Note: On the Reconstruction Formula for the Ridgelet Transform

Raian Suzuki [*]

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[*]Department of Physics, Faculty of Science and Engineering, Chuo University, Kasuga, Bunkyo-ku, Tokyo 112-8551, Japan;

1 Introduction

We define the ridgelet transform and its dual as follows:

$$\mathscr{R}_\psi[f](\boldsymbol{a}, b) := \int_{\mathbb{R}^m} f(x)\, \psi(\boldsymbol{a} \cdot \boldsymbol{x} - b)\, \|\boldsymbol{a}\|^s \, \mathrm{d}\boldsymbol{x} \tag{1}$$

$$\mathscr{R}^\dagger_\eta[T](x) := \int_{-\infty}^{\infty} \int_{\mathbb{R}^m} T(\boldsymbol{a}, b)\, \eta(\boldsymbol{a} \cdot \boldsymbol{x} - b)\, \|\boldsymbol{a}\|^{-s} \, \mathrm{d}\boldsymbol{a}\mathrm{d}b \tag{2}$$

We consider the reconstruction formula.

$$\mathscr{R}^\dagger_\eta[\mathscr{R}_\psi[f]](\boldsymbol{x}) = K_{\psi,\eta}\, f(x) \tag{3}$$

where

$$K_{\psi,\eta} := \int_{-\infty}^{\infty} \frac{\overline{\widehat{\psi}(\zeta)}\widehat{\eta}(\zeta)}{|\zeta|^m} \, \mathrm{d}\zeta \tag{4}$$

2 Reconstruction Formula

Let η and ψ be fourier-transformable functions on $\mathbb{R}$, and let $\widehat{\psi}$ and $\widehat{\eta}$ be their Fourier transforms, respectively. In what follows, to avoid notational conflict, we denote the evaluation point of the reconstruction as $\boldsymbol{y}$.

Throughout this section, we assume that all functions involved are sufficiently well-behaved (e.g., rapidly decaying and smooth, or belonging to appropriate L^1 or L^2 spaces) such that all interchanges of integration order are justified by Fubini's Theorem.

$$\mathscr{R}_\psi[f](\boldsymbol{a}, b) = \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{x}\, f(x)\, \psi(\boldsymbol{a}\cdot\boldsymbol{x} - b)\, \|\boldsymbol{a}\|^s \tag{5}$$

$$\begin{aligned}
\mathscr{R}_\eta^\dagger[\mathscr{R}_\psi[f]](\boldsymbol{y}) &= \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{a} \int_{-\infty}^{\infty} \mathrm{d}b \left(\int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{x}\, f(\boldsymbol{x})\, \psi(\boldsymbol{a}\cdot\boldsymbol{x} - b)\, \|\boldsymbol{a}\|^s \right) \eta(\boldsymbol{a}\cdot\boldsymbol{y} - b)\, \|\boldsymbol{a}\|^{-s} && (6)\\
&= \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{a} \int_{-\infty}^{\infty} \mathrm{d}b \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{x}\, f(\boldsymbol{x})\, \psi(\boldsymbol{a}\cdot\boldsymbol{x} - b)\, \eta(\boldsymbol{a}\cdot\boldsymbol{y} - b) && (7)\\
&= \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{x}\, f(\boldsymbol{x}) \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{a} \int_{-\infty}^{\infty} \mathrm{d}b\, \psi(\boldsymbol{a}\cdot\boldsymbol{x} - b)\, \eta(\boldsymbol{a}\cdot\boldsymbol{y} - b) && (8)
\end{aligned}$$

To evaluate the integral, we introduce a change of variables. Consider the transformation from $(\boldsymbol{a}, b)$ to $(z, \boldsymbol{w})$:

$$\begin{cases} z &= \boldsymbol{a}\cdot\boldsymbol{x} - b \\ \boldsymbol{w} &= (a_1, a_2, \cdots a_m) \end{cases} \iff \begin{cases} b &= \boldsymbol{a}\cdot\boldsymbol{x} - z \\ \boldsymbol{a} &= \boldsymbol{w} \end{cases}$$

Where $(a_1, a_2, \cdots a_m)$ are the components of $\boldsymbol{a}$.

The Jacobian matrix of this transformation is given by:

$$J = \begin{pmatrix} \partial z/\partial a_1 & \partial w_1/\partial a_1 & \cdots & \partial w_{m-1}/\partial a_1 & \partial w_m/\partial a_1 \\ \partial z/\partial a_2 & \partial w_1/\partial a_2 & \cdots & \partial w_{m-1}/\partial a_2 & \partial w_m/\partial a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \partial z/\partial a_m & \partial w_1/\partial a_m & \cdots & \partial w_{m-1}/\partial a_m & \partial w_m/\partial a_m \\ \partial z/\partial b & \partial w_1/\partial b & \cdots & \partial w_{m-1}/\partial b & \partial w_m/\partial b \end{pmatrix}$$

$$= \begin{pmatrix} x_1 & 1 & 0 & \cdots & 0 & 0 \\ x_2 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{m-1} & 0 & 0 & \cdots & 1 & 0 \\ x_m & 0 & 0 & \cdots & 0 & 1 \\ -1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Where $(x_1, x_2, \cdots x_m)$ are the components of $\boldsymbol{x}$. The determinant is computed by cofactor expansion along the last row. Let $C_{(i,j)}$ denote the cofactor corresponding to the element in the i-th row and j-th column. Since the only nonzero element in the last row is in position $(m+1, 1)$, its sign is given by $(-1)^{(m+1)+1}$.

The minor corresponding to this entry is the $m \times m$ identity matrix, so its cofactor is:

$$C_{((m+1),1)} = (-1)^{(m+1)+1} \cdot \det(I_m) = (-1)^{m+2} \cdot 1 = (-1)^{m+2}$$

Hence:

$$\det(J) = -1 \cdot C_{((m+1),1)} = -1 \cdot (-1)^{m+2} = (-1)^{m+3}, \quad \text{so} \quad |\det(J)| = 1$$

.

The integral after the transformation is as follows:

$$(\text{RHS}) = \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{x}\, f(\boldsymbol{x}) \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{w} \int_{-\infty}^{\infty} \mathrm{d}z\, \psi(z)\, \eta(\boldsymbol{w} \cdot (\boldsymbol{y} - \boldsymbol{x}) + z) \tag{9}$$

$$= \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{x}\, f(\boldsymbol{x}) \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{w} \int_{-\infty}^{\infty} \mathrm{d}z \left(\int_{-\infty}^{\infty} \mathrm{d}\zeta\, \widehat{\psi}(\zeta) e^{i\zeta z} \right) \eta(\boldsymbol{w} \cdot (\boldsymbol{y} - \boldsymbol{x}) + z) \tag{10}$$

$$= \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{x}\, f(\boldsymbol{x}) \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{w} \int_{-\infty}^{\infty} \mathrm{d}z \int_{-\infty}^{\infty} \mathrm{d}\zeta\, \widehat{\psi}(\zeta) e^{i\zeta z}\, \eta(\boldsymbol{w} \cdot (\boldsymbol{y} - \boldsymbol{x}) + z) \tag{11}$$

$$= \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{x}\, f(\boldsymbol{x}) \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{w} \int_{-\infty}^{\infty} \mathrm{d}\zeta\, \widehat{\psi}(\zeta) \int_{-\infty}^{\infty} \mathrm{d}z\, e^{i\zeta z}\, \eta(\boldsymbol{w} \cdot (\boldsymbol{y} - \boldsymbol{x}) + z) \tag{12}$$

In equation (10), we apply the Fourier inversion formula to ψ, which yields the integral representation of ψ via $\widehat{\psi}$.

Next, consider the following transformation:

$$z' = \boldsymbol{w} \cdot (\boldsymbol{y} - \boldsymbol{x}) + z \tag{13}$$

$$\iff z = z' - \boldsymbol{w} \cdot (\boldsymbol{y} - \boldsymbol{x}) \tag{14}$$

$$\text{(RHS)} = \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{x}\, f(\boldsymbol{x}) \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{w} \int_{-\infty}^{\infty} \mathrm{d}\zeta\, \widehat{\psi}(\zeta) \int_{-\infty}^{\infty} \mathrm{d}z'\, e^{i\zeta z'}\, e^{-i\zeta \boldsymbol{w}\cdot(\boldsymbol{y}-\boldsymbol{x})}\, \eta(z') \tag{15}$$

$$= \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{x}\, f(\boldsymbol{x}) \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{w} \int_{-\infty}^{\infty} \mathrm{d}\zeta\, \widehat{\psi}(\zeta)\, e^{-i\zeta \boldsymbol{w}\cdot(\boldsymbol{y}-\boldsymbol{x})} \int_{-\infty}^{\infty} \mathrm{d}z'\, \eta(z')\, e^{i\zeta z'} \tag{16}$$

$$= \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{x}\, f(\boldsymbol{x}) \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{w} \int_{-\infty}^{\infty} \mathrm{d}\zeta\, \widehat{\psi}(\zeta)\, e^{-i\zeta \boldsymbol{w}\cdot(\boldsymbol{y}-\boldsymbol{x})}\, \widehat{\eta}(\zeta) \tag{17}$$

$$= \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{x}\, f(\boldsymbol{x}) \int_{-\infty}^{\infty} \mathrm{d}\zeta\, \widehat{\psi}(\zeta)\, \widehat{\eta}(\zeta) \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{w}\, e^{-i\zeta \boldsymbol{w}\cdot(\boldsymbol{y}-\boldsymbol{x})} \tag{18}$$

$$\tag{19}$$

Considering the transformation from $\zeta\boldsymbol{w}$ to $\boldsymbol{w}'$, and noting that its Jacobian is $|\zeta|^m$:

$$\text{(RHS)} = \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{x}\, f(\boldsymbol{x}) \int_{-\infty}^{\infty} \mathrm{d}\zeta\, \widehat{\psi}(\zeta)\, \widehat{\eta}(\zeta) \int_{\mathbb{R}^m} \frac{\mathrm{d}\boldsymbol{w}}{|\zeta|^m}\, e^{-i\boldsymbol{w}'\cdot(\boldsymbol{y}-\boldsymbol{x})} \tag{20}$$

$$= \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{x}\, f(\boldsymbol{x}) \int_{-\infty}^{\infty} \mathrm{d}\zeta\, \frac{\widehat{\psi}(\zeta)\, \widehat{\eta}(\zeta)}{|\zeta|^m} \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{w}\, e^{-i\boldsymbol{w}'\cdot(\boldsymbol{y}-\boldsymbol{x})} \tag{21}$$

$$= \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{x}\, f(\boldsymbol{x}) \int_{-\infty}^{\infty} \mathrm{d}\zeta\, \frac{\widehat{\psi}(\zeta)\, \widehat{\eta}(\zeta)}{|\zeta|^m}\, \delta(\boldsymbol{y} - \boldsymbol{x}) \tag{22}$$

$$= \int_{-\infty}^{\infty} \mathrm{d}\zeta\, \frac{\widehat{\psi}(\zeta)\, \widehat{\eta}(\zeta)}{|\zeta|^m} \int_{\mathbb{R}^m} \mathrm{d}\boldsymbol{x}\, f(\boldsymbol{x})\, \delta(\boldsymbol{y} - \boldsymbol{x}) \tag{23}$$

$$= f(\boldsymbol{y}) \int_{-\infty}^{\infty} \mathrm{d}\zeta\, \frac{\widehat{\psi}(\zeta)\, \widehat{\eta}(\zeta)}{|\zeta|^m}. \tag{24}$$

Thus, it is shown that the reconstruction formula is possible.

3 Examples

We choose the hyperbolic tangent function as the activation function η. The Fourier transform of the hyperbolic tangent function is given by:

$$\widehat{\eta}(\zeta) = -i\pi \operatorname{sech}\left(\frac{\pi}{2}\zeta\right) \tag{25}$$

We now calculate why its Fourier transform takes this form.

$$\begin{aligned}\widehat{\eta}(\zeta) &= \text{p.v.} \int_{-\infty}^{\infty} \tanh(z)\, e^{-i\zeta z}\, \mathrm{d}z \\ &= \lim_{R\to\infty} \int_{-R}^{R} \tanh(z)\, e^{-i\zeta z}\, \mathrm{d}z\end{aligned}$$

Here, p.v. denotes the Cauchy principal value. The necessity of taking the Cauchy principal value arises from the fact that $\tanh(z)$ converges to a constant at infinity, whereas the exponential term exhibits oscillatory behavior.

As direct evaluation of this integral is difficult, we resort to considering a complex integral along a contour $\mathcal{C}$.

$$\int_{\mathcal{C}} \tanh(z)\, e^{-i\zeta z}\, \mathrm{d}z = \int_{\gamma_R} \tanh(z)\, e^{-i\zeta z}\, \mathrm{d}z + \int_{\ell_R} \tanh(z)\, e^{-i\zeta z}\, \mathrm{d}z.$$

Where $\mathcal{C}$ is a closed contour in the complex plane, and γ_R and ℓ_R are specific segments of this contour. For $\zeta > 0$, we choose $\mathcal{C}$ to be the semi-circular contour in the upper half-plane, consisting of the real axis segment $[-R, R]$ (denoted by ℓ_R) and the semi-circular arc $\gamma_R^+ = \{z = Re^{i\theta} \mid 0 \leqslant \theta \leqslant \pi\}$. Conversely, for $\zeta < 0$, we choose the semi-circular contour in the lower half-plane $\gamma_R^- = \{z = Re^{-i\theta} \mid 0 \leqslant \theta \leqslant \pi\}$.

Now, we will utilize the Residue Theorem to evaluate this complex integral. The next steps involve:

1. Identifying the contribution from the straight line segment.
2. Calculating the residues.
3. Showing the integral along the semi-circular arc vanishes.

Next, we will calculate the residues at the poles within our chosen contour. The fact that poles of $\tanh(z)e^{-i\zeta z}$ are located at $z_k = \pm i(/2 + k)\pi$ for $k \in \mathbb{Z}$ can be readily seen from the following lemma:

Lemma 3.1 (Infinite Product Expansion of Hyperbolic Cosine)**.** *The hyperbolic cosine function* $\cosh(z)$ *has the infinite product expansion:*

$$\cosh(z) = \prod_{k=0}^{\infty} \left(1 + \frac{z^2}{\left(k + \frac{1}{2}\right)^2 \pi^2} \right)$$

Since these poles are simple pole, we can use the following lemma to calculate the residues:

Lemma 3.2 (Residue at a Simple Pole)**.** *Let* $f(z) = \frac{p(z)}{q(z)}$ *be a complex function where* $p(z)$ *and* $q(z)$ *are analytic functions. If* $q(z_0) = 0$ *and* $q'(z_0) \neq 0$ *(implying* z_0 *is a simple zero of* $q(z)$*), and* $p(z_0) \neq 0$*, then* $f(z)$ *has a simple pole at* $z = z_0$*, and its residue is given by:*

$$Res(f(z), z_0) = \frac{p(z_0)}{q'(z_0)}$$

Case 1: $\zeta > 0$ For $\zeta > 0$, we choose the semi-circular contour in the lower half-plane. Therefore, we need to consider the poles $z_k = -i(\pi/2 + k\pi)$ for $k \in \mathbb{N}$. The residues at these poles can be computed using Lemma 3.2. Here, $p(z) = \sinh(z)e^{-i\zeta z}$ and $q(z) = \cosh(z)$. The residue at the pole $z_k = -i(\pi/2 + k\pi)$ is given by:

$$\begin{aligned}
\mathrm{Res}(\tanh(z)e^{-i\zeta z}, z_k) &= \frac{\sinh(z_k)e^{-i\zeta z_k}}{\cosh'(z_k)} \\
&= \frac{\sinh(z_k)e^{-i\zeta z_k}}{\sinh(z_k)} \\
&= e^{-i\zeta z_k} \\
&= e^{-i\zeta(-i(\pi/2+k\pi))} \\
&= e^{\zeta(\pi/2+k\pi)}.
\end{aligned}$$

As $R \to \infty$, the contour enclose an infinite number of poles in the lower half-plane, specifically $z_j = -i(j + 1/2)\pi$ for $j \in \mathbb{N}$. Thus, the total sum of residues will be an infinite seriese:

$$
\begin{aligned}
\sum_{j=0}^{\infty} \operatorname{Res}(\tanh(z)e^{-i\zeta z}, z_j) &= \sum_{j=1}^{\infty} e^{\zeta(j+1/2)\pi} \\
&= e^{\zeta\pi/2} \sum_{j=1}^{\infty} e^{j\zeta\pi} \\
&= e^{\zeta\pi/2} \frac{e^{\zeta\pi}}{1-e^{\zeta\pi}} \\
&= \frac{i\pi}{2\sinh\left(\frac{\pi}{2}\zeta\right)}.
\end{aligned}
$$

It's important to note that when integrating along a contour in the lower half-plane, the contour is typically traversed anticlockwise. Since we are integrating clockwise this time, a minus sign is attached when using the residue theorem.

Case 2: $\zeta < 0$ By similar calculations, the infinite sum of the residues for $\zeta < 0$ is given by:

$$
\begin{aligned}
\sum_{j=0}^{\infty} \operatorname{Res}(\tanh(z)e^{-i\zeta z}, z_j) &= \sum_{j=1}^{\infty} e^{-\zeta(j+1/2)\pi} \\
&= e^{-\zeta\pi/2} \sum_{j=1}^{\infty} e^{-j\zeta\pi} \\
&= e^{-\zeta\pi/2} \frac{e^{-\zeta\pi}}{1-e^{-j\zeta\pi}} \\
&= \frac{-i\pi}{2\sinh\left(\frac{\pi}{2}\zeta\right)}
\end{aligned}
$$

Next, we will show that the integral over a semicircle is zero.

Case 1: $\zeta > 0$

For $\zeta > 0$, we consider the semicircular contour in the lower half-plane, denoted as $\gamma_R^- = \{z = Re^{-i\theta} \mid 0 \leqslant \theta \leqslant \pi\}$. We need to show that the integral over this semicircle vanishes as $R \to \infty$.

We have:

$$\begin{aligned}
\left|\int_{\gamma_R^-} \tanh(z)e^{-i\zeta z}\,\mathrm{d}z\right| &= \left|\int_0^\pi \tanh(Re^{-i\theta})e^{-i\zeta Re^{-i\theta}}(-iRe^{-i\theta})\,\mathrm{d}\theta\right| \\
&\leqslant \int_0^\pi \left|\tanh(Re^{-i\theta})e^{-i\zeta Re^{-i\theta}}(-iRe^{-i\theta})\right|\,\mathrm{d}\theta \\
&= R\int_0^\pi \left|\tanh(Re^{-i\theta})\right| e^{-\zeta R\sin\theta}\,\mathrm{d}\theta \\
&\leqslant R\int_0^\pi e^{-\zeta R\sin\theta}\,\mathrm{d}\theta.
\end{aligned}$$

Here, we use the following lemma:

Lemma 3.3 (Laplace's Method)**.** *Let $f(x)$ be a twice continuously differentiable function on the interval $[a,b]$, such that it attains its maximum value at a unique point $x_0 \in (a,b)$, i.e., $f(x_0) = \max_{a\leqslant x\leqslant b} f(x)$, and $f''(x_0) < 0$. Under these conditions, we have:*

$$\int_a^b e^{nf(x)}\,\mathrm{d}x \sim \sqrt{\frac{2\pi}{n\,|f''(x_0)|}}e^{nf(x_0)} \quad \text{as } n \to \infty.$$

Here, $\sim$ denotes that the ratio of the two sides approaches 1 as $n \to \infty$.

Using Laplace's method, we can evaluate the integral as follows:

$$\begin{aligned}
\int_0^\pi e^{-\zeta R\sin\theta}\,\mathrm{d}\theta &\sim \sqrt{\frac{2\pi}{\zeta R\,|-\cos(0)|}}e^{-\zeta R\sin(\frac{\pi}{2})} \\
&= \sqrt{\frac{2\pi}{\zeta R}}e^{-\zeta R} \\
&= \sqrt{\frac{2\pi}{\zeta R}}e^{-\zeta R}.
\end{aligned}$$

Hence, we calculate the integral over the semicircle as follows:

$$\begin{aligned}
R\int_0^\pi e^{-\zeta R\sin\theta}\,\mathrm{d}\theta &\leqslant R\sqrt{\frac{2\pi}{\zeta R}}e^{-\zeta R} \\
&= \sqrt{\frac{2\pi R}{\zeta}}e^{-\zeta R}.
\end{aligned}$$

Therefore, we conclude that:

$$\int_{\gamma_R^+} \tanh(z)e^{-i\zeta z}\,\mathrm{d}z \to 0 \quad \text{as } R \to \infty.$$

Case 2: $\zeta < 0$ For $\zeta < 0$, we consider the semicircular contour in the upper half-plane, denoted as $\gamma_R^+ = \{z = Re^{i\theta} \mid 0 \leqslant \theta \leqslant \pi\}$. We need to show that the integral over this semicircle vanishes as $R \to \infty$. We have:

$$\begin{aligned}
\left| \int_{\gamma_R^+} \tanh(z) e^{-i\zeta z} \, \mathrm{d}z \right| &= \left| \int_0^\pi \tanh(Re^{i\theta}) e^{-i\zeta Re^{i\theta}} (iRe^{i\theta}) \, \mathrm{d}\theta \right| \\
&\leqslant \int_0^\pi \left| \tanh(Re^{i\theta}) e^{i\zeta Re^{i\theta}} (iRe^{i\theta}) \right| \, \mathrm{d}\theta \\
&\leqslant \int_0^\pi \left| \tanh(Re^{i\theta}) e^{i\zeta Re^{i\theta}} (iRe^{i\theta}) \right| \, \mathrm{d}\theta \\
&= R \int_0^\pi \left| \tanh(Re^{i\theta}) \right| e^{i\zeta R \sin\theta} \, \mathrm{d}\theta \\
&\leqslant R \int_0^\pi e^{\zeta R \sin\theta} \, \mathrm{d}\theta.
\end{aligned}$$

Here, using Laplace's lemma, the same calculation yields 0. Finally, we can conclude that the integral over the semicircle vanishes as $R \to \infty$.

Thus, we have:

$$\widehat{\eta}(\zeta) = \frac{-i\pi}{2\sinh\left(\frac{\pi}{2}\zeta\right)}.$$

References

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