AdS/CFT correspondence and Integrability

Ryo Suzuki (Shing-Tung Yau Center of Southeast University)

2nd Summer School on String theory and Holography
August 2021

Plan of Lectures

1. Introduction
2. N=4 Supersymmetric Yang-Mills Theory
3. Hagedorn transition in gauge theory
4. Integrability and TBA

Notice

If you have any questions, stop me at any moment

Let me know if you think I speak too fast
(Probably I don’t understand Chinese)

Lecture materials will be uploaded; rest assured

I assume the audience knows basics of QFT and string theory

Introduction

explain title : AdS / CFT correspondence

Physics of gravity on AdS_{d+1} spacetime

$\parallel$

Physics of conformal field theory in d-dim.

Proposed by Juan Maldacena   Hep-th/9711200

Bulk boundary corr. ($CFT_d \longleftrightarrow \partial AdS_{d+1}$)

Realizes large N duality proposed by 't Hooft

Maldacena's setup :

$N \gg 1$ coincident D3- branes on the flat space

closed string description :

IIB superstring on $AdS_5 \times S^5$ spacetime

with N units of RR flux $\sqrt{\lambda} = \frac{R^2}{\alpha'} \gg 1$

open string description :

$D=4$ $\mathcal{N}=4$ super Yang Mills theory

with gauge group $SU(N)$

strong - weak duality → hard to 'prove' $\lambda = N g_{YM}^2 \ll 1$

Comments :

1) Hard to quantize covariantly the superstring theories on $\underbrace{\text{the spacetime w. RR-flux}}_{\text{coming from D-branes}}$

2) $\mathcal{N}=4$ SYM is a superconformal gauge theory with the central charge $\frac{\dim SU(N)}{4} = \frac{N^2-1}{4}$

3) Many versions of AdS / CFT proposed

other versions may also be right, but Maldacena's version was studied precisely

Integrability (title of this lecture)

Liouville (or classical) integrability ;

The system has as many conserved charges as the degrees of freedom.

$$q_i(t) - q_i(0) = \int_0^t ds \, \underbrace{\dot{q}_i(s)}_{\text{constants of motion}}$$

Quantum integrability ;

The system (typically with a mass gap) has ∞-dim symmetry related to Yang-Baxter relation

- Integrable systems are often solved by Ansatz

(= working hypothesis)

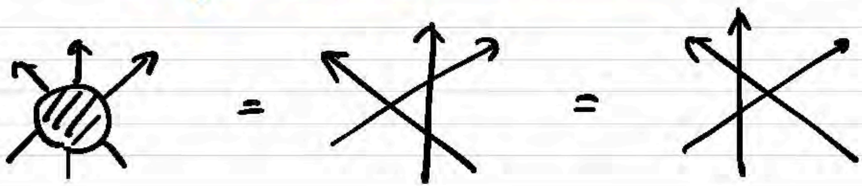
e.g. Bethe Ansatz

- Integrable QFT is often found in 1+1 dim,

whose many-body S-matrix factorizes

$$S_{12\cdots n}\,|a_1 a_2 \cdots a_n\rangle_{in} = |a_1' a_2' \cdots a_n'\rangle_{out}$$

$n=3$: Yang-Baxter relation



YB relations $\longrightarrow$ Physical observables

(like energy spectrum)

Yangian algebra $\longrightarrow$ ∞ conserved quantities

Well understood (Quantum inverse scattering)

Q: Is a given system integrable?

A: Hard question; check phase space trajectory

or level-spacing statistics (Berry-Tabor conjecture)

The (maximally supersymmetric) AdS / CFT pair is believed to be integrable at $N = \infty$

$\mathcal{N}=4$ SYM

Superstring on $AdS_5 \times S^5$

?

$\Delta_\alpha(\lambda) \longleftrightarrow E_\alpha(\lambda)$

conformal dim. of a single-trace operator

energy of a closed string

Integrable system

$\lambda \ll 1$

$E_\alpha^{(int)}(\lambda)$

$\lambda \gg 1$

Review: 1012.3982

Comments :

- Integrability → AdS / CFT dictionary

- Application to QCD

  $\mathcal{N}=4$ SYM predicts "part of" QCD data

- Why integrable ? Other integrable AdS / CFT pairs ?

  good questions, hard

- Exactly solvable $\neq$ Exactly solved

  needs explicit computation (→ Mathematica)

- I discuss partition functions, not spectral problem

$\mathcal{N}=4$ Super Yang-Mills

cf. Sohnius, Physics Reports 128 (1985) 39; D'Hoker Freedman, hep-th/0201253; Minahan, 1012.3983

Lagrangian

trivial dimensional reduction of $D=10$ $\mathcal{N}=1$ SYM $\longrightarrow$ $D=4$ $\mathcal{N}=4$ SYM

$$\mathcal{L}_{10d} = -\frac{1}{4g_{YM}^2}\,\mathrm{tr}\left(F_{MN}F^{MN}\right) + \text{fermions}$$

$$F_{MN} = [\nabla_M, \nabla_N], \qquad M,N = 0,1,\cdots,9$$

$$A_N = (A_\mu, \phi_I) \qquad \begin{array}{l} \mu = 0,1,2,3 \\ I = 4 \sim 9 \end{array}$$

Neglecting ∂_I, $\qquad F_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$

$$F_{\mu I} = -F_{I\mu} = \nabla_\mu \phi_I \,, \qquad F_{IJ} = [\phi_I, \phi_J]$$

$$\mathrm{tr}\, F_{MN}^2 = \mathrm{tr}\left(F_{\mu\nu}^2 + 2 (\nabla_\mu \phi_I)^2 + [\phi_I, \phi_J]^2 \right)$$

$$\rightsquigarrow \mathcal{L}_{4d}$$

Properties

- All fields are adjoint under $SU(N)$

$$(A_\mu)_{ij} \qquad (\Phi_I)_{ij} \qquad (\lambda_{a\alpha})_{ij} \qquad (\bar{\lambda}_{a\dot{\alpha}})_{ij}$$

$$4(N^2-1) \qquad 6(N^2-1) \qquad 4\times 2\,(N^2-1) \qquad 4\times 2\,(N^2-1)$$

- Maximally supersymmetric in 4d

after requiring $|h| \leq 2$, $h =$ helicity of fields

- Interacting 4d CFT at UV

scale invariant theory $\beta(g_{YM}) = 0$

conformal field theory $\langle T_\mu{}^\mu \rangle = 0$

$\Rightarrow$ $\mathcal{N}=4$ SYM has the global superconformal symmetry

$$psu(2,2|4) \supset \underbrace{su(2,2)}_{\text{4d conformal symm}} \times su(4)_R$$

cf. $$AdS_5 \times S^5 = \frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)} \subset \frac{PSU(2,2|4)}{USp(2,2) \times USp(4)}$$

Cartan subalgebra of $psu(2,2|4)$

$$\underbrace{\Delta}_{\text{conformal dim}} \qquad \underbrace{\underbrace{S_1 \quad S_2}_{\text{Lorentz spin}} \qquad \underbrace{J_1 \quad J_2 \quad J_3}_{\text{R-charges}}}_{}$$

non-cpt direction of AdS_5

cpt directions of $AdS_5 \times S^5$ quantized to integers

Review of Day 1

AdS/CFT correspondence

Integrability

N=4 super Yang-Mills

Superconformal symmetry PSU(2,2|4)

Plan of Day 2

Hagedorn transition in gauge theory

Hagedorn temperature in string theory
(Hopefully I have time to talk about it)

Hagedorn transition in gauge theory

We compute thermal partition function of

$\mathcal{N}=4$ SYM on $S^1 \times S^3$ at zero coupling

$$\mathbb{R}^4 \xrightarrow{\text{polar coord.}} \mathbb{R} \times S^3 \longrightarrow \underbrace{S^1}_{\text{radius } \beta} \times \underbrace{S^3}_{\substack{\text{set radius}=1 \\ \text{using conformal symmetry}}}$$

compactify radial direction as "time"

"Hamiltonian" = Dilatation D

$$Z_{S^1\times S^3} = \int DA_\mu D\phi_z D\lambda_A D\bar{\lambda}_A \, e^{-S} = \mathrm{tr}\left(e^{-\beta D}\right)$$

We should sum over "gauge invariant" states

$\longleftarrow$ Gauss law constraint on S^3

Derivation SU(N) current conservation around $g_{YM} = 0$

$$j^0 = \nabla_\mu F^{\mu 0} = \partial_k \partial^k A^0 - \underbrace{\partial_k \partial^0 A^k}_{=0 \text{ if } \partial_k A^k = 0} + O(g_{YM})$$

Integrate this over S^3

$$Q = \int_{S^3} d^3x \; \partial_k \partial^k A^0 = 0$$

(false for $\mathbb{R}^3$ due to extra charge at ∞)

- Here we say "total charge over S^3 is zero" but non-trivial SU(N) charge distribution $\rho(x)$ may exist

- It is a non-trivial question whether elementary degrees of freedom are $\underline{\text{SU(N)-singlet}}$ or $\underline{\text{adjoint}}$

$$\mathrm{tr}(\Phi_1 \Phi_2 \cdots) = \sum_{a,b,\cdots} (\Phi_1)_{ab} (\Phi_2)_{bc} \cdots$$

$$(\Phi_I)_{ab} \qquad a, b = 1, 2, \cdots, N$$

- This "color confinement" is a result of S^3 topology, and not related to strong coupling dynamics of QCD

Define free energy $Z_{S^1\times S^3} = e^{-\beta F}$ $\left(\beta = \frac{1}{T}\right)$

Expectation: 0 — T_H — ∞ T

Confined	Deconfined
$F \sim \mathcal{O}(1)$	$F \sim \mathcal{O}(N^2)$
States $\sim$ multi-trace operators	Color flux tubes

Phase transition at T_H
(Hagedorn temperature)

Compute $\mathrm{tr}(e^{-\beta D})$ in the confined phase, tree-level

Sundborg, hep-th/9908001; Polyakov, hep-th/0110196; Aharony et al. hep-th/0310285

How to count $\mathcal{O} = \mathrm{tr}(W^{A_1} W^{A_2} \cdots W^{A_L})$?

W^A : adjoint of $SU(N)$, single letter of $\mathcal{N}=4$ SYM

$$W^A \in \left\{ \begin{array}{l} D_{(\mu_1} D_{\mu_2} \cdots D_{\mu_s)} \phi^I \\ \\ D \cdots D \lambda_{a\alpha} , \quad D \cdots D \bar{\lambda}_{a\dot{\alpha}} \\ \\ D \cdots D F_{\mu\nu} \end{array} \right\}$$

Since $[D_\mu, D_\nu] = F_{\mu\nu}$, we can symmetrize $D_{\mu_1} \cdots D_{\mu_s}$

Define $\quad x \equiv e^{-\beta}$

<u>Scalars</u> $\quad \phi^I - \underbrace{D^2\phi^I}_{\text{eom for } \phi^I} \quad (\Delta_0 = 1, \ I = 1 \sim 6)$

contribute to Z by $\quad 6x - 6x^3$

Conformal descendants

$$\phi \longrightarrow D_{(\mu_1} D_{\mu_2} \cdots D_{\mu_s)} \phi \ , \quad \mu_k = 0 \sim 3$$

$$Z_{scalar} = \frac{6x - 6x^3}{(1-x)^4} \ , \quad \frac{1}{(1-x)^4} = 1 + \underset{D_\mu}{4x} + \underset{D_{(\mu}D_{\nu)}}{10x^2} + \cdots$$

Fermions

$$\lambda_{a\alpha} - \not{D}\,\lambda_{a\alpha}, \qquad \bar{\lambda}_{a\dot{\alpha}} - \not{D}\,\bar{\lambda}_{a\dot{\alpha}}$$

$(a = 1 \sim 4, \quad \alpha, \dot{\alpha} = 1, 2)$

$$Z_{\text{fermions}} = \frac{16\, x^{3/2} - 16\, x^{5/2}}{(1-x)^4}$$

note: fermions are anti-periodic by going around S^1_β.

$$\left(e^{2\pi i} x\right)^{1/2} = - x^{1/2}$$

Gauge fields

$$F_{\mu\nu} = [\nabla_\mu, \nabla_\nu] \quad (\mu, \nu = 0 \sim 3) \rightsquigarrow \frac{6x^2}{(1-x)^4}$$

Subtract equations of motion & Bianchi identities

$$\begin{cases} D^\mu F_{\mu\nu} = 0 \\ D^\mu \widetilde{F}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} D^\mu F^{\rho\sigma} = 0 \end{cases}$$

Since both eqs have one D, $\rightsquigarrow \; - \dfrac{8x^3}{(1-x)^3}$

Consider trivial identities

$$X_{\mu\nu} \equiv (D^\mu D^\nu + D^\nu D^\mu) F_{\mu\nu} = 0$$

$$\widetilde{X}_{\mu\nu} \equiv (D^\mu D^\nu + D^\nu D^\mu) \widetilde{F}_{\mu\nu} = 0$$

no sum over μ, ν

$$X_{\mu\nu} = -X_{\nu\mu} \longrightarrow 6 \text{ constraint eqs} \quad ①$$

However, we have already subtracted $D(\text{EoM})$ or

$$\sum_\mu X_{\mu\nu} = \sum_\mu (D^\mu D^\nu + D^\nu D^\mu) F_{\mu\nu} \quad (\nu = 0 \sim 3) \quad ②$$

$\rightarrow$ Doubly count 4 eqs in ①

Note that X itself is antisymmetric

$$\sum_{\mu,\nu} X_{\mu\nu} = 0$$

$\rightarrow$ Doubly count 1 eq in ②

$\therefore)$ $6 - 4 + 1 = 3$, same for $\tilde{X}_{\mu\nu} \longrightarrow \dfrac{-6x^4}{(1-x)^4}$

$$Z_{gauge} = \frac{6x^2}{(1-x)^4} - \frac{8x^3}{(1-x)^3} - \frac{6x^4}{(1-x)^4}$$

Now define the single letter function

$$\zeta(x) = \underbrace{\zeta_B(x)}_{\text{scalar + gauge}} + \underbrace{\zeta_F(x)}_{\text{fermions}}$$

$$= \frac{2x(3-\sqrt{x})}{(1-\sqrt{x})^3}$$

$$= \underset{\phi^I}{6x} + \underset{\lambda_a,\ \bar{\lambda}_{\dot{a}}}{16x^{3/2}} + \underset{D_\mu\phi^I,\ F_{\mu\nu}}{(24+6)x^2} + \cdots$$

The $\mathcal{N}=4$ SYM partition fn on $S^1_\beta \times S^3$ at $g_{YM} \sim 0$

$\sim$ Partition fn of multi-trace operators

$=$ Plethystic exponential of $\underbrace{\text{single-trace partitions}}$

$=$ Polya counting of single letter function

- We can apply combinatorial theory (for zero chem. pot)
- Partition fn $\neq$ Superconformal index (SUSY-protected)

Pólya Enumeration Theorem

Domain $= \{1, 2, \cdots, L\}$, Range $= \{Z, Y\}$

Single-trace op

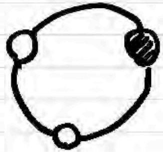
$$\mathrm{tr}\,(\underbrace{ZZ\cdots YY\cdots}_{L}) \longleftrightarrow \mathrm{Map}\left(\mathbb{Z}_L \backslash \mathrm{Domain} \rightarrow \mathrm{Range}\right)$$

weight function $c(x, y) = x + y$, such that

$x^m y^n$ correspond to $Z^m Y^n$ $(m+n=L)$

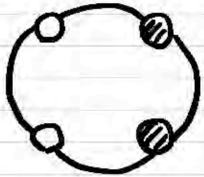
In our setup $c(x, y) \longrightarrow \zeta(x)$ with $x = e^{-\beta}$

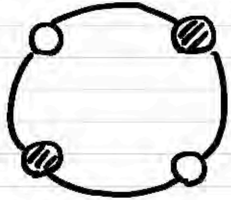
Examples $\quad z^2 W$



same for zW^2

$z^2 W^2$





$$Z = 1 + (x+y) + (x^2 + xy + y^2)$$
$$+ (x^3 + x^2 y + x y^2 + y^3)$$
$$+ (x^4 + x^3 y + 2x^2y^2 + xy^3 + y^4) + \cdots$$

The generating fn for the number of single-trace ops made of (Z, Y) is

$$Z^{\text{Polya}}(x,y) = \sum_{L=1} \underbrace{\frac{1}{L} \sum_{h|L} \text{Tot}(h)\, c(x^h, y^h)^{L/h}}_{\text{cycle index for } \mathbb{Z}_L}$$

$$\text{Tot}(h) = \text{Euler Totient's function}$$

$$= \sum_{d=1}^{h} \delta(\gcd(h,d), 1) \quad (= 1, 1, 2, 2, 4, \dots)$$

$$\Rightarrow Z^{\text{Polya}}(x,y) = -\sum_{h=1}^{\infty} \frac{\text{Tot}(h)}{h} \log\left(1 - c(x^h, y^h)\right)$$

In $\mathcal{N}=4$ SYM,

$$Z_{\text{single}}(x) = \underbrace{-z(x)}_{\text{tr}\,W=0 \text{ for } SU(N)} - \sum_{n=1}^{\infty} \frac{\text{Tot}(n)}{n} \log\left[1 - z(\underbrace{(-1)^{n+1} x^n}_{})\right]$$

fermions are anti-periodic by going around S^1_β

($(-1)^n$ or $(-1)^{n+1}$ is non-trivial)

$$= 21 x^2 + 96 x^{5/2} + 376 x^3 + \cdots$$

$\text{tr}(\phi^I \phi^J)$	$\text{tr}(\phi^I \lambda_a)$	$\text{tr}(\phi^I \phi^J \phi^K)$ 76
	$\text{tr}(\phi^I \bar{\lambda}_a)$	$\text{tr}(\phi^I D_\mu \phi^J)$ 144
		$\text{tr}(\phi^I F_{\mu\nu})$ 36

Multi-trace partition function is

$$Z_{multi}(x) = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} Z_{single}(x^m) \right)$$

$$= \exp\left(- \sum_{m} \frac{1}{m} z(x^m)\right) \cdot \prod_{m=1}^{\infty} \frac{1}{1-z(x^m)}$$

$$\leftrightarrow \quad \mathrm{tr}(e^{-\beta D})$$

Z_{single} and Z_{multi} diverges if $z(x_H) = 1$.

$$x_H = \underline{7 \pm 4\sqrt{3}} = \{0.07 \,,\ 13.9\}$$

$-$ is correct because $e^{-\beta} < 1$ for $\beta > 0$

Further references

Spradling, Volovich, hep-th/0408178
Bianchi, hep-th/0409304
Yaffe, Yamada, hep-th/0602074
Bianchi et al., hep-th/0609179

Hagedorn temperature in string theory

Sundborg, Nucl. Phys. B254 (1985) 583

$$\mathrm{tr}\,(x^D) \sim \sum_m C_{m/2}\, x^{m/2}\,, \qquad x \equiv e^{-\beta} < 1$$

Naively large m terms are small because $x < 1$

However, if the number of states grows exponentially

$C_{m/2} \sim \exp\left(\alpha \frac{m}{2}\right)$ with $\alpha \geq \beta$, then

$$\mathrm{tr}\,(x^D) \sim \sum_m e^{\frac{m}{2}(\alpha-\beta)} \to \infty$$

Hagedorn behavior

Spacetime (not worldsheet) partition function of

IIB superstring theory on $S^1_\beta \times \mathbb{R}^9$ (at $g_s = 0$)

$\simeq$ Partition fn of supergravity on $S^1_\beta \times \mathbb{R}^9$ with

∞ many particles with mass $\alpha' m^2 = 0, 4, 8, \cdots$

$$Z = \int \prod_n [d\phi_n \, d\psi_n] \, e^{-\sum_n S[\phi_n, \psi_n]}$$

$$\ln Z = \int dm \, \underbrace{\rho(m)}_{\text{density of states w. mass } m} \int \frac{d^9 k}{\sqrt{k^2+m^2}} \ln\left(\frac{1+e^{-\beta\sqrt{k^2+m^2}}}{1-e^{-\beta\sqrt{k^2+m^2}}} \right)$$

Laine, Vuorinen, 1701.01554

Density of states from string

usual CFT techniques $\quad |0\rangle, \ \alpha_{-1}^{\mu}|0\rangle, \ \psi_{-1}^{\mu}|0\rangle,$

$\alpha_{-1}^{\mu}\alpha_{-1}^{\nu}|0\rangle, \ \alpha_{-2}^{\mu}|0\rangle, \ \cdots$

in NS-NS sector

Bosons $\quad [\alpha_m^{\mu}, \alpha_n^{\nu}] = m\,\eta^{\mu\nu}\,\delta_{m+n,0} \quad$ + right mover

Fermions $\quad \{\psi_m^{\mu}, \psi_n^{\nu}\} = \eta^{\mu\nu}\,\delta_{m+n,0} \quad$ + right mover

mass $\quad \alpha' m^2 = 4N_{tot} = 4\left(N_B + \bar{N}_B + N_F + \bar{N}_F\right)$

We are interested in $N_{tot} \gg 1$

Bosons

$$(\alpha_{-m})^{p}(\bar{\alpha}_{-n})^{q}|0\rangle \quad \text{for all} \quad p, q \in \mathbb{Z}_{\geq 0}$$

$$\sum_{p}\sum_{q} x^{mp}\,\bar{x}^{nq} = \frac{1}{1-x^{m}}\,\frac{1}{1-\bar{x}^{n}} \qquad (m, n \in \mathbb{Z}_{\geq 1})$$

- We impose the level matching conditions later
- There are 8 physical directions α^{μ}_{-m} $(\mu = 2\sim 9)$

Fermions

$$(\psi_{-m})^{p}(\bar{\psi}_{-n})^{q}|0\rangle, \qquad p, q = 0, 1$$

$$\sum_{p}\sum_{q} x^{mp}\,\bar{x}^{nq} = (1+x^{m})(1+\bar{x}^{n})$$

Summing over $8_B + 8_F$ directions

$$\mathrm{tr}\left(x^N \bar{x}^{\bar{N}}\right) = \prod_{m=1}^{\infty} \left(\frac{1+x^m}{1-x^m}\right)^8 \left(\frac{1+\bar{x}^m}{1-\bar{x}^m}\right)^8$$

$$\equiv \sum_{n,\bar{n}} d_{n,\bar{n}}\, x^n \bar{x}^{\bar{n}}$$

Impose level matching $(n = \bar{n})$

$$\Pi(x) = \sum_N d_{N,N}\, x^{2N} = \frac{1}{\vartheta_4(0|\tau)^{16}}$$

Elliptic Theta, $x = e^{\pi i \tau}$

The physical partition function is

$$\sim \Pi_{NS,NS} + \underbrace{\Pi_{NS,R} + \Pi_{R,NS} + \Pi_{R,R}}_{\text{not important for computing } P(m)}$$

$$d_{N,N} = \oint_{z=0} \frac{dz}{2\pi i} \frac{\Pi(z)}{z^{2N+1}} \quad , \quad \Pi(z) \text{ is singular at } z=1$$

$$\Pi(z) \sim \left(\frac{1-z}{4\pi} \right)^{\gamma} \exp\left(2\pi^2 \frac{1+z}{1-z} \right)$$

The saddle point is shifted as $z \sim 1 - \pi \sqrt{\frac{2}{N}}$

Thus

$$d_{N,N} \sim \frac{e^{4\pi\sqrt{2N}}}{4096\, N^4} \quad , \quad \text{using} \quad \alpha' m^2 = 4N_{tot}$$

$$\rho(m) \sim \frac{e^{m\sqrt{8\pi^2\alpha'}}}{16\,(\alpha' m^2)^4}$$

Substitute this result into the spacetime partition fn.

$$\ln Z = \int dm\, \rho(m) \underbrace{\int \frac{d^9 k}{\sqrt{k^2+m^2}} \ln\left(\frac{1+e^{-\beta\sqrt{k^2+m^2}}}{1-e^{-\beta\sqrt{k^2+m^2}}} \right)}_{= \mathrm{vol}(S^9) \sum_{n:\,\mathrm{odd}} \frac{2}{n} \underbrace{\int \frac{dk \cdot k^8}{\sqrt{k^2+m^2}} e^{-n\beta\sqrt{k^2+m^2}}}_{\left(\frac{m}{n}\right)^4 K_4(n\beta m)}}$$

modified Bessel fn. K_ν

$$\ln Z \sim \sum_n \int dm\, \rho(m)\, K_4(n\beta m)$$

$$\sim \sum_n \int_0^\infty dm\, \exp\left[m\sqrt{8\pi^2\alpha'} - mn\beta \right]$$

Diverges at $n=1$ if $\beta \leq \beta_H = \sqrt{8\pi^2\alpha'}$; Hagedorn temperature

Review of Day 2

Hagedorn transition in N=4 SYM

Hagedorn temperature in superstring

Plan of Day 3

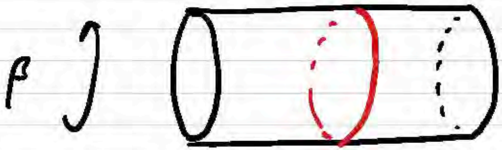
Thermodynamics of AdS gravity

Integrability in N=4 SYM

Reply to questions in Day 2

Atick, Witten, Nucl. Phys. B310 (1988) 291
Kruczenski, Lawrence, hep-th/0508148

Hagedorn transition in string theory



closed string winding modes around thermal circle become tachyonic

closed string tachyon condensation

$\rightarrow$ deforms background spacetime

Consistent with Hawking-Page transition (AdS $\rightarrow$ AdS-BH)

Thermodynamics of AdS gravity

References:
Hawking, Page, Comm. Math. Phys. 87 (1983)
Witten, hep-th/9803131

Reviews:
Natsuume, 1409.3575
Hartman, Lectures on Quantum Gravity and Black Holes

$AdS_5 \times S^5$ superstring

$\alpha' \to 0$
neglect massive string modes

$AdS_5 \times S^5$ IIB supergravity

neglect towers of KK modes
neglect fermions

AdS_5 gravity

Gravity action

$$S = -\frac{1}{16\pi G_N}\int d^5x \sqrt{-g}\,(R - 2\Lambda)$$

$$\Lambda = -\frac{6}{L^2} \quad \text{for } AdS_5 \qquad (L : \text{AdS radius})$$

EoM : $$R_{\mu\nu} - \underbrace{\frac{R}{2} g_{\mu\nu}}_{} + \Lambda g_{\mu\nu} = 0$$

Einstein spent ~10 years to find this term (?)

AdS is non-cpt ⇒ Need to regularize S by adding various boundary (& boundary counter) terms

Black hole temperature (conical defect trick)

Write the BH metric

$$ds^2 = -V dt^2 + \frac{dr^2}{V} + r^2 d\Omega_3$$

$g_{tt} = 0$ or $V(r_0) = 0$ is the event horizon

Introduce imaginary time $t = i\tau$, and expand the metric $r = r_0 + \epsilon$, $\epsilon \ll 1$

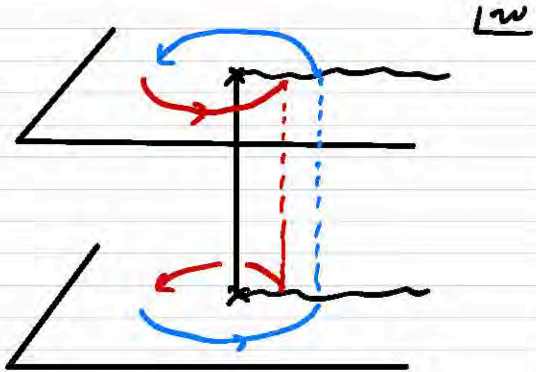
$$ds^2 \sim \epsilon V'(r_0)\, d\tau^2 + \frac{d\epsilon^2}{\epsilon V'(r_0)} + \cdots$$

Requires that $ds^2 = dR^2 + R^2 d\Theta^2 + \cdots$

$$= dz\, d\bar{z} + \cdots \ , \quad z = Re^{i\Theta}$$

No conical singularity at $z=0$ if $\Theta \sim \Theta + 2\pi$

$z = \sqrt{w}$



The branch point "x" has curvature in GR

(also in CFT "defect operator")

$$ds^2 = \epsilon V' d\tau^2 + \frac{d\epsilon^2}{\epsilon V'} + \cdots = R^2 d\Theta^2 + dR^2 + \cdots$$

$$\frac{d\epsilon}{dR} = \sqrt{\epsilon V'}\ , \quad \epsilon(R=0) = 0 \quad \Rightarrow \quad \epsilon = \frac{R^2 V'}{4}$$

then $$ds^2 = \left(\frac{RV'}{2}\right)^2 d\tau^2 + dR^2 + \cdots$$

$d\tau = \beta = \dfrac{1}{T}$ corresponds to $d\Theta = 2\pi$

$\Rightarrow \quad T = \dfrac{V'}{4\pi}$ (Bekenstein - Hawking temperature)

CFT_4 on $S^1_\beta \times S^3$

$\longleftrightarrow$ AdS_5 BH with spherical (not planar) horizon

$$d\tilde{s}^2 = -f\,dt^2 + \frac{dr^2}{f} + \frac{r^2}{L^2}\sum dx_i^2 \qquad \text{(planar)}$$

$$f = \frac{r^2}{L^2}\left(1 - \frac{r_0^4}{r^4}\right)$$

- $$ds^2 = -V\,dt^2 + \frac{dr^2}{V} + r^2\,d\Omega_3 \qquad \text{(spherical)}$$

$$V = 1 + \frac{r^2}{L^2} - \frac{\mu}{r^2} \qquad \mu \sim \text{BH mass}$$

$d\hat{s}^2$ is invariant under the rescaling

$$(x_i, r, r_0) \longrightarrow (a x_i, a^{-1} r, a^{-1} r_0)$$

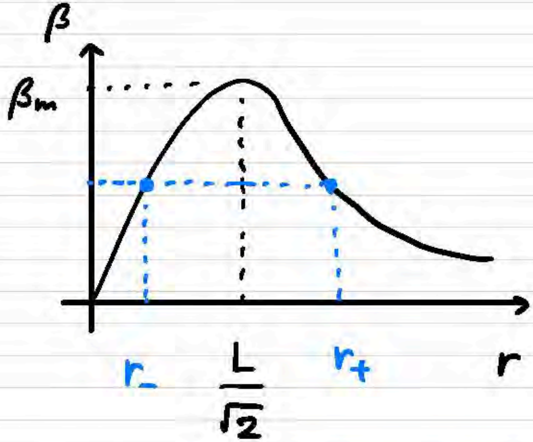
corresponding to the scale invariance in CFT_4

ds^2 is <u>not</u> invariant under the rescaling

$\rightsquigarrow$ possible phase transition at a finite β

conical defect trick $\Rightarrow$

$$\beta = \frac{4\pi}{V'} = \frac{2\pi r_+ L^2}{L^2 + 2r_+^2}$$



$$\beta_m = \frac{\pi L}{\sqrt{2}}$$

minimum BH temp.

For $\beta < \beta_m$, small BH (r_-) and large BH (r_+) exist

Large BH ~ dominant contribution to $Z = e^{-S}$

new object in AdS (specific heat > 0)

Small BH ~ similar to flat space BH

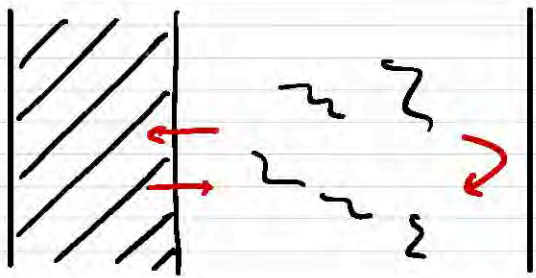
(specific heat < 0)

Physically AdS is a finite-size box

BH and thermal radiation → equilibrium

BH free energy <u>increases</u> by absorbing radiation

Radiation reflects back by AdS boundary



On-shell Euclidean Action

Compute $Z = e^{-S} \equiv e^{-\beta F}$ for

various solutions $\begin{cases} \text{small / large BH,} \\ \text{thermal AdS if } \mu \sim r_+^2 \to 0 \end{cases}$

Bulk : Einstein - Hilbert term (cutoff at $r = \frac{1}{\epsilon} \gg 1$)

Boundary : Gibbons - Hawking - York term $\left(\text{from } \delta g_{\mu\nu} \Big|_{\frac{1}{\epsilon}} = 0 \right)$

AdS counter terms

"holographic renormalization"

$$I = -\frac{1}{16\pi G_N} \int d^5x \sqrt{g} \left(R(g) - 2\Lambda \right)$$ EH

$$+ \frac{1}{8\pi G_N} \int_{r=1/\epsilon} d^4x \sqrt{h} \left(K_{AdS} - K_{flat} \right)$$ GHY

$$+ \frac{1}{8\pi G_N} \int_{r=1/\epsilon} d^4x \sqrt{h} \left(\frac{3}{L} + \frac{L}{4} R(h) \right)$$ c.t.

h_{ab} : metric on the horizon ∂M at $r = r_*$

K : extrinsic curvature on ∂M (n_ν : normal to ∂M)

$$2K = h^{\mu\nu} \nabla_\mu n_\nu = h^{\mu\nu} \partial_r h_{\mu\nu} = \frac{\partial_r \sqrt{h}}{\sqrt{h}}$$

By substituting the metric

$$I_{AdS-Sch} = \frac{\pi \beta}{32 G_N L^2} \left\{ 3L^4 + 4L^2 r_+^2 - 4r_+^4 \right\}$$

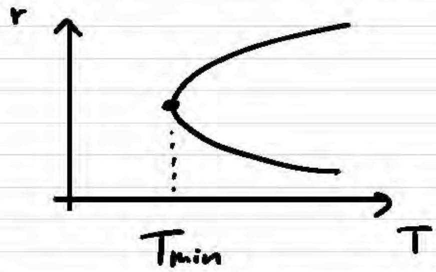
$$\mu = \left(\frac{r_+}{L}\right)^2 (L^2 + r_+^2)$$

large r_+ $\Rightarrow$ $Z = e^{-I}$ becomes large

Compare $I_{AdS-Sch}$ and I_{th} (thermal AdS, $r_+ = 0$)

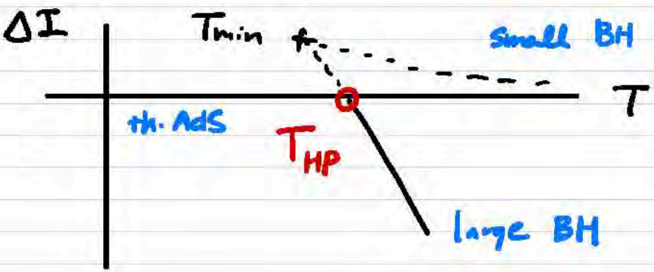
$$\Delta I \equiv I_{AdS-Sch} - I_{th} = \frac{\pi \beta}{8 G_N} \left(\frac{r_+}{L}\right)^2 (L^2 - r_+^2)$$

Recall $\quad \beta_{BH} = \dfrac{2\pi r_+ L^2}{L^2 + 2r_+^2}$



$$T_{min} = \frac{\sqrt{2}}{\pi L}$$

Express ΔI as a function of T



$$\Delta I(r_+ = L) = 0$$

$$T_{HP} = \frac{3}{2\pi L}$$

$T < T_{HP}$: thermal AdS is stable

$T > T_{HP}$: large BH is stable (Hawking - Page transition)

Comment

Hagedorn temp. of flat superstring, $T_H = \frac{1}{2\sqrt{2}\,\alpha' \pi}$

Hawking - Page temperature, $T_{HP} = \frac{3}{2\pi L}$

$$\left(\frac{T_H}{T_{HP}}\right)^2 = \frac{L^2}{18\alpha'} = \frac{\sqrt{\lambda}}{18}$$

If $\sqrt{\lambda} < 18$, $T_H < T_{HP}$

massive string modes contribute to BH transition

more than gravitons (massless string modes)

Half BPS states in $\mathcal{N}=4$ SYM

Introduce $SU(3)$ notation

$$Z = \phi^5 + i\phi^6 , \quad \bar{Z} = \phi^5 - i\phi^6 , \quad Y = \phi^3 + i\phi^4$$

$$\bar{Y} = \phi^3 - i\phi^4 , \quad X = \phi^1 + i\phi^2 , \quad \bar{X} = \phi^1 - i\phi^2$$

$SU(N)$ Wick rule, with $i, j, k, l = 1, 2, \cdots, N$

$$(\phi^I)_i{}^j \, (\phi^J)_k{}^l = \delta^{IJ} \left(\delta_i^l \delta_k^j - \frac{\delta_i^j \delta_k^l}{N} \right)$$

$$XX = YY = ZZ = 0 \quad \text{but} \quad Z\bar{Z} \neq 0$$

Half-BPS $\sim$ $[0, L, 0]$ irrep of $SU(4)_R$

symmetric traceless scalars

$$\mathcal{O}^{IJ} = \mathrm{tr}(\phi^I \phi^J) - \frac{\delta^{IJ}}{6} \mathrm{tr}(\phi^k \phi^k)$$

$$\underbrace{\mathcal{O}^{ZZ} = \mathrm{tr}\, Z^2 , \quad \mathcal{O}^{ZZ\cdots Z} = \mathrm{tr}\, Z^L}_{\text{HWS (highest weight states) of } SU(4)_R}$$

$$su(4) \ni j \sim \begin{pmatrix} X\partial_X & Y\partial_X & Z\partial_X \\ X\partial_Y & Y\partial_Y & Z\partial_Y \\ X\partial_Z & Y\partial_Z & Z\partial_Z \end{pmatrix}$$

Annihilates $\mathrm{tr}\, Z^L$

Introduce Γ-matrices for SUSY transformations

4d : $\gamma_\mu = \begin{pmatrix} & (\sigma_\mu)_{\alpha\dot\beta} \\ (\bar\sigma_\mu)_{\dot\alpha\beta} & \end{pmatrix}$ $\qquad \alpha, \beta, \dot\alpha, \dot\beta = 1, 2$

$\mu = 0 \sim 3$

6d : $(\gamma_I)_{ab} = -(\gamma_I)_{ba}$ $\qquad a, b = 1, 2, 3, 4$

$I = 1 \sim 6$

$$\gamma_{\mu\nu} \equiv [\gamma_\mu, \gamma_\nu] = \begin{pmatrix} (\sigma_{\mu\nu})_{\alpha\beta} & \\ & (\bar\sigma_{\mu\nu})_{\dot\alpha\dot\beta} \end{pmatrix}$$

Then $\quad F_{\mu\nu} \to F^+_{\alpha\beta} = F^+_{\beta\alpha}, \quad F^-_{\dot\alpha\dot\beta} = F^-_{\dot\beta\dot\alpha} \qquad \underline{3 \oplus 3}$

$\qquad \phi_I \to \phi_{ab} = -\phi_{ab} \qquad \underline{6}$

$\mathcal{N}=4$ supercharges $\quad Q_{a\alpha}, \quad \tilde{Q}^{b}_{\dot\alpha} \quad (a,b=1\sim4)$

$[0,2,0] \quad \mathcal{O}^{abcd} = \mathrm{tr}(\phi^{ab}\phi^{cd}) - \dfrac{\epsilon^{abcd}\epsilon_{efgh}}{24}\mathrm{tr}(\phi^{ef}\phi^{gh})$

Then

$$Q_{a\alpha}\,\phi_{bc} = \epsilon_{abcd}\,\lambda^{d}_{\alpha}$$

$(\phi_{12}, \phi_{34}) = (\bar{Z}, Z) \Rightarrow \quad Q_{1\alpha}\bar{Z} = Q_{2\alpha}\bar{Z} = 0$ BPS

Also, $\begin{cases} Q_{a\alpha}Q_{b\beta}Q_{c\gamma}Q_{d\delta}\,\mathcal{O}^{abcd} \sim F^{+}_{\alpha\beta}F^{+}_{\gamma\delta} \\ \bar{Q}\;\bar{Q}\;\bar{Q}\;\bar{Q}\;\mathcal{O}^{abcd} \sim F^{-}_{\dot\alpha\dot\beta}F^{-}_{\dot\gamma\dot\delta} \end{cases}$

$\mathrm{tr}\,F_{\mu\nu}^{2} \in \mathcal{N}=4$ Lagrangian $\Rightarrow g_{\mathrm{YM}}$ is SUSY protected.

Spin Chain

Operators close to $\mathrm{tr}\, Z^L$;

$$\mathrm{tr}(Z^{L-2}\chi^2) \sim \mathrm{tr}(ZZ \cdots Z\chi Z \cdots Z\chi Z \cdots ZZ)$$

χ : any single letter, adjoint rep of $SU(N)$

These operators are generally non-BPS, and mix under renormalization

$\rightarrow$ Need to diagonalize quantum dilatation operator

OR anomalous dimension matrix

$$\Delta \mathcal{O}_a = M_{ab}\,\mathcal{O}_b \rightarrow \Delta \mathcal{O}_\alpha = \gamma_\alpha \mathcal{O}_\alpha$$

Case of $\mathcal{X} = Y$

1-loop dilatation : $\Delta_1 = \dfrac{\lambda}{8\pi^2} \operatorname{tr}\left([Y, Z][\partial_Y, \partial_Z] \right)$

Δ_1 interchanges Y and Z

$$\cdots\ Y\ \ Z\ \cdots$$

$$\cdots\ Z\ \ Y\ \cdots$$

$\longrightarrow$ Identical to the spectral problem of 1D spin chain

$(Z, Y) \longleftrightarrow (\uparrow, \downarrow)$

$\operatorname{tr}(ZZ\cdots YY\cdots) \longleftrightarrow |\uparrow\uparrow\cdots\downarrow\downarrow\cdots\rangle$ with PBC

$\Delta_{1\text{-loop}} \longleftrightarrow (\lambda/8\pi^2)\, \mathcal{H}_{XXX}$

The XXX Hamiltonian

$$\mathcal{H}_{XXX} = \sum_{k=1}^{L} \left(I_{k,k+1} - P_{k,k+1} \right)$$

$$I_{k,k+1} \,|\cdots a_k \; b_{k+1} \rangle = |\cdots a_k \; b_{k+1} \cdots \rangle$$

$$P_{k,k+1} \,|\cdots a_k \; b_{k+1} \rangle = |\cdots b_k \; a_{k+1} \cdots \rangle$$

$$(a_k, b_k) \in (\uparrow, \downarrow)$$

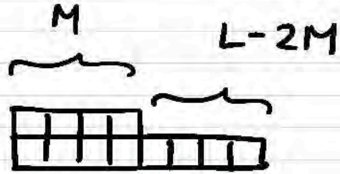
If we introduce 2x2 Pauli matrices $\{\sigma_k^+, \sigma_k^-, \sigma_k^z\}$

$$\mathcal{H}_{XXX} = \frac{1}{2} \sum_{k=1}^{L} \left(I_{k,k+1} - \vec{\sigma}_k \cdot \vec{\sigma}_{k+1} \right)$$

XXX model is integrable, and its energy spectrum is given by solving Bethe Ansatz Equations

cf. Faddeev, hep-th/9605187; Doikou et al., 0912.3350

$$\begin{cases} \mathcal{H}_{XXX}\, \Psi_\alpha = E_\alpha \Psi_\alpha \\ \Psi_\alpha \text{ is HWS of } su(2) \text{ for} \end{cases}$$



$\updownarrow$ "equivalence" is tricky (Mathematica)

$$\begin{cases} E = \sum_{j=1}^{M} \epsilon(u_j), \qquad \epsilon(u) = \dfrac{1}{4u^2+1} \\ \left(\dfrac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k \neq j}^{M} \dfrac{u_j - u_k + i}{u_j - u_k - i} \end{cases}$$

BAE for $XXX_{1/2}$

Review of Day 3

Thermodynamics of AdS gravity

Integrability in N=4 SYM

Plan of Day 4

All-loop asymptotic spin chain

TBA for Hagedorn temperature

Free energy (if I have an extra time)

$\mathcal{N}=4$ SYM has all-loop integrable structure.

The integrability method typically gives the prediction:

$$\boxed{(\text{Observable}) = (\text{asymptotic}) + (\text{wrapping})}$$

(Observable)	(asymptotic)	(wrapping)
2pt, 3pt, gluon scattering, ...	mostly determined by symmetry	∞ sum over virtual particles

"asymptotic" $\longleftrightarrow$ ∞ long background

e.g. $L \to \infty$ in $\mathrm{tr}\, Z^L$

"wrapping" $\longleftrightarrow$ finite size corrections

This structure is valid at <u>any λ</u> (and $N=\infty$)

Asymptotic spin chain

Beisert, hep-th/0511082, nlin/0610017
Komatsu, Wang, 2004.09514

$\mathcal{N}=4$ SYM exhibits perturbative integrability

(also superstring on $AdS_5 \times S^5$ is classically integrable)

$\wr$

All-loop integrability (at large N)

The idea is to take the limit $L \to \infty$ in $\mathrm{tr}\, Z^L$

~~Periodicity~~ $\longrightarrow$ central extension of global symmetry

$\cap$

Part of ∞-dim algebra (Yangian)

$$\mathcal{X} \in \begin{pmatrix} D_{\alpha\dot{\alpha}} Z & \lambda_{a\alpha} \\ \bar{\lambda}_{a\dot{\alpha}} & \underbrace{\Phi_{ab}}_{X,\,Y,\,\bar{X},\,\bar{Y}} \end{pmatrix}$$

$a, b, \alpha, \dot{\alpha} = 1, 2$

$$(\phi_a \,|\, \psi_\alpha)_L \otimes (\bar{\phi}_b \,|\, \bar{\psi}_{\dot{\beta}})_R \sim \mathcal{X}$$

$$(2|2)_L \otimes (2|2)_R \sim 8_B \oplus 8_F$$

- $\mathcal{X}$ satisfies $\Delta_0 - J_3 = 1$
- Other letters (eg. $\bar{Z}$, $D\Phi$, $D\lambda$) have $\Delta_0 - J_3 = 2$

$\to$ bound states of the asymptotic spin chain

Algebra $\quad psu(2|2) \ltimes \mathbb{R}$

$$J \in \{ \underbrace{R^a{}_b ,\ L^\alpha{}_\beta}_{su(2)^2 \text{ rotation}} ,\ \underbrace{Q^\alpha{}_a}_{\text{super}} ,\ \underbrace{S^b{}_\beta}_{\text{superconformal}} ,\ \underline{\hat{C}} \}$$

center $(\Delta - J_3)$

satisfying

$$\{ Q^\alpha{}_a , S^b{}_\beta \} = \delta^\alpha_\beta R^b{}_a + \delta^b_a L^\alpha{}_\beta + \delta^b_a \delta^\alpha_\beta \hat{C}$$

Central extension : $\quad psu(2|2) \ltimes \mathbb{R}^3$

$$\{ Q^\alpha{}_a , Q^\beta{}_b \} = \epsilon^{\alpha\beta} \epsilon_{ab} \hat{P}$$

$$\{ S^a{}_\alpha , S^b{}_\beta \} = \epsilon^{ab} \epsilon_{\alpha\beta} \hat{K}$$

One-particle states

$$|\chi(p)\rangle = \sum e^{ipn} \underbrace{|ZZ \cdots \overset{n}{\chi} \cdots ZZ\rangle}_{L \text{ fields}}$$

$|ABC\cdots\rangle$ means $\cancel{\mathrm{tr}}(ABC\cdots)$, forgetting trace cyclicity

In the limit $L \to \infty$, this state is invariant (up to const) under the addition / removal of Z's. Then, the additional centers $\hat{P}, \hat{K}$ can be non-trivial on $|\chi(p)\rangle$

Caution: In $\mathcal{N}=4$ SYM $\mathrm{tr}(ZZ\cdots\chi\cdots ZZ)$ is BPS descendant

"one-particle state" is a mathematical building block.

Write $|\mathcal{X}(p)\rangle = |\mathcal{X}_L\rangle \otimes |\mathcal{X}_R\rangle$, $\mathcal{X}_{L,R} \in (2|2)$

The charges $\{Q, S\}$ act on $|\mathcal{X}_L\rangle$ as (same for $\mathcal{X}_R$)

$Q^{\alpha}_{\ a}|\phi^b\rangle = A\,\delta^b_a\,|\psi^\alpha\rangle$, $\quad Q^{\alpha}_{\ a}|\psi^\beta\rangle = B\,\epsilon^{\alpha\beta}\epsilon_{ab}\,|Z^+\phi^b\rangle$

$S^{a}_{\ \alpha}|\phi^b\rangle = C\,\epsilon^{ab}\epsilon_{\alpha\beta}\,|Z^-\psi^\beta\rangle$, $\quad S^{a}_{\ \alpha}|\psi^\beta\rangle = D\,\delta^\beta_\alpha\,|\phi^a\rangle$

Commutation relations of $psu(2|2) \ltimes \mathbb{R}^3$ give

$$\hat{C}|\mathcal{X}\rangle = \frac{AD+BC}{2}|\mathcal{X}\rangle, \quad \underbrace{\hat{P}}_{=\{Q,Q\}}|\mathcal{X}\rangle = AB\,|\mathcal{X}\rangle, \quad \underbrace{\hat{K}}_{=\{S,S\}}|\mathcal{X}\rangle = CD\,|\mathcal{X}\rangle$$

$$\underbrace{\&\quad 1 = AD - BC}_{=\{Q,S\}}$$

Based on perturbative data, introduce variables

$$A = \sqrt{f}\,\gamma\,, \qquad B = \frac{\sqrt{f}}{\gamma}\left(1 - \frac{x^+}{x^-}\right)$$

$$C = \frac{i\sqrt{f}\,\gamma}{x^+}\,, \qquad D = \frac{\sqrt{f}\,x^+}{i\gamma}\left(1 - \frac{x^-}{x^+}\right)$$

$$AD - BC = 1 \iff x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{f}$$

We can solve it by Zhukovski map $x = x(u)$,

$$x(u) + \frac{1}{x(u)} = u\,, \qquad x^{\pm}(u) = x\left(u \pm \frac{i}{2f}\right)$$

With $\frac{x^+}{x^-} \equiv e^{ip}$ (from $x(p)$), the center becomes

$$2\hat{C} = \frac{1+\frac{1}{x^+x^-}}{1-\frac{1}{x^+x^-}} = \sqrt{1+\underline{\underline{4f^2}}\sin^2\frac{p}{2}}$$

($= \Delta - J_3 = E(p)$, elliptic dispersion relation w. mass $=1$)

f is an arbitrary function of the 't Hooft coupling

In $\mathcal{N}=4$ SYM, $4f^2 = \frac{\lambda}{\pi^2}$ $(\equiv 4g^2)$

In ABJM, comparison w. BPS Wilson lines (localization) fixes this function

Gromov, Sizov, 1403.1894

Two-particle states

$$|\chi_a(p_1)\,\chi_b(p_2)\rangle$$

$$= \sum_{n_1 \ll n_2} \Big(e^{ip_1 n_1 + ip_2 n_2} \,|Z \cdots Z \overset{n_1}{\overset{\vee}{\chi_a}} Z \cdots Z \overset{n_2}{\overset{\vee}{\chi_b}} Z \cdots \rangle$$

$$+ e^{ip_1 n_2 + ip_2 n_1} \underbrace{S_{ab}^{cd}(p_1, p_2)}_{} |Z \cdots Z \chi_c Z \cdots Z \chi_d Z \cdots \rangle \Big)$$

$+ (n_1 \sim n_2)$ this is 2-particle S-matrix

2-particle states form 16^2-dimensional (reducible) representation of $psu(2|2)^2 \ltimes \mathbb{R}^3$

- We impose that the S-matrix commutes with all generators of $psu(2|2)^2 \ltimes \mathbb{R}^3$

  $\rightsquigarrow$ $S_{ab}{}^{cd}(p_1, p_2)$ determined up to overall factor (dressing factor)

- This $psu(2|2)^2$-invariant S-matrix satisfies Yang-Baxter relations, unitarity and crossing

- The entire algebra is part of Yangian of $psu(2|2)$

Arutyunov, Frolov, 0710.1568; Beisert, 0704.0400

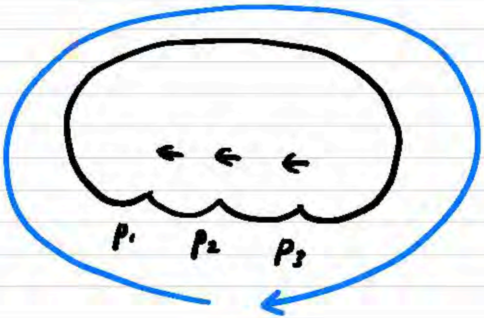
Asymptotic Bethe Ansatz Equations

$$\Delta - J_3 = \sum_{k=1}^{M} \sqrt{1 + f(g) \underbrace{\sin^2 \frac{p_k}{2}}}$$

momentum is determined by BAE

Bethe Ansatz ~ Periodic boundary conditions

→ Now $L (= J_3)$ is large but finite, neglect finite L corrections



We go around the spin chain

$$\text{PBC} \rightarrow \text{“} 1 = e^{-ip_j L} \prod_{k \neq j} S(p_j, p_k) \text{”}$$

Explicit form of $psu(2|2)^2$ BAE is complicated

$$x_s(u) \equiv \frac{u}{2}\left(1+\sqrt{1-\frac{4}{u^2}}\right),$$

$$x_k^{\pm} \equiv x_s\left(u_k \pm \frac{i}{g}\right), \qquad g = \frac{\sqrt{\lambda}}{2\pi} \quad (= \text{string tension})$$

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{l \neq k}^{K^{I}} S_0(u_k, u_l)\, \frac{x_k^+ - x_l^-}{x_k^- - x_l^+} \sqrt{\frac{x_l^+ x_k^-}{x_l^- x_k^+}} \times$$

$$\prod_{\alpha=L,R} \prod_{m=1}^{K_\alpha^{II}} \frac{x_k^- - y_m^{(\alpha)}}{x_k^+ - y_m^{(\alpha)}} \sqrt{\frac{x_k^+}{x_k^-}}$$

+ 4 more equations for auxiliary Bethe roots $(y^{(\alpha)}, w^{(\alpha)})$

Dressing factor

$$S(x_1^{\pm}, x_2^{\pm}) = \underbrace{S_0(x_1^{\pm}, x_2^{\pm})}_{\text{scalar}} \underbrace{\left[\hat{S}_{su(2|2)_L} \otimes \hat{S}_{su(2|2)_R} \right]}_{\text{matrix determined by symmetry}}$$

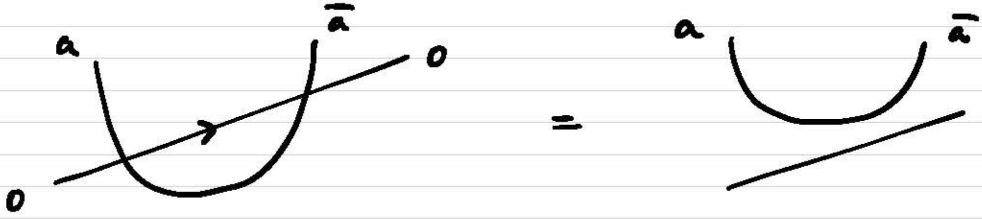
$$x_j^{\pm} = x\left(u_j \pm \frac{i}{2g}\right)$$

Perturbative dilatation / energy spectra $\Rightarrow$ $S_0 \neq 1$

eg. Higher loop computation of QFT predicts highly transcendental numbers $\zeta(3), \zeta(5), \cdots$

But BAE (without phase) gives irrational but algebraic numbers (polynomial roots)

One way to obtain S_0 is to solve crossing relations



$$S_{0a} S_{0\bar{a}} = 1$$

If a test particle (0) scatters against a pair of particle & anti-particle $(a, \bar{a})$, then S-matrix is trivial

anti-particle $(E<0)$ is mathematically given by the crossing transformation, $x^{\pm} \to 1/x^{\pm}$

Janik, hep-th/0603038

- Note that "crossing symmetry" is natural only from the viewpoint of worldsheet theory (2d QFT).

  From $\mathcal{N}=4$ SYM / spin chain, this symmetry is mysterious

- Solutions of the crossing equations are not unique, due to the ambiguity of periodic (crossing-inv.) functions $\rightsquigarrow$ CDD factor

  Beisert, Eden, Staudacher, hep-th/0610251

- We don't need an explicit form of $S_0(x_1^\pm, x_2^\pm)$ in Quantum Spectral Curve (= rewriting of TBA)

TBA for Hagedorn temperature

Harmark, Wilhelm, 1706.03074 and 1803.04416

Recall the $\mathcal{N}=4$ SYM partition fn on $S^1_\beta \times S^3$

$$\mathrm{tr}(e^{-\beta D}) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} Z_{\text{single}}(e^{-n\beta}) \right)$$

We look for the largest $\beta = \beta_*(\lambda)$ (smallest T_*)

s.t. the partition function diverges at any λ

Perturbative computation $\rightarrow$ can focus on $n=1$

Rewrite dilatation $D = D(\lambda=0) + \delta D$

Expand the single-trace partition as

$$Z_{\text{single}}(e^{-\beta}) = \sum_{m=2} \operatorname{tr} e^{-\beta\left[\frac{m}{2} + \delta D\left(\frac{m}{2}\right)\right]} \equiv \sum_{m}{}' e^{-\beta F_m}$$

$F_m \sim \left\{ \begin{array}{l} \text{Sum of anomalous dim. of all single-traces} \\ \text{with the canonical dim. } D_0 = m/2 \end{array} \right\}$

Cauchy's root test: $\sum_n{}' a_n$ converges / diverges if

$$\lim_{n\to\infty} (a_n)^{1/n} = r \quad \text{satisfies} \quad r<1 \;/\; r>1$$

$\because$) roughly $\sum_n{}' a_n \sim \sum_n{}' n^c r^n$ for a fixed c

In our case.

$$\lim_{m\to\infty} \exp\left(-\frac{\beta}{m} F_m\right) \longrightarrow r \overset{?}{=} 1$$

$$\log r \sim -\frac{\beta}{m}\left\{ \frac{m\beta}{2} + \underbrace{\log \mathrm{tr}\left[e^{\beta\,\delta D\left(\frac{m}{2}\right)} \right]}_{\equiv \frac{m\beta}{2}\mathcal{F}(T)} \right\} \overset{?}{=} 0$$

Hagedorn temperature : $\mathcal{F}(T_H) = -1$

$\mathcal{F} \longleftrightarrow$ sum of anomalous dim. of very long operators

$\longleftrightarrow$ thermodynamic limit of $psu(2|2)^2$ Bethe Ansatz

TBA equations

Zamolodchikov, Nucl.Phys.B342(1990)695

We sum the (free) energies over the states which solve Bethe Ansatz Equations $\underbrace{\text{in a thermodynamic limit}}_{L\to\infty,\ M\to\infty,\ \frac{M}{L}\ \text{fixed}}$

Free energy

$$F = E - TS \underbrace{- T \cdot i\gamma N}_{\text{chemical potential for fermions}}$$

$$E - T i\gamma N = \sum_a \int \underbrace{du}_{\text{rapidity}} \left(e_a(u) - T i \gamma_a \right) \underbrace{\rho_a(u)}_{\text{density of type a particles}}$$

$$S = \sum_a \log \frac{(N_a + \bar{N}_a)!}{N_a! \ \bar{N}_a!}$$

$$\to \sum_a \int du \left\{ (\rho_a + \bar{\rho}_a) \log (\rho_a + \bar{\rho}_a) - \rho_a \log \rho_a - \bar{\rho}_a \log \bar{\rho}_a \right\}$$

$N_a + \bar{N}_a$ = # (possible energy levels)

N_a = # (occupied ")

$\bar{N}_a$ = # (unoccupied ")

We extremize $F[\rho_a, \bar{\rho}_a]$ w.r.t. $\rho_a, \bar{\rho}_a$

$$\delta F = \frac{\partial F}{\partial \rho_a} \delta \rho_a + \frac{\partial F}{\partial \bar{\rho}_a} \delta \bar{\rho}_a = 0$$

Use BAE to relate $\delta\rho_a$ and $\delta\bar{\rho}_a$

$$\log(\text{BAE}) : \quad 2\pi i\, n_{a,k} = i p(u_k) L_a + \sum_b{}' \sum_{j \neq k} \log S_{ab}(u_k, u_j)$$

$n_{a,k} \in \mathbb{Z}$ has upper & lower bounds because

$\operatorname{Im} \log z \in (-\pi, \pi]$ and RHS is a finite sum.

Expectation For each (appropriate) choice of $\{n_{a,k}\}$

there is a consistent solution of $\{u_{a,k}\}$

Continuum limit of BAE

$$\rightarrow \quad \rho_a + \bar{\rho}_a = \frac{p(u) L_a}{2\pi} + \sum_b \int dv \, \frac{1}{2\pi i} \partial_v \log S_{ba}(v,u) \cdot \rho_b(u)$$

After a little algebra,

"TBA eq"

$\delta F = 0 \iff$

$$\log Y_b(u) = \underbrace{\frac{e_b(u)}{T}}_{\text{Source term}} - \sum_a \int dv \, \log\left(1 + e^{i\gamma_a} Y_a(v)\right) K_{ab}(v,u)$$

$$e^{i\gamma_a} Y_a(u) = \frac{\rho_a(u)}{\bar{\rho}_a(u)} \, , \quad K_{ab}(v,u) = \frac{1}{2\pi i} \frac{\partial}{\partial v} \log S_{ab}(v,u)$$

Free energy at the extremum is

$$F = -T \underbrace{\sum_a}_{\infty \text{ sum for } \mathcal{N}=4 \text{ SYM}} \int \frac{du}{2\pi} \log\left(1 + e^{i\gamma_a} Y_a(u)\right)$$

TBA equations from asymptotic spin chains

$\hat{=}$ Y-system + Discontinuity Relations

$$Y_{a,s}^{+} Y_{a,s}^{-} = \frac{(1+Y_{a,s-1})(1+Y_{a,s+1})}{\left(1+\frac{1}{Y_{a-1,s}}\right)\left(1+\frac{1}{Y_{a+1,s}}\right)}, \quad f^{\pm} = f\left(u \pm \frac{i}{g}\right)$$

If we introduce $Y_{a,s} = \dfrac{T_{a,s+1}\,T_{a,s-1}}{T_{a+1,s}\,T_{a-1,s}}$

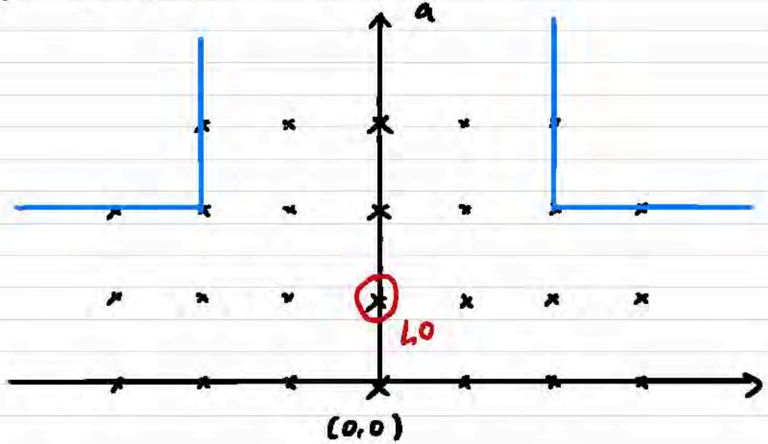
Y-system $\Leftrightarrow$ $T_{a,s}^{+} T_{a,s}^{-} = T_{a,s-1} T_{a,s+1} + T_{a-1,s} T_{a+1,s}$

T-system or Hirota equations

If $T_{a,s}(u) =$ constant (indep. of u),

$$T^{\pm}_{a,s} \rightarrow T_{a,s} = \text{PSU}(2,2|4) \text{ character for the}$$
"rectangular representation" $a \times s$

PSU(2,2|4) T-hook



$T_{a,s} = 0$ outside T-hook

$T_{0,s} = 1$ and

$T_{2,\pm a} = T_{a,\pm 2}$

by gauge choice

Hirota eq is invariant under

$$T_{a,s} \rightarrow g_1^{[a+s]} \, g_2^{[a-s]} \, g_3^{[-a+s]} \, g_4^{[-a-s]} \, T_{a,s}$$

$$f^{[n]} = f\left(u + \frac{in}{g}\right)$$

$Y_{a,s}$ is the gauge-invariant variable

At $g = \frac{\sqrt{\lambda}}{2\pi} = 0$, TBA equations indeed have constant

solutions given by the $psu(2,2|4)$ characters $\chi_R(g)$

Gromov, Kazakov, Vieira, 0901.3753
Gromov, Kazakov, Tsuboi, 1002.3981

PSU(2,2|4) characters

Generating fn for GL(M|N) characters

$$w_{M|N}(t,g) = \mathrm{Sdet}\,\frac{1}{1-gt} = \frac{\prod_{n=1}^{N}(1-y_n t)}{\prod_{m=1}^{M}(1-x_m t)}$$

$$\equiv \sum_{s=1}^{\infty} t^s \, T_{1,s}^{(M|N)}(g)$$

where $g = \mathrm{diag}\left(x_1, x_2 \cdots x_M \,\middle|\, y_1, y_2, \cdots y_N\right)$

$[1,s] \longleftrightarrow$ s-th symmetric representation

Clearly, we can decompose

$$W_{4|4}(t, g_L \times g_R) = W_{2|2}(t, g_L) \times W_{2|2}(t, g_R)$$

$$\rightarrow \quad T^{(4|4)}_{1,s}(g_L \times g_R) = \sum_{j=0}^{s} T^{(2|2)}_{1,s-j}(g_L)\, T^{(2|2)}_{1,j}(g_R)$$

We need $\underbrace{GL(2,2|4)}_{\text{non-cpt}}$ character, not $\underbrace{GL(4|4)}_{\text{cpt}}$

$\rightarrow$ expand $GL(2|2)_R$ in $1/t$, g_R^{-1}

$$T^{(2,2|4)}_{1,s}(g_L \times g_R) = \frac{y_3 y_4}{x_3 x_4} \sum_{j=\max(0,-s)}^{\infty} T^{(2|2)}_{1,s+j}(g_L)\, T^{(2|2)}_{1,j}(g_R^{-1})$$

$$PSU(2,2|4) \ni g = \mathrm{diag}\,(\underbrace{x_1, x_2, x_3, x_4}_{SU(2,2)} \,|\, \underbrace{y_1, y_2, y_3, y_4}_{SU(4)_R})$$

$$x_1 x_2 x_3 x_4 = y_1 y_2 y_3 y_4 = 1$$

We set $x_1 = x_2 = \dfrac{1}{x_3} = \dfrac{1}{x_4} = -e^{-\beta/2}$, $y_{1\sim 4} = 1$

then $T_{1,0} = z(x)$ with $x = e^{-\beta}$

single-letter function of $\mathcal{N}=4$ SYM

$\because$) singleton rep. of $psu(2,2|4)$

$[0, 0 | 0, 1, 0 | 0, 0]$

Solving TBA

Pick up one of ∞ equations ;

$$\log Y_{1,1} Y_{2,2} = \sum_{Q=1}^{\infty} \log\left(1 + Y_{Q,0}\right) * K_{Qy}$$

$$f * K(u) = \int_{-\infty}^{\infty} dv\, f(v) K(v,u)$$

All Y-functions are constant

$$\Rightarrow \quad \text{RHS becomes} \quad \sim \int_{-\infty}^{\infty} dv\, K(v,u) = 0$$

$$1 = Y_{1,1} Y_{2,2} = \frac{T_{1,0}}{T_{0,1}} \frac{T_{2,3}}{T_{3,2}} = T_{1,0} \quad \text{(gauge choice)}$$

Solving TBA at $g_{YM}=0$ $\Rightarrow$ $T_{1,0}=1$

This is consistent with $\mathcal{Z}(x_H)=1$

$\mathcal{Z}(x)=T_{1,0}(x)$

Unfortunately, the free energy vanishes under the constant (= character) solutions ; $\mathcal{F}\neq -1$

$\rightarrow$ need to go to higher order in g_{YM}

We can solve TBA numerically starting from the character solution at $\lambda=0$ to any $\lambda>0$

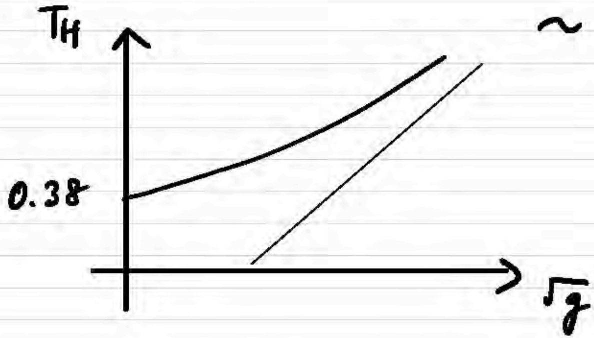
Quantum Spectral Curve is useful for this computation.

Gromov, Kazakov, Laurent, Volin, 1305.1939

Numerical Solution

$$\sim \sqrt{\frac{g}{4\pi}} = \frac{R}{\sqrt{8\pi^2\alpha'}}$$

$(x_H = 7 - 4\sqrt{3})$



$T_H (g \gg 1)$ approaches the Hagedorn temp. of the flat space superstring $\left(T_H = \frac{1}{\sqrt{8\pi^2\alpha'}} \right)$ in the unit of AdS radius R.

Comments

- We can include chemical potential
- Not known how to quantize (covariantly) superstring on $AdS_5 \times S^5$
- Hagedorn TBA gives T_H but not $Z(T < T_H)$
- Several calculations exist for $Z(T > T_H)$ free energy $\sim \mathcal{O}(N^2)$

Free energy

Compute the free energy of $\mathcal{N}=4$ SYM at $g_{YM}=0$

To examine $F = \mathcal{O}(N^2)$ or $\mathcal{O}(1)$

Sundborg, hep-th/9908001, Skagerstam, Z. Phys. C24 (1987) 97, Wilhelm et al. 2005.06480

Step 1

$$\mathrm{tr}(e^{-\beta D_0}) = \int dg \prod_n \frac{\det(1+e^{-\beta\omega_n} R_g)}{\det(1-e^{-\beta\omega_n} R_g)}$$

which can be derived e.g. by inserting the coherent states & projecting into $SU(N)$ singlets

dg : Haar measure of $SU(N)$

R_g : (adjoint) representation of $g \in SU(N)$

Step 2 The $psu(2,2|4)$ structure is hidden in the index "n". We replace the sum over bosons & fermions by $\zeta(x)$

$$\mathrm{tr}(e^{-\beta D_0}) =$$

$$\int dg \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \left[\zeta_B(x^n) \chi_{adj}(g^n) + (-1)^{n+1} \zeta_F(x^n) \chi_{adj}(g^n) \right] \right)$$

Since $N \otimes \bar{N} = \mathrm{adj} + 1$

$$\chi_{\mathrm{adj}}(g) = \chi_N(g)\,\chi_{\bar{N}}(g) - 1$$

If $\begin{cases} g \sim \mathrm{diag}\,(e^{i\alpha_1}, e^{i\alpha_2}, \cdots, e^{i\alpha_N}) \\ \sum_{j=1}^{N} \alpha_j = 2\pi k \, , \end{cases}$

$$\chi_{\mathrm{adj}}(g) = \sum_{j,k} e^{i(\alpha_j - \alpha_k)} - 1$$

$$= \sum_{j \leq k} 2\cos(\alpha_j - \alpha_k) - 1$$

Step 3

Evaluate Haar measure on $SU(N)$

Mehta, Random matrices

Unitary matrix is diagonalizable,

$$g = M \Lambda M^\dagger, \quad \Lambda = \mathrm{diag}\left(e^{i\alpha_1}, e^{i\alpha_2}, \cdots, e^{i\alpha_N}\right)$$

Invariant measure of $U(N)$ is $\quad dg = \dfrac{\prod_{i,j=1}^{N} dg_{ij}}{(\det g)^N}$

& we substitute the above decomposition

Since the integrand depends only on Λ,

the integral over angular directions is trivial

except for Jacobian.

$$dg = \prod_{j=1}^{N} d\alpha_j \prod_{j<k} \sin^2\left(\frac{\alpha_j - \alpha_k}{2}\right) \underbrace{d\Omega}_{\text{angular part}}$$

For SU(N) we add $\delta\left(\sum_j \alpha_j - 2\pi k\right)$

In summary,

$$\mathrm{tr}\left(e^{-\beta D}\right) = \int \prod_j d\alpha_j \underbrace{\prod_{j<k} \sin^2\left(\frac{\alpha_j - \alpha_k}{2}\right)} \delta\left(\sum_j \alpha_i - 2\pi k\right) \times$$

$$\exp\left[\sum_{n=1}^{\infty} \frac{\zeta_B(x^n) - (-1)^n \zeta_F(x^n)}{n} \left(\sum_{j\leqslant k} 2\cos\left(n(\alpha_j - \alpha_k)\right) - 1\right)\right]$$

$$\sim = \exp\left[\sum_{j\neq k} \log\left|\sin\frac{\alpha_j - \alpha_k}{2}\right|\right]$$

Step 4 Take large N limit, introduce the density of eigenvalues $\rho(\alpha)$, look for saddle points

$$\sum_n f(\alpha_n) \longrightarrow \underline{\underline{N}} \int_{-\alpha_c}^{\alpha_c} d\alpha \; \rho(\alpha)\, f(\alpha)$$

$$\sum_{m<n} f(\alpha_m, \alpha_n) \longrightarrow \underline{\underline{\frac{N^2}{2}}} \; \mathrm{p.v.} \int_{-\alpha_c}^{\alpha_c} \int_{-\alpha_c}^{\alpha_c} d\alpha\, d\beta \; \rho(\alpha)\, \rho(\beta)\, f(\alpha, \beta)$$

Add Lagrange multiplier $\quad \lambda \left(\int d\alpha\, \rho(\alpha) - 1 \right)$

These factors are the origin of $\mathcal{F} \sim \mathcal{O}(N^2)$

unless the coefficient vanishes ...

$$\mathrm{tr}\left(e^{-\beta D}\right) = \int d\rho d\lambda e^{-S(\rho)+\lambda\left(\int d\alpha \rho(\alpha)-1\right)}$$

$$S = -N^2 \int d\alpha d\beta \rho(\alpha)\rho(\beta) K(\alpha,\beta) + \sum_{n=1}^{\infty} \frac{\zeta_B(x^n) - (-1)^n \zeta_F(x^n)}{n}$$

$$K = \log\left|2\sin\frac{\alpha-\beta}{2}\right| + \sum_{n=1}^{\infty} \frac{\zeta_B(x^n) - (-1)^n \zeta_F(x^n)}{n} \cos(n(\alpha-\beta))$$

Since all eigenvalues show up in the difference

$(\alpha-\beta)$, ρ = constant is a saddle point ($S = \mathcal{O}(1)$)

again we expect that $n=1$ gives dominant term

After focusing on the $n=1$ term,

this becomes Gross-Witten-Wadia matrix model

Gross, Witten, Phys. Rev. D21 (1980) 446; Wadia, Phys. Lett. B93 (1980) 403;
Russo, Tierz, 2007.08515

If $\rho(\alpha) \neq 0$ over $-\pi \leq \alpha \leq \pi$, then

$$\rho(\alpha) \sim c_0 + c_1 \cos\alpha, \quad S = \mathcal{O}(1)$$

If $\rho(\alpha) \neq 0$ over $-\alpha_c \leq \alpha \leq \alpha_c$ $(\alpha_c < \pi)$, then

$$\rho(\alpha) = \frac{\cos\frac{\alpha}{2}}{\pi \sin^2\frac{\alpha_c}{2}} \sqrt{\sin^2\frac{\alpha_c}{2} - \sin^2\frac{\alpha}{2}}, \quad S = \mathcal{O}(N^2)$$

Singular Integral Eq. / Mann, Polchinski hep-th/0508232

We need to solve Riemann-Hilbert problem

with the periodicity conditions on α-variables...

According to Sundborg,

$$F_{>} \sim -\frac{N^2}{2}\left\{\underbrace{\zeta - 1 + \sqrt{\zeta^2-\zeta} - \log\left(\zeta + \sqrt{\zeta^2 - \zeta}\right)}_{\text{vanishes if } \zeta=1}\right\} + \zeta$$

(after Hagedorn)

$$F_{<} \sim \zeta \quad \text{(before Hagedorn)}$$