Integrability and Instability in AdS/CFT

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in collaboration with

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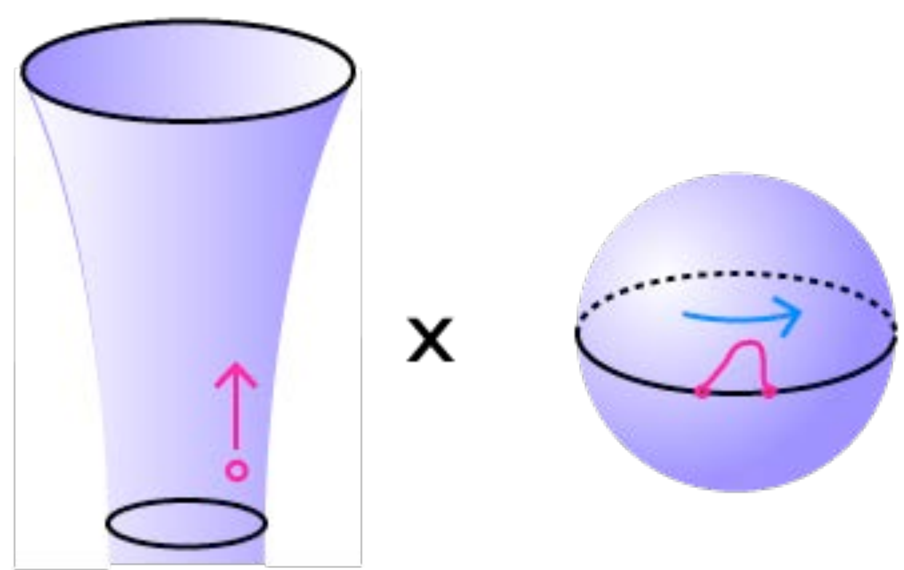
Brane-antibrane system

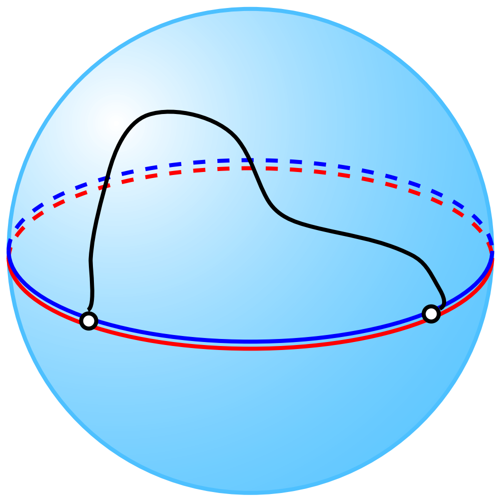
D-brane & D-antibrane (D-$\bar{\text{D}}$) system in the flat spacetime is an example of unstable state in string theory

D-brane & D-antibrane and open strings in between in the $AdS_5 \times S^5$ spacetime are less well-understood



AdS/CFT correspondence

In AdS/CFT, the energy of an open string ending on a pair of "giant-graviton" D-$\bar{\text{D}}$ branes in $AdS_5 \times S^5$



should be dual to the dimension of a determinant-like operator in 4D $SU(N)$ $\mathcal{N}$=4 super Yang-Mills theory

$$\mathcal{O} = \epsilon_{a_1 a_2 \ldots a_{2N}} \epsilon^{b_1 b_2 \ldots b_{2N}} \begin{pmatrix} Y & 0 \\ 0 & \bar{Y} \end{pmatrix}^{a_1}_{b_1} \cdots \begin{pmatrix} Y & 0 \\ 0 & \bar{Y} \end{pmatrix}^{a_{2N-2}}_{b_{2N-2}} \begin{pmatrix} 0 & V \\ W & 0 \end{pmatrix}^{a_{2N-1}}_{b_{2N-1}} \begin{pmatrix} 0 & V \\ W & 0 \end{pmatrix}^{a_{2N}}_{b_{2N}}$$

[Balasubramanian, Berkooz, Naqvi, Strassler (2001)] [Balasubramanian, Huang, Levi, Naqvi (2002)]

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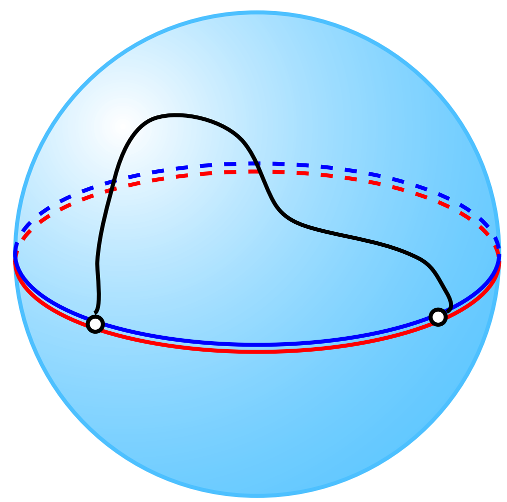

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$\boxed{\text{Test the duality using integrability}}$

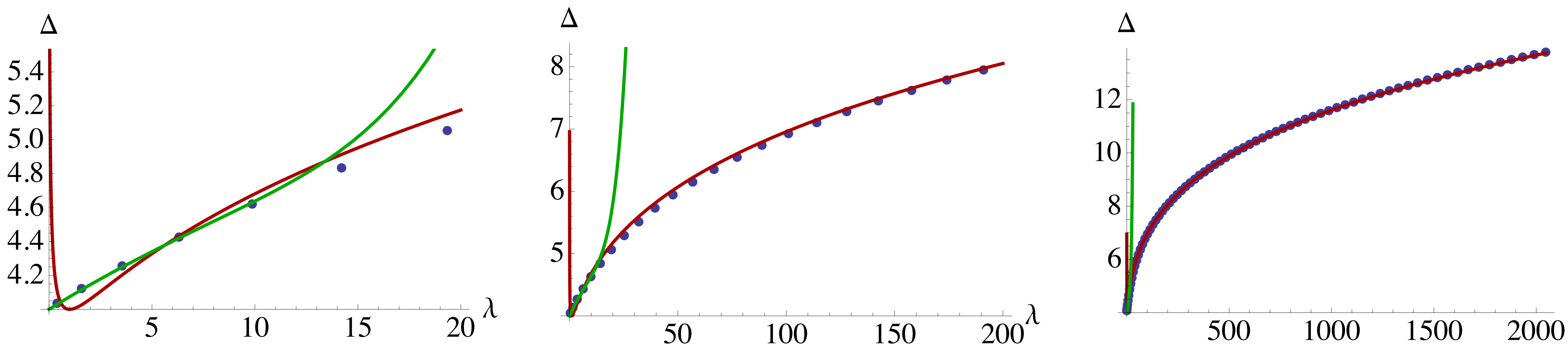
Integrability Methods

The spectral problem at large N is now “solvable” through (Asymptotic/Thermodynamic) Bethe Ansatz

$$E_{\text{string}}(\lambda) \xleftarrow{\ \sim\ } E_{\text{ABA}}(\lambda) \ \text{ or } \ E_{\text{TBA}}(\lambda) \xrightarrow{\ \sim\ } \Delta_{\text{SYM}}(\lambda)$$

We want to solve TBA; i.e. obtain $E_{\text{TBA}}(\lambda)$

Example: the exact dimension of Konishi operator



Green: SYM, weak 5-loop Blue: TBA, numerics Red: String, strong 1-loop

[Gromov, Kazakov, Vieira (2009)] [Frolov (2010)] and others

Dimension of Konishi (descendant) operator from integrability

$$\mathcal{O}_{\text{Konishi}} = \mathrm{tr}\,[D_+^2 Z^2 - (D_+ Z)^2], \qquad g \equiv \frac{\sqrt{\lambda}}{2\pi} = \frac{\sqrt{N g_{\text{YM}}^2}}{2\pi}$$

$$\boxed{\begin{aligned}\Delta_{\text{Konishi}} = {}& 4 + 3g^2 - 3g^4 + \frac{21 g^6}{4} + \frac{3g^8}{8}(-26 + 6\,\zeta_3 - 15\,\zeta_5) \\ & - \frac{3g^{10}}{32}(-158 + 54\,\zeta_3^2 - 72\,\zeta_3 + 90\,\zeta_5 - 315\,\zeta_7)\end{aligned}}$$

Perturbatively checked

$$\begin{aligned}
& - \frac{3g^{12}}{256}(160 + 432\,\zeta_3^2 - 72(45\,\zeta_5 - 76)\,\zeta_3 - 2340\,\zeta_5 - 1575\,\zeta_7 + 10206\,\zeta_9) \\
& + \frac{3g^{14}}{1024}\Big(- 44480 + 2592\,\zeta_3^3 - 8784\,\zeta_3^2 + 24(357\,\zeta_5 - 1680\,\zeta_7 + 4540)\,\zeta_3 \\
& \qquad -20700\,\zeta_5^2 - 4776\,\zeta_5 - 26145\,\zeta_7 - 17406\,\zeta_9 + 152460\,\zeta_{11}\Big) \\
& + \frac{3g^{16}}{4096}\Big(1133504 - 36(2520\,\zeta_5 + 3605\,\zeta_7 - 2178\,\zeta_8 - 13440\,\zeta_9 + 48320)\,\zeta_3 \\
& \qquad -288(285\,\zeta_5 - 574)\,\zeta_3^2 + 41472\,\zeta_3^3 + 72\,\zeta_5(1683\,\zeta_6 + 6440\,\zeta_7 - 5694) \\
& \qquad +49680\,\zeta_5^2 + 178200\,\zeta_4\,\zeta_7 + 455598\,\zeta_7 + 263736\,\zeta_2\,\zeta_9 + 194328\,\zeta_9 \\
& \qquad -555291\,\zeta_{11} - 2208492\,\zeta_{13} - 14256\,\boxed{\zeta_{1,2,8}}\Big) \\
& + \mathcal{O}(g^{18})
\end{aligned}$$

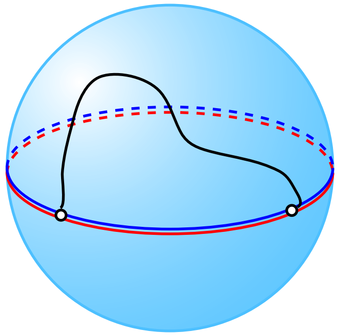
Multiple Zeta Value

[Leurent, Volin] arXiv:1302.1135 and Volin's talk in IGST2013

To do

$$E_{\text{string}}(\lambda) \xleftarrow{\sim} E_{\text{ABA}}(\lambda) \text{ or } E_{\text{TBA}}(\lambda) \xrightarrow{\sim} \Delta_{\text{SYM}}(\lambda)$$

We propose BTBA equations
(Boundary Thermodynamic Bethe Ansatz)
and solve them numerically



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However,

D-$\bar{D}$ states are unstable

Integrability vs. Instability

Plan of Talk

✓ Introduction

- Integrability and AdS/CFT
- Determinants and giant-gravitons
- BTBA equations and energy bound
- Summary and outlook

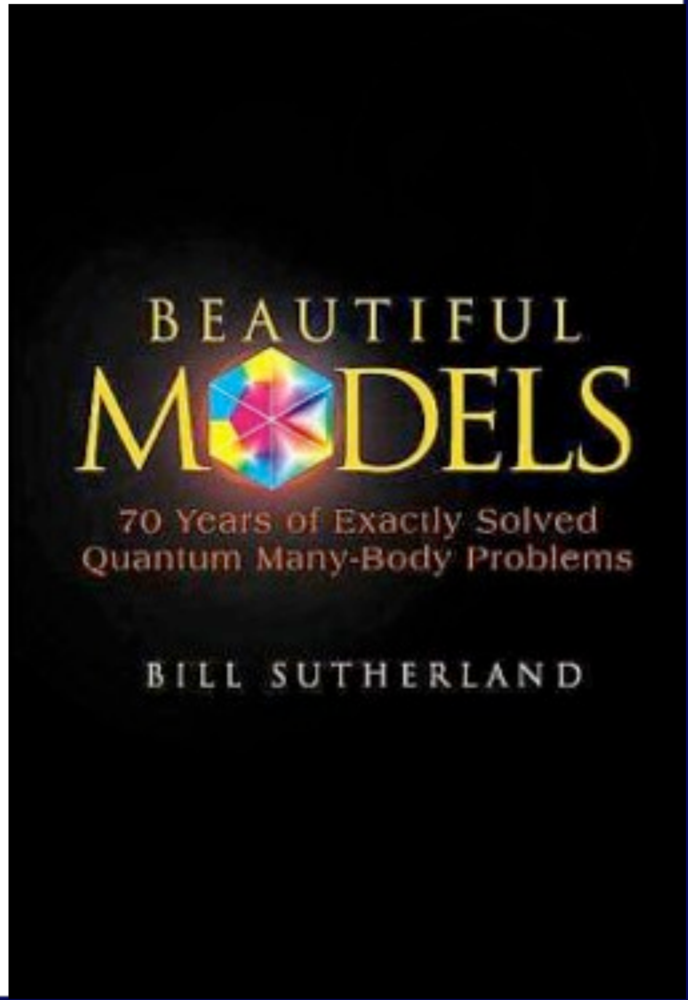
Integrability and AdS/CFT

What is integrability?

Textbook definition

A system is classically integrable
if it has the maximal set of Poisson-commuting invariants.

A system is quantum integrable
if the multi-body S-matrix factorizes into a product of two-body Smatrices.



What is integrability?

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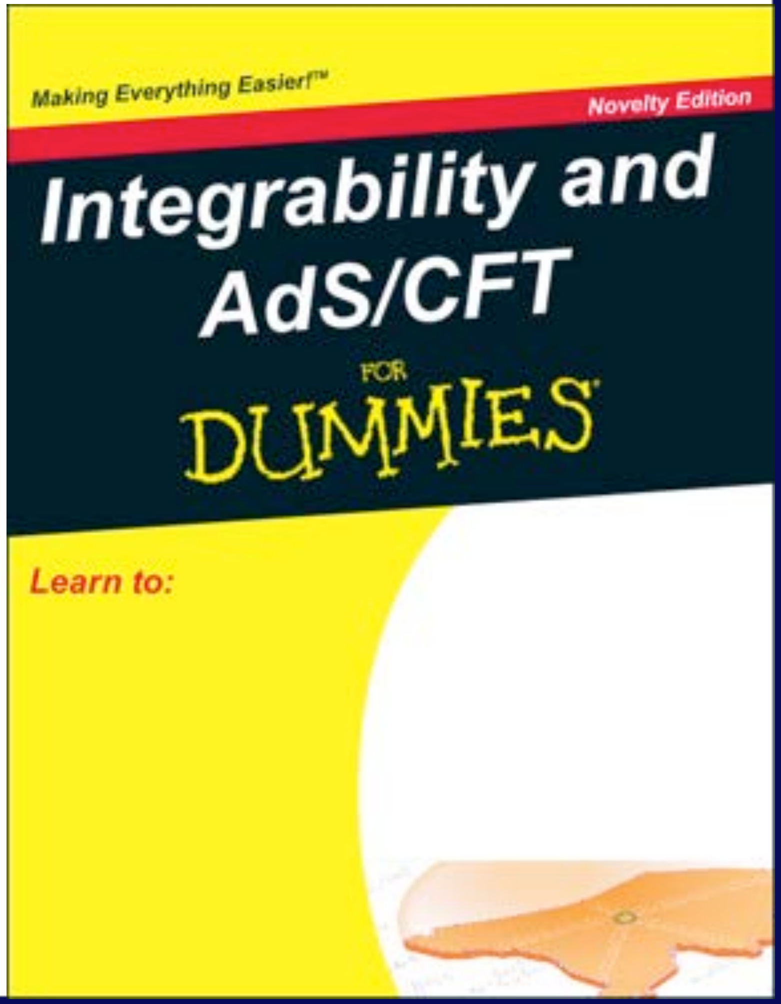
Working definition

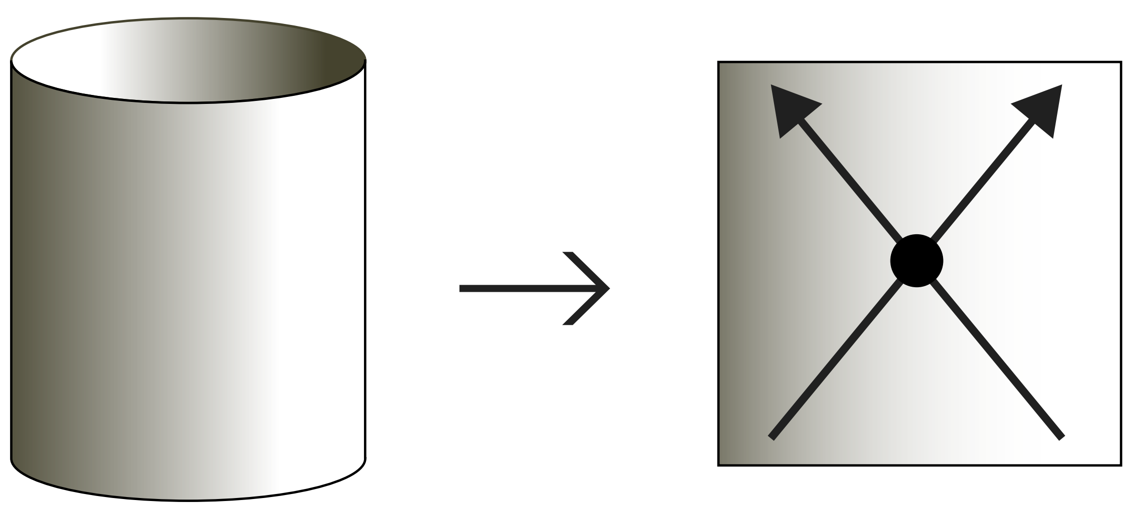
A system is *integrable* if the following works:

1. Compute new physical quantities
2. Find infinite-dimensional symmetry
3. Conjecture all-loop "Bethe-Ansatz" formula
4. Check your proposal -- agreement!



Integrability in the σ-model on $AdS_5 \times S^5$

- This σ-model is classically integrable;
  the target space can be written as a supercoset
- We break worldsheet conformal symmetry by a gauge choice
- By taking the large-volume (asymptotic) limit,
  we can define asymptotic states and their S-matrix
- The worldsheet S-matrix is (believed to be) integrable



[Bena Polchinski Roiban] (2003) [Hofman Maldacena] (2006) and others

$\mathcal{N}$=4 SYM and spin chain

Dilatation operator of $\mathcal{N}$=4 SYM = Hamiltonian of spin chain

Half BPS operator, tr Z^J = Ground state

Spectrum in the asymptotic limit, $J \to \infty$

- One-particle

$$\sum_k e^{ipk}(\dots ZZZ\chi ZZZ\dots) \sim A^\dagger_\chi(p)|0\rangle$$

- Two-particle

$$\sum_{k<k'} e^{ikp_1+ik'p_2}\,\mathrm{tr}\,(Z\dots Z\chi ZZ\chi' Z\dots Z) \sim A^\dagger_\chi(p_1)A^\dagger_{\chi'}(p_2)\,|0\rangle$$

This choice of vacuum breaks the global symmetry

$$\mathfrak{psu}(2,2|4) \to \mathfrak{psu}(2|2)^2 \ltimes \mathbb{R} \sim (E=\Delta, S_1, S_2, J_1, J_2, J)$$

[Minahan Zarembo (2002)] [Beisert (2005)] and others

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The residual global symmetry enhances in the asymptotic limit

$$\mathrm{tr}\left(Z^{J-m}\chi Z^m\right) \quad \to \quad (\dots ZZZ\dots Z\chi Z\dots ZZZ\dots)$$

$$\mathfrak{psu}(2|2)^2 \ltimes \mathbb{R} \quad \to \quad \mathfrak{su}(2|2)^2 \ltimes \mathbb{R}$$

[Minahan Zarembo (2002)] [Beisert (2005)] and others

Infinite-dimensional symmetry

The centrally-extended su(2|2) is powerful:
it determines the asymptotic dispersion and S-matrix
of fundamental representations almost uniquely

$$\Delta - J = \sum_{j=1}^{N} \sqrt{1 + 4f(g)^2 \sin^2 \boxed{\frac{p_j}{2}}}, \quad f(g) = g \equiv \frac{\sqrt{\lambda}}{2\pi} \text{ in } \mathcal{N} = 4 \,\text{SYM}$$

?

$$A_a^\dagger(p_1) A_b^\dagger(p_2) = \mathbb{S}_{ab}^{cd}(p_1, p_2) A_c^\dagger(p_2) A_d^\dagger(p_1), \quad \mathbb{S} = S_0[\hat{S}_{\mathfrak{su}(2|2)} \otimes \hat{S}_{\mathfrak{su}(2|2)}]$$

This S-matrix of AdS/CFT satisfies Yang-Baxter relation

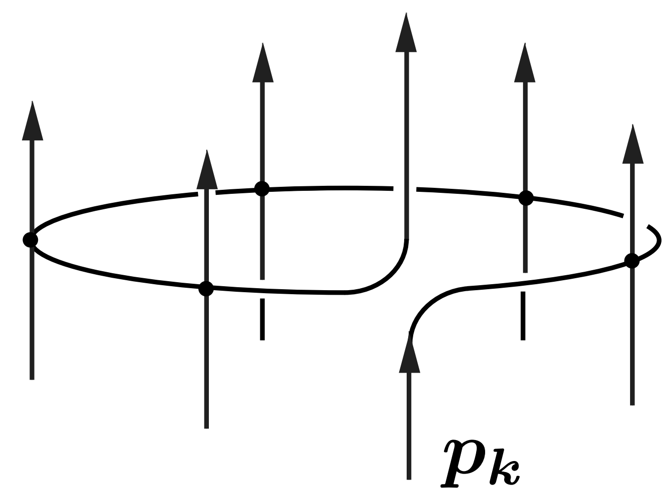
$$\mathbb{S}_{12}\,\mathbb{S}_{13}\,\mathbb{S}_{23} = \mathbb{S}_{23}\,\mathbb{S}_{13}\,\mathbb{S}_{12} \equiv \mathbb{S}_{123}$$

and all the algebraic relations extend
to the Yangian of su(2|2)

[Beisert (2007)] and others

Bethe-Yang equation (BYE)

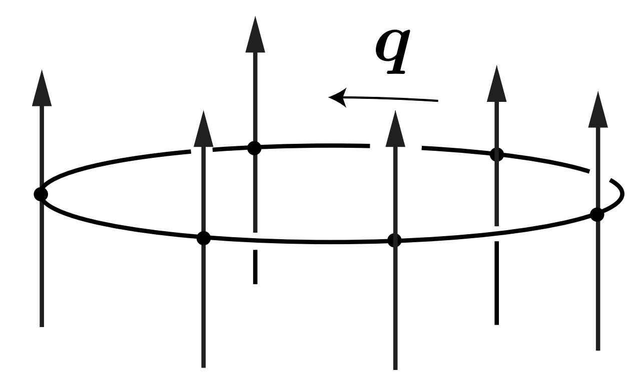
For a large and finite J, momenta of the particles are determined by the Bethe-Yang (or Bethe Ansatz) equation



$$-1 = e^{-iJp_k} \prod_{j=1}^{N} S(p_j, p_k)$$

$$S(p,p) = -1$$

BYE in terms of transfer matrix



$$T_a(q|\vec{p}) \equiv (\mathrm{s})\mathrm{tr}_{V_a}\Big[\mathbb{S}_{a1}(q,p_1)\cdots\mathbb{S}_{aN}(q,p_N)\Big]$$

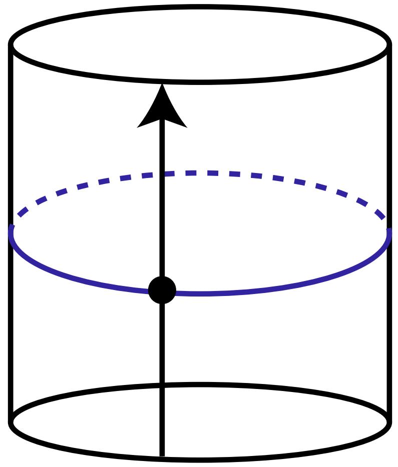
Yang-Baxter relation for integrable S-matrices $\Rightarrow \quad [T_a(q_a|\vec{p}), T_b(q_b|\vec{p})] = 0$

$$\textbf{BYE} \quad \Leftrightarrow \quad -1 = e^{-iJq}\, T(q|\vec{p})\Big|_{q=p_k}$$

Wrapping corrections

- The dimension Δ of SYM operator with a finite R-charge J receives exponentially small "wrapping" corrections
- The leading wrapping correction is related to the transfer matrix via the Lüscher formula

$$\Delta_{\text{Lüscher}} \sim \sum_Q \int_{-\infty}^{\infty} d\tilde{p}_Q \, e^{-\tilde{\mathcal{E}}_Q(\tilde{p}_Q)J}$$



$$(\mathcal{E}_Q, p_Q) = (-i\tilde{p}_Q, -i\tilde{\mathcal{E}}_Q), \quad \tilde{\mathcal{E}}_Q = 2\text{arcsinh}\left(\sqrt{Q^2 + \tilde{p}_Q^2}/(2g)\right)$$

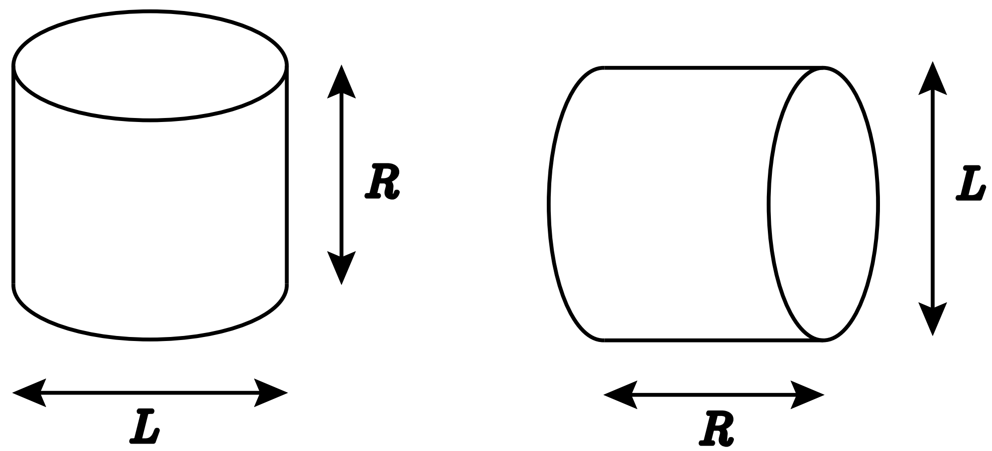
$$\Delta_{\text{Lüscher}} = -\sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \frac{d\tilde{p}_Q}{2\pi} Y_Q^{\bullet}(\tilde{p}_Q), \quad Y_Q^{\bullet}(\tilde{p}_Q) = e^{-\tilde{\mathcal{E}}_Q J} \underline{T_Q(\tilde{p}_Q|\vec{p})}$$

[Lüscher (1986)] [Janik Łukowski (2007)] and others

Exact dimension/energy

Begin with the equivalence of Euclidean worldsheet partition functions

[Zamolodchikov (1990)]
[Arutyunov, Frolov (2007)]



$$Z_E(L,R) = \int [dX]\, e^{-S_E} = \int [d\tilde{X}]\, e^{-\tilde{S}_E} = \tilde{Z}(R,L)$$

In Hamiltonian formalism, $\quad \mathrm{tr}\, e^{-RH(L)} = \mathrm{tr}\, e^{-L\tilde{H}(R)}$

Take the large R limit, $\quad e^{-RE_0(L)} = \lim_{R\to\infty} e^{-\boxed{\tilde{\mathcal{F}}(\mathcal{R})}}$

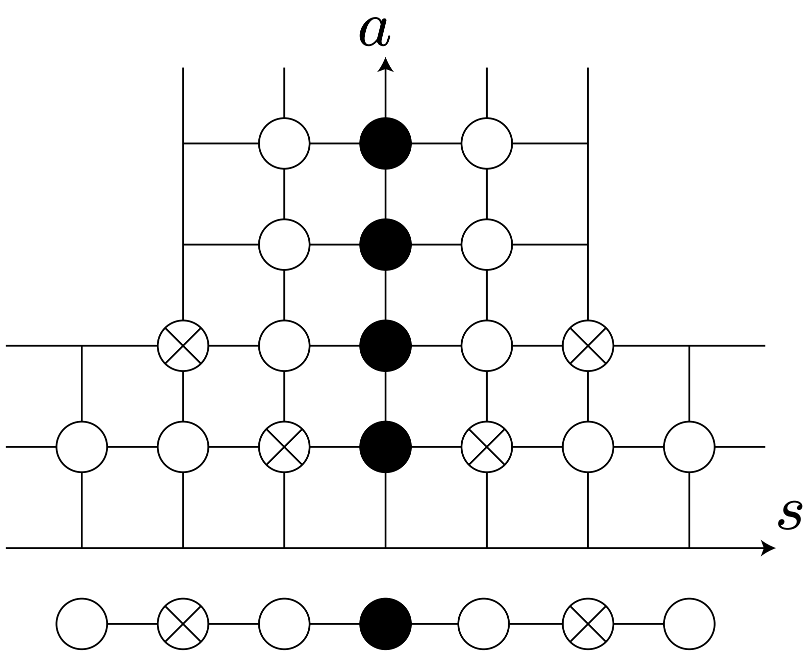
The "mirror" free energy can be computed by the "mirror" asymptotic Bethe Ansatz equations in the thermodynamic limit

⇨ Thermodynamic Bethe Ansatz equations (TBA)

TBA in $AdS_5 \times S^5$ = Y-system + discontinuity

TBA (schematically): $$\log Y_A = V_A + \boxed{\sum_B} \log(1 \pm Y_B) \star K_{BA}$$

$\mathfrak{psu}(2,2|4)$-hook



$$\log(1+Y) * K = \int dt \, \log(1+Y(t))K(t,v)$$

[Gromov, Kazakov, Vieira (2009)] [Bombardelli, Fioravanti, Tateo (2009)] [Gromov, Kazakov, Kozak, Vieira (2009)] [Arutyunov, Frolov (2009)] and others

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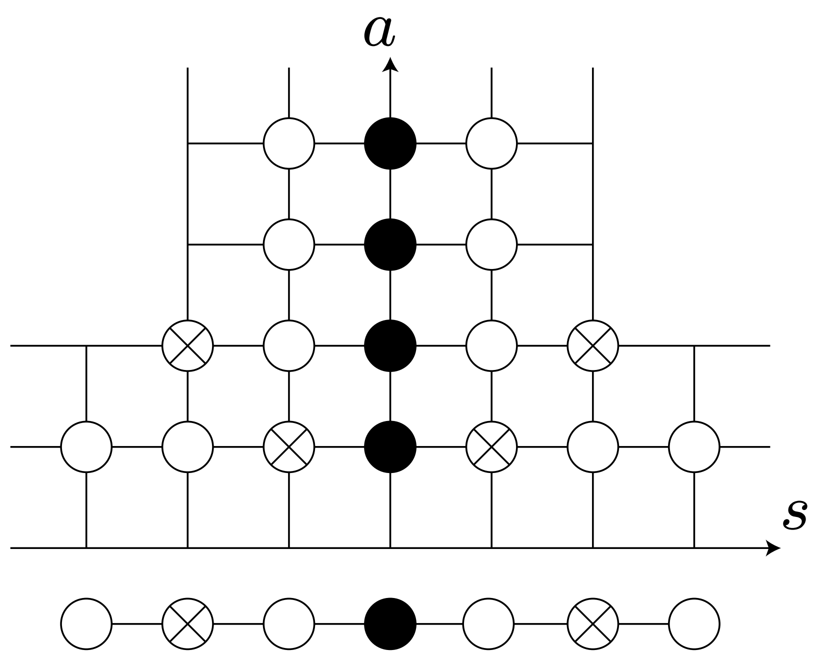
Y-system: $$\frac{Y_{a,s}^+ \, Y_{a,s}^-}{Y_{a-1,s} \, Y_{a+1,s}} = \frac{(1+Y_{a,s-1})(1+Y_{a,s+1})}{(1+Y_{a-1,s})(1+Y_{a+1,s})}$$

$\mathfrak{psu}(2,2|4)$-hook



$$\log(1+Y) * K = \int dt \, \log(1+Y(t))K(t,v)$$

$$Y^{\pm}(v) = Y(v \pm i/g)$$

Y-functions have various branch cuts in the v-plane

Exact energy: $$E - J = -\sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \frac{d\tilde{p}_Q}{2\pi} \log(1+Y_Q), \quad Y_Q = Y_{Q,0}$$

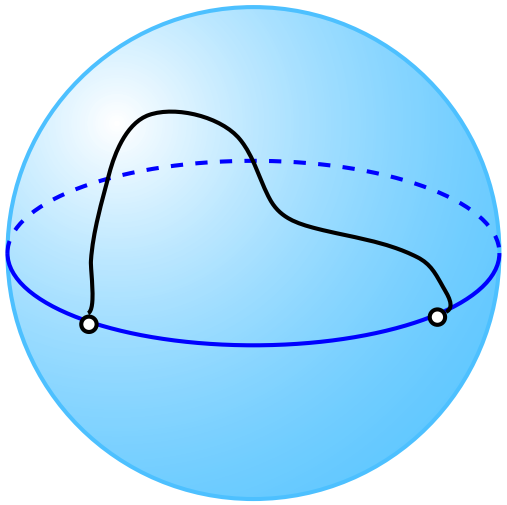
[Gromov, Kazakov, Vieira (2009)] [Bombardelli, Fioravanti, Tateo (2009)] [Gromov, Kazakov, Kozak, Vieira (2009)] [Arutyunov, Frolov (2009)] and others

Determinants and giant-gravitons

Spherical Maximal Giant Gravitons (SMGG's)

D3-brane solution of the DBI+CS action on $\mathrm{AdS}_5 \times \mathrm{S}^5$

with a large angular momentum $J = \mathcal{O}(N)$



Spherical $\Leftrightarrow$ **"wrap" on $\mathrm{S}^3 \subset \mathrm{S}^5$**

with the angular momentum bound $J \leq N$

Maximal $\Leftrightarrow$ $J = N$

Half-BPS state

SMGG's are classified by the choice:

$$\mathrm{S}^3 \subset \mathrm{S}^5 = \{|X|^2 + |Y|^2 + |Z|^2 = R^2_{\mathrm{sphere}}\}$$

$$X = 0 \ \text{ or } \ Y = 0 \ \text{ or } \ Z = 0 \ \cdots$$

Y=0 brane with the opposite chirality: $\overline{Y} = 0$

[McGreevy, Susskind, Toumbas (2000)]

Giant graviton is determinant

SMGG's are dual to determinants

$$\det \Phi^N = \epsilon^{i_1 \cdots i_N} \, \epsilon_{j_1 \cdots j_N} \, \Phi^{j_1}_{i_1} \cdots \Phi^{j_N}_{i_N}$$

Open strings on the Y=0 brane are dual to det-like operator

$$\det \left(Y^{N-1} V \right) = \epsilon^{i_1 \cdots i_N} \, \epsilon_{j_1 \cdots j_N} \, Y^{j_1}_{i_1} \cdots Y^{j_{N-1}}_{i_{N-1}} \, V^{j_N}_{i_N}$$

A pair of open strings on Y=0 and Ybar=0 is dual to:

$$\mathcal{O} = \epsilon_{a_1 a_2 \ldots a_{2N}} \epsilon^{b_1 b_2 \ldots b_{2N}} \begin{pmatrix} Y & 0 \\ 0 & \bar{Y} \end{pmatrix}^{a_1}_{b_1} \cdots \begin{pmatrix} Y & 0 \\ 0 & \bar{Y} \end{pmatrix}^{a_{2N-2}}_{b_{2N-2}} \begin{pmatrix} 0 & V \\ W & 0 \end{pmatrix}^{a_{2N-1}}_{b_{2N-1}} \begin{pmatrix} 0 & V \\ W & 0 \end{pmatrix}^{a_{2N}}_{b_{2N}}$$

[Balasubramanian, Berkooz, Naqvi, Strassler (2001)] [Balasubramanian, Huang, Levi, Naqvi (2002)]

GG as boundary condition

GG is a boundary condition for an asymptotic open spin chain

Y=0 brane: $\epsilon^{i_1\cdots i_N}\,\epsilon_{j_1\ldots j_N}\boxed{Y^{j_1}_{i_1}\cdots Y^{j_{N-1}}_{i_{N-1}}}\boxed{(ZZ\ldots ZZ)^{j_N}_{i_N}}$

The Y=0 preserves the symmetry psu(1|2)^2 in psu(2|2)^2

This psu(1|2) determines the reflection matrix

$$[\mathbb{R}^{\pm}_{Y}\,, J] = 0, \quad \forall J \in \mathfrak{psu}(1|2) \quad \Rightarrow \quad \mathbb{R}^{\pm}_{Y} \ \textbf{is diagonal}$$

$$\mathbb{R}^{-}_{Y}(p) = R^{-}_{0}(p)^2 \begin{pmatrix} e^{-ip/2} & & & \\ & -e^{ip/2} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}^{\otimes 2}$$

$R^{-}_{0}(p)^2 = -e^{-ip}\,\sigma(p,-p)$ **obeys boundary crossing relation**

[Hofman, Maldacena (2007)] [Chen, Correa (2007)]

The Y_θ=0 brane

New reflection amplitudes can be found by rotating R_Y

- $\mathcal{N}$=4 SYM: **Field redefinition:** $\det Y^N \rightarrow \det\left(Y\cos\theta + \overline{Y}\sin\theta\right)^N$
- Integrable system:

$$\textbf{Rotation } T: \begin{pmatrix}1\\2\end{pmatrix} \rightarrow \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}, \quad \textbf{same for } (\dot{1},\dot{2})$$

$$\mathbb{R}_\theta^-(p) \equiv T R_Y^- T^{-1} = R_0^-(p)^2 \begin{pmatrix} \cos^2\theta\, e^{-ip/2} - \sin^2\theta e^{ip/2} & \sin\theta\cos\theta\left(e^{-ip/2}+e^{ip/2}\right) & & \\ \sin\theta\cos\theta\left(e^{-ip/2}+e^{ip/2}\right) & \sin^2\theta\, e^{-ip/2} - \cos^2\theta e^{ip/2} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}^{\otimes 2}$$

- R_θ still solves boundary Yang-Baxter relation!

$$\mathbb{S}(-p_2,-p_1)\,\mathbb{R}(p_1)\,\mathbb{S}(p_1,-p_2)\,\mathbb{R}(p_2) = \mathbb{R}(p_2)\,\mathbb{S}(p_2,-p_1)\,\mathbb{R}(p_1)\,\mathbb{S}(p_1,p_2)$$

- $\theta = \pi/2$ corresponds to the Ybar=0 brane

Asymptotic Bethe Ansatz and Lüscher formula
can be generalized to boundary integrable models

YbarY determinant-like operator

A pair of open strings on Y=0 and Ybar=0 should be dual to:

$$\mathcal{O}=\epsilon_{a_1a_2\ldots a_{2N}}\epsilon^{b_1b_2\ldots b_{2N}}\begin{pmatrix}Y & 0\\ 0 & \bar{Y}\end{pmatrix}^{a_1}_{b_1}\cdots\begin{pmatrix}Y & 0\\ 0 & \bar{Y}\end{pmatrix}^{a_{2N-2}}_{b_{2N-2}}\begin{pmatrix}0 & V\\ W & 0\end{pmatrix}^{a_{2N-1}}_{b_{2N-1}}\begin{pmatrix}0 & V\\ W & 0\end{pmatrix}^{a_{2N}}_{b_{2N}}$$

In the 't Hooft limit, its dimension takes the factorized form

$$\boxed{\Delta=2N-2+\Delta[V]+\Delta[W]}$$

The simplest case is $V=Z^L,\ W=Z^{L'}$

$$\Delta[V]=L+\textbf{wrapping},\quad \Delta[W]=L'+\textbf{wrapping}$$

The energy of a corresponding open string should be

$$\boxed{E=2N+E_{\rm open}[V]+E_{\rm open}[W]}$$

$$E_{\rm open}[V]=-1+L+\textbf{wrapping}$$

Two-point functions

Consider the two-point function of a YbarY operator

$$\mathcal{O} = \epsilon_{a_1 a_2 \ldots a_{2N}} \epsilon^{b_1 b_2 \ldots b_{2N}} \begin{pmatrix} Y & 0 \\ 0 & \bar{Y} \end{pmatrix}^{a_1}_{b_1} \cdots \begin{pmatrix} Y & 0 \\ 0 & \bar{Y} \end{pmatrix}^{a_{2N-2}}_{b_{2N-2}} \begin{pmatrix} 0 & Z^L \\ Z^{L'} & 0 \end{pmatrix}^{a_{2N-1}}_{b_{2N-1}} \begin{pmatrix} 0 & Z^L \\ Z^{L'} & 0 \end{pmatrix}^{a_{2N}}_{b_{2N}}$$

$$= \epsilon^{i_1 \cdots i_N} \epsilon_{j_1 \cdots j_N} \epsilon^{k_1 \cdots k_N} \epsilon_{l_1 \cdots l_N} Y^{j_1}_{i_1} \cdots Y^{j_{N-1}}_{i_{N-1}} (Z^L)^{l_N}_{i_N} \overline{Y}^{l_1}_{k_1} \cdots \overline{Y}^{l_{N-1}}_{k_{N-1}} (Z^{L'})^{j_N}_{k_N}$$

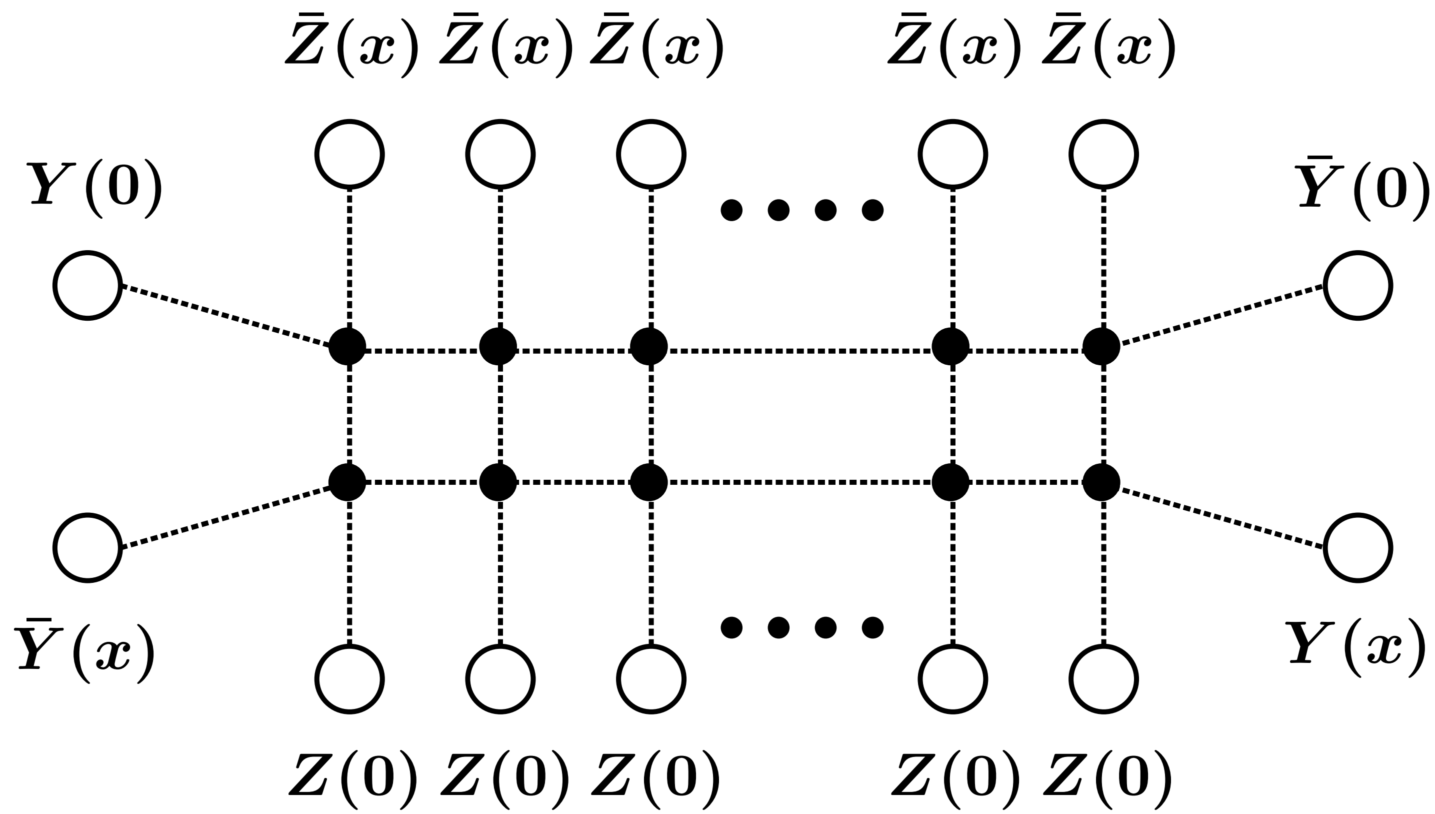
Dilatation acts in almost the same way as on a YY operator

$$\mathcal{O}_{\mathrm{BPS}} = \epsilon_{a_1 a_2 \ldots a_{2N}} \epsilon^{b_1 b_2 \ldots b_{2N}} \begin{pmatrix} Y & 0 \\ 0 & Y \end{pmatrix}^{a_1}_{b_1} \cdots \begin{pmatrix} Y & 0 \\ 0 & Y \end{pmatrix}^{a_{2N-2}}_{b_{2N-2}} \begin{pmatrix} 0 & Z^L \\ Z^{L'} & 0 \end{pmatrix}^{a_{2N-1}}_{b_{2N-1}} \begin{pmatrix} 0 & Z^L \\ Z^{L'} & 0 \end{pmatrix}^{a_{2N}}_{b_{2N}}$$

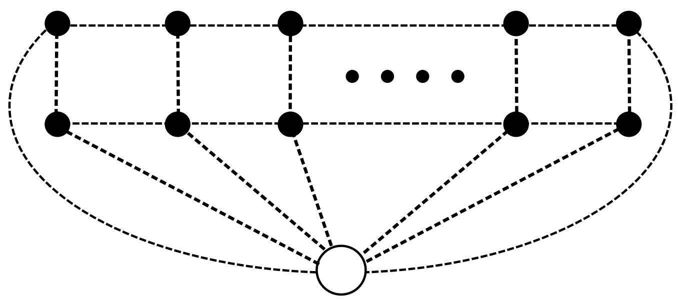
- This degeneracy can be lifted only when the interaction propagates from one boundary to another; from the loop-order $L+1$ at least
- However, most boundary-to-boundary interactions are same for both
- The difference appears first at the loop-order $2L$

Wrapping diagram

After a lot of tree-level contractions between Y-$\bar{Y}$, we obtain



Spacetime structure (amputated)

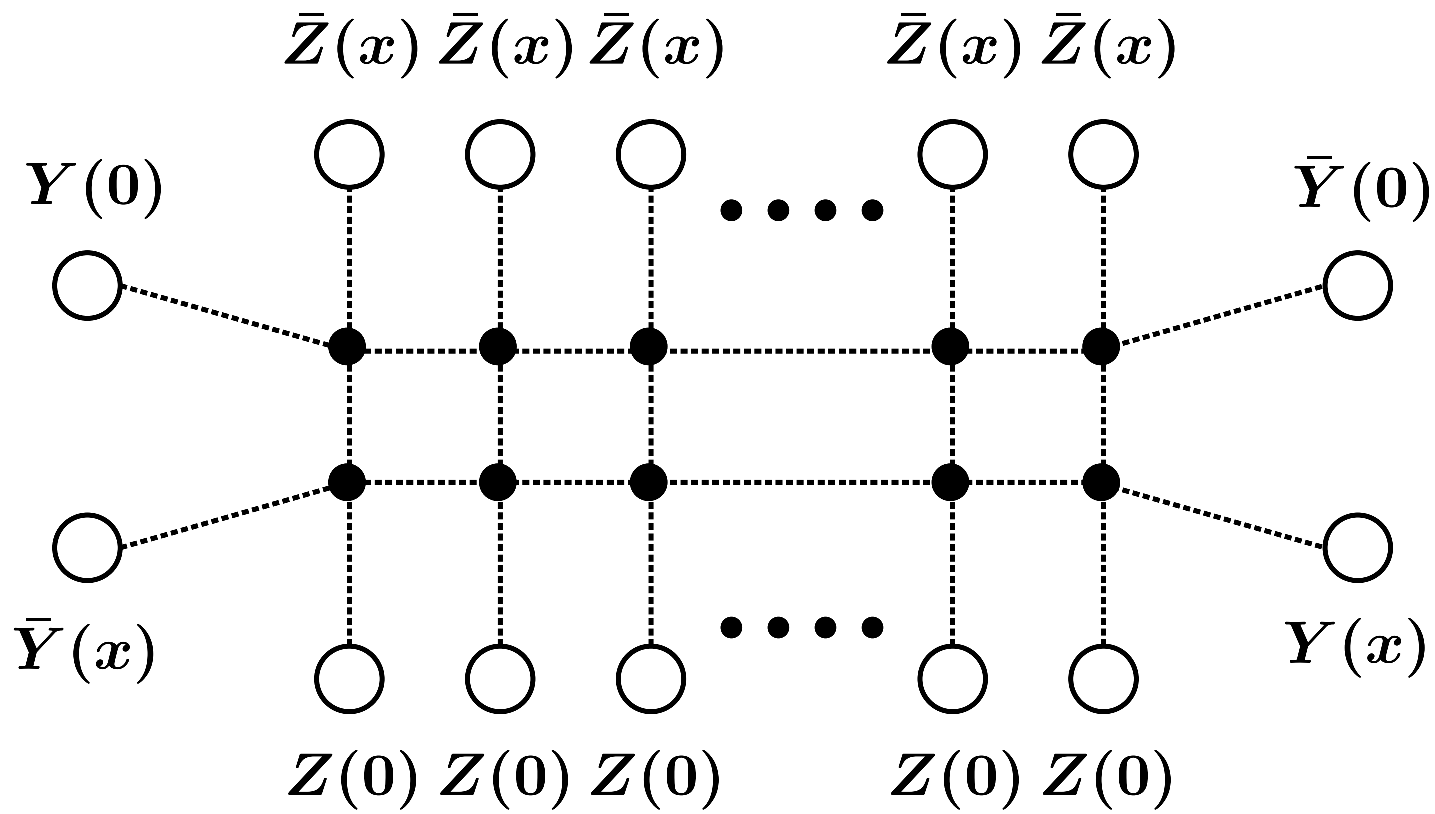


this is same as the so-called zig-zag diagram

cf. [Brown, Schnetz] arXiv:1208.1890, [Schnetz] arXiv:1210.5376

Wrapping diagram

After a lot of tree-level contractions between Y-$\bar{Y}$, we obtain



The result is

$$\delta\Delta_L = -\frac{4(g/2)^{4L}}{4L-1}\binom{4L}{2L}\zeta(4L-3) + \mathcal{O}(g^{4L+2}), \quad g \ll 1$$

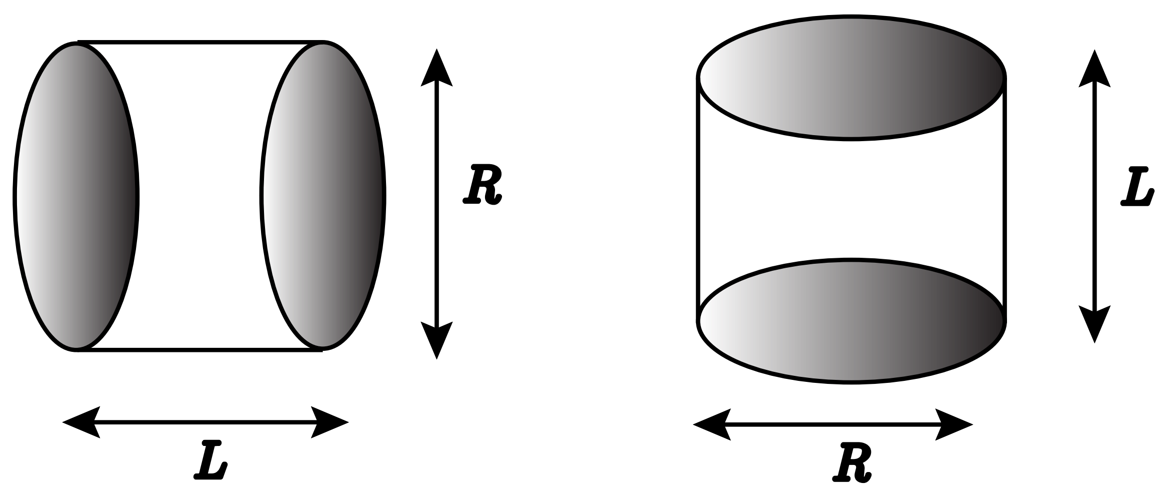
Agree with the boundary Lüscher formula for $L = J > 1$

cf. [Brown, Schnetz] arXiv:1208.1890, [Schnetz] arXiv:1210.5376

BTBA equations and energy bound

Mirror trick with boundary

A simple generalization is to change boundary conditions



$$Z_E^{(\alpha\beta)}(L,R) = \int [dX]_{\alpha\beta}\, e^{-S_E} = \int [d\tilde{X}]_{\alpha\beta}\, e^{-\tilde{S}_E} = \tilde{Z}^{(\alpha\beta)}(R,L)$$

$$\mathrm{tr}\, e^{-RH_{\alpha\beta}(L)} = \langle B_\alpha |\, e^{-L\tilde{H}(R)}\, | B_\beta \rangle = \sum_\psi \frac{\langle B_\alpha|\psi\rangle\langle\psi|B_\beta\rangle}{\langle\psi|\psi\rangle}\, e^{-L\tilde{\mathcal{E}}_\psi(R)}$$

Take the large R limit, $e^{-RE_{\alpha\beta,0}(L)} = \lim_{R\to\infty} e^{-\tilde{\mathcal{F}}(\mathcal{R}) + \boxed{B_{\alpha\beta}(R)}}$

Difficult to derive the boundary factor $B_{\alpha\beta}$
in integrable models with non-diagonal S-matrix

BTBA for YbarY

We conjecture the BTBA of the Y=0 & Ybar=0 as follows:

$$\log Y_a = \log(1 \pm Y_b) \star K_{ba} + V_a$$

- Boundary just introduces a momentum-dependent chemical potential which just changes the source term V_a
- As a result, the Y-system is same as in the periodic case
- The BTBA must be consistent with the Lüscher formula at large L
- The Y-system and Lüscher Y_Q almost completely determine BTBA

$$\boxed{Y_Q^{\bullet}(v) = \frac{16Q^2 v^2}{v^2 + Q^2/g^2}\left(\frac{x(v - iQ/g)}{x(v + iQ/g)}\right)^{2L}, \quad x(v) = \frac{1}{2}\left(v - i\sqrt{4 - v^2}\right)}$$

Calculated from the double-row transfer matrix with Y=0 & Ybar=0 boundaries

BTBA for YbarY

We conjecture the BTBA of the Y=0 & Ybar=0 as follows:

$$\log Y_a = \log(1 \pm Y_b) \star K_{ba} + V_a$$

In other words, we define source terms V_a by the asymptotic Y's

$$V_a \equiv \log Y_a^\circ - \log(1 \pm Y_b^\circ) * K_{ba}$$

$$Y_{\mathrm{aux}}^\circ = \text{asymptotitc Y-functions}, \quad Y_Q^\circ = 0$$

Our ground-state BTBA takes the form

$$\log \frac{Y_a}{Y_a^\circ} = \log\left(\frac{1 \pm Y_b}{1 \pm Y_b^\circ}\right) \star K_{ba} \quad \text{for auxiliary Y}$$

$$\log \frac{Y_Q}{Y_Q^\bullet} = \log\left(\frac{1 \pm Y_b}{1 \pm Y_b^\circ}\right) \star K_{bQ}$$

Energy of YbarY states

YbarY BTBA: $$\log \frac{Y_a}{Y_a^\circ} = \log\left(\frac{1 \pm Y_b}{1 \pm Y_b^\circ}\right) \star K_{ba}$$

BTBA energy: $$E_{\mathrm{BTBA}}(L,g) = -\sum_{Q=1}^{\infty} \int_0^{\infty} \frac{d\tilde{p}_Q}{2\pi} \log(1+Y_Q)$$

Our BTBA describes Δ of the determinant-like operator:

$$\mathcal{O}(L,L') = \epsilon^{i_1 \cdots i_N}\, \epsilon_{j_1 \cdots j_N}\, \epsilon^{k_1 \cdots k_N}\, \epsilon_{l_1 \cdots l_N} \times$$
$$Y_{i_1}^{j_1} \cdots Y_{i_{N-1}}^{j_{N-1}} (Z^L)_{i_N}^{l_N}\, \overline{Y}_{k_1}^{l_1} \cdots \overline{Y}_{k_{N-1}}^{l_{N-1}} (Z^{L'})_{k_N}^{j_N}$$

$$\Delta = 2N - 2 + L + L' + \underbrace{E_{\mathrm{BTBA}}(L,g) + E_{\mathrm{BTBA}}(L',g)}_{\text{all wrapping corrections}}$$

However, there exists a lower bound for the (B)TBA energy

YQ(v) at large v

BTBA equation for YQ in the large v limit

$$\log \frac{Y_Q(v)}{Y_Q^{\bullet}(v)} = -2 \int_{-\infty}^{\infty} dt \, \log(1 + Y_{Q'}(t)) \, K_{\Sigma}^{Q'Q}(t,v) + \dots$$

$$\sim -4E_{BTBA} \log(v), \quad v \gg 1$$

$$\Leftrightarrow \quad \log Y_Q(v) \sim -\left(4L + 4E_{\mathrm{BTBA}}\right) \log(v)$$

However, the integrals in BTBA energy diverges if $Y_Q(v) \sim 1/v$

$$\int_0^{\infty} \frac{dv}{2\pi} \frac{d\tilde{p}_Q}{dv} \log(1 + Y_Q(v)) \sim (\mathrm{const}) \int^{\infty} dv \, v^{-4L-4E_{\mathrm{BTBA}}}$$

The BTBA energy cannot be negative and large

$$4L + 4E_{\mathrm{BTBA}} > 1 \quad \Leftrightarrow \quad E_{\mathrm{BTBA}} > 1/4 - L$$

YQ(v) at large Q

BTBA equation for YQ in the large Q limit

$$\Leftrightarrow \quad \log Y_Q(v) \sim (3 - 4L - 4E_{\mathrm{BTBA}}) \log(Q)$$

However, the sum in BTBA energy diverges if $Y_Q(v) \sim 1/Q$

$$E_{\mathrm{BTBA}} = -\sum_{Q=1}^{\infty} \int_0^{\infty} \frac{d\tilde{p}_Q}{2\pi} \log(1 + Y_Q)$$

$$\sim \sum_{Q=1}^{\infty} (\mathrm{const})\, Q^{3-4L-4E_{\mathrm{BTBA}}}$$

The BTBA energy cannot be negative and large

$$4L + 4E_{\mathrm{BTBA}} > 4 \quad \Leftrightarrow \quad E_{\mathrm{BTBA}} > 1 - L$$

Energy lower bound

The stronger bound is

$$E_{\mathrm{open}}[Z^L] = L - 1 + E_{\mathrm{BTBA}}(L, g) > 0$$

It is not possible to saturate the lower bound.

Suppose $E_{\mathrm{BTBA}} = 1 - L$

then BTBA dictates $Y_Q(v) \sim 1/Q$

This implies E_{BTBA} diverges, which is a contradiction

A sign of divergences can also be seen at numerical analysis

CPU resources

120

Mars Beowulf cluster
(Utrecht University)

Mathematica

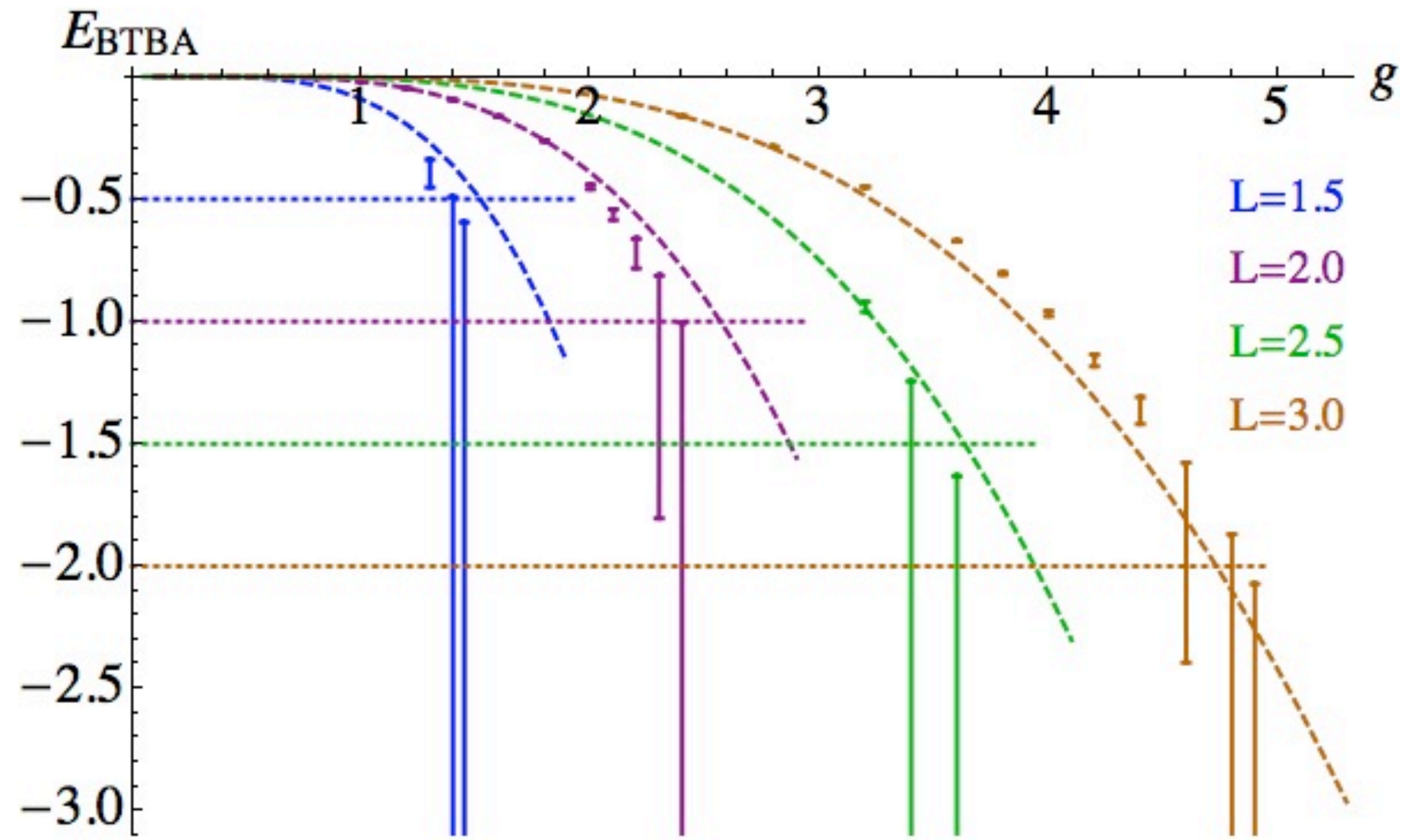
4

Laptop
(Personal)

Sushiki server
(Yukawa Institute)

24

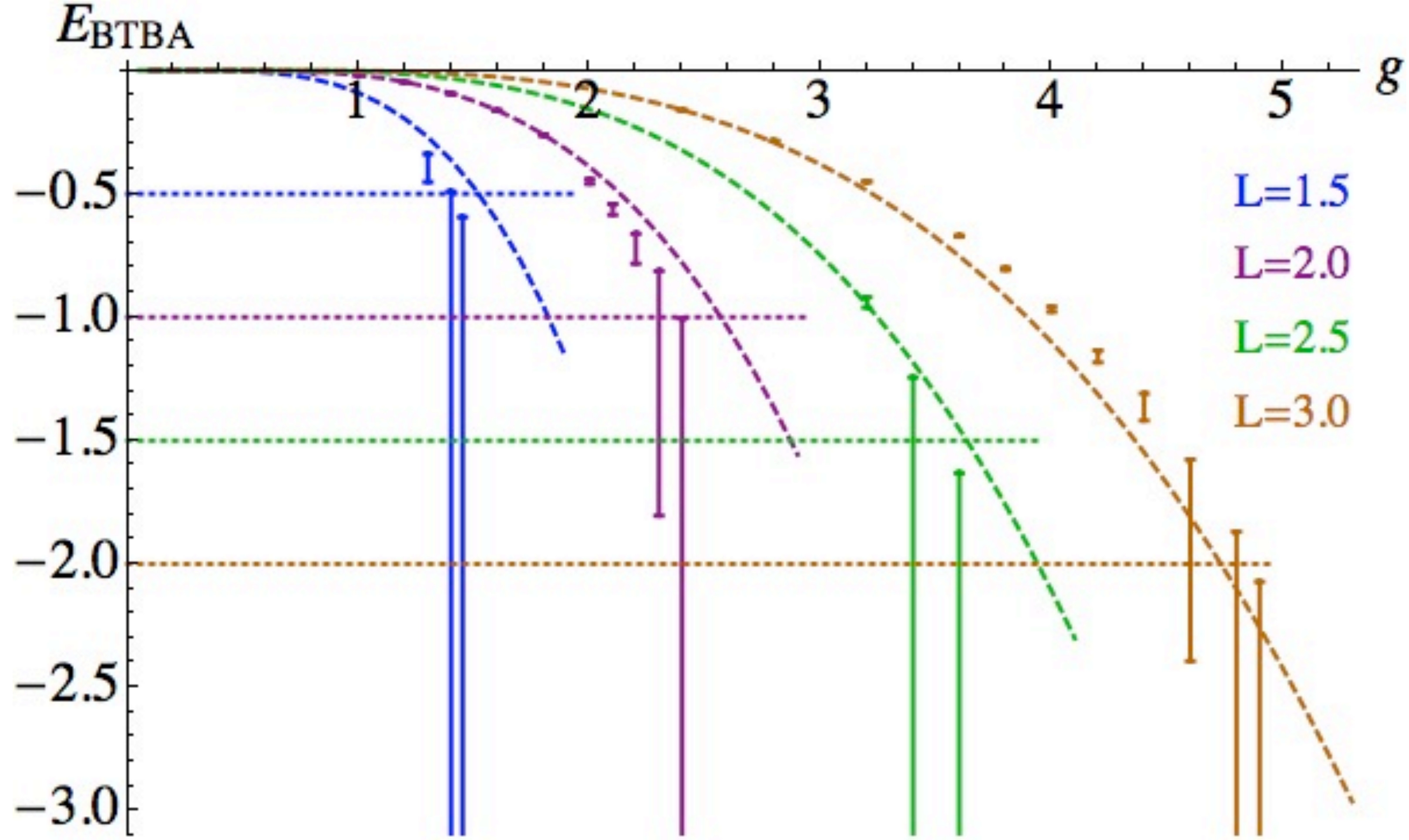
Numerical Results



Solid: BTBA solution, Dashed: Lüscher formula, Dotted: Lower bound

$$E_{\mathrm{BTBA}}^{(\mathrm{num})}(J,g) = -\sum_{Q=1}^{Q_{\max}} \int_0^\infty \frac{d\widetilde{p}_Q}{2\pi} \log(1+Y_Q) - \sum_{Q=Q_{\max}+1}^{100} \int_0^\infty \frac{d\widetilde{p}_Q}{2\pi} \log(1+Y_Q^{\bullet})$$

Numerical Results



- Cannot go further just by a brute-force computation
- Not clear how to go beyond the critical coupling analytically
- Consistent with open string tachyon at strong coupling

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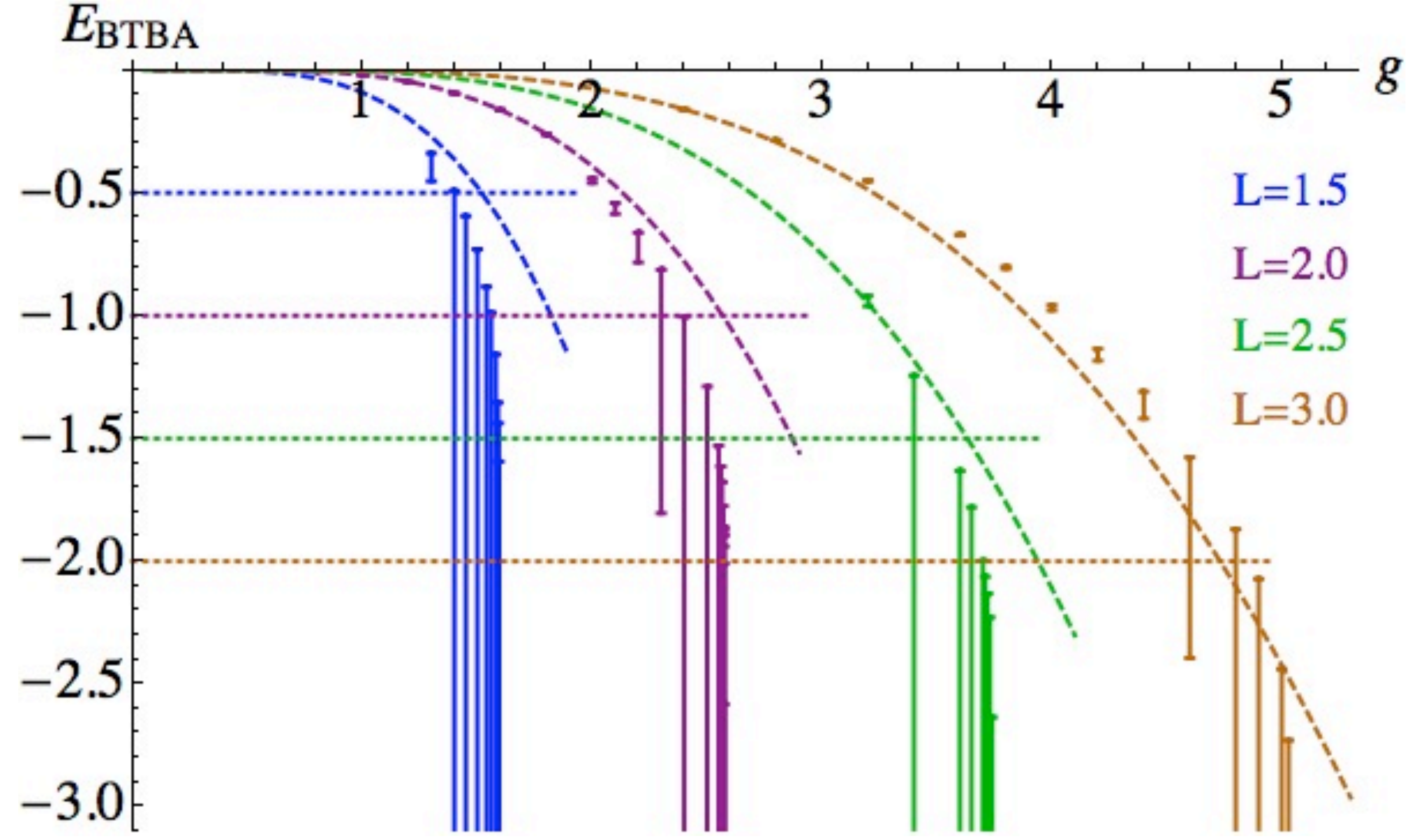

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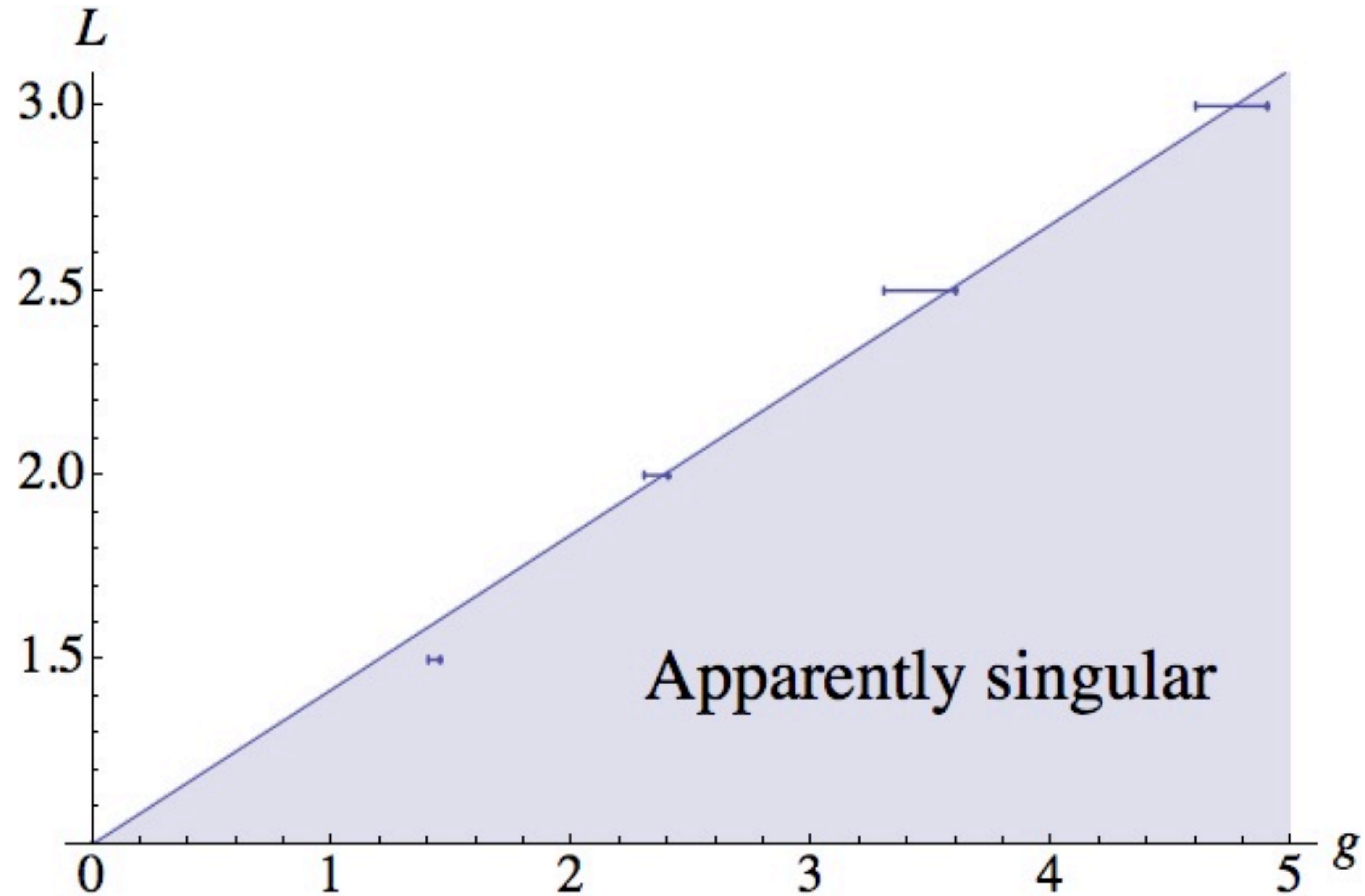
Phase diagram



under the assumption that the $L = 1$ energy diverges at $g = 0$

Summary and outlook

Summary

- Studied the spectrum of determinant-like operators dual to open strings ending on giant gravitons
- Wrapping corrections from $\mathcal{N}$=4 SYM agree with the Lüscher formula
- Proposed and solved BTBA equations for Y=0 & Ybar=0
- Found the lower-bound for the (B)TBA energy

Future works

- Beyond the critical coupling? Compare with string theory?
- How to compute the dimension of the L=1 state?
- AdS/CFT for unstable systems?

Thank you for attention

Infinite-dimensional symmetry

An N-particle state and its dimension/energy is

$$|p_1, \ldots p_N\rangle = A_1^\dagger(p_1)\ldots A_N^\dagger(p_N)|0\rangle, \quad \Delta - J = \sum_{j=1}^{N} \sqrt{1 + 4g^2 \sin^2 \frac{p_j}{2}}$$

The creation-annihilation operators have a free-field-like representation (Zamolodchikov-Faddeev algebra)

$$A_1^\dagger A_2^\dagger = A_2^\dagger A_1^\dagger \mathbb{S}_{12}, \quad A_1 A_2 = \mathbb{S}_{12} A_2 A_1, \quad A_1 A_2^\dagger = A_2^\dagger A_1 \mathbb{S}_{12} + \delta_{12}$$

The centrally-extended su(2|2) extends further to the Hopf-algebra with a non-trivial co-product

$$\Delta \mathfrak{J}^A = \mathfrak{J}^A \otimes 1 + e^{ip[A]} \otimes \mathfrak{J}^A, \quad \mathfrak{J}^A \; : \; \mathfrak{su}(2|2) \text{ generators}$$

$$[\Delta \mathfrak{J}^A, \mathbb{S}] = 0$$

eventually to the Yangian of su(2|2) [Beisert (2005)] and others

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Symmetry in the asymptotic limit

Global symmetry of $\mathbf{AdS_5 \times S^5}$**:** $\mathfrak{psu}(2,2|4) \sim (E, S_1, S_2, J_1, J_2, J)$

The "uniform light-cone gauge" imposes the relation

$$E - J \leftrightarrow \mathcal{H}_{\mathrm{ws}}\,, \quad J \leftrightarrow r \quad (-r \le \sigma \le r,\ \tau \in \mathbb{R})$$

In the large-volume (asymptotic) limit, we observe the worldsheet spectrum

$$E - J \sim \mathcal{H}_{\mathrm{ws}} = \text{finite}, \quad J \sim r \to \infty \quad (E \to \infty)$$

- Ground state : $E - J = 0$
- First excited states: $E - J = 1$ (at $\lambda = 0$)

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- Ground state : $\boldsymbol{E} - \boldsymbol{J} = \mathbf{0}$
- First excited states: $\boldsymbol{E} - \boldsymbol{J} = \mathbf{1}\ \ (\mathbf{at}\ \boldsymbol{\lambda} = \mathbf{0})$

The residual symmetry is now

$$\mathfrak{psu}(2,2|4) \;\to\; \mathfrak{psu}(2|2)^2 \ltimes \mathbb{R} \;\sim\; (\boldsymbol{E}, \boldsymbol{S_1}, \boldsymbol{S_2}, \boldsymbol{J_1}, \boldsymbol{J_2}, \boldsymbol{J})$$

~~Periodicity condition~~ $\longrightarrow$ Extra central charges

$$\mathfrak{psu}(2|2)^2 \ltimes \mathbb{R} \quad \longrightarrow \quad \mathfrak{su}(2|2)^2 \ltimes \mathbb{R}$$

Mirror trick

Lorentzian mirror theory is defined by the Wick rotation

$$(\mathcal{E}_Q, p_Q) \to (-i\widetilde{p}_Q, i\widetilde{\mathcal{E}}_Q)$$

which is also a different real section of complexified su(2|2)^2

$\boxed{\text{Mirror BAE is analytic continuation of string-theory BAE}}$

Asymptotic mirror Bethe Ansatz equation

$$-1 = e^{iR\tilde{p}_k} \prod_{j=1}^{N} S(\tilde{p}_k, \tilde{p}_j) \quad \Leftrightarrow \quad \pi i(2I_k + 1) = iR\,\tilde{p}_k + \sum_{j=1}^{N} \log S(\tilde{p}_k, \tilde{p}_j)$$

If we specify the mode numbers $\{I_k\}$ to some integers
and look for the solution, one of the followings happen:

- No solution
- The solution $\{p_k\}$ exists and unique (Fermi statistics)

Thermodynamic Bethe Ansatz (TBA) equations

Partition function in terms of the Bethe root/hole densities:

$$\mathrm{tr}\, e^{-L\tilde{H}(R)} = \sum_N \sum_{\{I_j\}\in\mathbb{Z}^N} e^{-L\tilde{E}_N(R)}$$

thermodynamic limit $\rightarrow \int [d\rho][d\bar{\rho}]\, \delta(\mathrm{BAE}[\rho,\bar{\rho}])\, e^{-L\tilde{E}[\rho,\bar{\rho}]+S[\rho,\bar{\rho}]}$

This can be evaluated by the saddle-point approximation

$$\rho = \rho_* + r, \quad \bar{\rho} = \bar{\rho}_* + \bar{r}, \quad e^{-\tilde{\mathcal{F}}} = e^{-\tilde{\mathcal{F}}_*} \int [dr][d\bar{r}]\, e^{-L\delta\tilde{E}[\rho,\bar{\rho}]+\delta S[\rho,\bar{\rho}]}\Big|_{\mathrm{BAE}}$$

The saddle-point is $\mathrm{O}(R)$ and contributes to $RE_0(L)$

The fluctuation is $\mathrm{O}(1)$ and negligible

[Woynarovich (2004,2010)], [Pozsgay (2010)]

TBA = saddle-point condition: $Y_a = \rho_a/\bar{\rho}_a$

$$\log Y_a = \boxed{\sum_b} \log(1 \pm Y_b) \star K_{ba} + V_a$$

Why SMGG = determinant?

- Matching of the residual symmetry

$$\left[\det Y^N \leftrightarrow \mathrm{S}^3 \subset \mathrm{S}^5\right] : SO(6) \to SO(4) \times SO(2)$$

- Single-trace vs. multi-trace operators

✓ Multi-trace large operators can mix even at large N

✓ determinant is a linear combination of multi-traces

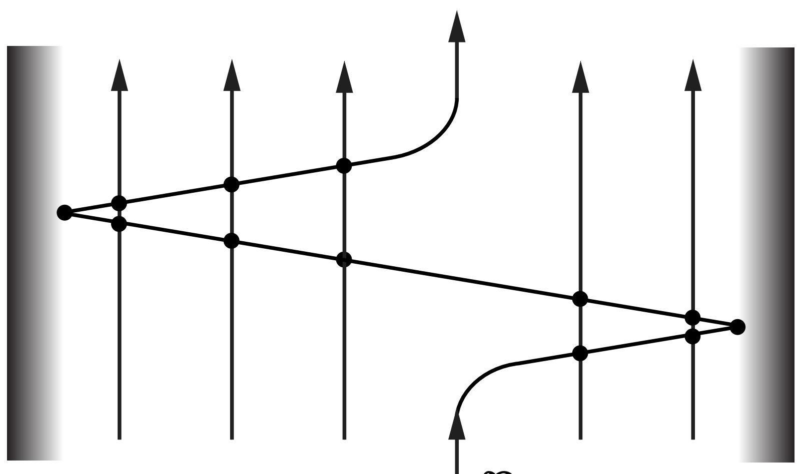
$$\det Y^N = c[1^N](\mathrm{tr}\, Y)^N + \cdots + c[N]\,\mathrm{tr} Y^N, \quad c[r] = \text{constant}$$

- Determinant and sub-determinant do not correlate, nor do maximal and non-maximal giant gravitons

[Witten (1998)] [Balasubramanian, Berkooz, Naqvi, Strassler (2001)] [Corley, Jevicki, Ramgoolam (2001)]

Boundary Bethe-Yang equation

Integrable open spin chains obey boundary BYE

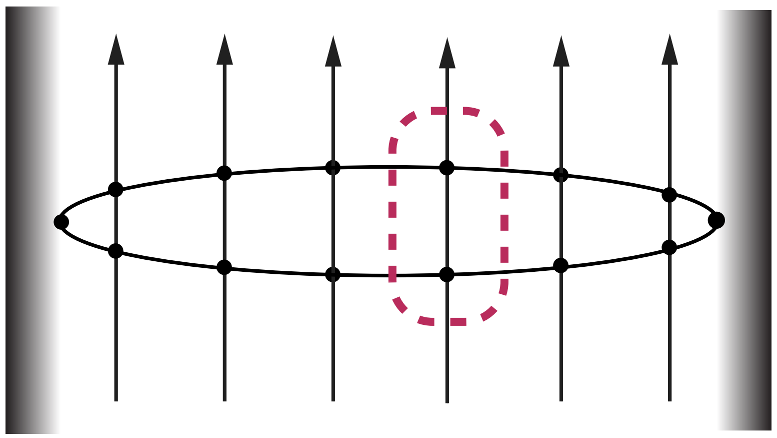


$$1 = e^{-i2Jp_K} \prod_{j \neq k}^{N} S(p_k, p_j) R^-(p_k) \times \prod_{j \neq k}^{N} S(p_j, -p_k) R^+(-p_k)$$

BBYE from double-row transfer matrix

$$D_a = \mathrm{tr}_a \left[\mathbb{S}_{aN} \cdots \mathbb{S}_{a1} \, \mathbb{R}^- \mathbb{S}_{1a} \cdots \mathbb{S}_{Na} \, \tilde{\mathbb{R}}^+ \right]$$

$\mathbb{R}^\pm$: **reflection matrix**



Boundary Yang-Baxter for $\mathbb{R}^\pm \quad \Rightarrow \quad [D_a, D_b] = 0$

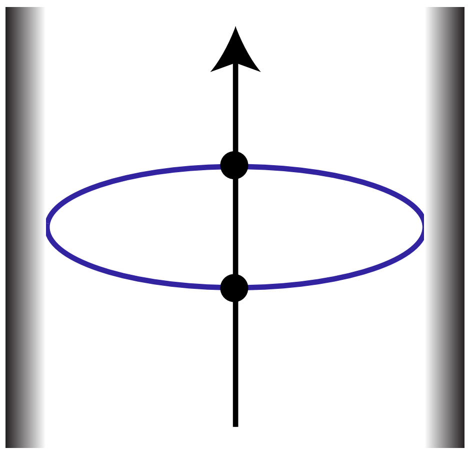
BBYE $\quad \Leftrightarrow \quad -1 = e^{-2iqJ} D_a(q|\vec{p})\Big|_{q=p_k}$

[Sklyanin (1988)]

Boundary wrapping corrections

- Boundary Lüscher formula has been conjectured and tested

$$\Delta_{\text{Lüscher}} \sim \sum_Q \int_0^\infty d\tilde{p}_Q \, e^{-\tilde{\mathcal{E}}_Q(\tilde{p}_Q)2J}$$



- In terms of the double-row transfer matrix

$$\Delta_{\text{Lüscher}} = -\sum_{Q=1}^{\infty} \int_0^\infty \frac{d\tilde{p}_Q}{2\pi} Y_Q^\bullet, \quad Y_Q^\bullet = e^{-\tilde{\mathcal{E}}_Q 2J} D_Q$$

Agree with $\mathcal{N}$=4 SYM perturbation at weak coupling for simple states

[Correa, Young (2009)] [Bajnok, Palla (2010)]

cf. Dressing phase kernel

$$K^{\Sigma}_{Q'Q}(t,v) = \frac{1}{2\pi i}\frac{\partial}{\partial t}\log \Sigma^{Q'Q}(t,v)$$

$$\begin{aligned}
\frac{1}{i}\log \Sigma^{Q'Q}(t,v) = {} & \Phi(y_1^+,y_2^+) - \Phi(y_1^+,y_2^-) - \Phi(y_1^-,y_2^+) + \Phi(y_1^-,y_2^-) \\
& + \frac{1}{2}\Big(\Psi(y_2^+,y_1^+) + \Psi(y_2^-,y_1^+) - \Psi(y_2^+,y_1^-) - \Psi(y_2^-,y_1^-)\Big) \\
& - \frac{1}{2}\Big(\Psi(y_1^+,y_2^+) + \Psi(y_1^-,y_2^+) - \Psi(y_1^+,y_2^-) - \Psi(y_1^-,y_2^-)\Big) \\
& + \frac{1}{i}\log \frac{i^{Q'}\,\Gamma\big[Q - \frac{i}{2}g\big(y_1^+ + \frac{1}{y_1^+} - y_2^+ - \frac{1}{y_2^+}\big)\big]}{i^{Q}\,\Gamma\big[Q' + \frac{i}{2}g\big(y_1^+ + \frac{1}{y_1^+} - y_2^+ - \frac{1}{y_2^+}\big)\big]}\,\frac{1 - \frac{1}{y_1^+ y_2^-}}{1 - \frac{1}{y_1^- y_2^+}}\sqrt{\frac{y_1^+ y_2^-}{y_1^- y_2^+}}
\end{aligned}$$

$$\Phi(x_1,x_2) = i\oint \frac{dw_1}{2\pi}\oint \frac{dw_2}{2\pi}\,\frac{1}{(w_1-x_1)(w_2-x_2)}\,\log\frac{\Gamma[1+\frac{ig}{2}\big(w_1+\frac{1}{w_1}-w_2-\frac{1}{w_2}\big)]}{\Gamma[1-\frac{ig}{2}\big(w_1+\frac{1}{w_1}-w_2-\frac{1}{w_2}\big)]}$$

$$\Psi(x_1,x_2) = i\oint \frac{dw}{2\pi}\,\frac{1}{w-x_2}\,\log\frac{\Gamma[1+\frac{ig}{2}\big(x_1+\frac{1}{x_1}-w-\frac{1}{w}\big)]}{\Gamma[1-\frac{ig}{2}\big(x_1+\frac{1}{x_1}-w-\frac{1}{w}\big)]}$$

$$x(v) = \frac{1}{2}\Big(v - i\sqrt{4-v^2}\Big), \quad y_1^{\pm} = x\Big(t \pm \frac{iQ'}{g}\Big), \quad y_2^{\pm} = x\Big(v \pm \frac{iQ}{g}\Big)$$

Error bars

We put Qmax=6 to draw the solid line

$$E^{(\mathrm{num})}_{\mathrm{BTBA}}(J,g) = -\sum_{Q=1}^{Q_{\max}} \int_0^{\infty} \frac{d\tilde{p}_Q}{2\pi} \log(1+Y_Q) - \sum_{Q=Q_{\max}+1}^{100} \int_0^{\infty} \frac{d\tilde{p}_Q}{2\pi} \log(1+Y_Q^{\bullet})$$

The error from the truncation of YQ is huge around the critical value

$$E_{\mathrm{BTBA}} = \sum_Q \mathrm{E}(Q), \quad \mathrm{E}(Q) = -\int \frac{d\tilde{p}_Q}{2\pi} \log(1+Y_Q) \sim Q^{-4J-4E_{\mathrm{BTBA}}}$$

We extrapolate the BTBA energy from Qmax=6 to Qmax=100 using the large Q asymptotics of $\mathrm{E(Q)}$

$$\tilde{E}_{\mathrm{BTBA}} = \sum_{Q=1}^{6} \mathrm{E}^{(\mathrm{original})}(Q) + \sum_{Q=7}^{100} \mathrm{E}^{(\mathrm{fit})}(Q) \ \left(< E^{(\mathrm{num})}_{\mathrm{BTBA}}\right)$$

Estimate of truncation error: $\delta E_{\mathrm{BTBA}} \equiv E^{(\mathrm{num})}_{\mathrm{BTBA}} - \tilde{E}_{\mathrm{BTBA}}$