Project #13 Richardson's Arms Model

Introduction Lewis Richardson (1881-1953) was a meteorlogist in Britain. A man of wide interests and abilities, he made contributions to science in the areas of meteorology, fluid dynamics, fractals and chaos theory .During World War I, he served for France in their medical corps and saw first hand the horrors of warfare. After the war, he began to analytically think about the arms buildups going on in Europe, being concerned that it would lead to another global conflict. The data he gathered and the mathematical model he developed are the subject of this project.

Background familiarity with solutions of systems and matrix algebra; the notion of a discrete dynamical system.

The Model Suppose for sake of discussion we study the behavior of three nations; A,B and C. Suppose nation A is quite aggressive and war prone, nation B a fairly neutral and passive nation (like Switzerland much of this century), and nation C is a reluctant foe of nation A. Suppose we assign variables x, y and z to them respectively, which indicate the amount of arms that each nation has. A convenient unit of measurement is money.

The arms level that each nation has at time t=k+1; one unit of time from now; may depend on four general things:

(1) the amount of arms they already had at time t = k

(2) the amount of arms they might build in response to the other nations arms levels

(3) the amount of arms they might have gotten rid of due to their internal tendencies

(as we have seen in the US, maintaining armed forces can be expensive and sometimes is the subject of cutbacks in peacetimes due to other priorities or budget deficits)

(4) if they are particularly warlike, the amount of arms they would build anyway, even if no other nations presented a threat

These four factors allow us to consider a system of three equations for our three hypothetical nations:

$$x(k+1)= f_1\, x(k) + a_{12}\, y(k) + a_{13}\, z(k) + g_1$$

$$y(k+1) = f_2\, y(k) + a_{21}\, x(k) + a_{23}\, z(k) + g_2$$

$$z(k+1) = f_3\, z(k) + a_{31}\, x(k) + a_{32}\, y(k) + g_3$$

where, in general, the f_i are "fatigue" coefficients described in (3), above, the g_i are

"grievances" described in (4), and the a_{ij} would represent the response of nation i to the arms level of nation j.

These equations might be for any three nations; for our three hypothetical nations, one might assume that $g_2 = 0$,g_1 is positive, a_{32} =0 (since B and C are not enemies), and perhaps that a_{12} and a_{13} are greater than 1 (indicating that A overcompensates for everything the other nations have). Possibly f_1=1, f_2 might be zero and f_3 <1 indicating that in the absence of other armed nations, A keeps all its anyway, B gets rid of all of its, and C keeps some of its. Possibly a_{31} might be 1 indicating that for every arm that A has, C will build one. On the other hand, if a_{13} = 1.2, this would indicate that for every arm that C has, A will build 20% more.

In matrix form, we have

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + \mathbf{g} \quad k = 0,1,2,3,... \qquad \mathbf{x}(0) \text{ given}$$

where in the above example,

$$A = \begin{pmatrix} f_1 & a_{12} & a_{13} \\ a_{21} & f_2 & a_{23} \\ a_{31} & a_{32} & f_3 \end{pmatrix} \qquad g = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}$$

In general, one should see that that matrix A has response terms in the off-diagonal entries and (1-fatigue factors) on the diagonal.

Steady State It is possible to have an equilibrium situation. This would be where $\mathbf{x}(k+1) = \mathbf{x}(k)$ or that

$$\mathbf{x}_S = A\mathbf{x}_S + \mathbf{g}$$

which is algebraically equivalent to $(I_n - A)\mathbf{x}_S = \mathbf{g}$ **.**

Several considerations are in order here. This is a nonhomogeneous system so

* there might be no solution if the rank of I - A is less than n

* there might be a unique solution

* the solution might have some negative components, which would physically make no sense

If a nonnegative steady state occurs, it might suggest that an "uneasy peace" existed.

A Solution to the System

As in other projects, we may iteratively solve this

$\mathbf{x}(1) = A\mathbf{x}(0) + \mathbf{g}$

$\mathbf{x}(2) = A\mathbf{x}(1) + \mathbf{g} = A(A\mathbf{x}(0) + \mathbf{g}) + \mathbf{g} = A^2\mathbf{x}(0) + A\mathbf{g} + \mathbf{g}$

$\mathbf{x}(3) = A\mathbf{x}(2) + \mathbf{g} = A(A^2\mathbf{x}(0) + A\mathbf{g} + \mathbf{g}) + \mathbf{g} = A^3\mathbf{x}(0) + A^2\mathbf{g} + A\mathbf{g} + \mathbf{g}$

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and apparently in general, that

$$\mathbf{x}(k) = A^k\mathbf{x}(0) + (A^{k-1} + A^{k-2} + \ldots + A + I_n)\,\mathbf{g}$$

(one might note that the term in front of g in parentheses is a partial geometric series in A)

As earlier in the course, this solution, for k large, may either

* tend to infinity (unstable)

* tend to the steady state (stable)

* go to zero

At this point, we only have enough tools to determine this by direct simulation.

Richardson's Model of the World in 1935

Richardson spent considerable time and effort after WWI gathering data to describe 10 nations and their arms dynamics. He published the 1935 version which includes the following values for the matrix A:

Czech	.5	0	0	.1	0	0	0	.05	0	0	.05
China	0	.05	0	0	0	0	.2	0	0	.1	.05
France	0	0	0	.1	.2	0	.2	0	0	0	.05
Germany	.2	0	.2	5	.1	0	0	.05	0	.04	.15
England	0	0	0	.2	.25	.3	.1	0	0	0	g = .05
Italy	0	0	.1	0	.2	.75	0	0	0	.1	.10
Japan	0	.2	0	0	0	0	.5	0	.2	.2	.15
Poland	.05	0	0	.05	0	0	0	.5	0	.05	.05
US	0	0	0	.1	.1	.1	.2	0	.65	.1	.05
USSR	0	.1	0	.4	.1	.1	.2	.05	0	.5	.10

(here the time interval is .05 years)

Problems

1. In the basic model, argue why the fatigue coefficients, f_i, must be in the range

$$0 < f_i < 1$$

what would the $f_i = 1$ mean? $f_i = 0$?

2.a. Suppose the coefficient matrix for a group of 4 nations

$$\begin{pmatrix} .2 & 1.05 & 1.03 & 0 \\ .5 & .4 & 0 & 0 \\ .4 & 0 & 1 & 0 \\ 0 & 0 & 0 & .22 \end{pmatrix}$$

what can you say about the relationships of the 4 nations?

b. Assuming no grievances and equal initial arms levels $(1\ 1\ 1\ 1)^t$, simulate the above arms race and decide if it is stable or not. If it is stable, what is the equilibrium? If it is not stable, find the factor by which it is eventually geometrically growing.

3. Suppose we have a situation where all n nations fall into two groups with i of them in the first group and n-i in the second group. Suppose we have organized them so the first group is variables x_1 through x_i and the others in

x_{i+1} through xn . Further assume that all those in the first group are allied with each other against all those in the second group, and vice versa.

What can you say about how the matrix A will look (no actual numbers, just +,- or 0 for individual entries) ?

4. In terms of the entries in the matrix A, how would you go about spotting nations that are only defensive in their posture? How would a neutral nation like Switzerland appear?

5. In Richardson's 1935 matrix above,

a. what nation is the most aggressive? Why/how did you decide?

b. what nation is most disliked or feared? Why/how did you decide?

6. In general, neglecting diagonal entries, what would

a. what would **row** sums represent?

b. what would **column** sums represent?

7. Consider Richardson's 1935 model of the world.

a. By simulating it, decide if it is stable or not. If it is stable, what is the steady state? If it is unstable, what is the **factor** by which it geometrically grows? (1 decimal place in either) What nation has the most arms in the long run?

b. What is the steady state for this system?

8. a. There have been many attempts in this century in the area of "arms controls", especially after the development and escalation of nuclear weapons following World War II and more recently, the existence of biological and chemical weapons. Suppose you work for the United Nations. Describe in words a scheme you might find realistic for arms control. Once you have described it, interpret it mathematically and modifiy Richardson's model accordingly.

b. Apply your arms control scheme to the problem in **2a** and show that it does indeed do what you have intended it to.

c. Is your scheme "fair"? Will some nations feel discriminated against?

References

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