Lecture 014

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EXAMPLE: Compute the determinant of the following matrix A:

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 5 & 6 \\ 7 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 4 & 5 & 6 \\ 3 & -1 & 2 \\ 7 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 6 & 4 \\ 3 & -1 & 2 \\ 7 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 6 & 4 \\ 0 & -19 & -10 \\ 0 & -41 & -26 \end{bmatrix} \to \dots$$

$$\to \boxed{84}$$

We will begin with a new way of computing determinants, the cofactor expansion formula.

DEF: Let A by a square matrix. The minor M_{ij} is the matrix obtained by deleting the ith row and jth column of A. For example,

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 5 & 6 \\ 7 & 1 & 2 \end{bmatrix}, M_{1.1} = \begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix}, M_{2.2} = \begin{bmatrix} 3 & 2 \\ 7 & 2 \end{bmatrix}, M_{2.3} = \begin{bmatrix} 3 & -1 \\ 7 & 1 \end{bmatrix}$$

DEF: The cofactor at a_{ij} is defined as as

$$A_{ij} = (-1)^{i+j} \det(M_{ij})$$

To compute the determinant, take any row or column, take out all the indices (i,j) in this row/columnm then sum up $a_{ij}A_{ij}$. e.g. on a 3x3 matrix, if we compute $\det(A)$ by exapnding from the first row, then we do

$$a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

EXAMPLE: Compute the determinant of the matrix using cofactor expansion.

$$\begin{bmatrix} 3 & -1 & 2 \\ 4 & 5 & 6 \\ 7 & 1 & 2 \end{bmatrix}$$

$$a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$
$$= 3 \det \begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix} - (-1) \det \begin{bmatrix} 4 & 6 \\ 7 & 2 \end{bmatrix} + 2 \det \begin{bmatrix} 4 & 5 \\ 7 & 1 \end{bmatrix}$$
$$= 3 \cdot 4 - 34 + 2(-31) = 12 - 34 - 62 = \boxed{84}$$

EXAMPLE: Compute the determinant of the following matrix:

$$\begin{bmatrix} 4 & 1 & 3 \\ 2 & 3 & 0 \\ 1 & 3 & 2 \end{bmatrix} \to 4 \det \begin{bmatrix} 3 & 0 \\ 3 & 2 \end{bmatrix} - 1 \det \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} + 3 \det \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}$$

$$= 24 - 4 + 9 = \boxed{29}$$

THEOREM: If two matrices A and A' are row equivalent, that means $Ax = 0$ and $A'x = 0$ have the same solutions. Additionally, a homogenous equation

$Ax = 0$ is always consistent and has infinitely many solutions if and only if it has a free variable. Finally, if A is a square matrix, then the following are equivalent:

$$Ax = 0 \text{ has at least one free variable}$$

$$\text{RREF}(A) \text{ has a row of zeroes}$$

$$A \text{ is not invertible}$$

$$\det(A) = 0$$

(basically, if you know one of these statements is true, all of them are)