Lecture 7: Row Echelon Forms

Svadrut Kukunooru

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A matrix is said to be in *reduced row echelon form* (RREF) if:

1. Rows of zeroes appear at bottom of matrix
2. First nonzero entry at each row is one
3. Nonzero entries are placed into an echelon form
4. All entries above leading ones are zero

If a matrix is not in RREF, we can make it into one by using elementary row operations, e.g.

- Exchange two rows
- Multiply row by scalar
- Add one row to another

0.0.1 Example Problem

Simplify the following matrix to RREF:

$$\begin{bmatrix} -1 & 2 & 5 \\ 2 & -1 & 6 \\ 2 & -2 & 7 \end{bmatrix}.$$

First, we can subtract the 2nd and 3rd row by two copies of row one, e.g.

$$\begin{bmatrix} 1 & -2 & 5 \\ 0 & 3 & -4 \\ 0 & 2 & -3 \end{bmatrix}.$$

Then, subtract the 2nd row by two copies of the 3rd row:

$$\begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -1 \\ 0 & 2 & -3 \end{bmatrix}.$$

Subtract the third row by two copies of the second row:

$$\begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Some more manipulations:

$$\begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}.$$

The procedure is this:

1. Make the leading term of 1st row 1 by scaling
2. Kill the 1st entries of all rows below
3. Make the leading form of 2nd row 1 by scaling
4. Kill the leading terms below the second leading 1
5. Repeat the above until you get row echelon form
6. Kill the entries below leading 1's by going bottom-up

DEF: Two matrices are said to be row equivalent if one is obtained from another using elementary row operations.

THEOREM: Any matrix is row equivalent to one in RREF, which is unique (rank? I can't read). A matrix can be row equivalent to multiple row echelon forms, e.g.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

0.0.2 Example Problem

Reduce the following matrix to RREF:

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The only reason this matrix isn't already in RREF is that the 5 and 1 are nonzero entries above the 3rd leading 1. Therefore,

$$\boxed{\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}.$$