The primary goal of this introduction is to demonstrate how to implement spectral element method in 2d. We seek to produce the displacement field caused by an earthquake in a finite earth model with volume Ω. In order to simplify calculation, we just use stress-free boundary condition. We will talk about artificial absorbing boundary condition in detail later. The displacement field $\boldsymbol{u}$ is produced by an earthquake is governed by the momentum equation,

$$\rho\, \partial_t^2 \boldsymbol{u} = \boldsymbol{\nabla} \cdot \boldsymbol{T} + \boldsymbol{f} \tag{1}$$

,where $\boldsymbol{T} = (\boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3)^T$, and $\boldsymbol{T}_i$ is vector function. The weak form of above equation is produced by dotting the momentum equation with an arbitrary test vector $\boldsymbol{w} = (w_1(\boldsymbol{x}), w_2(\boldsymbol{x}), w_3(\boldsymbol{x}))^T$, integrating by the model volume Ω. Since we use stress-free boundary condition, which is easy to derive the weak form of wave equation, we have $\boldsymbol{T}|_{\partial\Omega} = 0$

$$\begin{aligned}
&\int_\Omega \boldsymbol{\nabla} \cdot \boldsymbol{T} \cdot \boldsymbol{w}(\boldsymbol{x})\ \mathrm{d}^3\boldsymbol{x} \\
&= \sum_{i=1}^{3} \int_\Omega w_i(\boldsymbol{x}) \nabla \cdot \boldsymbol{T}_i\ \mathrm{d}^3\boldsymbol{x} \\
&= \sum_{i=1}^{3} \int_\Omega \nabla \cdot (w_i(\boldsymbol{x}) \boldsymbol{T}_i) - \nabla w_i(\boldsymbol{x}) \cdot \boldsymbol{T}_i\ \mathrm{d}^3\boldsymbol{x} \\
&= \sum_{i=1}^{3} \left(\int_{\partial\Omega} w_i(\boldsymbol{x}) \boldsymbol{T}_i \cdot\ \mathrm{d}\boldsymbol{s} - \int_\Omega \nabla w_i(\boldsymbol{x}) \cdot \boldsymbol{T}_i\ \mathrm{d}^3\boldsymbol{x} \right) \\
&= -\sum_{i=1}^{3} \int_\Omega \nabla w_i(\boldsymbol{x}) \cdot \boldsymbol{T}_i\ \mathrm{d}^3\boldsymbol{x} \\
&= -\int_\Omega \boldsymbol{\nabla} \boldsymbol{w}(\boldsymbol{x}) : \boldsymbol{T}\ \mathrm{d}^3\boldsymbol{x}
\end{aligned} \tag{2}$$

The Source function is

$$\boldsymbol{f} = -\boldsymbol{M} \cdot \boldsymbol{\nabla} \delta(\boldsymbol{x} - \boldsymbol{x}_s) S(t) \tag{3}$$

Using the properties of Dirac delta distribution,after integration, it transformed in the following way,

$$\begin{aligned}
&\int_\Omega \boldsymbol{f} \cdot \boldsymbol{w}\ \mathrm{d}^3\boldsymbol{x} \\
&= -S(t) \int_\Omega \boldsymbol{M} \cdot \boldsymbol{\nabla} \delta(\boldsymbol{x} - \boldsymbol{x}_s) \cdot \boldsymbol{w}\ \mathrm{d}^3\boldsymbol{x} \\
&= \boldsymbol{M} : \boldsymbol{\nabla} \boldsymbol{w}(\boldsymbol{x}_s) S(t)
\end{aligned} \tag{4}$$

In 2 dimension, We can derive this using Green Formula.

$$\int_\Omega \rho\, \boldsymbol{w} \cdot \partial_t^2 \boldsymbol{u}\ \mathrm{d}^2\boldsymbol{x} = -\int_\Omega \boldsymbol{\nabla} \boldsymbol{w}(\boldsymbol{x}) : \boldsymbol{T}\ \mathrm{d}^2\boldsymbol{x} + \boldsymbol{M} : \boldsymbol{\nabla} \boldsymbol{w}(\boldsymbol{x}_s) S(t) \tag{5}$$

In the following parts, we will explain the general procedure of specfem.

First, we divide the region into a number of non-overlapping elements, Ω_e, $e = 1, \ldots, n_e$, such that $\Omega = \bigcup_e^{n_e} \Omega_e$, as shown in Fig.1. In SEM, the shape of element is restricted to quadrilateral. The integration in the whole region become the sum of integration over every element, as shown below.

$$\sum_e^{n_e} \int_{\Omega_e} \rho\, \boldsymbol{w} \cdot \partial_t^2 \boldsymbol{u} \, \mathrm{d}^2\boldsymbol{x} = -\sum_e^{n_e} \int_{\Omega_e} \boldsymbol{\nabla} \boldsymbol{w}(\boldsymbol{x}) : \boldsymbol{T} \, \mathrm{d}^2\boldsymbol{x} + \boldsymbol{M} : \boldsymbol{\nabla} \boldsymbol{w}(\boldsymbol{x}_s) S(t) \tag{6}$$

Since the equation holds for any test function $\boldsymbol{w}$, we can use $\boldsymbol{w}$ to extract the value of displacement$\boldsymbol{u}$ at one point by setting w equals 1 at this point and equals 0 at other points. It's undoubted that this kind of $\boldsymbol{w}$ did exist. Using Gauss-Lobatto-Legendre points, the integration become the forms like below:

$$M\ddot{U} + K = F, \tag{7}$$

where M denotes the global mass matrix, K the global stiffness matrix and F the source term. Let's build the mass matrix M first. According the quadrature, we can know that the mass matrix is diagonal.

$$\begin{aligned}
&\int_{\Omega_e} \rho \boldsymbol{w} \cdot \partial_t^2 \boldsymbol{s} \, \mathrm{d}\boldsymbol{x} \\
&\approx \sum_{\alpha',\beta'} \omega_{\alpha'} \omega_{\beta'} \boldsymbol{J}_e^{\alpha'\beta'} \rho^{\alpha'\beta'} \sum_{i,j=1}^{2} \hat{\boldsymbol{x}}_i \cdot \hat{\boldsymbol{x}}_j \\
&\qquad \times \sum_{\alpha,\beta} w_i^{\alpha\beta} \ell_\alpha(\xi_{\alpha'}) \ell_\beta(\eta_{\beta'}) \sum_{\sigma\tau} \ddot{s}_j^{\sigma\tau}(t) \ell_\sigma(\xi_{\alpha'}) \ell_\tau(\eta_{\beta'}) \\
&= \sum_{\alpha',\beta'} \omega_{\alpha'} \omega_{\beta'} \boldsymbol{J}_e^{\alpha'\beta'} \rho^{\alpha'\beta'} \sum_{i,j}^{3} \delta_{ij} \sum_{\alpha,\beta} w_i^{\alpha\beta} \delta_{\alpha\alpha'} \delta_{\beta\beta'} \\
&\qquad \times \sum_{\sigma\tau} \ddot{s}_j^{\sigma\tau}(t) \delta_{\sigma\alpha'} \delta_{\tau\beta'} \\
&= \sum_{\alpha,\beta} \omega_\alpha \omega_\beta \boldsymbol{J}_e^{\alpha\beta} \rho^{\alpha\beta} \sum_{i=1}^{2} w_i^{\alpha\beta} \ddot{s}_i^{\alpha\beta}(t)
\end{aligned} \tag{8}$$

Now, Let's consider how to build the stiffness matrix. We can start from a local integration. The first step is to construct the stress tensor $\boldsymbol{T}$. To compute this term, we need the strain tensor first.

$$\begin{aligned}
\partial_i s(\boldsymbol{x}(\xi_\alpha, \eta_\beta), t) &\approx \partial_i \left[s_j^{\alpha\beta} \sum_{\alpha,\beta}^{n_\ell} \ell_\alpha(\xi_\alpha) \ell_\beta(\eta_\beta) \right] \\
&= \left[\sum_{\sigma}^{n_\ell} s_j^{\sigma\beta}(t) \ell'_\sigma(\xi_\alpha) \right] \partial_i \xi(\xi_\alpha, \eta_\beta) \\
&\quad + \left[\sum_{\sigma}^{n_\ell} s_j^{\alpha\sigma}(t) \ell'_\sigma(\eta_\beta) \right] \partial_i \eta(\xi_\alpha, \eta_\beta)
\end{aligned} \tag{9}$$

And the stress tensor $\boldsymbol{T}$ is obtained by the following.

$$\boldsymbol{T}(\boldsymbol{x}(\xi_\alpha, \eta_\beta), t) = \boldsymbol{c} : \boldsymbol{\nabla s}(\boldsymbol{x}(\xi_\alpha, \eta_\beta), t). \tag{10}$$

where $\boldsymbol{c}$ denotes the stress-strain tensor, is a forth order tensor.