— 数 III 積分 —

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以下，C を積分定数とする。

1. $$\int dx = x + C$$

2. $$\int x^n \, dx = \begin{cases} \log|x| + C & (n = -1) \\ \dfrac{1}{n+1}x^{n+1} + C & (n \neq -1) \end{cases}$$

3. $$\int 2^x \, dx = \frac{2^x}{\log 2} + C$$

4. $$\begin{aligned} \int \log x \, dx &= x \log x - \int dx \\ &= x \log x - x + C \end{aligned}$$

5. $$\begin{aligned} \int x \log x \, dx &= \int (\frac{1}{2}x^2)' \log x \, dx \\ &= \frac{1}{2}x^2 \log x - \int \frac{1}{2}x \, dx \\ &= \frac{1}{2}x^2 \log x - \frac{1}{4}x^2 + C \end{aligned}$$

6. $$\begin{aligned} \int \sin 2x \sin 9x \, dx &= \int -\frac{1}{2}(\cos 11x - \cos 7x) \, dx \\ &= \frac{1}{22} \sin 11x + \frac{1}{14} \sin 7x + C \end{aligned}$$

7. $$\int \tan x \, dx = -\log|\cos x| + C$$

8. $$\begin{aligned} \int \frac{dx}{\sin x + 1} &= \int \frac{(\sin x - 1)}{(\sin x + 1)(\sin x - 1)} \, dx \\ &= \int \frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} \, dx \\ &= \int \frac{1}{\cos^2 x} - \left(-\frac{(\cos x)'}{\cos^2 x} \right) dx \\ &= \tan x - \frac{1}{\cos x} + C \end{aligned}$$

別解：三角関数を含む分数関数の積分→ $t=\tan\dfrac{x}{2}$ の置換

$t=\tan\dfrac{x}{2}$ とおくと

$\tan x=\dfrac{2t}{1-t^2}$　(∵倍 角の公式)

$\tan^2\dfrac{x}{2}=\dfrac{1-\cos x}{1+\cos x}$　(∵半 角の公式)

$\therefore \cos x=\dfrac{1-t^2}{1+t^2}$

$\sin x=\tan x\cos x=\dfrac{2t}{1+t^2}$

$dt=\dfrac{dx}{2\cos^2\dfrac{x}{2}}=\dfrac{1+t^2}{2}\,dx \Rightarrow dx=\dfrac{2}{1+t^2}\,dt$

$$\begin{aligned}\int\frac{dx}{\sin x+1}&=\int\frac{\dfrac{2}{1+t^2}}{\dfrac{2t}{1+t^2}+1}\,dt\\&=\int\frac{2}{t^2+2t+1}\,dt\\&=\int\frac{2}{(t+1)^2}\,dt\\&=-\frac{1}{t+1}+C=-\frac{2}{\tan\dfrac{x}{2}+1}+C\end{aligned}$$

9.
$$\begin{aligned}\int\frac{dx}{\cos x}&=\int\frac{\cos x}{\cos^2 x}\,dx\\&=\int\frac{\cos x}{1-\sin^2 x}\,dx\\&=\int\frac{1}{2}\left(\frac{\cos x}{1-\sin x}+\frac{\cos x}{1+\sin x}\right)dx\\&=-\frac{1}{2}\log|1-\sin x|+\frac{1}{2}\log|1+\sin x|+C\\&=\frac{1}{2}\log\left|\frac{1+\sin x}{1-\sin x}\right|+C\end{aligned}$$

10.
$$\begin{aligned}\int\frac{dx}{x^2+2x+1}&=\int\frac{dx}{(x+1)^2}\\&=-\frac{1}{x+1}+C\end{aligned}$$

11.
$$\begin{aligned}\int\frac{dx}{x^2-4x+3}&=\int\frac{1}{2}\left(\frac{1}{x-3}-\frac{1}{x-1}\right)dx\\&=\frac{1}{2}\log\left|\frac{x-3}{x-1}\right|+C\end{aligned}$$

12. $$\begin{aligned}\int_{-2}^{\sqrt{5}-2} \frac{dx}{x^2+4x+9} &= \int_{-2}^{\sqrt{5}-2} \frac{dx}{(x+2)^2+5} \\ &= \int_0^{\frac{\pi}{4}} \frac{\sqrt{5}}{5(1+\tan^2\theta)} \times \frac{1}{\cos^2\theta}\,d\theta \quad (x=\sqrt{5}\tan\theta-2) \\ &= \frac{\sqrt{5}}{20}\pi\end{aligned}$$

例題：

$$\begin{aligned}&\int_{-1}^{\sqrt{3}} \frac{dx}{x^2+4x+5} \\ &= \int_{-1}^{\sqrt{3}} \frac{dx}{(x+2)^2+1} \\ &= \int_{\frac{\pi}{4}}^{\frac{5}{12}\pi} \frac{1}{1+\tan^2\theta}\cdot\frac{1}{\cos^2\theta}\,d\theta \quad (x+2=\tan\theta) \\ &= \int_{\frac{\pi}{4}}^{\frac{5}{12}\pi} d\theta \\ &= \frac{5}{12}\pi-\frac{\pi}{4} \\ &= \frac{\pi}{6}\end{aligned}$$

補遺：
$\tan\frac{5}{12}\pi = 2+\sqrt{3}$ はわからなかったかもしれませんが，知っておいて損はありません。$\frac{\pi}{12}, \frac{\pi}{8}$ などは有名角に加えて導出はできるようにしましょう。$\frac{\pi}{12}=\frac{\pi}{3}-\frac{\pi}{4}$ から加法定理です。また，$\frac{\pi}{8}$ のように加法定理が使えなさそうな場合は $\frac{\pi}{8}=\frac{\pi}{4}\times\frac{1}{2}$ から半角の公式を用いて導出します。自分の手を動かして求めないとわからないこともあります。例えば $\frac{\pi}{8}$ については半角の公式を用いるためその三角関数の 2 乗が求まるのでその平方根を求める必要があります。そのとき，二重根号を外せるものとそうでないものがあるので注意してください。https://mathsuke.jp/trigonometric-ratio/ が参考になります。

13. $$\begin{aligned}\int \frac{2x^2+12x+7}{x^2+5x+1}\,dx &= \int\left(2+\frac{2x+5}{x^2+5x+1}\right)dx \\ &= 2x+\log|x^2+5x+1|+C\end{aligned}$$

14. $$\begin{aligned}\int \frac{3x+1}{(x+2)^2}\,dx &= \int \frac{(x+2)\times 3-5}{(x+2)^2}\,dx \\ &= \int\left(\frac{3}{x+2}-\frac{5}{(x+2)^2}\right)dx \quad (\text{部分分数分解}) \\ &= 3\log|x+2|+\frac{5}{x+2}+C\end{aligned}$$

15. $$\int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{\frac{1-x}{1+x}}\,dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{6}} -2\tan\theta\sin 2\theta\,d\theta \quad (x=\cos 2\theta)$$
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 4\sin^2\theta\,d\theta \quad (\because \sin 2\theta = 2\sin\theta\cos\theta)$$
$$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1-\cos 2\theta)\,d\theta$$
$$= \frac{\pi}{3}$$

16. $$\int x^x(1+\log x)\,dx = x^x + C$$

17. $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}\,dx$, $J = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x+\cos x}\,dx$とおく。

J に対して $x = \dfrac{\pi}{2} - t$ とおくと,

$$J = \int_{\frac{\pi}{2}}^{0} \frac{\cos\left(\frac{\pi}{2}-t\right)}{\sin\left(\frac{\pi}{2}-t\right)+\cos\left(\frac{\pi}{2}-t\right)}\,dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{\sin t}{\sin t+\cos t}\,dt$$
$= I$ 　であり,

$$I+J = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$$
$$\therefore I = \frac{\pi}{4}$$

上の $I=J$ の導出は $I-J=0$ を示すことと同値なので次のようにもできる。

別解：

$$I-J = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{\sin x+\cos x}\,dx$$
$$= [\log(\sin x+\cos x)]_0^{\frac{\pi}{2}}$$
$$= 0-0 = 0$$
$$\therefore I = J$$

18. $$\int \frac{dx}{e^x+1} = \int \frac{e^{-x}}{1+e^{-x}}\,dx$$
$$= -\log(1+e^{-x}) + C$$

別解：

$$\int \frac{dx}{e^x+1} = \int \left(1 - \frac{e^x}{e^x+1}\right) dx$$
$$= x - \log(1+e^x) + C \quad (= \log e^x - \log(1+e^x) + C = -\log(1+e^{-x}) + C)$$

19. $\displaystyle\int \sqrt{e^x} + 1\,dx = \int\left(e^{\frac{x}{2}} + 1\right) dx$

$$= 2e^{\frac{x}{2}} + x + C$$

20. $\displaystyle\int \frac{dx}{\sqrt{x^2+1}}$

$t = x + \sqrt{x^2+1}$とおくと

$$dt = \left(1 + \frac{x}{\sqrt{x^2+1}}\right) dx$$

$$= \frac{t}{\sqrt{x^2+1}}\,dx \text{ より}$$

$$J = \int \frac{dt}{t}$$

$$= \log|t| + C$$

$$= \log(x + \sqrt{x^2+1}) + C$$

21. $\displaystyle\int \sqrt{x^2+1}\,dx = \int (x)'\sqrt{x^2+1}\,dx$

$$= x\sqrt{x^2+1} - \int \frac{x^2}{\sqrt{x^2+1}}\,dx$$

$$= x\sqrt{x^2+1} - \int \sqrt{x^2+1}\,dx + \int \frac{dx}{\sqrt{x^2+1}}$$

$$\therefore \int \sqrt{x^2+1}\,dx = x\sqrt{x^2+1} + \log(x + \sqrt{x^2+1}) + C \quad (\because\ 20.)$$

別解 1：$t = x + \sqrt{x^2+1} \Leftrightarrow x = \dfrac{1}{2}\left(t - \dfrac{1}{t}\right)$ $(t > 0)$ とおく。

$$x^2 + 1 = \frac{1}{4}\left(t^2 - 2 + \frac{1}{t^2}\right) + 1 = \frac{1}{4}\left(t + \frac{1}{t}\right)^2$$

$$dx = \frac{1}{2}\left(1 + \frac{1}{t^2}\right) dt$$

であるので，

$$\int \sqrt{x^2+1}\,dx = \int \frac{1}{2}\left(t + \frac{1}{t}\right) \cdot \frac{1}{2}\left(1 + \frac{1}{t^2}\right) dt$$

$$= \int \frac{1}{4}\left(t + \frac{2}{t} + \frac{1}{t^3}\right) dt$$

$$= \frac{1}{2} \cdot \frac{1}{2}\left(t + \frac{1}{t}\right) \cdot \frac{1}{2}\left(t - \frac{1}{t}\right) + \frac{1}{2}\log t + C$$

$$= \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\log(x + \sqrt{x^2+1}) + C$$

別解 2：$(\log(x+\sqrt{x^2+1}))' = \dfrac{1}{\sqrt{x^2+1}}$ より，$t = \log(x+\sqrt{x^2+1}) \Leftrightarrow x = \dfrac{e^t - e^{-t}}{2}$ とおく。

$$\begin{aligned}\sqrt{x^2+1} &= \frac{e^t+e^{-t}}{2}\\ dx &= \frac{e^t+e^{-t}}{2}\,dt\end{aligned}$$

$$\begin{aligned}\int \sqrt{x^2+1}\,dx &= \int \left(\frac{e^t+e^{-t}}{2}\right)^2 dt\\ &= \frac{1}{8}(e^{2t}-e^{-2t}) + \frac{1}{2}t + C\\ &= \frac{1}{2}\cdot\frac{e^t+e^{-t}}{2}\cdot\frac{e^t-e^{-t}}{2} + \frac{1}{2}t + C\\ &= \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\log(x+\sqrt{x^2+1}) + C\end{aligned}$$

22. $$\begin{aligned}\int \frac{x}{\sqrt{x+1}+1}\,dx &= \int \frac{x(\sqrt{x+1}-1)}{(\sqrt{x+1}+1)(\sqrt{x+1}-1)}\,dx\\ &= \int (\sqrt{x+1}-1)\,dx\\ &= \frac{2}{3}(x+1)^{\frac{3}{2}} - x + C\end{aligned}$$

23. $$\begin{aligned}I = \int_{-1}^{1} \frac{x^2}{1+e^x}\,dx &= \int_{1}^{-1} \frac{t^2}{1+e^{-t}}(-1)\,dt \quad (x=-t)\\ &= \int_{-1}^{1} \frac{t^2e^t}{1+e^t}\,dt\ (=J)\\ I+J &= \int_{-1}^{1} t^2\,dt\\ &= \frac{2}{3}\\ \therefore I = J &= \frac{1}{3}\end{aligned}$$

$\longrightarrow$cf.)King Property

24. $$\begin{aligned}I = \int e^x\sin x\,dx &= e^x\sin x - \int e^x\cos x\,dx\\ &= e^x\sin x - e^x\cos x - \int e^x\sin x\,dx\\ &= e^x(\sin x - \cos x) - I\end{aligned}$$

$$\therefore I = \frac{1}{2}e^x(\sin x - \cos x) + C$$

25. $$\int_\alpha^\beta (x-\alpha)^n(x-\beta)\,dx = \int_\alpha^\beta \left\{\frac{1}{n+1}(x-\alpha)^{n+1}\right\}'(x-\beta)\,dx$$
$$= \left[\frac{1}{n+1}(x-\alpha)^{n+1}(x-\beta)\right]_\alpha^\beta - \int_\alpha^\beta \frac{1}{n+1}(x-\alpha)^{n+1}\,dx$$
$$= 0 - \left[\frac{1}{(n+1)(n+2)}(x-\alpha)^{n+2}\right]_\alpha^\beta$$
$$= -\frac{1}{(n+1)(n+2)}(\beta-\alpha)^{n+2}$$

26. $$I = \int_0^\pi \frac{x\sin x}{3+\sin^2 x}\,dx$$

ある積分$J = \displaystyle\int_0^\pi xf(\sin x)\,dx$ に対して $x = \pi - t$ とおくと

$$J = \int_\pi^0 (\pi - t)f(\sin(\pi - t))(-1)\,dt$$
$$= \int_0^\pi (\pi - x)f(\sin x)\,dx$$
$$= \pi\int_0^\pi f(\sin x)\,dx - J$$
$$\therefore J = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$$

よって,

$$I = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{3+\sin^2 x}\,dx$$
$$= \frac{\pi}{2}\int_0^\pi \frac{\sin x}{3+(1-\cos^2 x)}\,dx$$
$$= \frac{\pi}{2}\int_1^{-1} \frac{-1}{4-t^2}\,dt \;(\cos x = t)$$
$$= \frac{\pi}{8}\int_{-1}^1 \left(\frac{1}{2-t} + \frac{1}{2+t}\right)dt$$
$$= \frac{\pi}{8}\left[\log\frac{2+t}{2-t}\right]_{-1}^1$$
$$= \frac{\pi}{8}\left(\log 3 - \log\frac{1}{3}\right)$$
$$= \frac{\pi}{4}\log 3$$

補遺：わざわざ上の積分 J を導入する必要はありません。普通に $x = \pi - t$ と置換積分して $J = (\pi$ が前についた積分$) - J$ という関係を導ければ解けます。