Homework 1

Problem 1

Show that $\forall x P(x) \vee \forall x Q(x)$ and $\forall x(P(x) \vee Q(x))$ are not logically equivalent, in general.

Say you have predicates $P(x)$ and $Q(x)$ such that

$$P(x) : x \text{ is even}$$
$$Q(x) : x \text{ is odd}$$

For the first expression, $\forall x P(x) \vee \forall x Q(x)$, the expression would be false, since $\forall x$ for each expression includes both even AND odd numbers, which is impossible, since a number cannot be even or odd at the same time.

For the second expression, $\forall x(P(x) \vee Q(x))$, the expression would be true, since this expression allows both even AND odd numbers to be included in a true result.

Therefore, since these two expressions have a different result for the same predicates, they are not logically equivalent. ∎

Problem 2

Prove that if x is a nonzero rational number and y is irrational, then their product xy is irrational.

We will use **proof by contradiction**.

Let x be irrational and y be rational; then, assume xy is rational.

This implies that

$$y = \frac{a}{b},\ xy = \frac{c}{d}$$

for some integers a, b, c, d.

$$x \cdot y = xy \rightarrow \frac{xy}{y} = x \rightarrow \frac{cb}{da} = x$$

This is a contradiction, since it implies that an irrational number x can be represented as a ratio of 2 integers. ∎

Problem 3

Disprove the following statement: If x and y are irrational, then $x+y$ is irrational.

We will use **proof by counterexample**.

Say $x = \pi$ and $y = 4 - \pi$.

$$x + y = \pi + 4 - \pi = 4$$

Since 4 is rational, the statement is false. ∎

Problem 4

Assume that n is an integer. Prove that if $3n+2$ is even, then n is even as well.

We will use **proof by contrapositive**.

The contrapositive of this expression is

$$\text{If } n \text{ is odd, then } 3n+2 \text{ is odd.}$$

Therefore, $n = 2k+1$, for some integer k. Substituting that in $3n+2$,

$$3(2k+1)+2 = 6k+5 = 2(3k+2)+1$$

Making $3k+2 = k'$,

$$3n+2 = 2k'+1$$

where $k' \in \mathbb{Z}$. Therefore, $3n+2$ is odd. If the contrapositive of the expression is true, then the original expression is true as well. ∎