Lecture 013

Svadrut Kukunooru

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Recall that last timed we showed that if A is a 2×2 matrix w/ row vectors $\vec{r_1}, \vec{r_2}, \ldots$, or

$$A = \begin{bmatrix} r_1 \\ r_2 \\ \cdots \end{bmatrix}$$

then the determinant of A is the signed area of $P(r_1, r_2)$ (the parallelogram formed by the two vectors).

DEFINITION: for an $n \times n$ matrix A, $\det(A)$ is defined to be the signed volume of $P(r_1, r_2, \ldots r_n)$ (the parallelotope; basically a 3-D version of a parallelogram defined by the row vectors $r_1, r_2, \ldots r_n$).

NOT TESTED ON EXAM, BUT GOOD TO KNOW: The sign of $\det(A)$ reflects whether the matrix function $f_A : \mathbb{R}^n \implies \mathbb{R}^n$ preserves the orientation of $\mathbb{R}^n$ orientation.

For R_1, the orientation is defined as either positive or negative. For R_2, the orientation is defiend as either counterclockwise or clockwise. In R_3, the orientation is defined as either right-handed or left-handed.

We can conclude that whenever E is an elementary matrix,

$$\det(EA) = \det(E) \cdot \det(A)$$

This rule continues to hold for higher dimensional parallelotopes. METHOD: perform elementary row operations to simplify A to one of the base cases, keep track of the determinants of the elementary matrices.

Recall for an $n \times n$ matrix A,

$$\text{RREF}(A) = \begin{cases} I_n & \Rightarrow \text{(A is invertible)} \\ \text{has a row of zeroes} & \Rightarrow \text{(A is not invertible)} \end{cases}$$

EXAMPLE: Compute the determinant of A if

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & -1 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -4 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to I_3$$

Therefore, the determinant is 1, since the determinant of an identity matrix must always be 1.