

# Note: On the Reconstruction Formula for the Ridgelet Transform

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# 1 Introduction

We define the ridgelet transform and its dual as follows:

$$\mathcal{R}_\psi[f](\mathbf{a}, b) := \int_{\mathbb{R}^m} f(x) \psi(\mathbf{a} \cdot \mathbf{x} - b) \|\mathbf{a}\|^s \, d\mathbf{x} \quad (1)$$

$$\mathcal{R}_\eta^\dagger[T](x) := \int_{-\infty}^{\infty} \int_{\mathbb{R}^m} T(\mathbf{a}, b) \eta(\mathbf{a} \cdot \mathbf{x} - b) \|\mathbf{a}\|^{-s} \, d\mathbf{a} db \quad (2)$$

We consider the reconstruction formula.

$$\mathcal{R}_\eta^\dagger[\mathcal{R}_\psi[f]](\mathbf{x}) = K_{\psi, \eta} f(x) \quad (3)$$

where

$$K_{\psi, \eta} := \int_{-\infty}^{\infty} \frac{\widehat{\psi}(\zeta) \widehat{\eta}(\zeta)}{|\zeta|^m} \, d\zeta \quad (4)$$

## 2 Reconstruction Formula

Let  $\eta$  and  $\psi$  be fourier-transformable functions on  $\mathbb{R}$ , and let  $\hat{\psi}$  and  $\hat{\eta}$  be their Fourier transforms, respectively. In what follows, to avoid notational conflict, we denote the evaluation point of the reconstruction as  $\mathbf{y}$ .

Throughout this section, we assume that all functions involved are sufficiently well-behaved (e.g., rapidly decaying and smooth, or belonging to appropriate  $L^1$  or  $L^2$  spaces) such that all interchanges of integration order are justified by Fubini's Theorem.

$$\mathcal{R}_\psi[f](\mathbf{a}, b) = \int_{\mathbb{R}^m} d\mathbf{x} f(\mathbf{x}) \psi(\mathbf{a} \cdot \mathbf{x} - b) \|\mathbf{a}\|^s \quad (5)$$

$$\mathcal{R}_\eta^\dagger[\mathcal{R}_\psi[f]](\mathbf{y}) = \int_{\mathbb{R}^m} d\mathbf{a} \int_{-\infty}^{\infty} db \left( \int_{\mathbb{R}^m} d\mathbf{x} f(\mathbf{x}) \psi(\mathbf{a} \cdot \mathbf{x} - b) \|\mathbf{a}\|^s \right) \eta(\mathbf{a} \cdot \mathbf{y} - b) \|\mathbf{a}\|^{-s} \quad (6)$$

$$= \int_{\mathbb{R}^m} d\mathbf{a} \int_{-\infty}^{\infty} db \int_{\mathbb{R}^m} d\mathbf{x} f(\mathbf{x}) \psi(\mathbf{a} \cdot \mathbf{x} - b) \eta(\mathbf{a} \cdot \mathbf{y} - b) \quad (7)$$

$$= \int_{\mathbb{R}^m} d\mathbf{x} f(\mathbf{x}) \int_{\mathbb{R}^m} d\mathbf{a} \int_{-\infty}^{\infty} db \psi(\mathbf{a} \cdot \mathbf{x} - b) \eta(\mathbf{a} \cdot \mathbf{y} - b) \quad (8)$$

To evaluate the integral, we introduce a change of variables. Consider the transformation from  $(\mathbf{a}, b)$  to  $(z, \mathbf{w})$ :

$$\begin{cases} z &= \mathbf{a} \cdot \mathbf{x} - b \\ \mathbf{w} &= (a_1, a_2, \dots, a_m) \end{cases} \iff \begin{cases} b &= \mathbf{a} \cdot \mathbf{x} - z \\ \mathbf{a} &= \mathbf{w} \end{cases}$$

Where  $(a_1, a_2, \dots, a_m)$  are the components of  $\mathbf{a}$ .

The Jacobian matrix of this transformation is given by:

$$J = \begin{pmatrix} \partial z / \partial a_1 & \partial w_1 / \partial a_1 & \cdots & \partial w_{m-1} / \partial a_1 & \partial w_m / \partial a_1 \\ \partial z / \partial a_2 & \partial w_1 / \partial a_2 & \cdots & \partial w_{m-1} / \partial a_2 & \partial w_m / \partial a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \partial z / \partial a_m & \partial w_1 / \partial a_m & \cdots & \partial w_{m-1} / \partial a_m & \partial w_m / \partial a_m \\ \partial z / \partial b & \partial w_1 / \partial b & \cdots & \partial w_{m-1} / \partial b & \partial w_m / \partial b \end{pmatrix}$$

$$= \begin{pmatrix} x_1 & 1 & 0 & \cdots & 0 & 0 \\ x_2 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{m-1} & 0 & 0 & \cdots & 1 & 0 \\ x_m & 0 & 0 & \cdots & 0 & 1 \\ -1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Where  $(x_1, x_2, \cdots x_m)$  are the components of  $\mathbf{x}$ . The determinant is computed by cofactor expansion along the last row. Let  $C_{(i,j)}$  denote the cofactor corresponding to the element in the  $i$ -th row and  $j$ -th column. Since the only nonzero element in the last row is in position  $(m+1, 1)$ , its sign is given by  $(-1)^{(m+1)+1}$ .

The minor corresponding to this entry is the  $m \times m$  identity matrix, so its cofactor is:

$$C_{((m+1),1)} = (-1)^{(m+1)+1} \cdot \det(I_m) = (-1)^{m+2} \cdot 1 = (-1)^{m+2}$$

Hence:

$$\det(J) = -1 \cdot C_{((m+1),1)} = -1 \cdot (-1)^{m+2} = (-1)^{m+3}, \quad \text{so} \quad |\det(J)| = 1$$

The integral after the transformation is as follows:

$$(\text{RHS}) = \int_{\mathbb{R}^m} d\mathbf{x} f(\mathbf{x}) \int_{\mathbb{R}^m} d\mathbf{w} \int_{-\infty}^{\infty} dz \psi(z) \eta(\mathbf{w} \cdot (\mathbf{y} - \mathbf{x}) + z) \quad (9)$$

$$= \int_{\mathbb{R}^m} d\mathbf{x} f(\mathbf{x}) \int_{\mathbb{R}^m} d\mathbf{w} \int_{-\infty}^{\infty} dz \left( \int_{-\infty}^{\infty} d\zeta \hat{\psi}(\zeta) e^{i\zeta z} \right) \eta(\mathbf{w} \cdot (\mathbf{y} - \mathbf{x}) + z) \quad (10)$$

$$= \int_{\mathbb{R}^m} d\mathbf{x} f(\mathbf{x}) \int_{\mathbb{R}^m} d\mathbf{w} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} d\zeta \hat{\psi}(\zeta) e^{i\zeta z} \eta(\mathbf{w} \cdot (\mathbf{y} - \mathbf{x}) + z) \quad (11)$$

$$= \int_{\mathbb{R}^m} d\mathbf{x} f(\mathbf{x}) \int_{\mathbb{R}^m} d\mathbf{w} \int_{-\infty}^{\infty} d\zeta \hat{\psi}(\zeta) \int_{-\infty}^{\infty} dz e^{i\zeta z} \eta(\mathbf{w} \cdot (\mathbf{y} - \mathbf{x}) + z) \quad (12)$$

In equation (10), we apply the Fourier inversion formula to  $\psi$ , which yields the integral representation of  $\psi$  via  $\hat{\psi}$ .

Next, consider the following transformation:

$$z' = \mathbf{w} \cdot (\mathbf{y} - \mathbf{x}) + z \quad (13)$$

$$\iff z = z' - \mathbf{w} \cdot (\mathbf{y} - \mathbf{x}) \quad (14)$$

$$(\text{RHS}) = \int_{\mathbb{R}^m} d\mathbf{x} f(\mathbf{x}) \int_{\mathbb{R}^m} d\mathbf{w} \int_{-\infty}^{\infty} d\zeta \hat{\psi}(\zeta) \int_{-\infty}^{\infty} dz' e^{i\zeta z'} e^{-i\zeta \mathbf{w} \cdot (\mathbf{y} - \mathbf{x})} \eta(z') \quad (15)$$

$$= \int_{\mathbb{R}^m} d\mathbf{x} f(\mathbf{x}) \int_{\mathbb{R}^m} d\mathbf{w} \int_{-\infty}^{\infty} d\zeta \hat{\psi}(\zeta) e^{-i\zeta \mathbf{w} \cdot (\mathbf{y} - \mathbf{x})} \int_{-\infty}^{\infty} dz' \eta(z') e^{i\zeta z'} \quad (16)$$

$$= \int_{\mathbb{R}^m} d\mathbf{x} f(\mathbf{x}) \int_{\mathbb{R}^m} d\mathbf{w} \int_{-\infty}^{\infty} d\zeta \hat{\psi}(\zeta) e^{-i\zeta \mathbf{w} \cdot (\mathbf{y} - \mathbf{x})} \hat{\eta}(\zeta) \quad (17)$$

$$= \int_{\mathbb{R}^m} d\mathbf{x} f(\mathbf{x}) \int_{-\infty}^{\infty} d\zeta \hat{\psi}(\zeta) \hat{\eta}(\zeta) \int_{\mathbb{R}^m} d\mathbf{w} e^{-i\zeta \mathbf{w} \cdot (\mathbf{y} - \mathbf{x})} \quad (18)$$

$$(19)$$

Considering the transformation from  $\zeta \mathbf{w}$  to  $\mathbf{w}'$ , and noting that its Jacobian is  $|\zeta|^m$ :

$$(\text{RHS}) = \int_{\mathbb{R}^m} d\mathbf{x} f(\mathbf{x}) \int_{-\infty}^{\infty} d\zeta \hat{\psi}(\zeta) \hat{\eta}(\zeta) \int_{\mathbb{R}^m} \frac{d\mathbf{w}}{|\zeta|^m} e^{-i\mathbf{w}' \cdot (\mathbf{y} - \mathbf{x})} \quad (20)$$

$$= \int_{\mathbb{R}^m} d\mathbf{x} f(\mathbf{x}) \int_{-\infty}^{\infty} d\zeta \frac{\hat{\psi}(\zeta) \hat{\eta}(\zeta)}{|\zeta|^m} \int_{\mathbb{R}^m} d\mathbf{w} e^{-i\mathbf{w}' \cdot (\mathbf{y} - \mathbf{x})} \quad (21)$$

$$= \int_{\mathbb{R}^m} d\mathbf{x} f(\mathbf{x}) \int_{-\infty}^{\infty} d\zeta \frac{\hat{\psi}(\zeta) \hat{\eta}(\zeta)}{|\zeta|^m} \delta(\mathbf{y} - \mathbf{x}) \quad (22)$$

$$= \int_{-\infty}^{\infty} d\zeta \frac{\hat{\psi}(\zeta) \hat{\eta}(\zeta)}{|\zeta|^m} \int_{\mathbb{R}^m} d\mathbf{x} f(\mathbf{x}) \delta(\mathbf{y} - \mathbf{x}) \quad (23)$$

$$= f(\mathbf{y}) \int_{-\infty}^{\infty} d\zeta \frac{\hat{\psi}(\zeta) \hat{\eta}(\zeta)}{|\zeta|^m}. \quad (24)$$

Thus, it is shown that the reconstruction formula is possible.

### 3 Examples

#### 3.1 Fourier Transform of the active function $\eta(z)$

We choose the hyperbolic tangent function as the activation function  $\eta$ . The Fourier transform of the hyperbolic tangent function is given by:

$$\hat{\eta}(\zeta) = -i\pi \operatorname{sech}\left(\frac{\pi}{2}\zeta\right) \quad (25)$$

We now calculate why its Fourier transform takes this form.

$$\begin{aligned} \hat{\eta}(\zeta) &= \text{p.v.} \int_{-\infty}^{\infty} \tanh(z) e^{-i\zeta z} dz \\ &= \lim_{R \rightarrow \infty} \int_{-R}^R \tanh(z) e^{-i\zeta z} dz \end{aligned}$$

Here, p.v. denotes the Cauchy principal value. The necessity of taking the Cauchy principal value arises from the fact that  $\tanh(z)$  converges to a constant at infinity, whereas the exponential term exhibits oscillatory behavior.

As direct evaluation of this integral is difficult, we resort to considering a complex integral along a contour  $\mathcal{C}$ .

$$\int_{\mathcal{C}} \tanh(z) e^{-i\zeta z} dz = \int_{\gamma_R} \tanh(z) e^{-i\zeta z} dz + \int_{\ell_R} \tanh(z) e^{-i\zeta z} dz.$$

Where  $\mathcal{C}$  is a closed contour in the complex plane, and  $\gamma_R$  and  $\ell_R$  are specific segments of this contour. For  $\zeta > 0$ , we choose  $\mathcal{C}$  to be the semi-circular contour in the upper half-plane, consisting of the real axis segment  $[-R, R]$  (denoted by  $\ell_R$ ) and the semi-circular arc  $\gamma_R^+ = \{z = Re^{i\theta} \mid 0 \leq \theta \leq \pi\}$ . Conversely, for  $\zeta < 0$ , we choose the semi-circular contour in the lower half-plane  $\gamma_R^- = \{z = Re^{-i\theta} \mid 0 \leq \theta \leq \pi\}$ .

Now, we will utilize the Residue Theorem to evaluate this complex integral. The next steps involve:

1. Identifying the contribution from the straight line segment.
2. Calculating the residues.
3. Showing the integral along the semi-circular arc vanishes.

Next, we will calculate the residues at the poles within our chosen contour. The fact that poles of  $\tanh(z)e^{-i\zeta z}$  are located at  $z_k = \pm i/(2 + k)\pi$  for  $k \in \mathbb{Z}$  can be readily seen from the following lemma:

**Lemma 3.1** (Infinite Product Expansion of Hyperbolic Cosine). *The hyperbolic cosine function  $\cosh(z)$  has the infinite product expansion:*

$$\cosh(z) = \prod_{k=0}^{\infty} \left( 1 + \frac{z^2}{\left(k + \frac{1}{2}\right)^2 \pi^2} \right)$$

Since these poles are simple pole, we can use the following lemma to calculate the residues:

**Lemma 3.2** (Residue at a Simple Pole). *Let  $f(z) = \frac{p(z)}{q(z)}$  be a complex function where  $p(z)$  and  $q(z)$  are analytic functions. If  $q(z_0) = 0$  and  $q'(z_0) \neq 0$  (implying  $z_0$  is a simple zero of  $q(z)$ ), and  $p(z_0) \neq 0$ , then  $f(z)$  has a simple pole at  $z = z_0$ , and its residue is given by:*

$$\text{Res}(f(z), z_0) = \frac{p(z_0)}{q'(z_0)}$$

**Case 1:**  $\zeta > 0$  For  $\zeta > 0$ , we choose the semi-circular contour in the lower half-plane. Therefore, we need to consider the poles  $z_k = -i(\pi/2 + k\pi)$  for  $k \in \mathbb{N}$ . The residues at these poles can be computed using Lemma 3.2. Here,  $p(z) = \sinh(z)e^{-i\zeta z}$  and  $q(z) = \cosh(z)$ . The residue at the pole  $z_k = -i(\pi/2 + k\pi)$  is given by:

$$\begin{aligned} \text{Res}(\tanh(z)e^{-i\zeta z}, z_k) &= \frac{\sinh(z_k)e^{-i\zeta z_k}}{\cosh'(z_k)} \\ &= \frac{\sinh(z_k)e^{-i\zeta z_k}}{\sinh(z_k)} \\ &= e^{-i\zeta z_k} \\ &= e^{-i\zeta(-i(\pi/2 + k\pi))} \\ &= e^{\zeta(\pi/2 + k\pi)}. \end{aligned}$$

As  $R \rightarrow \infty$ , the contour enclose an infinite number of poles in the lower half-plane, specifically  $z_j = -i(j + 1/2)\pi$  for  $j \in \mathbb{N}$ . Thus, the total sum of residues will be an infinite series:

$$\begin{aligned}
\sum_{j=0}^{\infty} \text{Res}(\tanh(z)e^{-i\zeta z}, z_j) &= \sum_{j=1}^{\infty} e^{\zeta(j+1/2)\pi} \\
&= e^{\zeta\pi/2} \sum_{j=1}^{\infty} e^{j\zeta\pi} \\
&= e^{\zeta\pi/2} \frac{e^{\zeta\pi}}{1 - e^{\zeta\pi}} \\
&= \frac{i\pi}{2 \sinh\left(\frac{\pi}{2}\zeta\right)}.
\end{aligned}$$

It's important to note that when integrating along a contour in the lower half-plane, the contour is typically traversed anticlockwise. Since we are integrating clockwise this time, a minus sign is attached when using the residue theorem.

**Case 2:**  $\zeta < 0$  By similar calculations, the infinite sum of the residues for  $\zeta < 0$  is given by:

$$\begin{aligned}
\sum_{j=0}^{\infty} \text{Res}(\tanh(z)e^{-i\zeta z}, z_j) &= \sum_{j=1}^{\infty} e^{-\zeta(j+1/2)\pi} \\
&= e^{-\zeta\pi/2} \sum_{j=1}^{\infty} e^{-j\zeta\pi} \\
&= e^{-\zeta\pi/2} \frac{e^{-\zeta\pi}}{1 - e^{-j\zeta\pi}} \\
&= \frac{-i\pi}{2 \sinh\left(\frac{\pi}{2}\zeta\right)}
\end{aligned}$$

Next, we will show that the integral over a semicircle is zero.

**Case 1:**  $\zeta > 0$

For  $\zeta > 0$ , we consider the semicircular contour in the lower half-plane, denoted as  $\gamma_R^- = \{z = Re^{-i\theta} \mid 0 \leq \theta \leq \pi\}$ . We need to show that the integral over this semicircle vanishes as  $R \rightarrow \infty$ .



We have:

$$\begin{aligned}
\left| \int_{\gamma_R^-} \tanh(z) e^{-i\zeta z} dz \right| &= \left| \int_0^\pi \tanh(Re^{-i\theta}) e^{-i\zeta Re^{-i\theta}} (-iRe^{-i\theta}) d\theta \right| \\
&\leq \int_0^\pi \left| \tanh(Re^{-i\theta}) e^{-i\zeta Re^{-i\theta}} (-iRe^{-i\theta}) \right| d\theta \\
&= R \int_0^\pi |\tanh(Re^{-i\theta})| e^{-\zeta R \sin \theta} d\theta \\
&\leq R \int_0^\pi e^{-\zeta R \sin \theta} d\theta.
\end{aligned}$$

Here, we use the following lemma:

**Lemma 3.3** (Laplace's Method). *Let  $f(x)$  be a twice continuously differentiable function on the interval  $[a, b]$ , such that it attains its maximum value at a unique point  $x_0 \in (a, b)$ , i.e.,  $f(x_0) = \max_{a \leq x \leq b} f(x)$ , and  $f''(x_0) < 0$ . Under these conditions, we have:*

$$\int_a^b e^{nf(x)} dx \sim \sqrt{\frac{2\pi}{n|f''(x_0)|}} e^{nf(x_0)} \quad \text{as } n \rightarrow \infty.$$

Here,  $\sim$  denotes that the ratio of the two sides approaches 1 as  $n \rightarrow \infty$ .

Using Laplace's method, we can evaluate the integral as follows:

$$\begin{aligned}
\int_0^\pi e^{-\zeta R \sin \theta} d\theta &\sim \sqrt{\frac{2\pi}{\zeta R |-\cos(0)|}} e^{-\zeta R \sin(\frac{\pi}{2})} \\
&= \sqrt{\frac{2\pi}{\zeta R}} e^{-\zeta R} \\
&= \sqrt{\frac{2\pi}{\zeta}} e^{-\zeta R}.
\end{aligned}$$

Hence, we calculate the integral over the semicircle as follows:

$$\begin{aligned}
R \int_0^\pi e^{-\zeta R \sin \theta} d\theta &\leq R \sqrt{\frac{2\pi}{\zeta R}} e^{-\zeta R} \\
&= \sqrt{\frac{2\pi R}{\zeta}} e^{-\zeta R}.
\end{aligned}$$

Therefore, we conclude that:

$$\int_{\gamma_R^+} \tanh(z) e^{-i\zeta z} dz \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

**Case 2:**  $\zeta < 0$  For  $\zeta < 0$ , we consider the semicircular contour in the upper half-plane, denoted as  $\gamma_R^+ = \{z = Re^{i\theta} \mid 0 \leq \theta \leq \pi\}$ . We need to show that the integral over this semicircle vanishes as  $R \rightarrow \infty$ . We have:

$$\begin{aligned}
\left| \int_{\gamma_R^+} \tanh(z) e^{-i\zeta z} dz \right| &= \left| \int_0^\pi \tanh(Re^{i\theta}) e^{-i\zeta Re^{i\theta}} (iRe^{i\theta}) d\theta \right| \\
&\leq \int_0^\pi \left| \tanh(Re^{i\theta}) e^{i\zeta Re^{i\theta}} (iRe^{i\theta}) \right| d\theta \\
&\leq \int_0^\pi \left| \tanh(Re^{i\theta}) e^{i\zeta Re^{i\theta}} (iRe^{i\theta}) \right| d\theta \\
&= R \int_0^\pi |\tanh(Re^{i\theta})| e^{i\zeta R \sin \theta} d\theta \\
&\leq R \int_0^\pi e^{\zeta R \sin \theta} d\theta.
\end{aligned}$$

Here, using Laplace's lemma, the same calculation yields 0. Finally, we can conclude that the integral over the semicircle vanishes as  $R \rightarrow \infty$ .

Thus, we have:

$$\hat{\eta}(\zeta) = \frac{-i\pi}{2 \sinh\left(\frac{\pi}{2}\zeta\right)}.$$

### 3.2 Inverse Fourier Transform of the dual function $\psi(z)$

We choose the following function as the dual function  $\psi$ :

$$\hat{\psi}(\zeta) = \zeta^{2k-1} e^{-\zeta^2/2}, \quad k \in \mathbb{N}. \quad (26)$$

Now, we calculate its inverse Fourier transform of  $\hat{\psi}(\zeta)$ :

$$\begin{aligned}
\psi(x) &= \int_{-\infty}^{\infty} \hat{\psi}(\zeta) e^{i\zeta x} d\zeta \\
&= \int_{-\infty}^{\infty} \zeta^{2k-1} e^{-\zeta^2/2} e^{i\zeta x} d\zeta.
\end{aligned}$$

Here, let  $I = \int_{-\infty}^{\infty} e^{-\zeta^2/2} e^{i\zeta x} d\zeta$ . From easily calculation, we have

$$\begin{aligned} I &= \int_{-\infty}^{\infty} e^{-\zeta^2/2} e^{i\zeta x} d\zeta \\ &= e^{-\frac{x^2}{4}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(\zeta - ix)^2} d\zeta \\ &= e^{-\frac{x^2}{4}} \sqrt{2\pi}. \end{aligned}$$

## References

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