

数Ⅲ積分 解答

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以下, C を積分定数とする。

$$1. \quad \int dx = x + C$$

$$2. \quad \int x^n dx = \begin{cases} \log|x| + C & (n = -1) \\ \frac{1}{n+1}x^{n+1} + C & (n \neq -1) \end{cases}$$

$$3. \quad \int 2^x dx = \frac{2^x}{\log 2} + C$$

$$4. \quad \begin{aligned} \int \log x dx &= x \log x - \int dx \\ &= x \log x - x + C \end{aligned}$$

$$\begin{aligned} 5. \quad \int x \log x dx &= \int \left(\frac{1}{2}x^2\right)' \log x dx \\ &= \frac{1}{2}x^2 \log x - \int \frac{1}{2}x dx \\ &= \frac{1}{2}x^2 \log x - \frac{1}{4}x^2 + C \end{aligned}$$

$$\begin{aligned} 6. \quad \int \sin 2x \sin 9x dx &= \int -\frac{1}{2}(\cos 11x - \cos 7x) dx \\ &= \frac{1}{22} \sin 11x + \frac{1}{14} \sin 7x + C \end{aligned}$$

$$7. \quad \int \tan x dx = -\log|\cos x| + C$$

$$\begin{aligned} 8. \quad \int \frac{dx}{\sin x + 1} &= \int \frac{(\sin x - 1)}{(\sin x + 1)(\sin x - 1)} dx \\ &= \int \frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} dx \\ &= \int \frac{1}{\cos^2 x} - \left(-\frac{(\cos x)'}{\cos^2 x}\right) dx \\ &= \tan x - \frac{1}{\cos x} + C \end{aligned}$$

別解：三角関数を含む分数関数の積分→ $t = \tan \frac{x}{2}$ の置換

$$t = \tan \frac{x}{2} \text{とおくと}$$

$$\tan x = \frac{2t}{1-t^2} \quad (\because \text{倍角の公式})$$

$$\tan^2 \frac{x}{2} = \frac{1-\cos x}{1+\cos x} \quad (\because \text{半角の公式})$$

$$\therefore \cos x = \frac{1-t^2}{1+t^2}$$

$$\sin x = \tan x \cos x = \frac{2t}{1+t^2}$$

$$dt = \frac{dx}{2 \cos^2 \frac{x}{2}} = \frac{1+t^2}{2} dx \Rightarrow dx = \frac{2}{1+t^2} dt$$

$$\begin{aligned} \int \frac{dx}{\sin x + 1} &= \int \frac{\frac{2}{1+t^2}}{\frac{2t}{1+t^2} + 1} dt \\ &= \int \frac{2}{t^2 + 2t + 1} dt \\ &= \int \frac{2}{(t+1)^2} dt \\ &= -\frac{1}{t+1} + C = -\frac{2}{\tan \frac{x}{2} + 1} + C \end{aligned}$$

$$\begin{aligned} 9. \quad \int \frac{dx}{\cos x} &= \int \frac{\cos x}{\cos^2 x} dx \\ &= \int \frac{\cos x}{1 - \sin^2 x} dx \\ &= \int \frac{1}{2} \left(\frac{\cos x}{1 - \sin x} + \frac{\cos x}{1 + \sin x} \right) dx \\ &= -\frac{1}{2} \log |1 - \sin x| + \frac{1}{2} \log |1 + \sin x| + C \\ &= \frac{1}{2} \log \left| \frac{1 + \sin x}{1 - \sin x} \right| + C \end{aligned}$$

$$\begin{aligned} 10. \quad \int \frac{dx}{x^2 + 2x + 1} &= \int \frac{dx}{(x+1)^2} \\ &= -\frac{1}{x+1} + C \end{aligned}$$

$$\begin{aligned} 11. \quad \int \frac{dx}{x^2 - 4x + 3} &= \int \frac{1}{2} \left(\frac{1}{x-3} - \frac{1}{x-1} \right) dx \\ &= \frac{1}{2} \log \left| \frac{x-3}{x-1} \right| + C \end{aligned}$$

$$\begin{aligned}
12. \quad \int_{-2}^{\sqrt{5}-2} \frac{dx}{x^2+4x+9} &= \int_{-2}^{\sqrt{5}-2} \frac{dx}{(x+2)^2+5} \\
&= \int_0^{\frac{\pi}{4}} \frac{\sqrt{5}}{5(1+\tan^2\theta)} \times \frac{1}{\cos^2\theta} d\theta \quad (x = \sqrt{5}\tan\theta - 2) \\
&= \frac{\sqrt{5}}{20}\pi
\end{aligned}$$

例題：

$$\begin{aligned}
&\int_{-1}^{\sqrt{3}} \frac{dx}{x^2+4x+5} \\
&= \int_{-1}^{\sqrt{3}} \frac{dx}{(x+2)^2+1} \\
&= \int_{\frac{\pi}{4}}^{\frac{5}{12}\pi} \frac{1}{1+\tan^2\theta} \cdot \frac{1}{\cos^2\theta} d\theta \quad (x+2 = \tan\theta) \\
&= \int_{\frac{\pi}{4}}^{\frac{5}{12}\pi} d\theta \\
&= \frac{5}{12}\pi - \frac{\pi}{4} \\
&= \frac{\pi}{6}
\end{aligned}$$

補遺：

$\tan \frac{5}{12}\pi = 2 + \sqrt{3}$ はわからなかったかもしれませんが、知っておいて損はありません。 $\frac{\pi}{12}$, $\frac{\pi}{8}$ などは有名角に

加えて導出はできるようにしましょう。 $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$ から加法定理です。また、 $\frac{\pi}{8}$ のように加法定理が使えな

さそうな場合は $\frac{\pi}{8} = \frac{\pi}{4} \times \frac{1}{2}$ から半角の公式を用いて導出します。自分の手を動かして求めないとわからない

こともあります。例えば $\frac{\pi}{8}$ については半角の公式を用いるためその三角関数の 2 乗が求まるのでその平方根を求める必要があります。そのとき、二重根号を外せるものとそうでないものがあるので注意してください。

<https://mathsuke.jp/trigonometric-ratio/> が参考になります。

$$\begin{aligned}
13. \quad \int \frac{2x^2+12x+7}{x^2+5x+1} dx &= \int \left(2 + \frac{2x+5}{x^2+5x+1} \right) dx \\
&= 2x + \log|x^2+5x+1| + C
\end{aligned}$$

$$\begin{aligned}
14. \quad \int \frac{3x+1}{(x+2)^2} dx &= \int \frac{(x+2) \times 3 - 5}{(x+2)^2} dx \\
&= \int \left(\frac{3}{x+2} - \frac{5}{(x+2)^2} \right) dx \quad (\text{部分分数分解}) \\
&= 3\log|x+2| + \frac{5}{x+2} + C
\end{aligned}$$

$$\begin{aligned}
15. \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{\frac{1-x}{1+x}} dx &= \int_{\frac{\pi}{3}}^{\frac{\pi}{6}} -2 \tan \theta \sin 2\theta d\theta \quad (x = \cos 2\theta) \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 4 \sin^2 \theta d\theta \quad (\because \sin 2\theta = 2 \sin \theta \cos \theta) \\
&= 2 \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1 - \cos 2\theta) d\theta \\
&= \frac{\pi}{3}
\end{aligned}$$

$$16. \quad \int x^x (1 + \log x) dx = x^x + C$$

$$17. \quad I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx, \quad J = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x} dx \text{ とおく。}$$

J に対して $x = \frac{\pi}{2} - t$ とおくと,

$$\begin{aligned}
J &= \int_{\frac{\pi}{2}}^0 \frac{\cos\left(\frac{\pi}{2} - t\right)}{\sin\left(\frac{\pi}{2} - t\right) + \cos\left(\frac{\pi}{2} - t\right)} dx \\
&= \int_0^{\frac{\pi}{2}} \frac{\sin t}{\sin t + \cos t} dt \\
&= I \quad \text{であり,}
\end{aligned}$$

$$\begin{aligned}
I + J &= \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2} \\
\therefore I &= \frac{\pi}{4}
\end{aligned}$$

上の $I = J$ の導出は $I - J = 0$ を示すことと同値なので次のようにもできる。

別解：

$$\begin{aligned}
I - J &= \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{\sin x + \cos x} dx \\
&= [\log(\sin x + \cos x)]_0^{\frac{\pi}{2}} \\
&= 0 - 0 = 0 \\
\therefore I &= J
\end{aligned}$$

$$\begin{aligned}
18. \quad \int \frac{dx}{e^x + 1} &= \int \frac{e^{-x}}{1 + e^{-x}} dx \\
&= -\log(1 + e^{-x}) + C
\end{aligned}$$

別解：

$$\begin{aligned}
\int \frac{dx}{e^x + 1} &= \int \left(1 - \frac{e^x}{e^x + 1}\right) dx \\
&= x - \log(1 + e^x) + C \quad (= \log e^x - \log(1 + e^x) + C = -\log(1 + e^{-x}) + C)
\end{aligned}$$

$$19. \quad \int \sqrt{e^x} + 1 \, dx = \int \left(e^{\frac{x}{2}} + 1 \right) dx \\ = 2e^{\frac{x}{2}} + x + C$$

$$20. \quad \int \frac{dx}{\sqrt{x^2 + 1}}$$

$$t = x + \sqrt{x^2 + 1} \text{とおくと}$$

$$dt = \left(1 + \frac{x}{\sqrt{x^2 + 1}} \right) dx \\ = \frac{t}{\sqrt{x^2 + 1}} dx \quad \text{より}$$

$$J = \int \frac{dt}{t} \\ = \log |t| + C \\ = \log(x + \sqrt{x^2 + 1}) + C$$

$$21. \quad \int \sqrt{x^2 + 1} \, dx = \int (x)' \sqrt{x^2 + 1} \, dx \\ = x\sqrt{x^2 + 1} - \int \frac{x^2}{\sqrt{x^2 + 1}} \, dx \\ = x\sqrt{x^2 + 1} - \int \sqrt{x^2 + 1} \, dx + \int \frac{dx}{\sqrt{x^2 + 1}} \\ \therefore \int \sqrt{x^2 + 1} \, dx = x\sqrt{x^2 + 1} + \log(x + \sqrt{x^2 + 1}) + C \quad (\because 20.)$$

$$\text{別解 1 : } t = x + \sqrt{x^2 + 1} \Leftrightarrow x = \frac{1}{2} \left(t - \frac{1}{t} \right) \quad (t > 0) \text{とおく。}$$

$$x^2 + 1 = \frac{1}{4} \left(t^2 - 2 + \frac{1}{t^2} \right) + 1 = \frac{1}{4} \left(t + \frac{1}{t} \right)^2$$

$$dx = \frac{1}{2} \left(1 + \frac{1}{t^2} \right) dt$$

であるので,

$$\int \sqrt{x^2 + 1} \, dx = \int \frac{1}{2} \left(t + \frac{1}{t} \right) \cdot \frac{1}{2} \left(1 + \frac{1}{t^2} \right) dt \\ = \int \frac{1}{4} \left(t + \frac{2}{t} + \frac{1}{t^3} \right) dt \\ = \frac{1}{2} \cdot \frac{1}{2} \left(t + \frac{1}{t} \right) \cdot \frac{1}{2} \left(t - \frac{1}{t} \right) + \frac{1}{2} \log t + C \\ = \frac{1}{2} x \sqrt{x^2 + 1} + \frac{1}{2} \log(x + \sqrt{x^2 + 1}) + C$$

別解 2 : $(\log(x + \sqrt{x^2 + 1}))' = \frac{1}{\sqrt{x^2 + 1}}$ より, $t = \log(x + \sqrt{x^2 + 1}) \Leftrightarrow x = \frac{e^t - e^{-t}}{2}$ とおく。

$$\begin{aligned}\sqrt{x^2 + 1} &= \frac{e^t + e^{-t}}{2} \\ dx &= \frac{e^t + e^{-t}}{2} dt\end{aligned}$$

$$\begin{aligned}\int \sqrt{x^2 + 1} dx &= \int \left(\frac{e^t + e^{-t}}{2} \right)^2 dt \\ &= \frac{1}{8}(e^{2t} - e^{-2t}) + \frac{1}{2}t + C \\ &= \frac{1}{2} \cdot \frac{e^t + e^{-t}}{2} \cdot \frac{e^t - e^{-t}}{2} + \frac{1}{2}t + C \\ &= \frac{1}{2}x\sqrt{x^2 + 1} + \frac{1}{2}\log(x + \sqrt{x^2 + 1}) + C\end{aligned}$$

$$\begin{aligned}22. \quad \int \frac{x}{\sqrt{x+1}+1} dx &= \int \frac{x(\sqrt{x+1}-1)}{(\sqrt{x+1}+1)(\sqrt{x+1}-1)} dx \\ &= \int (\sqrt{x+1}-1) dx \\ &= \frac{2}{3}(x+1)^{\frac{3}{2}} - x + C\end{aligned}$$

$$\begin{aligned}23. \quad I &= \int_{-1}^1 \frac{x^2}{1+e^x} dx = \int_1^{-1} \frac{t^2}{1+e^{-t}} (-1) dt \quad (x = -t) \\ &= \int_{-1}^1 \frac{t^2 e^t}{1+e^t} dt (= J) \\ I + J &= \int_{-1}^1 t^2 dt \\ &= \frac{2}{3} \\ \therefore I = J &= \frac{1}{3}\end{aligned}$$

→cf.)King Property

$$\begin{aligned}24. \quad I &= \int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx \\ &= e^x \sin x - e^x \cos x - \int e^x \sin x dx \\ &= e^x(\sin x - \cos x) - I \\ \therefore I &= \frac{1}{2}e^x(\sin x - \cos x) + C\end{aligned}$$

$$\begin{aligned}
25. \quad \int_{\alpha}^{\beta} (x - \alpha)^n (x - \beta) dx &= \int_{\alpha}^{\beta} \left\{ \frac{1}{n+1} (x - \alpha)^{n+1} \right\}' (x - \beta) dx \\
&= \left[\frac{1}{n+1} (x - \alpha)^{n+1} (x - \beta) \right]_{\alpha}^{\beta} - \int_{\alpha}^{\beta} \frac{1}{n+1} (x - \alpha)^{n+1} dx \\
&= 0 - \left[\frac{1}{(n+1)(n+2)} (x - \alpha)^{n+2} \right]_{\alpha}^{\beta} \\
&= -\frac{1}{(n+1)(n+2)} (\beta - \alpha)^{n+2}
\end{aligned}$$

$$26. \quad I = \int_0^{\pi} \frac{x \sin x}{3 + \sin^2 x} dx$$

ある積分 $J = \int_0^{\pi} x f(\sin x) dx$ に対して $x = \pi - t$ とおくと

$$\begin{aligned}
J &= \int_{\pi}^0 (\pi - t) f(\sin(\pi - t)) (-1) dt \\
&= \int_0^{\pi} (\pi - x) f(\sin x) dx \\
&= \pi \int_0^{\pi} f(\sin x) dx - J \\
\therefore J &= \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx
\end{aligned}$$

よって,

$$\begin{aligned}
I &= \frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{3 + \sin^2 x} dx \\
&= \frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{3 + (1 - \cos^2 x)} dx \\
&= \frac{\pi}{2} \int_1^{-1} \frac{-1}{4 - t^2} dt \quad (\cos x = t) \\
&= \frac{\pi}{8} \int_{-1}^1 \left(\frac{1}{2-t} + \frac{1}{2+t} \right) dt \\
&= \frac{\pi}{8} \left[\log \frac{2+t}{2-t} \right]_{-1}^1 \\
&= \frac{\pi}{8} \left(\log 3 - \log \frac{1}{3} \right) \\
&= \frac{\pi}{4} \log 3
\end{aligned}$$