Math 475 Homework 5

Due Monday 7th November at start of lecture

The following comments, restated from lecture, are for study purposes, and are considered part of the course materials relevant for homework and exams.

- Remember that if X is a random variable with density function $f(x) > 0$ for all $x \in \mathbb{R}$, then its cumulative distribution function $F(x)$ is monotone non-decreasing, and right-continuous, and it has inverse F^{-1}. If we generate $U \sim \text{Unif}(0,1)$ and let $X = F^{-1}(U)$, then

$$\begin{aligned}\mathbb{P}(X \le x) &= \mathbb{P}(F^{-1}(U) \le x) \\ &= \mathbb{P}(F(F^{-1}(U)) \le F(x)) \\ &= \mathbb{P}(U \le F(x)) \\ &= F(x),\end{aligned}$$

  which means that $X \sim F$, i.e. X is distributed according to F.

- The above doesn't always work. Suppose that $f(x) = 0$ for all $x \in [a,b]$, with $a < b$, and let $u = F(a)$. Then $F(x) = u$ for all $a \le x \le b$, and therefore F *does not* have a unique inverse at u.

- Another case that this doesn't work is when X is discrete, with $F(x) > \lim_{\epsilon \to 0^+} F(x - \epsilon) \equiv F(x-)$. Then for u in the interval $[F(x-), F(x))$, there is no value x' with $F(x') = u$.

- Both of the two problems above are solved by using the following definition for F^{-1}: for a probability $0 < u < 1$,

$$F^{-1}(u) = \inf\{x \in \mathbb{R} : F(x) \ge u\}.$$

  This set is never empty, and it always attains its infimum because F is right-continuous. The function $F^{-1}(u)$ is the **quantile function**.

Theorem 1. *Let F be a cumulative distribution function and let F^{-1} be its inverse (the 'quantile function' above). If $U \sim$ Unif$(0,1)$ and $X = F^{-1}(U)$, then $X \sim F$.*

Proof: It is sufficient to show that $F(x) \le u$ *if and only if* $x \le F^{-1}(u)$ for $0 < u < 1$. Both F and F^{-1} are non-decreasing. Because the infimum in the definition above of $F^{-1}(u)$ is attained, then $F^{-1}(u) \in \{x : F(x) \ge u\}$, and therefore $F(F^{-1}(u)) \ge u$. Similarly, $F^{-1}(F(x)) \le x$. Now, suppose that $F^{-1}(u) \le x$. Rewriting this, we have $x \ge F^{-1}(u)$, and thus $F(x) \ge F(F^{-1}(u)) \ge u$, from which we see $u \le F(x)$. Conversely, suppose that $u \le F(x)$. Then $F^{-1}(u) \le F^{-1}(F(x)) \le x$.

Generating Random Samples

Use the material above (and/or material from lecture) to answer these questions.

1. Suppose that X has a standard exponential distribution with density $f(x) = e^{-x}$ for $x > 0$. Show that, if you have the ability to sample $U \sim \text{Unif}(0,1)$, then you can sample from this exponential distribution by sampling $U \sim \text{Unif}(0,1)$, and then defining $X = -\log(1-U)$. That is to say, show that X defined in this way will have the claimed exponential distribution. (Hint: you need to find the CDF).

2. Suppose more generally that X has an exponential distribution with rate λ, i.e. $f(x) = \lambda \exp(-\lambda x)$ for $x > 0$. Show that you can sample from this exponential distribution by sampling $U \sim \text{Unif}(0,1)$, and then taking $X = -\log(1-U)/\lambda$.

Other Questions

3. Explain, in your own words (not just copying exactly from lecture), and to the best of your ability:

   (i) What does it mean to say that 'an n-point Gaussian quadrature rule is exact for polynomials of degree $2n-1$ or less'?

   (ii) Why is the above statement true?

4. Look into the `R` documentation and learn how `R` generates uniform random numbers with the default `runif` function. That is, tell me in your own words what the procedure is. You don't need to give every mathematical detail, but you should give enough details to show that you understand the concept.