

NAME _____ ROLL _____

Instructions

1. This is an OPEN BOOK examination.
2. The question paper has total 35 marks.
3. Student should write their roll number and name on the question paper.
4. Do not attach the rough work with answer sheet.
5. Total time for the examination is 1 hours.

Best of Luck

1. Let \mathcal{X} be a **countable** set, and let $f : 2^{\mathcal{X}} \rightarrow \mathbb{R}$ be **permutation-invariant**, meaning $f(\{x_1, \dots, x_M\})$ depends only on the set, not the order of listing its elements. Prove that there exist functions $\phi : \mathcal{X} \rightarrow \mathbb{R}$ and $\rho : \mathbb{R} \rightarrow \mathbb{R}$ such that for every finite set $S \subseteq \mathcal{X}$,

$$f(S) = \rho \left(\sum_{x \in S} \phi(x) \right).$$

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Since \mathcal{X} is countable, fix an injection $c : \mathcal{X} \rightarrow \mathbb{N}$. Define

$$\phi(x) := 4^{-c(x)} \quad \text{and} \quad E(S) := \sum_{x \in S} \phi(x) = \sum_{x \in S} 4^{-c(x)}$$

for every finite $S \subseteq \mathcal{X}$.

We claim that E is injective on finite subsets. Suppose $E(S) = E(T)$ for finite S, T . Then

$$0 = E(S) - E(T) = \sum_{n \in \mathbb{N}} \alpha_n 4^{-n},$$

where $\alpha_n \in \{-1, 0, 1\}$ indicates whether n appears in $c(S)$ but not $c(T)$, in $c(T)$ but not $c(S)$, or in neither/both. If $S \neq T$, let n_0 be the smallest index with $\alpha_{n_0} \neq 0$. Multiplying by 4^{n_0} gives

$$0 = \alpha_{n_0} + \sum_{n > n_0} \alpha_n 4^{-(n-n_0)}.$$

The tail satisfies

$$\left| \sum_{n > n_0} \alpha_n 4^{-(n-n_0)} \right| \leq \sum_{k \geq 1} 4^{-k} = \frac{1/4}{1 - 1/4} = \frac{1}{3},$$

so it cannot cancel $\alpha_{n_0} \in \{\pm 1\}$, a contradiction. Hence $S = T$, and E is injective.

Define ρ on $\text{range}(E)$ by $\rho(E(S)) := f(S)$. This is well-defined by injectivity. Extend ρ arbitrarily to a function $\rho : \mathbb{R} \rightarrow \mathbb{R}$. Then for every finite $S \subseteq \mathcal{X}$,

$$f(S) = \rho(E(S)) = \rho \left(\sum_{x \in S} \phi(x) \right),$$

as required.

2. Let B_n be the set of $n \times n$ **doubly stochastic** matrices:

$$B_n = \{A \in \mathbb{R}^{n \times n} : A_{ij} \geq 0, \sum_j A_{ij} = 1 \forall i, \sum_i A_{ij} = 1 \forall j\}.$$

Recall that a point x in a convex set C is an **extreme point** if in the convex combination of two distinct points in C ; i.e., $x = ty + (1-t)z$ for $y, z \in C$ and $t \in (0, 1)$ implies $x = y = z$. Show that every $n \times n$ permutation matrix is an extreme point of B_n .

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Let P be a permutation matrix and suppose

$$P = tA + (1-t)B$$

for some $A, B \in B_n$ and $t \in (0, 1)$. If $P_{ij} = 0$, then

$$0 = tA_{ij} + (1-t)B_{ij},$$

and since $A_{ij}, B_{ij} \geq 0$, it follows that $A_{ij} = B_{ij} = 0$.

Fix a row i . There is a unique column $\sigma(i)$ such that $P_{i\sigma(i)} = 1$ and $P_{ij} = 0$ for $j \neq \sigma(i)$. From the previous paragraph, $A_{ij} = 0$ for all $j \neq \sigma(i)$. Using the row-sum constraint on A ,

$$1 = \sum_j A_{ij} = A_{i\sigma(i)},$$

so $A_{i\sigma(i)} = 1$. The same argument gives $B_{i\sigma(i)} = 1$ and $B_{ij} = 0$ for $j \neq \sigma(i)$.

Thus every entry of A and B matches the corresponding entry of P , so $A = B = P$. Therefore P is an extreme point of B_n .

3. Say $X = \{x_1, \dots, x_n\}$ where $x_i \in \mathbb{R}$ and $Y = \{y_1, \dots, y_n\}$ with $y_i \in \mathbb{R}$. Prove that $\min_P \sum_{i,j} |x_i - y_j| P_{ij} = |\text{Sort}(X) - \text{Sort}(Y)|$, where P is a permutation matrix.

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Let $x_{(1)} \leq \dots \leq x_{(n)}$ and $y_{(1)} \leq \dots \leq y_{(n)}$ be the sorted lists. Matching via a permutation matrix P is equivalent to choosing a permutation π with cost

$$\sum_{i=1}^n |x_{(i)} - y_{(\pi(i))}|.$$

It suffices to show that for every π ,

$$\sum_{i=1}^n |x_{(i)} - y_{(\pi(i))}| \geq \sum_{i=1}^n |x_{(i)} - y_{(i)}|.$$

We use the inequality: if $a \leq b$ and $c \leq d$, then

$$|a - d| + |b - c| \geq |a - c| + |b - d|.$$

To see this, define $g(z) = |a - z| - |b - z|$. For $a \leq b$, the function g is nonincreasing in z (its slope is 0 for $z \leq a$, -2 for $a \leq z \leq b$, and 0 for $z \geq b$). Hence $g(d) \leq g(c)$, i.e.

$$(|a - d| - |b - d|) \leq (|a - c| - |b - c|),$$

which rearranges to the desired inequality.

If π is not increasing, it has an inversion: there exist $i < j$ with $\pi(i) > \pi(j)$. Then

$$x_{(i)} \leq x_{(j)} \quad \text{and} \quad y_{(\pi(j))} \leq y_{(\pi(i))}.$$

Applying the inequality with $a = x_{(i)}$, $b = x_{(j)}$, $c = y_{(\pi(j))}$, $d = y_{(\pi(i))}$ shows that swapping $\pi(i)$ and $\pi(j)$ does not increase the total cost. Repeating this inversion-removal procedure yields an increasing permutation, which must be the identity, and the cost can only decrease along the way. Therefore the minimum is achieved by matching in sorted order:

$$\min_P \sum_{i,j} |x_i - y_j| P_{ij} = \sum_{i=1}^n |x_{(i)} - y_{(i)}|.$$

By definition, the right-hand side is $|\text{Sort}(X) - \text{Sort}(Y)|$.

Total: 35
