Boundary states in curved backgrounds

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1. Introduction

CFT in string theory

- Powerful for covariant perturbation theory
- Applicable to quantization in curved backgrounds

$\Longrightarrow$ CFT in general backgrounds is interesting

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Nonperturbative aspects of strings

- D-branes: excitations nonperturbative in g_s
- CFT: powerful for perturbation in g_s

$\Longrightarrow$ CFT clarifies many features of D-branes

1. Introduction (continued)

How to describe D-branes in CFT language

[Open] Boundaries of worldsheet

Different boundary conditions, different D-branes

[Closed] Massive extended objects

Coupling to closed strings describe D-branes

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[Closed] Massive extended objects

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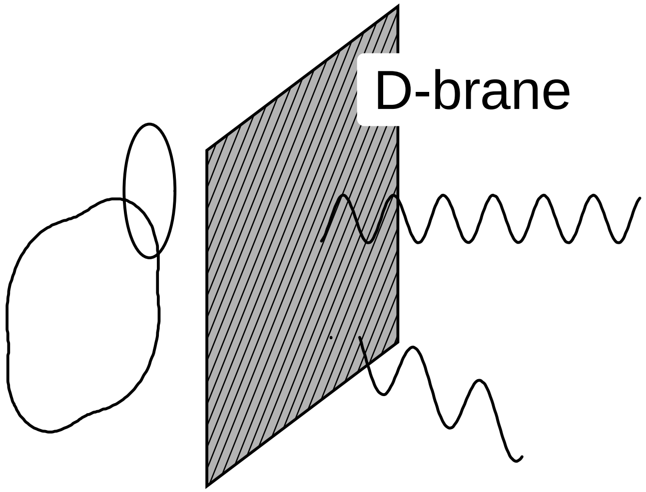

Boundary conditions (Open)
= Boundary states (Closed)

Contents

2. Boundary CFT

Boundary states

glue left- and right-moving modes

- Boundary conformal invariance

$$\left(L_n - \bar{L}_{-n}\right) |B\rangle = 0, \quad \forall\, n \in \mathbb{Z} \tag{1}$$

- Gluing condition (If conformal symmetry is extended)

$$\left[W_n - (-1)^h \,\Omega\, \bar{W}_{-n}\right] |B\rangle = 0, \quad \forall\, n \in \mathbb{Z} \tag{2}$$

2. Boundary CFT

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$$\left[W_n - (-1)^h\, \Omega\, \bar{W}_{-n}\right] |B\rangle = 0, \quad \forall\, n \in \mathbb{Z} \tag{2}$$

Ishibashi states solve these conditions [Ishibashi]

$$|I\rangle\!\rangle = \sum_{i \in \mathcal{V}_I} |i\rangle \otimes \left|\overline{Ui}\right\rangle . \tag{3}$$

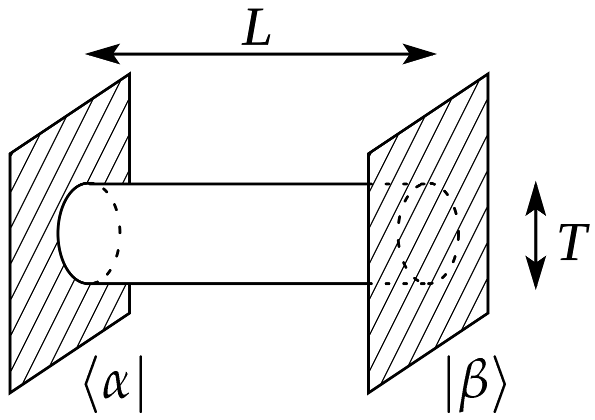
i.e. We have to specify the holomorphic part alone.

2. The Cardy condition

Identify boundary states as boundary conditions

The Cardy condition must hold

$$\mathrm{Tr}\, \mathrm{e}^{-TH^{o}_{\alpha\beta}} = \langle\alpha|\, \mathrm{e}^{-LH^{c}}\, |\beta\rangle \qquad (4)$$



α, β : open string boundary conditions

$|\alpha\rangle\,, |\beta\rangle$: corresponding boundary states

(A consistency condition for CFT's on a cylinder)

2. The Cardy condition

Identify boundary states as boundary conditions

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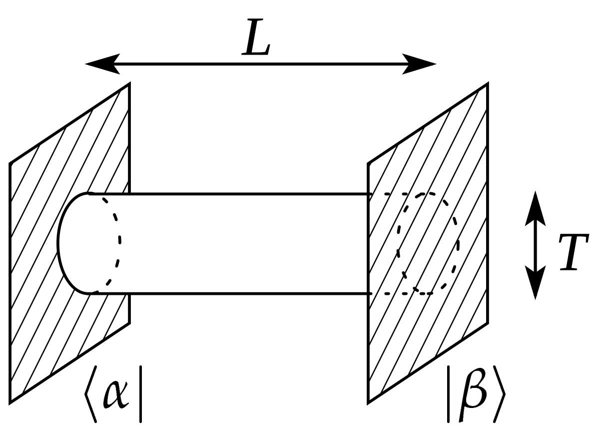

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α, β : open string boundary conditions

$|\alpha\rangle\,, |\beta\rangle$: corresponding boundary states

Cardy states solve it, if CFT is rational [Cardy]

$$\sum_I n^I_{\alpha\beta}\, S_{IJ} = \langle\alpha|J\rangle\rangle\langle\langle J|\beta\rangle, \quad n^I_{\alpha\beta} \geq 0. \qquad (5)$$

$\langle\alpha|J\rangle\rangle, \langle\langle J|\beta\rangle$ are expressed in terms of S-transformation matrices of Virasoro characters S_{IJ}

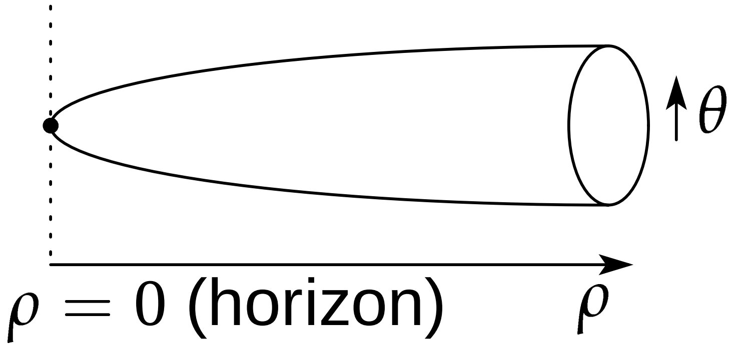
Continue . . .

3. The 2D black hole

The 2D BH $\leftarrow$ $SL(2,\mathbb{R})\,/\,U(1)_A$ WZW model

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \tanh^2\rho\, \mathrm{d}\theta^2$$

$$e^{\Phi} \propto \frac{1}{\cosh\rho}$$



This geometry is conformal at $\mathcal{O}\left(\alpha'\right)$

3. The 2D black hole

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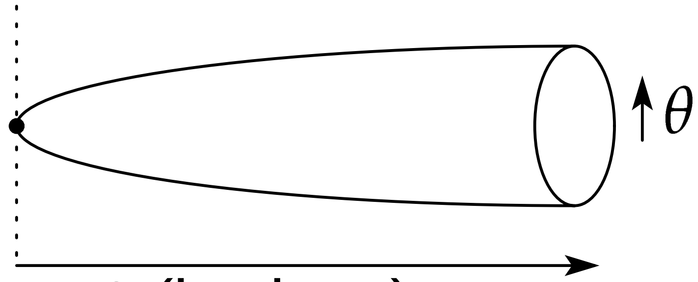

$\rho = 0$ (horizon)

This geometry is conformal at $\mathcal{O}(\alpha')$

Our interests

- The near-horizon limit of NS5 brane (blackhole solution of supergravity without RR charge)
- Noncompact, irrational and yet solvable CFT

3. Primary fields

Descend from Virasoro primaries of $\widehat{SL}_k(2,\mathbb{R})$

$$\left[J^3_n\,,\,J^3_m\right] = -\frac{kn}{2}\,\delta_{n+m}\,, \qquad \left[J^3_n\,,\,J^\pm_m\right] = \pm\,J^\pm_{n+m}$$

$$\left[J^+_n\,,\,J^-_m\right] = -2\,J^3_{n+m} + kn\,\delta_{n+m,0} \tag{6}$$

Primaries form unitary representations of $SL(2,\mathbb{R})$

- Principal Continuous series

$$\mathcal{C}^\alpha_j = \left\{|j,\alpha;m\rangle \,\middle|\, j = -\frac{1}{2} + \mathrm{i}\,\mathbb{R}\,,\ m = \alpha + \mathbb{Z}\,,\ -\frac{1}{2} \le \alpha < \frac{1}{2}\right\}.$$

- Discrete series

$$\mathcal{D}^\pm_j = \{|j;m\rangle \mid \pm\, m = j+1, j+2, \ldots\}\,.$$

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Gauging the timelike direction J^3 gives coset primaries.

- Descendants from affine primaries

$$\widehat{\mathcal{C}}^\alpha_j \quad \left(j = -\frac{1}{2} + \mathrm{i}\,\mathbb{R}\right)\;; \quad \widehat{\mathcal{D}}^\pm_j \quad \left(-\frac{k-1}{2} < j < -\frac{1}{2}\right) \tag{7}$$

Checked by path-integrating gauged WZW action [a]

[a] Hanany, Prezas and Troost hep-th/0202129; Fotopoulos, Niarchos and Prezas hep-th/0406017

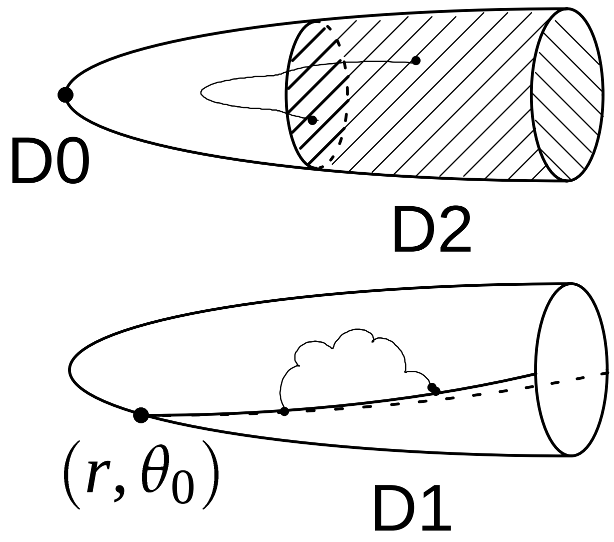
3. Boundary states

Semiclassical analysis, based on DBI action

[D0] Pointlike, Localize near the tip of the cigar

[D1] Extend along ρ direction

[D2] Carries 2-form field strength, covers the region away from the tip



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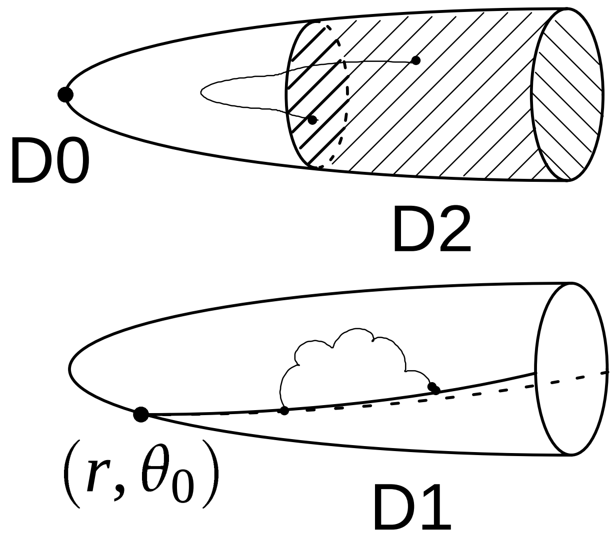

The exact solution is proposed first by [RS] [a]

Derivations by Conformal bootstrap or Modular bootstrap [b]

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[a] Ribault S. and Schomerus V. hep-th/0310024

[b] Fotopoulos, Niarchos and Prezas hep-th/0406017

4. Conformal bootstrap approach

Strategy : Descending method

- The gauging of J_n^3's yields boundary state candidates

  Boundary states in $H_3 \simeq SL(2,\mathbb{C})/SU(2)$ [a]

  $\longrightarrow$ Boundary states in the 2D black hole

  - S^2 branes $\rightarrow$ D0
  - AdS_2 branes $\rightarrow$ D1
  - Rotated AdS_2 branes $\rightarrow$ D2

- Check the Cardy condition

  Discard some of the candidates

[a] Ponsot, Schomerus and Teschner `hep-th/0112198`

4. Derivation (D0 case)

Gluing conditions for S^2 (spherical) branes

$$J^{\pm}(z) = \bar{J}^{\mp}(\bar{z}),\ J^3(z) = -\bar{J}^3(\bar{z}), \quad \text{on the boundary } \ z = \bar{z}$$

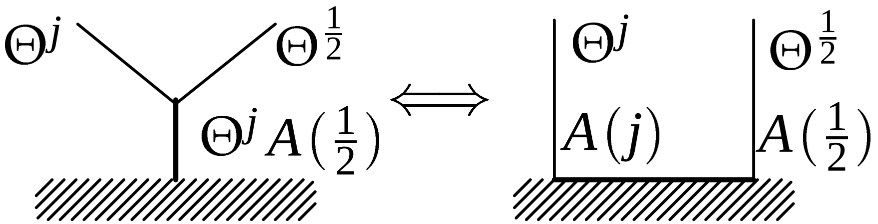
Coupling to a (normalized) bulk primary $\Theta^j(u|z)$

$$\left\langle \Theta^j(u|z) \right\rangle_{S^2} \equiv A(j)\,(1+u\bar{u})^{2j}\,|z-\bar{z}|^{-2h_j}$$

Use Teschner's trick

$$\partial_u^2 \Theta^{1/2}(u|z) = \partial_{\bar{u}}^2 \Theta^{1/2}(u|z) = 0 \qquad \text{(Degenerate field)}$$

$$\Theta^{1/2}(u_2|z_2)\Theta^j(u_1|z_1) \sim C_+(j)\left[\Theta^{j+1/2}(u_1|z_1)\right] + C_-(j)\left[\Theta^{j-1/2}(u_1|z_1)\right]$$



Cluster decomposition property

$\rightarrow$ A functional equation determines $A(j)$

4. Derivation (D0 case)

Annulus amplitude of open strings between S^2 branes

$$Z^{S^2}_{mm'}(q,z) = \tilde{q}^{-k\theta^2/4}\, {}_{S^2}\langle m|\, \tilde{q}^{L_0+\bar{L}_0-c/24} e^{2\pi i\tilde{\theta}J^3_0}\, |m'\rangle_{S^2}$$

$\leftarrow$ Insert a complete set of eigenstates $j=-1/2+iP,\ J^3_0 = ip/2$

Relation to D0 branes in the 2D black hole

$$Z^{D0}_{mm'}(q) \;=\; \eta(\tau)\int_{-\infty}^{\infty} d\theta \int_{-\infty}^{\infty} d\alpha\, z^{\alpha} q^{\alpha^2/k}\, Z^{S^2}_{mm'}(q,z)\,,\quad q=e^{2\pi i\tau}, z=e^{2\pi i\theta}$$

$\leftarrow$ Put $Z^{S^2}_{mm'}(q,z)$, shift the path of $\int dP$ and transform $\int d\theta \to \sum_{w\in\mathbb{Z}}$

Express $Z^{D0}_{mm'}(q)$ in terms of $\widehat{\mathfrak{sl}}(2,\mathbb{R})/\widehat{\mathfrak{u}}(1)$ characters

$$Z^{c}_{mm'}(q) = \int dP \sum_{w\in\mathbb{Z}} \chi^{c}_{(-\frac{1}{2}+iP,\frac{kw}{2})}(\tilde{q})\, \left|\Psi_m(-\frac{1}{2}+iP,w)\right|^2$$

4. Main results

- Expressed as coupling to bulk primaries

$$\langle \Phi^j_{nw}(z,\bar z)\rangle_B \equiv \langle 0|\, \Phi^j_{nw}(z,\bar z)\,|B\rangle = \langle 0|\, \Phi^j_{nw}(z,\bar z)\psi_B(0,0)\,|0\rangle$$

$\leftarrow$ Compute using the wavefunctions $\Psi_B\,(j,n,w) = \langle B|\, j,n,w\rangle\!\rangle$

- Consistent with the coset construction

$$\Phi^j_{n\,w}(z,\bar z) \equiv \exp\left[\frac{\mathrm{i}}{\sqrt{2k}}\left\{\,(n+kw)\,X(z) + (n-kw)\,\bar X(\bar z)\right\}\right]\,\Phi^{\mathrm{H},j}_{n,-\mathrm{i}kw}(z,\bar z)$$

[D0]

$$\langle \Phi^j_{nw}(z,\bar z)\rangle^{\mathrm{D0}}_m = \delta_{n,0}\,\mathcal{N}_m(b)\,(-1)^{mw}\left(\frac{k}{2}\right)^{\frac14}\frac{\Gamma(-j+\frac{kw}{2})\Gamma(-j-\frac{kw}{2})}{\Gamma(-2j-1)}\times$$

$$\times\;\frac{\sin\pi b^2}{\sin\pi b^2 m}\,\frac{\sin\pi b^2 m(2j+1)}{\sin\pi b^2(2j+1)}\,\frac{\Gamma(1+b^2)\nu_b^{j+1}}{\Gamma(1-b^2(2j+1))}\;\frac{1}{|z-\bar z|^{h^j_{nw}+\bar h^j_{nw}}}\,.$$

4. (continued)

[D1]

$$\langle \Phi^j_{nw}(z,\bar z)\rangle^{\rm D1}_{(r,\theta_0)} = \delta_{w,0}\,\mathcal{N}'(b)\, e^{in\theta_0}\,(2k)^{-\frac14}\,\frac{\Gamma(2j+1)}{\Gamma(1+j+\frac n2)\Gamma(1+j-\frac n2)}\times$$

$$\times\left(e^{-r(2j+1)}+(-1)^n e^{r(2j+1)}\right)\Gamma(1+b^2(2j+1))\,\nu_b^{j+\frac12}\,\frac{1}{|z-\bar z|^{h^j_{nw}+\bar h^j_{nw}}}\,.$$

[D2]

$$\langle \Phi^j_{nw}(z,\bar z)\rangle^{\rm D2}_{\sigma} = \delta_{n,0}\,\mathcal{N}'(b)\times$$

$$\times\left(\frac{\Gamma(-j+\frac{kw}{2})}{\Gamma(j+1+\frac{kw}{2})}\,\mathrm{e}^{\,\mathrm{i}\sigma(2j+1)}+\frac{\Gamma(-j-\frac{kw}{2})}{\Gamma(j+1-\frac{kw}{2})}\,\mathrm{e}^{\,-\mathrm{i}\sigma(2j+1)}\right)\times$$

$$\times\ (k/2)^{\frac14}\,\Gamma(2j+1)\,\Gamma(1+b^2(2j+1))\,\nu_b^{j+\frac12}\,\frac{1}{|z-\bar z|^{h^j_{nw}+\bar h^j_{nw}}}\,.$$

4. Modular bootstrap approach

Modular bootstrap

Modular transformation of open characters

$\Rightarrow$ Boundary state candidates in closed sector

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Modular transformation of open characters

$\Rightarrow$ Boundary state candidates in closed sector

Suppose $\chi_J^{\text{open}}(T) \equiv \langle\langle 0|q^{-LH^c}|J\rangle\rangle$, then we check

$$\langle\langle \alpha|q^{-LH^c}|\beta\rangle\rangle = \sum_J n_{\alpha\beta}^J \chi_J^{\text{open}}(T)\,, \qquad n_{\alpha\beta}^J \geq 0\,,\ \forall\,\alpha,\beta\,.$$

It is highly nontrivial, because $\sharp(freedom) \ll \sharp(constraints)$

In the 2D black hole, D0 is labeled by the identity representations, D1 by continuous and D2 by discrete ones

4. Modular bootstrap approach

An example: Boundary Liouville field theory

- FZZT branes : extended along ϕ, non-degenrate reps.
- ZZ branes : localized along ϕ, degenerate reps.

Define $\Psi_{m,n}(P) := \langle\langle P | m, n \rangle$ for (m, n) ZZ-brane by

$$\chi_{m,n}(-1/\tau) \equiv \int_{-\infty}^{\infty} \mathrm{d}P\, \chi_P(q)\, \Psi_{m,n}(P) \Psi_{1,1}(-P)\,, \quad \chi_P(q) = q^{P^2}/\eta(\tau)$$

$$\chi_{m,n}(\tau) = \eta^{-1}(\tau) \left[q^{-(m/b+nb)^2/4} - q^{-(m/b-nb)^2/4} \right] \text{ (Degenerate ch.)}$$

Compare with the following identity relation

$$\chi_{m,n}(\tau) = 2\sqrt{2} \int_{-\infty}^{\infty} \mathrm{d}P\, \chi_P(\tau) \sinh(2\pi mP/b) \sinh(2\pi nbP)$$

4. Modular bootstrap approach

Agree with the Conformal bootstrap results

Boundary 1 point functions :

$$\Psi_{1,1}(P) \propto \frac{\mathrm{i}\pi P}{\Gamma(1-2\mathrm{i}Pb)\Gamma(1-2\mathrm{i}P/b)} \left(\pi\mu\gamma(b^2)\right)^{-\mathrm{i}P/b}$$

$$\Psi_{m,n}(P) = \Psi_{1,1}(P)\,\frac{\sinh(2\pi mP/b)\sinh(2\pi nbP)}{\sinh(2\pi P/b)\sinh(2\pi bP)}$$

The Cardy condition

$$\begin{aligned} Z_{(m,n)(m',n')} &= \int_{-\infty}^{\infty} \mathrm{d}P\, \chi_P(q)\, \Psi_{m,n}(P)\Psi_{m',n'}(-P) \\ &= \sum_{k=0}^{\min(m,m')-1} \sum_{l=0}^{\min(n,n')-1} \chi_{m+m'-2k-1,n+n'-2l-1}(q') \end{aligned}$$

$\leftrightarrow$ The fusion algebra of degenerate representations

5. Conculsion and Outlook

What we studied

- The spectra of bulk primaries and boundary states in the 2D black hole
- Conformal and Modular bootstrap approaches

Outlook

- Study interactions [a]
- Extension to other backgrounds (the whole NS5)
- Check dualities (dual to sine-Liouville, holography) [b]

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[a] Hosomichi `hep-th/0408172`

[b] McGreevy and Verlinde `hep-th/0304224`

End.

Thank you for listening!

The $\widehat{\mathfrak{sl}}(2,\mathbb{R})/\widehat{\mathfrak{u}}(1)$ characters

- Identity representations $\mathcal{D}_{j,r}$ $(j=0,k/2\,;r\in\mathbb{Z})$

$$\lambda_r^I(\tau)=\eta(\tau)^{-2}q^{-\frac{1}{4(k-2)}}q^{|r|+\frac{r^2}{k}}\left[1+\sum_{s=1}^{\infty}(-1)^s q^{\frac{1}{2}(s^2+(2|r|+1)s-2|r|)}(1+q^{|r|})\right]$$

- Discrete representations $\mathcal{D}_{j.r}$ $(0<j<k/2\,;r\in\mathbb{Z})$

$$\lambda_{j,r}^d(\tau)=\eta(\tau)^{-2}q^{-\frac{(j-\frac{1}{2})^2}{k-2}}q^{\frac{(j+r)^2}{k}}\sum_{s=0}^{\infty}(-1)^s q^{\frac{1}{2}s(s+2r+1)}$$

- Continuous representations $\mathcal{C}_{s,\alpha+r}$ $(s\in\mathbb{R}\,;r\in\mathbb{Z})$

$$\lambda^c_{\frac{1}{2}+\mathrm{i}s,\alpha+r}=\eta(\tau)^{-2}q^{\frac{s^2}{k-2}}q^{\frac{(\alpha+r)^2}{k}}$$

J_3^0 charges are $r, j+r, \alpha+r$ respectively

The $\widehat{\mathfrak{sl}}(2,\mathbb{R})/\widehat{\mathfrak{u}}(1)$ characters

We embed $\widehat{\mathfrak{sl}}(2,\mathbb{R})/\widehat{\mathfrak{u}}(1)$ to $\mathcal{N}=2$ algebra [a]

1. $\widehat{\mathfrak{sl}}(2,\mathbb{R})$ currents $\leftrightarrow$ $\mathcal{N}=2$ currents

$$J^3 = -\mathrm{i}\sqrt{\frac{k}{2}}\,\partial Y, \quad J^{\pm} = \sqrt{k}\,\psi^{\pm}\mathrm{e}^{\pm\mathrm{i}\sqrt{\frac{k}{2}}Y}, \quad k = \frac{2c}{3-c}$$

$\psi^{\pm}$: primaries of weight $(k-1)/k$, Y: timelike boson $Y(z)Y(0) \sim \ln z$

Wick rotate $Y \to \phi$ (spacelike bosons) gives $\mathcal{N}=2$ algebra

2. Decomposition into $\widehat{\mathfrak{u}}(1) \times \widehat{\mathfrak{sl}}(2,\mathbb{R})/\widehat{\mathfrak{u}}(1)$

- Identity representations

$$\mathrm{ch}_I(\tau,z) = \eta^{-1}(\tau)\sum_{n\in\mathbb{Z}} y^n q^{\frac{k-2}{2k}n^2}\lambda^I_{-n}(\tau) \qquad (8)$$

[a] [Dixon, Peskin and Lykken]

The $\widehat{\mathfrak{sl}}(2,\mathbb{R})\,/\,\widehat{\mathfrak{u}}(1)$ characters

- Discrete representations

$$\mathrm{ch}_d(h_{j,\pm j}, Q_{\pm j}; \tau, z) = \eta^{-1}(\tau) \sum_{n\in\mathbb{Z}} y^{\pm\frac{2j}{k-2}+n} q^{\frac{k-2}{2k}(\pm\frac{2j}{k-2}+n)^2} \lambda^d_{j,-n}(\tau) \tag{9}$$

- Continuous representations

$$\mathrm{ch}_c(h_{\frac{1}{2}+\mathrm{is},\alpha+r}, Q_{\alpha+r}; \tau, z) = \eta^{-1}(\tau) \quad \times$$
$$\sum_{n\in\mathbb{Z}} y^{\frac{2(\alpha+r)}{k-2}+n} q^{\frac{k-2}{2k}(\frac{2(\alpha+r)}{k-2}+n)^2} \lambda^c_{\frac{1}{2}+\mathrm{is},n-(\alpha+r)}(\tau) \tag{10}$$

LHS: $\mathcal{N}=2$ characters, h: weight, Q: $U(1)_R$ charge
RHS: $\widehat{\mathfrak{sl}}(2,\mathbb{R})\,/\,\widehat{\mathfrak{u}}(1)$ characters