Stochastic calculus and risk-neutral pricing:

theory and computation

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Overview

- Stochastic process
- SDEs & their numerical solution
- An algeric way to risk-neutral pricing in discrete situation

Stochastic process

Preliminaries

Filtration

A probability space $(\Omega, \mathcal{F}, P)$
equipped with filtration $(\mathcal{F}_i)_{i \in I}$ of σ-algebra $\mathcal{F}$,
is a filtered probability space.

Adapted

> “ past & now ”

For a stochastic process $X : I \times \Omega \to S$ under a filtered probability space with filtration $(\mathcal{F}_i)_{i \in I}$,
X is called an adapted process if
$\forall i \in I, X_i : \Omega \to S$ is a $(\mathcal{F}_i, \Sigma)$-measurable function

Predicatable

“ past ”

For a stochastic process $X : I \times \Omega \to S$ under a filtered probability space with filtration $(\mathcal{F}_i)_{i \in I}$,
X is called a predictable process if
$\forall i \in I, j = max_{k \in I, k < i} k, X_i : \Omega \to S$ is a $(\mathcal{F}_j, \Sigma)$-measurable function

Markov

“ now ”

In a filtered probability space
$(\Omega, \mathcal{F}, (\mathcal{F}_i)_{i \in I}, P)$,
An adapted process $X : I \times \Omega \to S$ s.t.

$$\forall s < t \in I, P(X_t \in A | \mathcal{F}_s) = P(X_t \in A | X_s)$$

is called Markov process.

Stochastic Differential Equation

Ito integral

In Ito calculus, integral of stochastic process is defined as

$$\int_0^t H\,dW = \lim_{n\to+\infty} \sum_{[t_{i-1},t_i]\in\pi_n} H_{t_{i-1}}(W_{t_i} - W_{t_{i-1}})$$

where W is a semimartingale,
and H is a cadlag adapted process of locally bounded variation

SDE

For integral equation of the form

$$X_{t+s} - X_t = \sum_{i<n} \int_t^{t+s} H_i dW_i$$

its differential form

$$dX = \sum_{i<n} H_i dW_i$$

is called SDE.

Numerical Simulation

Discretisation

Basic idea of simulating numerical solution of a SDE is discretisation.

For SDE $dX_t = a(t, X_t)dt + b(t, X_t)dW_t$, chop the interval $[0, T]$ into N grid which has $\Delta t = \frac{T}{N}$ interval.

And via discretisation, continuous stochastic process becomes a discrete-time Markov chain in a given scheme D.

$$Y_{n+1} = Y_n + D(Y_n, \Delta t, \Delta W)$$

Schemes

Euler-Maruyama approximation:

$$D = D_L = a(X_n)\Delta t + b(X_n)\Delta W_n$$

Milstein approximation:

$$D = D_L +$$

$$\frac{1}{2}b(X_n)b'(X_n)((\Delta W_n)^2 - \Delta t)$$

Runge-Kutta approximation:

$$D = D_L +$$

$$\frac{1}{2}(b(\hat{X}_n) - b(X_n))((\Delta W_n)^2 - \Delta t)(\Delta t)^{-1/2}$$

where

$$\hat{X}_n = X_n + a(X_n)\Delta t + b(X_n)\sqrt{\Delta t}$$

Benchmarks

for SDEs in form of

$$dX = \mu(X,t)dt + \sigma(X,t)dW$$

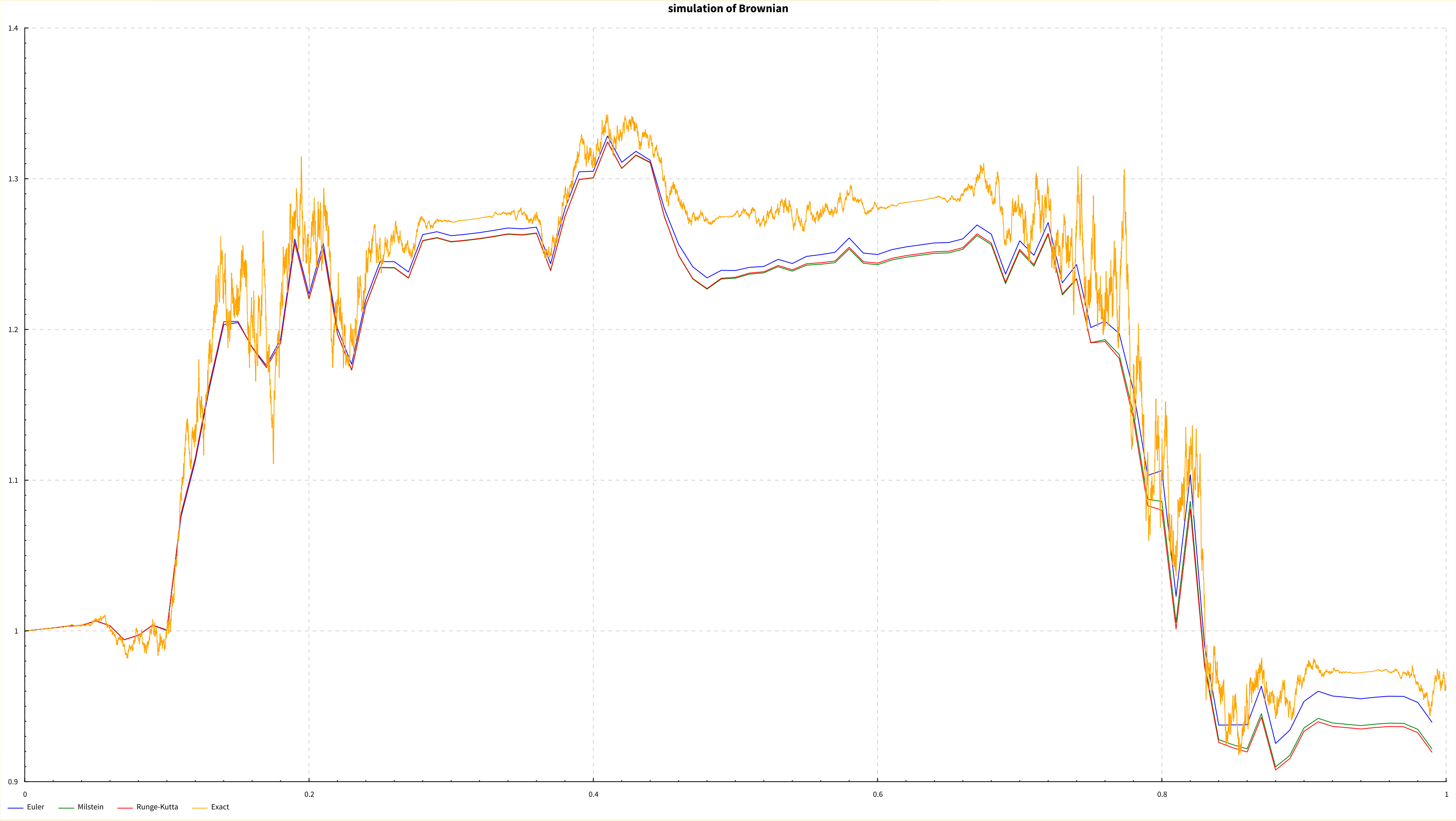
take simulation using Euler-Maruyama scheme in a finer grid as the exact solution

$$\mu = 0.1 * X$$
$$\sigma = 0.8 * X$$
$$X_0 = 1$$

$$\mu = 0.1 * X$$
$$\sigma = 0.8 * sin(10 * t) * X$$
$$X_0 = 1$$

$$\mu = 0.1 * X$$
$$\sigma = 0.8 * sin(10 * t) * \sqrt{X}$$
$$X_0 = 1$$

$$\mu = 0.1 * X$$
$$\sigma = 0.8 * sin(10 * t) * sin(5 * t * X)$$
$$X_0 = 1$$



Discretisation

SDE $\rightarrow$ discrete-time Markov chain

Trade-off: Local linearity

anlysis $\rightarrow$ algebra

Algebric Way to Risk Nutral Pricing in a discretised world

Motivation

- Rube-Goldberg machine
- overkill
- formalisation

Intuition

- commodities:
  - Algebra over a Field
- composable and programable transaction:
  - Bicartesian Closed Category
- diverable future:
  - Temporal Logic & Free Monad

Preliminaries

Algebra over a field:
Let K be a field, and let A be a vector space over K equipped with an additional binary operation $A \times A \longrightarrow A$

Then A is an algebra over K if :
Right distributivity: $(x + y) \cdot z = x \cdot z + y \cdot z$
Left distributivity: $x \cdot (y + z) = x \cdot y + x \cdot z$
Compatibility with scalars: $(ax) \cdot (by) = (ab) (x \cdot y)$

Bicartesian Closed Category:
A category closed under product, coproduct and exponential s.t.

```
(xy)z = (xz)y.
x×(y×z) = (x×y)×z
x×y = y×x
x×1 = x (here 1 denotes the terminal object of C)
1x = 1
x1 = x
(x×y)z = xz×yz
(xy)z = x(y×z)
x + y = y + x
(x + y) + z = x + (y + z)
x×(y + z) = x×y + x×z
x(y + z) = xy×xz
0 + x = x
x×0 = 0
x0 = 1
```

Temporal logic as Functor and Free Monad:

Functor:
pure : a -> f(a)
map: (a -> b) -> (f(a) -> f(b))
s.t. functor rules

Free: (f is a functor)
pure : a -> Free(f)(a)
impure : f (Free(f)(a)) -> Free(f)(a)

Then Free f is a monad s.t. monad rules

Formal Defination

Comodities:

Comodity category C is group generated with $\otimes$, $\oplus$ by base comoditiy object $O_i(i \in I)$ over field R

Comodity is bicartesian closed with exponential $\rightarrow$, product $\otimes$ with terminal object 1
and coproduct $\oplus$ with initial object 0
Comodity is equiped with Functor: Next:
next: (W -> C) -> Next C
pure c = next (const(c))
map f (next(g)) = next (f . g)

Then $M(a) = Free(Next(a))$ is a free monad we denote category $M(C)$ as general comodity category N

$\otimes$,$\oplus$ and $\rightarrow$ also works on N

Application

discretise SDE as time-discrete Markov Chain

for SDE :$dX = \mu dt + \sigma dW$

discretised as

$$Y_{n+1} = Y_n + D(Y_n, \Delta W)$$

represented as

$y_0 = pure(y_0)$

$y_n =$

$$next(w_1 \to$$

$$next(w_2 \to$$

$$\ldots$$

$$next(w_n \to$$

$$(\Sigma_i(D(y_i, w_i)) + y_0) \cdots))$$

Risk Neutral pricing

assuam O_1 is money and O_2 is a risky stock

European option expired in n steps later:

$$p_k : Free(Next)(C) \to Free(Next)(C)$$

$$p_k =$$

$$next(w_1 \to$$

$$next(w_2 \to$$

$$\ldots$$

$$next(w_n \to$$

$$(k \cdot O_1) \to (1 \cdot O_2)) \cdots))$$

"Realise" Future by paired Functor

$Observe:$

$observe:(\Omega\to W)\to Observe(W)$

$pair:Free(Next)(C)\to Observe(W)\to C$

$pair=impure(f)(f_0)\mapsto impure(g)(g_0)$

$\mapsto (f_0.g_0)+pair(f)(g))$

Question

Thanks