

The primary goal of this introduction is to demonstrate how to implement spectral element method in 2d. We seek to produce the displacement field caused by an earthquake in a finite earth model with volume Ω . In order to simplify calculation, we just use stress-free boundary condition. We will talk about artificial absorbing boundary condition in detail later. The displacement field \mathbf{u} is produced by an earthquake is governed by the momentum equation,

$$\rho \partial_t^2 \mathbf{u} = \nabla \cdot \mathbf{T} + \mathbf{f} \quad (1)$$

,where $\mathbf{T} = (\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3)^T$, and \mathbf{T}_i is vector function. The weak form of above equation is produced by dotting the momentum equation with an arbitrary test vector $\mathbf{w} = (w_1(\mathbf{x}), w_2(\mathbf{x}), w_3(\mathbf{x}))^T$, integrating by the model volume Ω . Since we use stress-free boundary condition, which is easy to derive the weak form of wave equation, we have $\mathbf{T}|_{\partial\Omega} = 0$

$$\begin{aligned} & \int_{\Omega} \nabla \cdot \mathbf{T} \cdot \mathbf{w}(\mathbf{x}) \, d^3\mathbf{x} \\ &= \sum_{i=1}^3 \int_{\Omega} w_i(\mathbf{x}) \nabla \cdot \mathbf{T}_i \, d^3\mathbf{x} \\ &= \sum_{i=1}^3 \int_{\Omega} \nabla \cdot (w_i(\mathbf{x}) \mathbf{T}_i) - \nabla w_i(\mathbf{x}) \cdot \mathbf{T}_i \, d^3\mathbf{x} \\ &= \sum_{i=1}^3 \left(\int_{\partial\Omega} w_i(\mathbf{x}) \mathbf{T}_i \cdot \mathbf{ds} - \int_{\Omega} \nabla w_i(\mathbf{x}) \cdot \mathbf{T}_i \, d^3\mathbf{x} \right) \\ &= - \sum_{i=1}^3 \int_{\Omega} \nabla w_i(\mathbf{x}) \cdot \mathbf{T}_i \, d^3\mathbf{x} \\ &= - \int_{\Omega} \nabla \mathbf{w}(\mathbf{x}) : \mathbf{T} \, d^3\mathbf{x} \end{aligned} \quad (2)$$

The Source function is

$$\mathbf{f} = -\mathbf{M} \cdot \nabla \delta(\mathbf{x} - \mathbf{x}_s) S(t) \quad (3)$$

Using the properties of Dirac delta distribution, after integration, it transformed in the following way,

$$\begin{aligned} & \int_{\Omega} \mathbf{f} \cdot \mathbf{w} \, d^3\mathbf{x} \\ &= -S(t) \int_{\Omega} \mathbf{M} \cdot \nabla \delta(\mathbf{x} - \mathbf{x}_s) \cdot \mathbf{w} \, d^3\mathbf{x} \\ &= \mathbf{M} : \nabla \mathbf{w}(\mathbf{x}_s) S(t) \end{aligned} \quad (4)$$

In 2 dimension, We can derive this using Green Formula.

$$\int_{\Omega} \rho \mathbf{w} \cdot \partial_t^2 \mathbf{u} \, d^2\mathbf{x} = - \int_{\Omega} \nabla \mathbf{w}(\mathbf{x}) : \mathbf{T} \, d^2\mathbf{x} + \mathbf{M} : \nabla \mathbf{w}(\mathbf{x}_s) S(t) \quad (5)$$

In the following parts, we will explain the general procedure of specfem.

First, we divide the region into a number of non-overlapping elements, Ω_e , $e = 1, \dots, n_e$, such that $\Omega = \bigcup_e \Omega_e$, as shown in Fig.1. In SEM, the shape of element is restricted to quadrilateral. The integration in the whole region become the sum of integration over every element, as shown below.

$$\sum_e^{n_e} \int_{\Omega_e} \rho \mathbf{w} \cdot \partial_t^2 \mathbf{u} \, d^2 \mathbf{x} = - \sum_e^{n_e} \int_{\Omega_e} \nabla \mathbf{w}(\mathbf{x}) : \mathbf{T} \, d^2 \mathbf{x} + \mathbf{M} : \nabla \mathbf{w}(\mathbf{x}_s) S(t) \quad (6)$$

Since the equation holds for any test function \mathbf{w} , we can use \mathbf{w} to extract the value of displacement \mathbf{u} at one point by setting w equals 1 at this point and equals 0 at other points. It's undoubted that this kind of \mathbf{w} did exist. Using Gauss-Lobatto-Legendre points, the integration become the forms like below:

$$M\ddot{\mathbf{U}} + \mathbf{K} = \mathbf{F}, \quad (7)$$

where M denotes the global mass matrix, K the global stiffness matrix and F the source term. Let's build the mass matrix M first. According the quadrature, we can know that the mass matrix is diagonal.

$$\begin{aligned} & \int_{\Omega_e} \rho \mathbf{w} \cdot \partial_t^2 \mathbf{s} \, d\mathbf{x} \\ & \approx \sum_{\alpha', \beta'} \omega_{\alpha'} \omega_{\beta'} \mathbf{J}_e^{\alpha' \beta'} \rho^{\alpha' \beta'} \sum_{i,j=1}^2 \hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j \\ & \quad \times \sum_{\alpha, \beta} w_i^{\alpha \beta} \ell_{\alpha}(\xi_{\alpha'}) \ell_{\beta}(\eta_{\beta'}) \sum_{\sigma \tau} \ddot{s}_j^{\sigma \tau}(t) \ell_{\sigma}(\xi_{\alpha'}) \ell_{\tau}(\eta_{\beta'}) \\ & = \sum_{\alpha', \beta'} \omega_{\alpha'} \omega_{\beta'} \mathbf{J}_e^{\alpha' \beta'} \rho^{\alpha' \beta'} \sum_{i,j}^3 \delta_{ij} \sum_{\alpha, \beta} w_i^{\alpha \beta} \delta_{\alpha \alpha'} \delta_{\beta \beta'} \\ & \quad \times \sum_{\sigma \tau} \ddot{s}_j^{\sigma \tau}(t) \delta_{\sigma \alpha'} \delta_{\tau \beta'} \\ & = \sum_{\alpha, \beta} \omega_{\alpha} \omega_{\beta} \mathbf{J}_e^{\alpha \beta} \rho^{\alpha \beta} \sum_{i=1}^2 w_i^{\alpha \beta} \ddot{s}_i^{\alpha \beta}(t) \end{aligned} \quad (8)$$