Homework 3

Problem 1

We attempt to define a function $f : \mathbb{Q}^+ \to \mathbb{Z}$ by

$$f(x) = p$$

where $x = p/q$ with p and q being positive integers.

Show that f is **not** well-defined.

This is false. Here is a counterexample:

Say you have a variable x, expressed as 2, where p is 4 and q is 2. Therefore,

$$f(2) = 4$$

Have another variable $x = 2$, where p is 8 and q is 4. Therefore,

$$f(2) = 8$$

One element in the domain maps to more than 1 element. Therefore, $f(x)$ is not well-defined. ■

Problem 2

Suppose $f : A \to B$ is a function which is one-to-one. Suppose C_1 and C_2 are subsets of A. Prove that

$$\{f(x) : x \in C_1\} \cap \{f(x) : x \in C_2\} \subseteq \{f(x) : x \in C_1 \cap C_2\}$$

Let

$$\begin{gathered}
y \in \{f(x) : x \in C_1\} \cap \{f(x) : x \in C_2\} \\
y \in \{f() : x \in C_1\}, y \in \{f(x) : x \in C_2\} \\
y = f(x_1) \text{ for } x_1 \in C_1, y = f(x_2) \text{ for some } x_2 \in C_2 \\
f(x_1) = f(x_2)
\end{gathered}$$

Since f is one-to-one, $f(x_1) = f(x_2) \to x_1 = x_2$

Since $x_1 \in C_1$ and $x_2 \in C_2$ and $x_1 = x_2$:

$$\begin{gathered}
x_1 = x_2 = C_1 \cap C_2 \\
y \in f(x_1), x_1 \in C_1 \cap C_2 \\
y \in \{f(x) : x \in C_1 \cap C_2\} \blacksquare
\end{gathered}$$

Problem 3

Throughout this question, we fix a set X.

1. Suppose $A \in P(x)$. Prove that $A = X - (X - A)$.

Since you are first removing the elements that are in A from set X, then subtracting this from X again, you are left with A.

2. Prove that for any set A, we have $X - A \in P(x)$

Since $P(x)$ is all the subsets of X, simply subtracting any amount of values from set X will still result in another subset inside $P(x)$.

3. Define $f : P(x) \to P(x)$ by $f(A) = X - A$. Prove that f is well-defined.

The definition of well-defined is if every element of the domain is mapped to an element in the target. Since we proved that $X - A \in P(x)$, we know that f is injective. Therefore, f is well-defined.

4. Prove that f is one-to-one.

See above.

5. Prove that f is onto.

Since $X - A$ is always in the range, it can always be expressed in terms of the domain. Therefore, the function is onto.

6. We know that f^{-1} is a function. Prove that f^{-1} equals f.

Since $f : P(x) \to P(x)$, f^{-1} must also be $P(x) \to P(x)$. Therefore, $f^{-1} = f$.

Problem 4

Suppose that A, B, C, D are sets such that $B \subseteq C$. Suppose $g : A \to B$ and $f : C \to D$ are functions. Prove that if $f \circ g$ is onto, then f is onto.

Let $d \in D$. Then since $f \circ g$ is onto, there exists $a \in A$ such that $(f \circ g)a = f(g(a)) = d$.

Therefore, if we let $y = g(a) \in B$, then $f(y) = c$. Thus, f is onto. ■