Homework 2

Problem 1

Prove that if A and B are sets, then

$$(A \oplus B) \oplus B = A$$

I will prove this using a **proof by cases**.

Say that A is an empty set (ϕ) and B is a non-empty set.

$$A \oplus B = B$$

$$B \oplus B = \varnothing$$

$\varnothing = A$, so the equation works for this combination.

Say that A is a non-empty set and B is an empty set ($\varnothing$).

$$A \oplus B = A$$

$$A \oplus B = A$$

$A = A$, so the equation works for this combination.

Say that both A and B are non-empty sets, with $A = \{1, 2, 3, 4\}$ and $B = \{5, 6\}$.

$$\{1, 2, 3, 4\} \oplus \{5, 6\} = \{1, 2, 3, 4, 5, 6\}$$

$$\{1, 2, 3, 4, 5, 6\} \oplus \{5, 6\} = \{1, 2, 3, 4\}$$

$\{1, 2, 3, 4\} = A$, so the equation works for this combination.

Say that both A and B are empty sets.

$$\varnothing \oplus \varnothing = \varnothing$$

$$\varnothing \oplus \varnothing = \varnothing$$

$\varnothing = A$, so the equation works for this combinations.

Since the equation works for all combinations, this statement is true. ■

Problem 2

Suppose A and B are subsets of C. Prove that $A \subseteq B$ if and only if $(C - A) \cup B = C$.

We will break this up into two proofs.

The first proof can be written as

$$\text{If } (C - A) \cup B = C, \text{then } A \subseteq B.$$

Since you first subtract the elements in A from C and then add the elements in B to get the same original set, it follows that A and B are made up of the same elements. Therefore, $A \subseteq B$.

The second proof can be written as

$$\text{If } A \subseteq B, \text{ then } (C - A) \cup B = C$$

If $A \subseteq B$, you can substitute one set for the other.

$$\begin{gathered}(C - B) \cup B = C \\ C = C\end{gathered}$$

Therefore, the statement is true. ■

Problem 3

Prove or disprove each of the following statements.

$$\text{For every set } A\text{, we have } A \in P(A).$$
$$\text{For every set } A\text{, we have } A \subseteq P(A).$$

The definition of a power set is a combination of all the subsets of a set. This includes the original elements in the set. Therefore, $A \in P(A)$.

However, the second statement is not true. Say we have a set $A = \{1, 2, 3\}$. The power set is $\{\{\}, 1, 2, 3, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. This is not the same elements as A. The statement is false. $\blacksquare$

Problem 4

Suppose A is a subset of B. Prove that $P(A) \subseteq P(B)$.

Suppose an element $X \in P(A)$. Then, $X \subseteq A$ (since by definition, $P(A)$ is all the subsets of A), and thus $X \subseteq B$. Therefore, $X \in P(B)$ and $P(A) \subseteq P(B)$. ■

Problem 5

Suppose A, B, C are sets such that A is **nonempty**. Prove that $B \subseteq C$ if and only if $A \times B \subseteq A \times C$.

Let an element $a \in A$. If $x \in B$, then $(a, x) \in A \times B$. Then $(a, x) \in A \times C$, so $x \in C$. Thus, $B \subseteq C$. $\blacksquare$