Homework 9

Question 1

Below is an invalid "proof" that if R is a relation on a set A which is symmetric and transitive, then R is reflexive.

"Proof"

i. Suppose $a \in A$.

ii. Take an element $b \in A$ such that $(a, b) \in R$.

iii. Because R is symmetric, we also have $(b, a) \in R$.

iv. Because R is transitive, $(a, b) \in R$, and $(b, a) \in R$, we have $(a, a) \in R$, as desired.

Explain what is wrong with the above.

Step 2 is wrong, since you wrongfully assume that there is another element that exists which will create a relation.

Write down an example of a nonempty relation R on $\{1, 2, 3\}$ which is symmetric and transitive, but not reflexive.

$$(1, 1), (1, 2), (2, 1), (2, 2)$$

This is not reflexive because 3 is not in R.

Question 2

Let S be the relation on $\mathbb{R}$ defined by $(x, y) \in S$ if and only if $x - y$ is rational. Is S symmetric? Anti-symmetric? Reflexive? Anti-reflexive? Transitive? Justify your answers with proofs.

(i)

$$(x, x) \in S$$

Since $x - x = 0$ is a rational number, (x, y) is a reflexive relation.

(ii)

$$(x, y) \in S$$

Take $x = 2, y = 1$.

$$x - y = 2 - 1 = 1$$

is a rational number. However, when we try $(y, x) \in S$:

$$y - x = 1 - 2 = -1$$

$$(x, y) \neq (y, x)$$

Therefore, it is not a symmetric relation.

(iii)

Say $(x, y) \in S$, $(y, z) \in S$. That means $(x, z) \in S$ is a rational number.

Take $x = 2, y = 1, z = 3$.

$$x - y = 2 - 1 = 1$$

is rational.

$$y - z = 1 - 3 = -2$$

is rational.

$$x - z = 2 - 3 = -1$$

is rational. Therefore, $(x, y) \in S$ is transitive.

(iv)

$$(x, y) \in S$$

$x - y$ is rational.

$$(y, x) \in S$$

$y - x$ is rational. However, $x \neq y$. Therefore the relation is not anti-symmetric. Finally the relation is reflexive, so it is not anti-reflexive.

Question 3

Suppose R is a strict order on a set A. Prove that R^{-1} is a strict order on A.

A strict order is a relation which is both transitive and anti-reflexive.

(i)

Assume that $(x, y) \in R^{-1}$ and $(y, z) \in R^{-1}$. Then by definition, you have $(y, x) \in R$ and $(z, y) \in R$. Since R is transitive, $(z, x) \in R$ and then $(x, z) \in R^{-1}$.

(ii)

Assume that R is anti-reflexive. Then, $(x, x) \notin R$. Assume R^{-1} is reflexive. Then, $(x, x) \in R^{-1}$. Then, $(x, x) \in R$. This is a contradiction, since R is supposed to be anti-reflexive. Therefore, R^{-1} is anti-reflexive.

Question 4

Prove or disprove: If R is a transitive relation on a set A, then $R \circ R$ is transitive.

Let $q \in R \circ R$ be arbitrary. Then, $q = (x, y) \in A$. There is a $z \in A$ such that $(x, z) \in R$ and $(z, y) \in R$. Since q is arbitrary. $R \in R$ is transitive.

Question 5

Prove that if R is a symmetric relation on a set A then R^+ is symmetric as well.

Let $(a, b) \in R$. That means $(b, a) \in R$. This means $(a, b) \in R^+$ as well, since $R \in R^+$.

Let (x, y) be an arbitrary element of R^+. Then,

$$(x, y) \in R^+, (y, x) \in R, (x, y) \in R$$

Therefore, R^+ is symmetric.

Question 6

Write down a topological sort of the following DAG.

3 2 6 4 1 5

Question 7

Prove that if G is a DAG and _ is a strict order, then _ is a topological sort for G iff _ is a topological sort for G^+.

Since $G^+ \in G$, $\prec$ is a topological sort for both. You can say the same thing vice versa.

Question 8

Let $G = (V, E)$ be a nonempty DAG. Our goal is to construct a topological sort for G.

(i) For each $v \in V$, we define $G - \{v\}$ to be the directed graph with vertex set $V - \{v\}$ and edge set $\{(u, w) \in E : u, w \neq v\}$. Prove that $G - \{v\}$ is acyclic.

Since this removes all edges with the vertex as well as the vertex itself, G remains a DAG. Therefore, G is acyclic.

(ii)

Prove that in G, there is a vertex v such that there are no edges from any vertex u to v.

Since G is a DAG, that means there has to be at least one vertex where no edges go to it.