)\right] = \mathbb{E}\left[\lim_{K\to\infty} \sum_{r=1}^{\tau_N(t)\wedge K} f(r)\right] = \mathbb{E}\left[\lim_{K\to\infty} \sum_{r=1}^{\tau_N(t)\wedge K} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right] = \mathbb{E}\left[\sum_{r=1}^{\tau_N(t)} \mathbb{E}[f(r) \mid \mathcal{F}_{r-1}]\right] \quad (71)$$

which concludes the proof. ∎

References

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