Homework 4

Question 1

Use induction to prove that $n^2 - 5n$ is even, for every $n \in \mathbb{N}$.

Try base case first:

For $n = 1$,

$$1 - 5 = -4,$$

which is even.

Let $k \in \mathbb{N}$ and suppose the expression is true for $n = k$. Then,

$$k^2 - 5k \text{ is even.}$$

We must prove that $(k+1)^2 - 5(k+1)$ is even as well.

$$k^2 + 2k + 1 - 5k - 5 = k^2 - 3k - 4$$

$$= (k^2 - 5k) + (2k - 4)$$

Since $2k - 4$ always ends in an even number, and adding two even numbers always results in an even number, the expression works for $(k+1)$. Therefore, the expression is true. ■

Question 2

Define a sequence $\{a_n\}_{n\in\mathbb{N}}$ by

$$a_0 = 0 \text{ and } a_n = a_{n-1} + n^3 \text{ for } n \geq 1$$

Prove that for each $n \in \mathbb{N}$,

$$a_n = \left(\frac{n(n+1)}{2}\right)$$

Try base case first:

$$a_0 = \left(\frac{0(0+1)}{2}\right) = 0 \ \checkmark$$

Let $a_k \in \{a_n\}$ and assume the expression is true for a_k. We must prove that

$$\begin{aligned}
a_{k+1} &= \left(\frac{(k+1)(k+2)}{2}\right) \\
&= \frac{k^2+3k+2}{2} \\
&= \frac{k^2+k+2k+2}{2} \\
&= \frac{k(k+1)}{2} + k + 1 \\
a_{k+1} &= a_k + k + 1 \ \checkmark
\end{aligned}$$

Question 3

(a) For each positive integer $n \leq 4$, compute whether $n! \geq n^2$ or not.

For $n = 4$, $4! \geq 4^2$.

For $n = 3$, $3! < 3^2$.

For $n = 2$, $2! < 2^2$.

For $n = 1$, $1! = 1^2$.

(b) Prove that for all integers $n \geq 4$, we have $n! \geq n^2$.

We have already proved the base case for $n = 4$ above.

Assume that $k = n$, and $k! \geq k^2$.

We must prove that $(k + 1)! \geq (k + 1)^2$.

$$\begin{gathered}(k + 1) \times k! \geq (k + 1)(k + 1) \\ k! \geq (k + 1)\end{gathered}$$

Since this is true, the expression is true as well.

Question 4

The post office sells an unlimited amount of 2-cent stamps and 5-cent stamps. Prove that for any integer $n \geq 4$, we can buy exactly n cents worth of stamps.

For base case $n = 4$,

$$5(2) - 3(2) = 4 \checkmark$$

Inductive hypothesis:

Any value j $(k \geq j \geq 4)$ can be expressed as $j = 2a + 5b$ with a and b being non-negative integers.

We must prove that we can express $k + 1$ as $2a + 5b$. We can use $k - 1$ since k works for the inductive hypothesis.

$$\begin{aligned} k - 1 &= 2a + 5b \\ k - 1 + 2 &= 2a + 5b + 2 \\ k + 1 &= 2(a + 1) + 5b \checkmark \end{aligned}$$

By the principle of strong induction, the statement is true for every n greater than 4.

Question 6

Let $k = 1$. $4 \times 1 + 1$ is 5.

Given that A starts,

5 A - 2 --> 3 B - 2 --> 1 A - 1 = 0, so B won the game. Let it be true for some n that $4n + 1$. Therefore, we have to show that $n + 1$ is true.

$$4(n + 1) + 1 = 4n + 1 + 4$$

We know that if B removes the stone the last stone is 1 for $4n$ stones are left after B remove the stone.

$$1 + 4 = 5$$

Now we have already shown if 5 is the number that B will win.

Therefore, we show that for $4k + 1$, $k \in N$, B will always win the game.