Practice Questions: Submodular functions

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1) Submodularity, marginal gains, and diminishing returns

Let V be a finite ground set and $f : 2^V \to \mathbb{R}$. Recall the following definitions:

- f is **submodular** if $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$ for all $A, B \subseteq V$.
- The **marginal gain** is $\Delta_f(e \mid S) := f(S \cup \{e\}) - f(S)$.
- f is **monotone** if $A \subseteq B \Rightarrow f(A) \leq f(B)$, and **normalized** if $f(\varnothing) = 0$.

(a) Prove that f is submodular if and only if it satisfies the **diminishing returns** property: for all $A \subseteq B \subseteq V$ and $e \in V \setminus B$,

$$\Delta_f(e \mid A) \geq \Delta_f(e \mid B).$$

(b) Using the diminishing returns property, determine whether the following functions $f : 2^V \to \mathbb{R}$ are submodular:

- $f(S) = |S|$
- $f(S) = \min\{|S|, k\}$ for some constant k.
- $f(S) = |S|^2$

2) Modular and Supermodular functions, and properties of submodular functions

(a) Prove the following **closure properties**: if f and g are submodular functions, then $f + g$, αf (for $\alpha \geq 0$), and $f + c$ (for constant c) are also submodular. Additionally, show that the restriction of a submodular function to a subset of the ground set preserves submodularity.
(b) A function m is **modular** if $m(S) = \sum_{i \in S} w_i$ for some weights w_i. Prove that a function is modular if and only if it is both submodular and supermodular.
(c) Let $g : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ be a nondecreasing concave function. Prove that the set function defined by $f(S) = g(|S|)$ is submodular.
(d) Define the **complement-transform** of f by $f^c(S) = f(V \setminus S)$. Prove that f is submodular if and only if f^c is submodular.
(e) Prove that if f is a normalized, monotone, and submodular function, then it is **subadditive**: $f(A \cup B) \leq f(A) + f(B)$ for all $A, B \subseteq V$.

3) Lovász extension

Let $V = \{1, 2, \ldots, n\}$ and $f : 2^V \to \mathbb{R}$ with $f(\varnothing) = 0$. For $x \in [0, 1]^n$, the **Lovász extension** is defined as:

$$\hat{f}(x) := \sum_{k=1}^{n} \left(x_{(k)} - x_{(k+1)}\right) f(S_k),$$

where $x_{(1)} \geq x_{(2)} \geq \cdots \geq x_{(n)}$ are the sorted coordinates, $x_{(n+1)} = 0$, and $S_k = \{(1), \ldots, (k)\}$.

(a) Show that $\hat{f}(\mathbf{1}_S) = f(S)$ for all $S \subseteq V$, where $\mathbf{1}_S$ is the indicator vector of S.
(b) Prove that f is submodular if and only if its Lovász extension $\hat{f}$ is convex on $[0, 1]^n$.

4) Greedy maximization under a cardinality constraint

Let $f : 2^V \to \mathbb{R}_{\geq 0}$ be a normalized, monotone, and submodular function. Consider the problem $\max\{f(S) : |S| \leq k\}$. Let $S^\star$ be an optimal solution and S_t be the set produced by the greedy algorithm after t iterations.

(a) Prove the key greedy inequality:

$$\max_{e \in V \setminus S_t} \Delta_f(e \mid S_t) \;\geq\; \frac{f(S^\star) - f(S_t)}{k}.$$

(b) Let $G_t = f(S_t)$. Using the inequality from part (a), derive a recurrence for G_t and solve it to show that:

$$f(S_k) \geq \left(1 - \left(1 - \frac{1}{k}\right)^k\right) f(S^\star).$$

Conclude that the greedy algorithm provides a $1 - 1/e$ approximation guarantee.

5) The Multilinear Extension

Let $f : 2^V \to \mathbb{R}$ be a submodular function. The **multilinear extension** $F : [0,1]^V \to \mathbb{R}$ is defined as $F(x) = \mathbb{E}[f(R_x)]$, where R_x is a random set including each element i independently with probability x_i.

(a) Prove the **Partial Derivative Identity**:

$$\frac{\partial F}{\partial x_i}(x) = \mathbb{E}[\Delta_f(i \mid R_x \setminus \{i\})].$$

(b) Prove that if f is monotone, then F is monotone in each coordinate.
(c) Prove that if f is submodular, F satisfies **DR-submodularity** (Continuous Diminishing Returns): for any $x \leq y$ coordinate-wise, $\nabla F(x) \geq \nabla F(y)$.