Homework 6

Question 1

Prove that $n^3 + 3n + 4$ is $\mathrm{O}(2n^3)$.

Since the highest degree in this equation is n^3 and $\mathrm{O}(2n^3)$ can be simplified to $\mathrm{O}(n^3)$, this statement is true.

Question 2

Prove that $n^{3/2}$ is not O(n).

Since $3/2 > 1$, and O(n) only works for equations that have a degree of 1, $n^{3/2}$ is not O(n).

Question 3

Consider the following recursive algorithm Square(n), which takes as input $n \in \mathbb{N}$:

If $n = 0$, output 0.

Otherwise, output Square($n-1$) + $n + n - 1$.

Use induction to prove that for all $n \in \mathbb{N}$, Square(n) terminates and outputs n^2.

Base Case: If we substitute $n = 1$,

$$\text{Square}(0) + 1 + 1 - 1 = 0 + 2 - 1 = 1$$

Since $1^2 = 1$.

Inductive Step: Assume that for $k \geq 0$, Square(k) returns k^2 for all $k \in \mathbb{N}$. Then we will prove that Square($k+1$) returns $(k+1)^2$ for all $k \in \mathbb{N}$.

$$\text{Square}(k+1) = \text{Square}(k) + 2k + 1$$

$$\text{Square}(k) = \text{Square}(k) + \text{Square}(k-1) + \ldots + \text{Square}(1) = k^2$$

$$\text{Square}(k+1) = k^2 + 2k + 1 = (k+1)^2 \quad \blacksquare$$

Let $T(n)$ denote the number of additions or subtraction operations performed by Square(n). Write down a recurrence relation for $T(n)$.

$$3 \cdot (n-1)$$

Question 4

Use strong induction to prove that for all lists L of positive integers, LocalMin(L) terminates, and its output (denoted by B) satisfies the following: Either $b = 0$ and L has no local minimum or $b \neq 0$ and b is a local minimum of L.

Base Case: We must prove that LocalMin works for $L = 2$ and $L = 3$.

If L has length 2 or less, it outputs zero. ✓

If L has length 2 or less, it will output zero or a local minimum. ✓

Inductive Step

Assume that for $k \geq 2$, LocalMin(k) is true. Then, we will try to prove that LocalMin($k + 1$) returns a local minimum or no local minimum.

Write down a recurrence relation for $T(n)$.

$$2T(|L|/2) + \theta(1)$$