BCS 12-GUESS PAPER

Q1. Show that

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & 0 \end{vmatrix} = 0$$

Where ω is a complex cube root of unity.

SOLUTION: -

Q1.

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0$$

$$= \begin{vmatrix} 1+\omega+\omega^2 & \omega & \omega^2 \\ 1+\omega+\omega^2 & \omega^2 & 1 \\ 1+\omega+\omega^2 & 1 & \omega \end{vmatrix} \quad (C_1 = C_1 + C_2 + C_3)$$

$$= \begin{vmatrix} 0 & \omega & \omega^2 \\ 0 & \omega^2 & 1 \\ 0 & 1 & \omega \end{vmatrix} \quad (\because 1+\omega+\omega^2 = 0)$$

$$= 0$$

Q2. If $A = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}$, Show that $A^2 - 4A + 5I_2 = 0$. Also, find A^4.

SOLUTION: -

Q2. $A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$ $\quad f(A) = A^2 - 4A + 5I_2 = 0$

$$f(A) = A^2 - 4A + 5I_2$$

$$= \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} - 4\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -4 \\ 8 & -1 \end{bmatrix} - \begin{bmatrix} 12 & -4 \\ 8 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -4 \\ 8 & 4 \end{bmatrix} - \begin{bmatrix} 12 & -4 \\ 8 & 4 \end{bmatrix}$$

$= 0$ proved.

$A^2 - 4A + 5I = 0 \Rightarrow A^2 = 4A - 5I$

$A^3 = A^2 . A$
$= (4A - 5I) . A$
$= 4A^2 - 5IA$
$= 4(4A - 5I) - 5A$
$= 16A - 20I - 5A$
$= 11A - 20I$

$A^4 = A^3 . A$
$= (11A - 20I)A$
$= 11A^2 - 20IA$
$= 11(4A - 5I) - 20A$
$= 44A - 55I - 20A$
$= 24A - 55I$
$= 24\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} - 55\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$= \begin{bmatrix} 17 & -24 \\ 48 & -31 \end{bmatrix}$ Ans.

Q4. If p^{th} term of an A.P is q and q^{th} term of the A.P. is p, find its r^{th} term.

SOLUTION: -

$a, a+d, a+2d \ldots$ a = first term, d = common diff

p^{th} term $= q$

$\Rightarrow q = a+(p-1)d$

q^{th} term $= p$

$p = a+(q-1)d.$

$p-q = (q-1)d - (p-1)d$

$p-q = d[q-1-p+1]$

$q = a-(p-1)$

$a = q+(p-1)$

$a = \underline{p+q-1}$

r^{th} term $= a+(r-1)d$

$\Rightarrow p+q-1 + (r-1)(-1)$

$\Rightarrow p+q-1-r+1$

$\Rightarrow p+q-r$

Q5. If 1, ω, ω^2 are cube roots of unity, show that $(2-\omega)(2-\omega^2)(2-\omega^{19})(2-\omega^{23}) = 49$.

SOLUTION: -

$(2-\omega)(2-\omega^2)(2-\omega^{19})(2-\omega^{23}) = 49$

$\omega^{19} = (\omega^3)^6\omega = \omega$ $\quad\quad$ $\omega^{23} = (\omega^3)^7\omega^2 = \omega^2$

Now, Thus

$(2-\omega)(2-\omega^2)(2-\omega^{19})(2-\omega^{23})$

$= (2-\omega)(2-\omega^2)(2-\omega)(2-\omega^2)$

$= [(2-\omega)(2-\omega^2)]^2$

$= [4 - 2\omega^2 - 2\omega + \omega^3]^2$

$= [4 - 2(\omega+\omega^2) + 1]^2$

$= [4 - 2(-1) + 1]^2$

$= [4 + 2 + 1]^2$

$= 49$

Q6. If α, β are roots of $x^2 - 3ax + a^2 = 0$, find the value(s) of a if $\alpha^2 + \beta^2 = \frac{7}{4}$.

SOLUTION: -

Q6. $x^2 - 3ax + a^2 = 0$

$\alpha + \beta = 3a \qquad \alpha\beta = a^2$

$\alpha^2 + \beta^2 = 7/4$

$(\alpha + \beta)^2 - 2\alpha\beta = 7/4$

$9a^2 - 2a^2 = 7/4$

$7a^2 = 7/4$

$a^2 = 1/4$

$a = \pm \frac{1}{2}$

Q7. If $y = ln\left(\frac{\sqrt{1+X}-\sqrt{1-X}}{\sqrt{1+X}+\sqrt{1-X}}\right)$, find $\frac{dy}{dX}$.

SOLUTION: -

$$\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} = \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\ \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}$$

$$= \frac{(\sqrt{1+x}-\sqrt{1-x})^2}{(1+x)-(1-x)}$$

$$= \frac{(1+x)+(1-x)-2\sqrt{1+x}\ \sqrt{1-x}}{2x}$$

$$= \frac{2-2\sqrt{1-x^2}}{2x} = \frac{1-\sqrt{1-x^2}}{x}$$

Therefore, $y = ln\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

$$= ln\left(\frac{1-\sqrt{1-x^2}}{x}\right)$$

$$= ln\ (1-\sqrt{1-x^2}) - lnx$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1-\sqrt{1-x^2}}\ \frac{d}{dx}\left(1-(1-x^2)^{-1/2}\right) - \frac{1}{x}$$

$$= \left[\frac{1}{1-\sqrt{1-x^2}}\left\{\frac{d}{dx}0 - \frac{1}{2}(1-x^2)^{-1/2}(-2x)\right\} - \frac{1}{x}\right]$$

$$= \frac{1}{1-\sqrt{1-x^2}}\ \frac{x}{\sqrt{1-x^2}} - \frac{1}{x}$$

$$= \frac{x^2 - [\sqrt{1-x^2}(1-\sqrt{1-x^2})]}{x\sqrt{1-x^2}(1-\sqrt{1-x^2})}$$

$$= \frac{x^2 - \sqrt{1-x^2} + (1-x^2)}{x\sqrt{1-x^2}(1-\sqrt{1-x^2})}$$

$$= \frac{1-\sqrt{1-x^2}}{x\sqrt{1-x^2}(1-\sqrt{1-x^2})} = \frac{1}{x\sqrt{1-x^2}}$$

Q9. Find the sum of all the integers between 100 and 1000 that are divisible by 9.

Solution : The first integer greater than 100 and divisible by 9 is 108

and the integer just smaller than 1000 and divisible by 9 is 999.

Thus, we have to find the sum of the series.

$108 + 117 + 126 + \ldots\ldots\ldots\ldots + 999.$

Here $t_1 = a = 108$, $d = 9$ and $l = 999$

Let n be the total number of terms in the series be n. Then

$999 = 108 + 9(n-1) \Rightarrow 111 = 12 + (n-1) \Rightarrow n = 100$

Hence, the required sum $= \frac{n}{2}(a+l) = \frac{100}{2}(108 + 999)$

$= 50(1107)$

$= 55350$

Q10. Write De Moivre's theorem and use it to find $(\sqrt{3} + i)^3$.

SOLUTION: -

De Moivre's Theorem (for Integral Index)

Taking $\theta_1 = \theta_2 \ldots\ldots\theta_n = \theta$ in (1) we obtain

$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

This proves the result for positive integral index.

However, it is valid for every integer n.

√3 + I IN THE POLAR FORM

Let $\sqrt{3} + i = r(\cos\theta + i\sin\theta)$

$\Rightarrow \quad \sqrt{3} = r\cos\theta$ and $1 = r\sin\theta$

$\Rightarrow \quad (\sqrt{3})^2 + 1^2 = r^2(\cos^2\theta + \sin^2\theta)$

$\Rightarrow \quad r^2 = 4 \quad \Rightarrow \quad r = 2$

Thus, $\sqrt{3} + i = 2(\cos\theta + i\sin\theta)$

$\Rightarrow \quad \sqrt{3} = 2\cos\theta$ and $1 = 2\sin\theta$

$\Rightarrow \quad 2\cos\theta = \frac{\sqrt{3}}{2}$ and $\sin\theta = \frac{1}{2}$

$\Rightarrow \quad \theta = 30°.$

Now, $(\sqrt{3} + i)^3 = [2\cos(30°) + i\sin(30°)]^3$

$= 2^3 [\cos(30°) + I\sin(30°)]^3$

$= 8 [\cos(3 \times 30°) + i\sin(3 \times 30°)]$ [De Moivre's theorem]

$= 8 (\cos 90° + i\sin 90°) = 8(0 + i)$

$= 8i$

Q11. Solve the equation $x^3 - 13x^2 + 15x + 189 = 0$. Given that one of the roots exceeds the other by 2.

$x^2 - 13x^2 + 15x + 189 = 0 \quad (1)$

being given that one root exceeds the other by 2.

As one root exceeds the other by 2, we may take the root as α, $\alpha+2$ and β.

Now.

$\alpha + (\alpha + 2) + \beta = 13 \quad (2)$

$\alpha(\alpha + 2) + \alpha\beta + (\alpha + 2)\beta = 15 \quad (3)$

and $\alpha(\alpha + 2)\beta = -189 \quad (4)$

From (2), $\beta = 11 - 2\alpha$.

Putting this in (3), we get

$\alpha(\alpha + 2) + (2\alpha + 2)(11 - 2\alpha) = 15$

$\Rightarrow \quad \alpha^2 + 2\alpha + 22\alpha + 22 - 4\alpha^2 - 4\alpha = 15$

$\Rightarrow \quad 3\alpha^2 - 20\alpha - 7 = 0$

$\Rightarrow \quad (3\alpha + 1)(\alpha - 7) = 0$

$\Rightarrow \quad \alpha = -1/3, 7$

when $\alpha = -\frac{1}{3}, \beta = 11 + \frac{2}{3} = \frac{35}{3}$.

But $\alpha = \frac{1}{3}$, $\beta = \frac{35}{3}$ does not satisfy equation (4)

When $\alpha = 7$, $\beta = -3$

These values satisfy equation (4)

Thus, roots of (1) are 7,9 and -3

Q12. Solve the inequality $\left|\frac{2}{X-1}\right| > 5$ and graph its solution.

SOLUTION: -

Q12. $\frac{2}{|x-1|} > 5$

Domain of the inequality is $\{x \mid x \neq 1\}$

For $x \neq 1$ $|x-1| > 0$

$\frac{2}{|x-1|} > 5$

$\Rightarrow 2 > 5|x-1|$

$\Rightarrow |x-1| < \frac{2}{5}$

$\Rightarrow -\frac{2}{5} < x-1 < \frac{2}{5}$

$1-\frac{2}{5} < x < \frac{2}{5}+1$

$\frac{3}{5} < x < \frac{7}{5}$

$\therefore$ the solution set is

$\{x \mid 3/5 < x < 7/5\} = (3/5, 7/5)$

graph:

Q13. Determine the values of x for which $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$ is increasing and for which it is decreasing.

SOLUTION: -

$f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21.$

$f'(x) = 4x^3 - 24x^2 + 44x - 24 = 4(x^3 - 6x^2 + 11x - 6)$

$f'(x) = 4(x-1)(x^2-5x+6)$

For $f(x)$ to be increasing, we must have

$f'(x) > 0$

$\Rightarrow 4(x-1)(x^2-5x+6) > 0$

$\Rightarrow (x-1)(x^2-5x+6) > 0$

$\Rightarrow (x-1)(x-2)(x-3) > 0 \qquad [\because 4 > 0]$

$\Rightarrow 1 < x < 2$ or $3 < x < \infty$

$\Rightarrow x \in (1, 2) \cup (3, \infty)$

F(X) IS INCREASING ON $(1,2) \cup (3, \infty)$

FOR F(X) TO BE DECREASING WE MUST HAVE

$f'(x) < 0$

$\Rightarrow 4(x-1)(x^2-5x+6) > 0$

$\Rightarrow (x-1)(x^2-5x+6) < 0$

$\Rightarrow (x-1)(x-2)(x-3) < 0$

$\Rightarrow 2 < x < 3$ or $x < 1$

$\Rightarrow x \in (2, 3) \cup (-\infty, 1)$

So f(x) is decreasing on $(2,3) \cup (-\infty,1)$

Q14. Find the points of local maxima and local minima of

$f(x) = x^3 - 6x^2 + 9x + 2014$.

Solution : $f'(x) = x^3 - 6x^2 + 9x + 2014$

Thus, $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$.
To obtain critical number of f, we set $f'(x) = 0$ this yields $x = 1, 3$.
Therefore, the critical number of f are $x = 1, 3$.
Now $f'(x) = 6x - 12 = 6(x-2)$
We have $f'(1) = 6(1-2) = -6 < 0$ and $f'(3) = 6(3-2) = 6 > 0$.

Using the second derivative test, we see that $f(x)$ has a local maximum at $x = 1$ and a local minimum at $x = 3$. The value of local maximum at $x = 1$ is $f(1) = 1 - 6 + 9 + 1 = 5$ and the value of local minimum at $x = 3$ is $f(3) = 3^3 - 6(3^2) + 9(3) + 1 = 27 - 54 + 27 + 1 = 1$.

Q15. Evaluate : $\int \frac{dx}{(e^x-1)^2}$

Solution :

Put $e^x - 1 = t$, so that $e^x dx = dt$, and

$$I = \int \frac{dt}{t^2(t+1}$$

We split $\frac{1}{t^2(t+1)}$ into partial fractions

$$\frac{1}{t^2(t+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t+1}$$

$$\Rightarrow 1 = At(t+1) + B(t+1) + Ct^2$$

Put $t = 0$, $t = -1$ to obtain

$1 = B \Rightarrow B = 1$

$1 = C \Rightarrow C = 1$

Comapring coefficient at t^2, we obtain

$0 = A + C \Rightarrow A = -C = -1$

Thus,

$$I = \int \left[-\frac{1}{t} + \frac{1}{t^2} + \frac{1}{t+1}\right] dt$$

$$= -\log|t| - \frac{1}{t} + \log|t+1| + c$$

$$= \log\left|\frac{t+1}{t}\right| - \frac{1}{t} + c$$

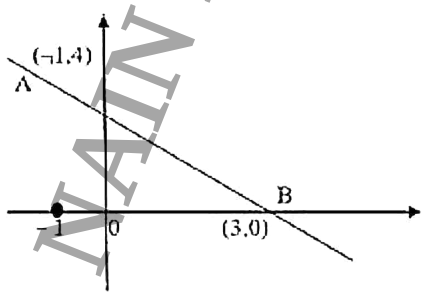
$$= \log\left(\frac{e^x+1}{e^x}\right) - \frac{1}{e^x} + c$$

Q16. Using integration, find length of the curve y = 3 – x from (-1, 4) to (3, 0).

Solution :

We have

$$\frac{dy}{dx} = -1$$



Required length

$$= \int_{-1}^{3} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

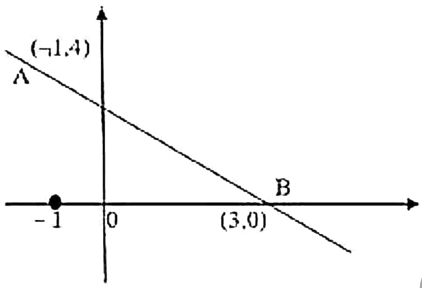
$$= \int_{1}^{3} \sqrt{1+1}\, dx = \sqrt{2}\, x\Big|_{-1}^{3} = \sqrt{2}\,(3+1) = 4\sqrt{2} \text{ units}$$

Q16. Using integration, find length of the curve y = 3 – x from (-1, 4) to (3, 0).

Solution :

We have

$$\frac{dy}{dx} = -1$$



Required length

$$= \int_{-1}^{3} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{1}^{3} \sqrt{1+1}\, dx = \sqrt{2}\, x \Big|_{1}^{3} = \sqrt{2}\,(3+1) = 4\sqrt{2} \text{ units}$$

Q17. Find the sum up to n terms of the series 0.4 + 0.44 + 0.444 + …

The sum of the series is

$0.4 + 0.44 + 0.444 + ... + \text{n terms} = \frac{4}{9}[n - \frac{1-(0.1)^n}{9}]$

Step-by-step explanation:

Given : Series 0.4+0.44+0.444+----- to n terms

To find : The sum of the series?

Solution :

Let the series be 'x'

$x = 0.4 + 0.44 + 0.444 + ... + \text{n terms}$

$x = 4(0.1 + 0.11 + 0.111 + ... + \text{n terms})$

Multiply and divide by 9,

$x = \frac{4}{9}(0.9 + 0.99 + 0.999 + .. + \text{n terms})$

$x = \frac{4}{9}[(1 - 0.1) + (1 - 0.01) + (1 - 0.001)... + \text{n terms}]$

$x = \frac{4}{9}[(1-0.1)+(1-(0.1)^2)+(1-(0.1)^3)...+\text{n terms}]$

$x = \frac{4}{9}[(1+1+1+...\text{n terms})-(0.1+(0.1)^2+(0.1)^3...+\text{n terms})]$

$x = \frac{4}{9}[(n)-(0.1(\frac{1-(0.1)^n}{1-0.1}))]$

$x = \frac{4}{9}[n-\frac{1-(0.1)^n}{9}]$

Therefore, The sum of the series is

$0.4+0.44+0.444+...+\text{n terms} = \frac{4}{9}[n-\frac{1-(0.1)^n}{9}]$

Q18. Show that the lines $\frac{x-5}{4}=\frac{y-7}{-4}=\frac{z-3}{-5}$ and $\frac{x-8}{4}=\frac{y-4}{-4}=\frac{z-5}{4}$ Intersect.

Q18 $\frac{x-5}{4}=\frac{y-7}{-4}=\frac{z-3}{-5}$ and $\frac{x-8}{4}=\frac{y-4}{-4}=\frac{z-5}{4}$

Converting these equations into vectors

$\vec{r} = (5\hat{i}+7\hat{j}+3k)+t(4\hat{i}-4\hat{j}-5k)$

$\vec{r} = (8\hat{i}+4\hat{j}+5k)+s(4\hat{i}-4\hat{j}+4k)$

comparing $\vec{r} = a+t\vec{b}$ and $r = \vec{c}+td$ respectively

$a = 5\hat{i}+7\hat{j}+3k$ $\quad b = 4\hat{i}-4\hat{j}-5k$ $\quad c = 8\hat{i}+4\hat{j}+5k$

$d = 4\hat{i}-4\hat{j}+4k$

$\therefore c-a = 3\hat{i}-3\hat{j}+2k$

$b \times d = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 4 & -4 & -5 \\ 4 & -4 & 4 \end{vmatrix} = -36\hat{i}-36\hat{j}$

$(c-a).(b\times d) = (3\hat{i}-3\hat{j}+2k).(-36\hat{i}-36\hat{j})$

$= 0$

Shortest distance $= \frac{(c-a).(b\times d)}{|b\times d|} = \frac{0}{|b\times d|} = 0$

Hence, it will intersect.

Q19. A tailor needs at least 40 large buttons and 60 small buttons. In the market, buttions are available in two boxes or cards. A box contains 6 large and 2 small buttons and a card contains 2 large and 4 small buttons. If the cost of a box is $ 3 and cost of a card is $ 2, find how many boxes and cards should be purchased so as to minimize the expenditure.

SOLUTION: -

A Box contains Buttons = 6 Large and 2 Small

A Card contains Buttons = 2 Large and 4 Small

We have to buy both these boxes in such a way that

x (Boxes) + y (Cards) = 40 Large + 60 Small

X (6l + 2s) + y (2L +4S) = 40 L + 40 S

6x +2y = 40 other 2x+ 4y = 60

Now

X = 2 , y = 14

Value of a box = 3$

Value of a card = 2 $

For x = 2 and y =14:

Total Cost = Cost of boxes + Cost of Cards

= 2(3) + 14(2) = 34 $

Q20. A manufacturer makes two types of furniture, chairs and tables. Both the products are processed on three machines A1, A2 and A3. Machine A1 requires 3 hours for a chair and 3 hours for a table, machine A2 requires 5 hours for a chair and 2 hours for a table and machine A3 requires 2 hours for a chair and 6 hours for a table. The maximum time available on machines A1, A2 and A3 is 36 hours, 50 hours and 60 hours respectively. Profits are $ 20 per chair and $ 30 per table. Formulate the above as a linear programming problem to maximize the profit and solve it.

SOLUTION: -

Chairs : x, Tables : y, then LPP is

Maximize

$$P = 20x + 30y$$

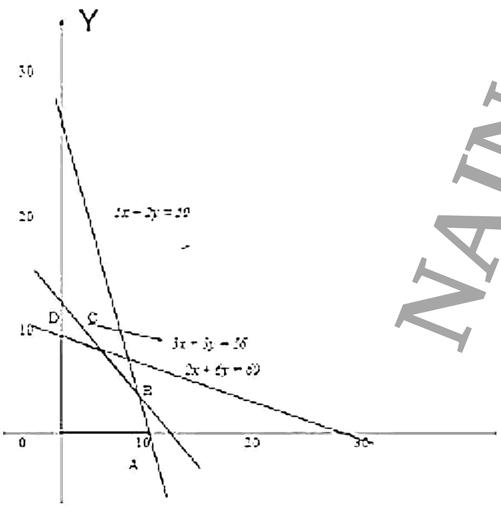
subject to

$$3x + 3y \le 36$$

$$5x + 2y \le 50$$

$$2x + 6y \le 60$$

$$x \ge 0, y \le 0$$



$P(A) = 20(10) + 30(0) = 200$

$$P(B) = 20\left(\frac{26}{3}\right) + 30\left(\frac{10}{3}\right) = \frac{820}{3}$$

$P(C) = 20(3) + 30(9) = 330$

$P(D) = 20(10) + 30(0) = 200$

$P(0) = 20(0) + 30(0) = 0$

Thus, profit is maximum

when $x = 3$, $y = 9$.

Maximum Profit = \$ 330

1. Evaluate the determinant given below, where ω is a cube root of unity.
(4 Marks)

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$$

Solution : $\begin{vmatrix} 1 & \omega & 2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$

$1 + \omega + \omega^2$

$$= \begin{vmatrix} 1+\omega+\omega^2 & \omega & \omega^2 \\ 1+\omega+\omega^2 & \omega^2 & 1 \\ 1+\omega+\omega^2 & 1 & \omega \end{vmatrix} \quad (\text{By } C_1 \to C_1 + C_2 + C_3)$$

$$= \begin{vmatrix} 0 & \omega & \omega^2 \\ 0 & \omega^2 & 1 \\ 0 & 1 & \omega \end{vmatrix} \quad (\because 1 + \omega + \omega^2 = 0)$$

$= 0$ [$\because C_1$ consists of all zero entries].

2. Using determinant, find the area of the triangle whose vertices are $(-3,\ 5)$, $(3, -6)$ and $(7,\ 2)$. (4 Marks)

Answer.

(b) $\Delta = \frac{1}{2} 1 \begin{vmatrix} -3 & 5 & 1 \\ 3 & -6 & 1 \\ 10 & 2 & 1 \end{vmatrix}$

$= \frac{1}{2} 1 \begin{vmatrix} -3 & 5 & 1 \\ 6 & -11 & 0 \\ 10 & -3 & 0 \end{vmatrix}$ (By applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$)

$= \frac{1}{2} 1 \,|-18 + 110)|$

$= \frac{1}{2} \times 92 = 46$ square units

3. Use the principle of mathematical induction to show that $2 + 2^2 + \ldots + 2^n = 2^{n+1} - 2$ for every natural number n. (4 Marks)

Solution : Let P_n denote the statement

$2 + 2^2 + \ldots\ldots\ldots\ldots + 2^n = 2^{n+1} - 2$

When $n = 1$, P_n becomes

$2 = 2^{1+1} - 2$ or $2 = 4 - 2$

This shows that the result holds for $n = 1$.

Assume that P_k is true for some $k \in N$.

That is, assume that

$2 + 2^2 + \ldots\ldots\ldots + 2^k = 2^{k+1} - 2$

We shall now show that truth of P_k implies the truth of P_{k+1} is

$2 + 2^2 + \ldots\ldots\ldots + 2^k + 2^{k+1} = 2^{k+1} - 2$ (1)

LHS of (1) $= 2 + 2^2 + \ldots\ldots\ldots + 2^k + 2^{k+1}$

$= (2^{k+1} - 2) + 2^{k+1}$ [induction assumption]

$= 2^{k+1}(1 + 1) - 2$

$= 2^{k+1} 2 - 2 = 2^{k+2} - 2$

$=$ RHS of (1)

This shows that the result holds for $n = k+1$; therefore, the truth of P_k implies the truth of P_{k+1}. The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number n.

4. Find the sum of all integers between 100 and 1000 which are divisible by 9. (4 Marks)

Solution : The first integer greater than 100 and divisible by 9 is 108 and the integer just smaller than 1000 and divisible by 9 is 999. Thus, we have to find the sum of the series.

108 + 117 + 126 + ...+ 999.

Here $t_1 = a = 108$, $d = 9$ and $l = 999$

Let n be the total number of terms in the series be n. Then

$999 = 108 + 9(n-1) \Rightarrow 111 = 12 + (n-1) \Rightarrow n = 100$

Hence, the required sum $= \frac{n}{2}(a+l) = \frac{100}{2}(108 + 999)$

$= 50(1107) = 55350.$

5. Check the continuity of the function f(x) at x = 0 : (4 Marks)

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Solution : (i) We have already seen in Example 7 that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

Hence, f is not continous at $x = 0$

6. If $y = \frac{\ln x}{x}$, show that $\frac{d^2y}{dx^2} = \frac{2\ln x - 3}{x^3}$ (4 Marks)

Solution : we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{lnx}{x}\right] = \frac{d}{dx}[x^{-1}lnx]$$

$= \frac{d}{dx}(x^{-1})lnx + x^{-1}\frac{d}{dx}(lnx)$ (product rule)

$= (-1)x^{-2}lnx + x^{-1}\frac{1}{x}$

$= x^{-2}[1 - \ln x]$

Differentiating both sides with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[x^{-2}][1 - lnx] + x^{-2}\frac{d}{dx}[1 - lnx]$$

$$= (-2)\,x^{-3}(1 - lnx) + x^{-2}\left(0 - \frac{1}{x}\right)$$

$$= -2x^{-3}(1 - lnx) + x^{-3}$$

$$= -x^{-3}(2 - 2\ln x + 1)$$

$$= \frac{2lnx - 3}{x^3}$$

7. If the mid-points of the consecutive sides of a quadrilateral are joined, then show (by using vectors) that they form a parallelogram. (4 Marks)

We have parallelogram ABCDABCD with midpoints P,Q,R,S.P,Q,R,S.
Draw diagonal DB.DB.

We
have: SP→=SA→+AP→=12DA→+12AB→=12(DA→+AB→)=12DB→SP→=SA→+AP→=12
DA→+12AB→=12(DA→+AB→)=12DB→
Hence: SP parallel DB and |SP|=12|DB||SP|=12|DB|

We
have: RQ→=RC→+CQ→=12DC→+12CB→=12(DC→+CB→)=12DB→RQ→=RC→+CQ→=1
2DC→+12CB→=12(DC→+CB→)=12DB→
Hence: RQ∥DBRQ∥DB and |RQ|=12|DB||RQ|=12|DB|

Then we have: SP∥RQSP∥RQ and |SP|=|RQ||SP|=|RQ|

Theorem: if two sides of a quadrilateral are parallel and equal,
the quadrilateral is a parallelogram.

Therefore, PQRSPQRS is a parallelogram.

8. Find the scalar component of projection of the vector (4 Marks)

$\vec{a} = 2\hat{i} + 3\hat{j} + 5\hat{k}$ on the vector $\vec{b} = 2\hat{i} - 2\hat{j} - \hat{k}$.

Solution : Scalar projection of $\vec{a}$ on $\vec{b}$ = $\dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

Here, $\vec{a}.\vec{b} = 2.2 + 3(-2) + 5(-1) = -7$

and $|\vec{b}| = \sqrt{2^2 + (-2)^2 + (-1)^2} = 3$

$\therefore$ Scalar projection of $\vec{a}$ on $\vec{b}$ $= \dfrac{-7}{3}$

9. Solve the following system of linear equations using Cramer's rule:
   $x + y = 0,\ y + z = 1,\ z + x = 3$ (4 Marks)

(b) Here,

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} \qquad [\text{Applying } C_2 \to C_2 - C_1]$$

$$= 2 \qquad (\text{Expanding along } R_1)$$

Since $\Delta \neq 0$, $\therefore$ the given system has unique solution.

$$\text{Now, } \Delta x = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix} = 2$$

$$\Delta y = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} = -2$$

$$\text{and } \Delta z = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{vmatrix} = 4$$

Hence by Cramer's Rule

$$x = \frac{\Delta x}{\Delta} = \frac{2}{2} = 1$$

$$y = \frac{\Delta y}{\Delta} = \frac{-2}{2} = -1 \text{ and}$$

$$z = \frac{\Delta z}{\Delta} = \frac{4}{2} = 2$$

10. If $A=\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$, $B=\begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A + B)^2 = A^2 + B^2$, Find a and b.

(4 Marks)

We have $(A+B)^2 = (A+B)(A+B)$

$= (A+B)A + (A+B)B$ (Distributive Law)

$= AA + BA + AB + BB$

$= A^2 + BA + AB + B^2$

Therefore, $(A+B)^2 = A^2 + B^2$

$\Rightarrow A^2 + BA + AB + B^2 = A^2 + B^2$

$\Rightarrow BA + AB = 0.$

Thus, we must find a and b such that BA + AB = 0.

We have BA $= \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} a+2 & -a-1 \\ b-2 & -b+1 \end{bmatrix}$

and AB $= \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} = \begin{bmatrix} a-b & 2 \\ 2a-b & 3 \end{bmatrix}$

Therefore,

BA + AB $= \begin{bmatrix} a+2 & -a-1 \\ b-2 & -b+1 \end{bmatrix} + \begin{bmatrix} a-b & 2 \\ 2a-b & 3 \end{bmatrix}$

$= \begin{bmatrix} 2a-b+2 & -a+1 \\ 2a-2 & -b+4 \end{bmatrix}$

But BA + AB = 0

$\Rightarrow 2a - b + 2 = 0, -a + 1 = 0, 2a - 2 = 0, -b + 4 = 0$

$\Rightarrow a = 1, b = 4$

11. Reduce the matrix A(given below) to normal form and hence find its rank.

(4 Marks)

$$A = \begin{bmatrix} 5 & 3 & 8 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Solution : $A = \begin{bmatrix} 5 & 3 & 8 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

Applying $R_1 \leftrightarrow R_3$, we have

$$A \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 5 & 3 & 8 \end{bmatrix}$$

Applying $R_3 \to R_3 - 5R_1$, we have

$$A \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 8 & 8 \end{bmatrix}$$

Applying elementary row operations $R_1 \to R_1 + R_2$ and $R_3 \to R_3 - 8R_2$, we have

$$A \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, we apply elementary column operation $C_3 \to C_3 - C_1$, to get

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Again, applying $C_1 \to C_1 - C_1$, we have

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We have thus reduced A to normal form.

Also, note that the rank of a matrix remains unaltered under elementary operations.

Thus, rank of A in above example is 2 because rank of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is 2.

In this regard, we state following theorem without proof.

12. Show that n(n+1) (2n+1) is a multiple of 6 for every natural number n.
(4 Marks)

The induction hypothesis - $P(k)$: $k(k+1)(2k+1)$ is divisible by 6, i.e. $k(k+1)(2k+1) = 6m$ for some m.

Now,

$$\begin{aligned}
&(k+1)\{(k+1)+1\}\{2(k+1)+1\}\\
&(k+1)(k+2)(2k+3)\\
&=k(k+1)(2k+3)+2(k+1)(2k+3)\\
&k(k+1)(2k+1)+2k(k+1)+2(k+1)(2k+3)\\
&6m+(k+1)(2k+4k+6)\\
&6m+6(k+1)^2
\end{aligned}$$

So $(k+1)\{(k+1)+1\}\{2(k+1)+1\}$ is divisible by 6 i.e. $P(k+1)$ is true.

13. Find the sum of an infinite G.P. whose first term is 28 and fourth term is $\frac{4}{49}$.
(4 Marks)

$a = 28, \quad ar^3 = \frac{4}{49}$

$\Rightarrow r^3 = \frac{4}{49} \times \frac{1}{28} = \frac{1}{7^3}$

$\Rightarrow r = 1/7$

Thus, $S = \frac{a}{1-r} = \frac{28}{1-1/7} = \frac{28 \times 7}{6} = \frac{98}{3}$

14. Use De Moivre's theorem to find $(\sqrt{3}+i)^3$. (4 Marks)

Solution : We first put $\sqrt{3}+i$ in the polar form.

Let $\sqrt{3}+i = r(\cos\theta + i\sin\theta)$

$\Rightarrow \sqrt{3} = r\cos\theta$ and $1 = r\sin\theta$

$\Rightarrow (\sqrt{3})^2 + 1^2 = r^2(\cos^2\theta + \sin^2\theta)$

$\Rightarrow r^2 = 4 \Rightarrow r = 2$

Thus, $\sqrt{3}+i = 2(\cos\theta + i\sin\theta)$

$\Rightarrow \sqrt{3} = 2\cos\theta$ and $1 = 2\sin\theta$

$\Rightarrow 2\cos\theta = \frac{\sqrt{3}}{2}$ and $\sin\theta = \frac{1}{2}$

$\Rightarrow \theta = 30°$.

Now, $(\sqrt{3}+i)^3 = [2\cos(30°) + i\sin(30°)]^3$

$= 2^3[\cos(30°) + I\sin(30°)]^3$

$= 8\,[\cos(3\times30°) + i\sin(3\times30°)]$ [De Moivre's theorem]

$= 8\,(\cos 90° + i\sin 90°) = 8(0 + i)$

$= 8i$

15. If 1, ω, ω^2 are cube roots unity, show that $(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11}) = 49$. (4 Marks)

(ii) Since $\omega^{10} = (\omega^3)^3\,\omega = \omega$

and $\omega^{11} = (\omega^3)^3\,\omega^2 = \omega^2$,

Thus $(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11})$

$= (2-\omega)(2-\omega^2)(2-\omega)(2-\omega^2)$

$= [(2-\omega)(2-\omega^2)]^2$

$= [4 - 2\omega - 2\omega^2 + \omega^3]^2$

$= [4 - 2(\omega + \omega^2) + 1]^2$

$= [4 - 2(-1) + 1]^2$ $[\because \omega + \omega^2 = -1]$

$= 7^2 = 49$

16. Solve the equation $2x^3 - 15x^2 + 37x - 30 = 0$, given that the roots of the equation are in A.P. (4 Marks)

Solution : Recall three numbers in A.P. can be taken as $\alpha - \beta, \alpha, \alpha + \beta$.

If $\alpha - \beta, \alpha, \alpha + \beta$ are roots of (1), then $(\alpha - \beta) + \alpha + (\alpha + \beta) = 15/2 \Rightarrow 3\alpha = 15/2$

$\Rightarrow \alpha = 5/2$

Next,

$\alpha(\alpha - \beta) + \alpha(\alpha + \beta) (\alpha - \beta)(\alpha + \beta) = 37/2$

$\Rightarrow \alpha^2 - \alpha\beta + \alpha^2 + \alpha\beta + \alpha^2 - \beta^2 = 37/2$

$\Rightarrow 3\alpha^2 - \beta^2 = 37/2$

$\Rightarrow \beta^2 = 3\alpha^2 - \frac{37}{2} = 3 \times \frac{25}{4} - \frac{37}{2} = \frac{1}{4}$

$\Rightarrow \beta = \pm\frac{1}{2}$

When $\beta = 1/2$, the roots are

$\frac{5}{2} - \frac{1}{2}, \frac{5}{2}, \frac{5}{2} + \frac{1}{2}$, or $2, \frac{5}{2}, 3$

When $\beta = -\frac{1}{2}$, the roots are 3, 5/2, 2

It is easily to check that these are roots of (1)

17. A young child is flying a kite which is at height of 50 m. The wind is carrying the kite horizontally away from the child at a speed of 6.5 m/s. How fast must the kite string be let out when the string is 130m ? (4 Marks)

Solution : Let h be the horizontal distance of the kite from the point directly over the child's head 5. Let l be the length of kite string from the child to the kite at time t. [See Fig. 1] Then

$$l^2 = h^2 + 50^2$$

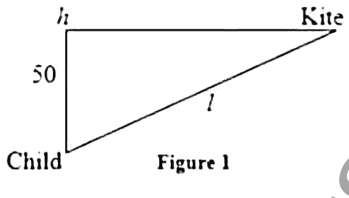


Figure 1

Differentiating both the sides with respect to t, we get

$$2l\frac{dl}{dt} = 2h\frac{dh}{dt} \text{ or } l\frac{dl}{dt} = h\frac{dh}{dt}.$$

We are given $\frac{dh}{dt} = 6.5\ m/s$. We are interested to find dl/dt when $l = 130$.
But when $l = 130$, $h^2 = l^2 - 50^2 = 130^2 - 50^2 = 14400$ or $h = 120$.

$$\text{Thus, } \frac{dl}{dt} = \frac{120}{130} \times 6.5 = 5\ m/s.$$

This shows that the string should be let out at a rate of 6 m/s.

18. Using first derivative test, find the local maxima and minima of the function $f(x) = x^3 - 12x$. (4 Marks)

2. (i) $f(x) = x^3 - 12x$

Differentiating w.r.t. x, we get

$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 2(x-2)(x+2)$

Setting $f'(x) = 0$, we obtain $x = 2, -2$ Thus, $x = -2$, and $x = 2$ are the only critical numbers of f. Fig. 35 shows the sign of derivative f' in three intervals.

Sign of $(x+2)$	$- - -$	$+ + +$	$+ + +$
Sign of $(x-2)$	$- - -$	$- - -$	$+ + +$

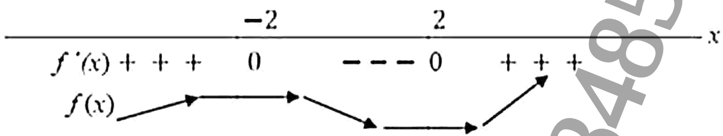


Figure 35

From figure 35 it is clear that if $x < -2$, $f'(x) > 0$; if $-2 < x < 2$, $f'(x) < 0$ and if $x > 2$, $f'(x) > 0$.

Using the first derivative test, we conclude that

$f(x)$ has a local maximum at $x = -2$ and a local minimum at $x = 2$.

Now, $f(-2) = (-2)^3 - 12(-2) = -8 + 24 = 16$ is the value of local maximum at $x = -2$ and $f(2) = 2^3 - 12(2) = 8 - 24 = -16$ is the value of the local minimum at $x = 2$.

19. Evaluate the integral $I = \int \frac{x^2}{(x+1)^3}\,dx$ (4 Marks)

Solution : To evaluate an integral of the form

$$\int \frac{P(x)}{(a+bx)^r}\,dx, \text{ we put } a + bx = t.$$

So, we put $x + 1 = t \Rightarrow dx = dt$

$$\text{and } I = \int \frac{(t+1)^2}{t^3}\,dt = \int \frac{t^2 + 2t + 1}{t^3}\,dt$$

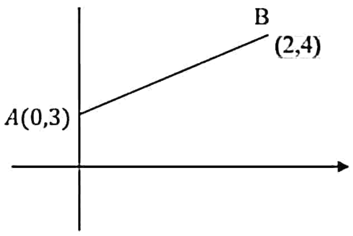
$$= \int \left(\frac{1}{t} - 2t^{-2} + t^{-3}\right) dt$$

$$= \log|t| - \frac{2t^{-1}}{-1} + \frac{t^{-2}}{-2} + c$$

$$= \log|t| + \frac{2}{t} - \frac{1}{2t^2} + c$$

$$= \log|x+1| - \frac{2}{x+1} + \frac{1}{2(x+1)^2} + c$$

20. Find the length of the curve $y = 3 + \frac{x}{2}$ from (0, 3) to (2, 4). (4 Marks)



. We have

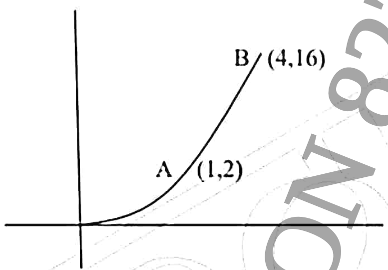
$$\frac{dy}{dx} = \frac{1}{2}$$

Required length

$$= \int_0^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^2 \sqrt{1 + \frac{1}{4}}\, dx = \frac{\sqrt{5}}{2}\int_0^2 dx = \frac{\sqrt{5}}{2}\, [x]_0^2 = \sqrt{5} \text{ units}$$

. We have



$$\frac{dy}{dx} = 3x^{1/2}$$

Required length

$$= \int_1^4 \sqrt{1 + 9x}\, dx = \left[\frac{(1+9x)^{3/2}}{9\left(\frac{3}{2}\right)}\right]_1^4 = \frac{2}{27}\left[37\sqrt{37} - 10\sqrt{10}\right] \text{units}$$

BCS-012 : BASIC MATHEMATICS

Q1.

1. (a) Show that 5

$$\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Ans 1.

Solution : Taking a, b, and c common four C_1, C_2 and C_3 respectively, we get

$$\Delta = abc\begin{vmatrix} -a & a & a \\ b & -b & b \\ c & c & -c \end{vmatrix}$$

Taking a, b and c common from R_1, R_2, R_3 respectively, we get.

$$\Delta = a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$ and $C_1 \to C_2 + C_3$, we get

$$\Delta = a^2b^2c^2\begin{vmatrix} 0 & 2 & 1 \\ 0 & 0 & 1 \\ 2 & 0 & -1 \end{vmatrix}$$

Expanding along C_1, we get $\Delta = a^2b^2c^2(4) = 4a^2b^2c^2$

Q2.

2. Construct a 2×2 matrix $A = [a_{ij}]_{2\times 2}$ where elements are given by
(a) $a_{ij} = \frac{1}{2}(i+2j)^2$ (b) $a_{ij} = \frac{1}{2}(i-j)^2$

Ans 2.

2. Note that a 2×2 matrix is given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

From the formulas given the elements, we have

(a) $A = \begin{bmatrix} 9/2 & 25/2 \\ 8 & 18 \end{bmatrix}$

(b) $A = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$

Q3 : Use the principle of mathematical induction to prove that

$1^3 + 2^3 + \dots\dots\dots\dots + n^3 = \frac{1}{4}n^2(n+1)^2$ for every natural number n.

Solution: Let P_n denote the statement

$$1^3 + 2^3 + \dots\dots\dots\dots + n^3 = \frac{1}{4}n^2(n+1)^2.$$

When $n = 1$, P_n becomes

$$1^3 + 2^3 + \dots\dots\dots\dots + n^3 = \frac{1}{4}(1^2)(1+1)^2 \text{ or } \mathbf{1} = 1$$

This shows that the result holds for $n = 1$. Assume that P_k is true for some $k \in \mathbf{N}$. That, is assume that

$$1^3 + 2^3 + \dots\dots\dots\dots + k^3 = \frac{1}{4}k^2(k+1)^2$$

We shall now show that the truth of P_k implies the truth of P_{k+1} where P_{k+1} is

$$1^3 + 2^3 + \dots\dots\dots\dots + k^3 = \frac{1}{4}(k+1)^2(k+1)^2 \qquad (1)$$

LHS of (1)

$= 1^3 + 2^3 + \dots\dots\dots\dots k^3 = + (k+1)^3$

$= \frac{1}{4}k^2(k+1)^2(k+1)^3$

$= \frac{1}{4}k^2(k+1)^2(k+1)^3$ [induction assumption]

$= \frac{1}{4}(k+1)^2[k^2 + 4(k+1)] = \frac{1}{4}(k+1)^2(k+2)^2$

= RHS of (1)

This shows that the result holds for $n = k+1$; therefore, the truth of P_k implies the truth of P_{k+1}. The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number n.

Q4. Find the Sum to n terms of the series 5

$5+55+555+\cdots\cdots+n$ Terms

4. Let $s_n = 5 + 55 + 555 + \ldots\ldots\ldots + $ upto n terms

$$= 5\,[1 + 11 + 111 + \ldots\ldots\ldots + \text{upto } n \text{ terms}]$$

$$= \frac{5}{9}[9 + 99 + 999 + \ldots\ldots\ldots + \text{upto } n \text{ terms}]$$

$$= \frac{5}{9}[(10-1) + (10_2 - 1) + (10_3 - 1) + \ldots\ldots\ldots + (10n - 1)]$$

$$= \frac{5}{9}[(10 + 10_2 + \cdots\ldots\ldots\ldots + 10_n]$$

$$= \frac{5}{9}\left[\frac{10(10^n - 1)}{10-1} - n\right] = \frac{5}{9}\left\{\frac{10(10^n - 1)}{10-1}\right\} - n]$$

Q5. : Find the points of local maxima and minima, if any, of each of the following functions. Find also the local maximum values and local minimum values.

(i) $f(x) = x^3 - 6x^2 + 9x + 1,$ $x \in \boldsymbol{R}$

(ii) $f(x) = \frac{1}{6}x^6 - 4x^5 + 25\,x^4,$ $x \in \boldsymbol{R}$

(iii) $f(x) = x^3 - 2ax^2 + a^2 x\ (a > 0),$ $x \in \boldsymbol{R}$

Solution : $f'(x) = x^3 - 6x^2 + 9x + 1$

Thus, $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$.

To obtain critical number of f, we set $f'(x) = 0$ this yields $x = 1, 3$.

Therefore, the critical number of f are $x = 1, 3$.

Now $f''(x) = 6x - 12 = 6(x-2)$

We have $f''(1) = 6(1-2) = -6 < 0$ and $f''(3) = 6(3-2) = 6 > 0$.

Using the second derivative test, we see that $f(x)$ has a local maximum at $x = 1$ and a local minimum at $x = 3$. The value of local maximum at $x = 1$ is $f(1) = 1 - 6 + 9 + 1 = 5$ and the value of local minimum at $x = 3$ is $f(3) = 3^3 - 6(3^2) + 9(3) + 1 = 27 - 54 + 27 + 1 = 1$.

(ii) We have $f'(x) = \frac{1}{6}x^6 - 4x^5 + 25x^4$

Thus, $f'(x) = x^5 - 20x^4 + 100x^3 = x^3(x^2 - 20x + 100) = x^3(x - 10)^2$

As $f'(x)$ is defined for every value of x, the critical number f are solutions of $f'(x) = 0$. Setting $f'(x) = 0$, we get $x = 0$ *or* $x = 10$.

Now $f'(x) = 3x^2(x - 10)^2 + x^3(2)(x - 10)$

$$= x^2(x - 10)[3(x - 10) + 2x] = 5x^2(x - 10)(x - 6)$$

We have $f'(0) = 0$ and $f'(10) = 0$. Therefore we cannot use the second derivative test to decide about the local maxima and minima. We, therefore use the first derivative test.

Figure 27 shows the sign of derivative f' in three intervals.

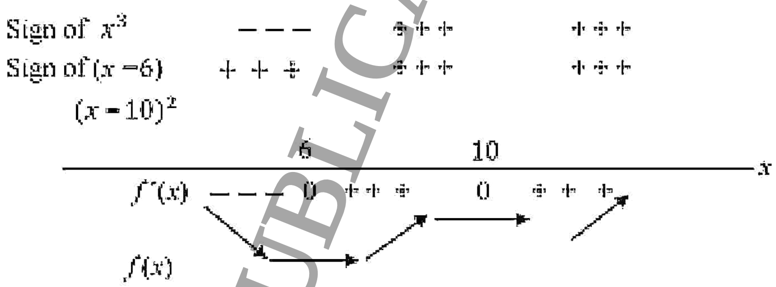


Figure 27

From figure 27 it is clear that $f'(x) < 0$ for $x < 0$, $f'(x) > 0$ for $0 < x < 10$ and $f'(x) > 0$ for $x > 10$

Thus, $f(x)$ has a local minimum at $x = 0$ and its value is $f(0) = 0$. But $f(x)$ has neither a local maximum nor a local minimum at $x = 10$.

(iii) We have $f(x) = x^3 - 2ax^2 + a^2 x \ (a > 0)$.

Thus, $f'(x) = 3x^2 - 4ax + a^2 = (3x - a)(x - a)$

As $f'(x)$ is defined for each $x \in R$, to obtain critical number of f we set $f'(x) = 0$. This yields $x = a/3$ or $x = a$. Therefore, the critical numbers of f and $a/3$ and a. Now, $f''(a) = 6x - 4a$.

We have $f''\left(\frac{a}{3}\right) = 6\left(\frac{a}{3}\right) - 4a = 2a - 4a = -2a < 0$ and
$f''(a) = 6a - 4a = 2a > 0$.

Using the second derivative test, we see that $f(x)$ has a local maximum at $x = a/3$ and a local minimum at $x = a$. The value of local maximum at

$$x = \frac{a}{3} \text{ is } f\left(\frac{a}{3}\right) = \left(\frac{a}{3}\right)^3 - 2a\left(\frac{a}{3}\right)^2 + a^2\left(\frac{a}{3}\right) = \frac{4}{27}a^3$$

and the value of local minimum at $x = a$ is $f(a) = a^3 - 2a.\ a^2 + a.a = 0$

Q6. : Evaluate the integral.

$$\int \frac{x\,dx}{(x-1)(x+5)(2x-1)}$$

Solution : We write

$$\frac{x}{(x-1)(x+5)(2x-1)} = \frac{A}{x-1} + \frac{B}{x+5} + \frac{C}{2x-1}$$

$\Rightarrow x = A(x+5)(2x-1) + B(x-1)(2x-1) + C(x-1)(x+5)$

Put $x = 1, -5$ and $\frac{1}{2}$ to obtain

$1 = 6A \Rightarrow A = 1/6$

$-5 = 66B \Rightarrow B = -5/66$

$\frac{1}{2} = -\frac{11}{4} C \Rightarrow C = -2/11$

Thus,

$$\int \frac{x}{(x-1)(x+5)(2x-1)} = \frac{1}{6}\int \frac{dx}{x-1} - \frac{5}{66}\int \frac{dx}{x+5} - \frac{2}{11}\int \frac{dx}{2x-1}$$

$$= \frac{1}{6}\log|x-1| - \frac{5}{66}\log|x+5| - \frac{1}{11}\log|2x-1| + c$$

Q7. : Find the value of λ for which the vectors $\vec{a} = \hat{\imath} - 4\hat{\jmath} + \hat{k}$, $\vec{b} = \lambda\hat{\imath} - 2\hat{\jmath} + \hat{k}$, and $\vec{c} = 2\hat{\imath} + 3\hat{\jmath} + 3\hat{k}$ are coplanar,

Solution : If $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar, we have

$= \vec{a}.(\vec{b} \times \vec{c}) = 0$ i.e.,

$$\begin{vmatrix} 1 & -4 & 1 \\ \lambda & -2 & 1 \\ 2 & 3 & 3 \end{vmatrix} = 0$$

$$\Rightarrow 1(-6-3) + 4(3\lambda - 2) + (3\lambda + 4) = 0$$

$$\Rightarrow \lambda = \frac{13}{15}$$

Q8. Find the equation of the line passing through the point $(-1,3,-2)$ and perpendicular to the two lines

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \text{ and } \frac{x+2}{-3} = \frac{y-1}{2} = \frac{z+1}{5}.$$

Solution : Let the equation of the given line be

$$\frac{x+2}{l} = \frac{y-3}{m} = \frac{z+2}{n}$$

Since (i) is perpendicular to both the lines therefore,

$l + 2m + 3n = 0$ ……………………(ii)

$-3l + 2m + 5n = 0$ …………………(iii)

Solving (ii) and (iii) we get

$$\frac{l}{10-6} = \frac{m}{-9-5} = \frac{n}{2+6}$$

Or

$$\frac{l}{4} = \frac{m}{-14} = \frac{n}{8}$$

$$\frac{l}{4} = \frac{m}{-7} = \frac{n}{8}$$

$\therefore$ The required equation is

$$\frac{x+2}{2} = \frac{y-3}{-7} = \frac{z+2}{4}$$

Q9. Solve the following system of linear equations using Cramer's rule.

(a) $x+2y-z=-1,\ 3x+8y+2z=28,\ 4x+9y+z=14$

(b) $x+y=0,\ \ y+z=1,\ \ z+x=3$

4. (a) We first evaluate Δ. We have

$$\Delta=\begin{vmatrix}1&2&-1\\3&8&2\\4&9&1\end{vmatrix}=\begin{vmatrix}1&0&0\\3&2&5\\4&1&5\end{vmatrix}\ \text{[Applying } C_2\to C_2-2C_1 \text{ and } C_3\to C_3+C_1]$$

$=10-5=5$ (expanding along R_1)

As $\Delta\neq 0$, the given system of equation has a unique solution. We shall now evaluate Δx, Δy and Δz. We have

$$\Delta x=\begin{vmatrix}-1&2&-1\\28&8&2\\14&9&1\end{vmatrix}=\begin{vmatrix}-1&2&-1\\26&12&0\\13&11&0\end{vmatrix}\ \text{(By applying } R_2\to R_2-2R_1 \text{ and } R_3\to R_3+R_1 \text{ we get)}$$

$$=-\begin{vmatrix}26&12\\13&11\end{vmatrix}\quad\text{(expanding along } C_3)$$

$=-130$

$$\Delta y=\begin{vmatrix}1&-1&-1\\3&28&2\\4&14&1\end{vmatrix}=\begin{vmatrix}1&-1&-1\\5&26&0\\5&13&0\end{vmatrix}\ \text{(By applying } C_2\to C_2-2C_1 \text{ and } C_3\to C_3+C_1)$$

$$=-\begin{vmatrix}5&26\\5&13\end{vmatrix}\quad\text{(expanding along } R_1)$$

$=65$

and $$\Delta z=\begin{vmatrix}1&2&-1\\3&8&28\\4&9&14\end{vmatrix}=\begin{vmatrix}1&0&0\\3&2&31\\4&1&18\end{vmatrix}\ \text{(Applying } C_2\to C_2-2C_1 \text{ and } C_3\to C_3+C_1)$$

$=5$ (expanding along R_1)

Hence by Cramer's Rule

$$x=\frac{\Delta x}{\Delta}=\frac{-130}{5}=-26$$

$$y=\frac{\Delta y}{\Delta}=\frac{65}{5}=13$$

$$z=\frac{\Delta z}{\Delta}=\frac{5}{5}=1$$

(b) Here,

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} \quad [\text{Applying } C_2 \to C_2 - C_1]$$

$$= 2 \quad (\text{Expanding along } R_1)$$

Since $\Delta \neq 0$, $\therefore$ the given system has unique solution,

Now, $$\Delta x = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix} = 2$$

$$\Delta y = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} = -2$$

and $$\Delta z = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{vmatrix} = 4$$

Hence by Cramer's Rule

$$x = \frac{\Delta x}{\Delta} = \frac{2}{2} = 1$$

$$y = \frac{\Delta y}{\Delta} = \frac{-2}{2} = -1 \text{ and}$$

$$z = \frac{\Delta z}{\Delta} = \frac{4}{2} = 2$$

Q10. If $A = \begin{bmatrix} 3 & 2 \\ 4 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 5 \\ 2 & 5 \end{bmatrix}$ 5

Verify $(AB)^{-1} = B^{-1}A^{-1}$

Solution : Since $|A| = -8 \neq 0$, $\therefore$ A is invertible.

Similarly, $|B| = 20 - 10 = 10 \neq 0$, $\therefore$ B is also invertible.

Let A_{ij} denote the cofactor of a_{ij} – the $(i,j)^{th}$ element of A. Then

$A_{11} = 0$, $A_{12} = -4$, $A_{21} = -2$ and $A_{22} = 3$.

Similarly, if B_{ij} is cofactor of $(i,j)^{th}$ element of B, then

$B_{11} = 5$, $B_{12} = -2$, $B_{21} = -5$ and $B_{22} = 4$

$\therefore$ adj A $= \begin{bmatrix} 0 & -2 \\ -4 & 3 \end{bmatrix}$ and adj B $= \begin{bmatrix} 5 & -5 \\ -2 & 4 \end{bmatrix}$

$\Rightarrow A^{-1} = \frac{1}{|A|}$ adj A $= \frac{-1}{8}\begin{bmatrix} 0 & -2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1/4 \\ 1/2 & -3/8 \end{bmatrix}$

and $B^{-1} = \frac{1}{|B|}$ adj B $= \frac{1}{10}\begin{bmatrix} 5 & -5 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ -1/5 & 2/5 \end{bmatrix}$

Let C = AB $= \begin{bmatrix} 3 & 2 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 12+4 & 15+10 \\ 16+0 & 20+10 \end{bmatrix} = \begin{bmatrix} 16 & 25 \\ 16 & 20 \end{bmatrix}$

We have

$C_{11} = 20$, $C_{12} = -16$, $C_{21} = -25$ and $C_{22} = 16$

Also, $|C| = -80 \neq 0$, $\therefore$ C is invertible.

Also, adj C $= \begin{bmatrix} 20 & -25 \\ -16 & 16 \end{bmatrix}$

$\Rightarrow C^{-1} = \frac{1}{|C|}$ adj C $= -\frac{1}{80}\begin{bmatrix} 20 & -25 \\ -16 & 16 \end{bmatrix} = \begin{bmatrix} -1/4 & 5/16 \\ 1/5 & -1/5 \end{bmatrix}$.

Hence, $B^{-1} A^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ -1/5 & 2/5 \end{bmatrix}\begin{bmatrix} 0 & 1/4 \\ 1/2 & -3/8 \end{bmatrix}$

$= \begin{bmatrix} -1/4 & 5/16 \\ 1/5 & -1/5 \end{bmatrix} = C^{-1} = (AB)^{-1}$

Q11. Reduce the matrix

$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 3 & 2 \\ 1 & 3 & 1 & 3 \end{bmatrix}$ to its normal form and hence determine its rank.

Solution : We have

$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 3 & 2 \\ 1 & 3 & 1 & 3 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -1 & 2 \end{bmatrix} \text{[by } R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 2R_1\text{]}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -1 & 2 \end{bmatrix} \text{[by } C_3 \to C_3 - 2C_1,\ C_4 \to C_4 - C_1\text{]}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix} \text{[by } R_3 \to R_3 - 3R_2\text{]}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{[by } R_3 \to \tfrac{1}{2}R_3\text{]}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{[by } C_3 \to C_3 + C_2\text{]}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{[by } C_4 \to C_4 - C_3\text{]}$$

Thus, A is reduced to normal form $[I_3\ 0]$ and hence rank of A is 3.

Q12. : If sum of three numbers in GP is 38 and their product is 1728, find the numbers.

Solution : Let the three number be $a/r,\ a,\ ar$

Then, $\frac{a}{r} + a + ar = 38$ (1)

and $\left(\frac{a}{r}\right)(a)(ar) = 1728$ (2)

We can write (2) as $a^3 = 1728 = 12^3 \Rightarrow a = 12$.

Putting $a = 12$ in (1), we get $\frac{12}{r} + 12 + 12r = 38$

$$\Rightarrow \frac{12}{r} + 12r = 26 \text{ or } \frac{1}{r} + r = \frac{26}{12} = \frac{13}{6}$$

$\Rightarrow \quad 6(r^2+1) = 13r \Rightarrow 6r^2 - 13r + 6 = 0 \Rightarrow 6r^2 - 9r - 4r + 6 = 0$

$\Rightarrow \quad 3r(2r-3) - 2(2r-3) = 0 \Rightarrow (3r-2)(2r-3) = 0 \Rightarrow r = 2/3$ or $3/2$.

When $a = 12$ and $r = \frac{2}{3}$, $\quad \frac{a}{r} = \frac{12}{2/3} = 18, a = 12, ar = 12\left(\frac{2}{3}\right) = 8.$

When $a = 12$ and $r = \frac{3}{2}$, $\quad \frac{a}{r} = \frac{12}{3/2} = 8, a = 12, ar = 12\left(\frac{3}{2}\right) = 8.$

Hence, the numbers are either 18, 12, 8 or 8, 12, 18.

Q13. If 1, w, w^2 are Cube roots of unity then show that. 5

$(1 - w + w^2)^5 + (1 + w - w^2)^5 = 32.$

ANS

$(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5$

$= (-\omega - \omega)^5 + (-\omega^2 - \omega^2)^5$

$= (-2)^5\omega^5 + (-2)^5(\omega^2)^5$

$= -32\omega^2 - 32\omega = -32(\omega^2 + \omega)$

$= (-32)(-1) = 32$

Q14. If α, β are the roots of the equation $2x^2 - 3x + 1 = 0$, form an equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$. 5

Ans. Since α and β are roots of $2x^2 - 3x + 1 = 0$, $\alpha + \beta = 3/2$ and $\alpha\beta = ½$.

We are to form an equation whose roots are α/β and β/α.

$$\text{Let S} = \text{Sum of roots} = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}.$$

$$= \frac{\left(\frac{3}{2}\right)^2 - 2\left(\frac{1}{2}\right)}{\frac{1}{2}}$$

$$= \left(\frac{5}{4}\right)\left(\frac{2}{1}\right) = \frac{5}{2}$$

$$\text{P} = \text{Product of roots} = \frac{\alpha}{\beta} \times \frac{\beta}{\alpha} = 1$$

Thus, the required quadratic equation is

$x^2 - (5/2)\,x + 1 = 0$ or $2\,x^2 - 5x + 2 = 0$

Q15. : Solve the inequality

$$-2 < \frac{1}{5}(4 - 3x) \le 8$$

and graph the solution set.

Solution : We first multiply the given inequality by 5 to obtain

$$(-2)\,(5) < 4 - 3x \le (8)\,(5)$$

$$\Leftrightarrow \quad -10 - 4 < -3x \le 40 - 4$$

$$\Leftrightarrow \quad \frac{-14}{-3} > x \ge \frac{-44}{4} \quad \Leftrightarrow \quad -11 \le x < 14/3$$

$$\therefore \text{ Solution set is } \{\, x \mid -11 \le x \ \frac{14}{3}\} = [-11, \frac{14}{3}\,]$$

Graph of the solution set is

Q16 If $x = a\left(t - \frac{1}{t}\right)$ and $y = a\left(t + \frac{1}{t}\right)$. 5

Find $\frac{dy}{dx}$.

Ans. $\frac{dx}{dt} = a\left(1 + \frac{1}{t^2}\right)$, $\frac{dy}{dt} = b\left(1 - \frac{1}{t^2}\right)$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b(1 - \frac{1}{t^2})}{a(1 + \frac{1}{t^2}),} = \frac{b(t^2 - 1)}{a(t^2 + 1)}$$

Q17 : Sand is being poured into a conical pile at the constant rate of 50 cubic centimetres per minute. Frictional forces in the sand are such that the height of the cone is always one half of the radius of its base. How fast is the height of the pile increasing when the sand is 5 cm deep ?

Solution : Let r the radius, h be height and V be the volume of the cone (Fig. 4) of sand at time t.

Then $V = \frac{1}{3}\pi r^2 h$

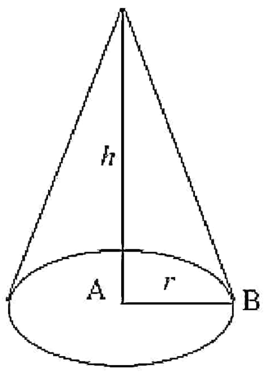


Figure 4

We are given that $h = \frac{1}{2}r$ or $r = 2h$. Thus,

$V = \frac{1}{3}\pi(2h)^2 h = \frac{4}{3}\pi h^3$.

Differentiating both the sides with respect to t, we get

$\frac{dV}{dt} = \frac{4}{3}\pi(3h^2)\frac{dh}{dt} = 4\pi h^2 \frac{dh}{dt}$.

we are given that $\frac{dV}{dt} = 50$, thus $50 = 4\pi h^2 \frac{dh}{dt}$.

$$\Rightarrow \frac{dh}{dt} = \frac{50}{4\pi h^2}$$

When $h = 5$. $\frac{dh}{dt} = \frac{50}{4\pi(5^2)} = \frac{50}{100\pi} = \frac{1}{2\pi}$.

Hence, the height of the cone is rising at the rate of (1/2π) cm/min.

Q18. : Evaluate

$$\int \frac{(a^x + b^x)^2}{a^x b^x} dx$$

Solution We have

$$\frac{(a^x + b^x)^2}{a^x b^x} = \frac{(a^x)^2 + (b^x)^2 + 2a^x b^x}{a^x b^x}$$

$$= \frac{(a^x)^2}{a^x b^x} + \frac{(b^x)^2}{a^x b^x} + \frac{2a^x b^x}{a^x b^x}$$

$$= \frac{a^x}{b^x} + \frac{b^x}{a^x} + 2$$

$$= \left(\frac{a}{b}\right)^x + \left(\frac{b}{a}\right)^x + 2$$

Thus,

$$\int \frac{(a^x + b^x)^2}{a^x b^x} dx = \int \left[\left(\frac{a}{b}\right)^x + \left(\frac{b}{a}\right)^x + 2\right] dx$$

$$= \frac{\left(\frac{a}{b}\right)^x}{\ln\left(\frac{a}{b}\right)} + \frac{\left(\frac{b}{a}\right)^x}{\ln\left(\frac{b}{a}\right)} + 2x + c$$

Q19. Find the area bounded by the curves $y = x^2$ and $y = x$.

Solution : To obtain point of intersection of $y = x^2$ and $y = x$, we set

$x^2 = x \Rightarrow x(x-1) = 0 \Rightarrow x = 0, 1.$

Thus, the two curves intersect in (0,0) and (1,1). See Fig 4.8

Required area

$= \int_0^1 (x - x^2)dx$

$= \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1$

$= \left(\frac{1}{2} - \frac{1}{3}\right)$ sq. units

$= \frac{1}{6}$ sq. units.

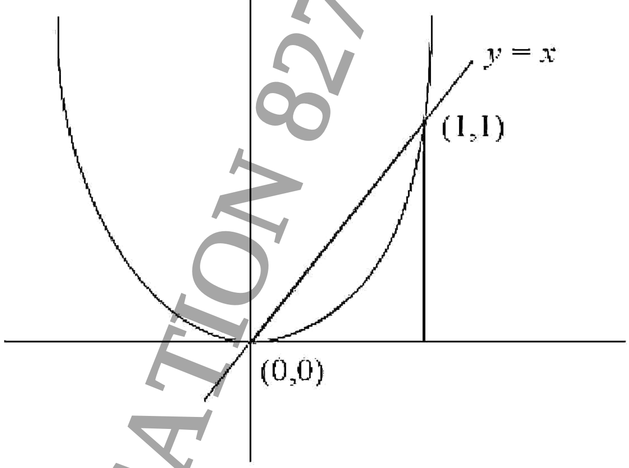


Figure 4.8

Q20. Find a unit vector perpendicular to each of the vector $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$, where $\vec{a} = \hat{i} + 2\hat{j} - 4\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + 2\hat{k}$

Ans. Here $\vec{a} + \vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$

$\vec{a} - \vec{b} = 3\hat{j} - 6\hat{k}$

Let $\vec{c} = (\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})$. Then $\vec{c}$ is vector perpendicular to both $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$.

Now $\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 0 & 3 & -6 \end{vmatrix}$

$= (-6 + 6)\hat{i} + (-12 + 0)\hat{j} + (6 + 0)\hat{k}$

$= 12\hat{j} + 6\hat{k}$

A unit vector in the direction of $\vec{c}$ is

$$\hat{c} = \frac{1}{|\vec{c}|}\vec{c} = \frac{1}{\sqrt{12^2+6^2}}(12\hat{j} + 6\hat{k})$$

$$= \frac{1}{6\sqrt{5}}(12\hat{j} + 6\hat{k})$$

$$= \frac{2}{\sqrt{5}}\hat{j} + \frac{1}{\sqrt{5}}\hat{k}$$

So, $\vec{c}$ is a unit vector perpendicular to both $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$.

Q21. Find k so that the lines

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2} \text{ and } \frac{x-1}{3k} = \frac{y-5}{1} = \frac{z-6}{-5}$$

are at right angles.

Ans. The given lines are

$$= \frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2} \quad \dots(1)$$

$$\text{and } \frac{x-1}{3k} = \frac{y-5}{1} = \frac{z-6}{-5} \quad \dots(2)$$

The direction ratios of line (1) are -3, $2k$, 2 and direction ratios of line (2) are $3k$, 1, -5. Since the lines (1) and (2) are at right angles, therefore

$(-3)(3k) + (2k)(1) + 2(-5) = 0$

or $-9k + 2k - 10 = 0$

or $-7k = 10$

$$or \quad k = \frac{10}{7}$$

Q22. Best Gift Packs company manufactures two types of gift packs, type A and type B. Type A requires 5 minutes each for cutting and 10 minutes assembling it. Type B requires 8 minutes each for cutting and 8 minutes each for assembling. There are at most 200 minutes available for cutting and at most 4 hours available for assembling. The profit is ₹ 50 each for type A and ₹ 25 each for type B. How many gift packs of each type should the company manufacture in order to maximise the profit ?

Ans. Type A : x, Type B : y then LPP is
Maximize

$$P = 50x + 25y$$

subject to

$5x + 8y \leq 200$ [Cutting constraint]

$10x + 8y \leq 240$ [Assembly constraint]

$x \geq 0$, $y \geq 0$ [Non- negativity]

P(A) = 1200
P (B) = 900
P (C) = 625
P(0) = 0

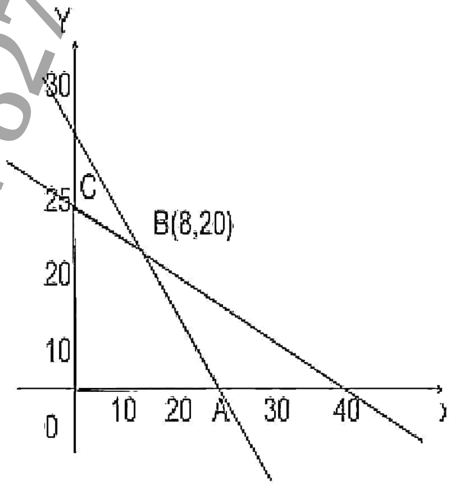


Figure 19

Thus, profit is maximum when $x = 24$, $y = 0$
Maximum profit = Rs. 1200

Q1: Use the principle of mathematical induction to show that $2 + 2^2 + ... + 2^n = 2^{n+1} - 2$ for every natural number n. **(4 Marks)**

Solution : Let P_n denote the statement

$$2 + 2^2 + ...+ 2^n = 2^{n+1} - 2$$

When $n = 1$, P_n becomes

$2 = 2^{1+1} - 2$ or $2 = 4 - 2$

This shows that the result holds for $n = 1$.

Assume that P_k is true for some $k \in N$.

That is, assume that

$$2 + 2^2 + ...+ 2^k = 2^{k+1} - 2$$

We shall now show that truth of P_k implies the truth of P_{k+1} is

$$2 + 2^2 + ...+ 2^k + 2^{k+1} = 2^{k+1} - 2 \quad (1)$$

$$\begin{aligned} \text{LHS of (1)} &= 2 + 2^2 + ...+ 2^k + 2^{k+1} \\ &= (2^{k+1} - 2) + 2^{k+1} \quad \text{[induction assumption]} \\ &= 2^{k+1}(1 + 1) - 2 \\ &= 2^{k+1}2 - 2 = 2^{k+2} - 2 \\ &= \text{RHS of (1)} \end{aligned}$$

This shows that the result holds for $n = k+1$; therefore, the truth of P_k implies the truth of P_{k+1}. The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number n.

Q2: Find the sum of all integers between 100 and 1000 which are divisible by 9. **(4 Marks)**

Solution : The first integer greater than 100 and divisible by 9 is 108 and the integer just smaller than 1000 and divisible by 9 is 999. Thus, we have to find the sum of the series.

108 + 117 + 126 + ...+ 999.

Here $t_1 = a = 108$, $d = 9$ and $l = 999$

Let n be the total number of terms in the series be n. Then

$999 = 108 + 9(n-1) \Rightarrow 111 = 12 + (n-1) \Rightarrow n = 100$

Hence, the required sum $= \frac{n}{2}(a+l) = \frac{100}{2}(108+999)$

$= 50\,(1107) = 55350.$

Q3: Reduce the matrix A(given below) to normal form and hence find its rank. **(4 Marks)**

$$A = \begin{bmatrix} 5 & 3 & 8 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Solution : $A \begin{bmatrix} 5 & 3 & 8 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

Applying $R_1 \leftrightarrow R_3$, we have

$$A \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 5 & 3 & 8 \end{bmatrix}$$

Applying $R_3 \to R_3 - 5R_1$, we have

$$A \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 8 & 8 \end{bmatrix}$$

Applying elementary row operations $R_1 \to R_1 + R_2$ and $R_3 \to R_3 - 8R_2$, we have

$$A \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, we apply elementary column operation $C_3 \to C_3 - C_1$, to get

$$A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Again, applying $C_3 \to C_3 - C_2$, we have

$$A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We have thus reduced A to normal form.

Also, note that the rank of a matrix remains unaltered under elementary operations.

Thus, rank of A in above example is 2 because rank of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is 2.

In this regard, we state following theorem without proof.

Q4: Show that n(n+1) (2n+1) is a multiple of 6 for every natural number n. **(4 Marks)**

Solution : Let P_n denote the statement $n(n + 1)(2n + 1)$ is a multiple of 6.

When $n = 1$, P_n becomes $1(1 + 1)((2)(1) + 1) = (1)(2)(3) = 6$ is a multiple of 6.

This shows that the result is true for $n = 1$.

Assume that P_k is true for some $k \in \mathbf{N}$. That is assume that $k(k + 1)(2k + 1)$ is a mutliple of 6.

Let $k(k + 1)(2k + 1) = 6m$ for some $m \in \mathbf{N}$.

We now show that the truth of P_k implies the truth of P_{k+1}, where P_{k+1} is $(k + 1)(k + 2)[2(k + 1) + 1] = (k + 1)(k + 2)(2k + 3)$ is a multiple of 6.

We have

$(k + 1)(k + 2)(2k + 3)$

$= (k + 1)(k + 2)[(2k + 1) + 2]$

$= (k + 1)[k(2k + 1) + 2(2k + 1) + 4)]$

$= (k + 1)[k(2k + 1) + 6(k + 1)]$

$= k(k + 1)(2k + 1) + 6(k + 1)^2$

$= 6m + 6(k + 1)^2 = 6[m + (k + 1)^2]$

Thus $(k + 1)(k + 2)(2k + 3)$ is multiple of 6.

This shows that the result holds for $n = k+1$; therefore, the truth of P_k implies the truth of P_{k+1}. The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number n.

Q5: Find the sum of an infinite G.P. whose first term is 28 and fourth term is $\frac{4}{49}$. **(4 Marks)**

Answer: -

G.P. : $a, ar, ar^2, ar^3, ar^4, \ldots\ldots\ldots a$

where, first term be 'a' and common ratio be 'r'.

According to the question,

$a = 28$ and

fourth term, $a_4 = ar^3 = \frac{4}{49}$

Substituting $a = 28$, we get

$\Rightarrow 28r^3 = \frac{4}{49}$

$\Rightarrow r^3 = \frac{\cancel{4}^1}{\cancel{28}^7 \times 49}$

$\Rightarrow r^3 = \left(\frac{1}{7}\right)^3$

$\therefore r = \frac{1}{7}$

According to the formula of sum of terms upto infinite in G.P., we have

$S_\infty = \frac{a}{1-r}$

Substituting the value of 'a' and 'r', we get

$S_\infty = \frac{28}{1-\frac{1}{7}} \Rightarrow \frac{28}{\frac{7-1}{7}} \Rightarrow \frac{\cancel{28}^{14} \times 7}{\cancel{6}^3} \Rightarrow \frac{98}{3} \Rightarrow 32\frac{2}{3}$

$\therefore$ The required sum $= 32\frac{2}{3}$ (Ans.)

Q6: Check the continuity of the function f(x) at x = 0 : **(4 Marks)**

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Solution : Let $f(x)= \frac{|x|}{x}$, $x \neq 0$.

Since $|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$

$\therefore f(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$

So, $\lim_{x \to 0+} f(x) = \lim_{x \to 0}(1) = 1$ and

$\lim_{x \to 0-} f(x) = \lim_{x \to 0}(-1) = -1$

Thus $\lim_{x \to 0} f(x)$ does not exist.

Q7: If $y = \frac{\ln x}{x}$, show that $\frac{d^2y}{dx^2} = \frac{2\ln x - 3}{x^3}$ **(4 Marks)**

Solution : we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{lnx}{x}\right] = \frac{d}{dx}[x^{-1}lnx]$$

$$= \frac{d}{dx}(x^{-1})lnx + x^{-1}\frac{d}{dx}(lnx) \quad \text{(product rule)}$$

$$= (-1)x^{-2}lnx + x^{-1}\frac{1}{x}$$

$$= x^{-2}[1 - \ln x]$$

Differentiating both sides with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[x^{-2}][1 - lnx] + x^{-2}\frac{d}{dx}[1 - lnx]$$

$$= (-2)\,x^{-3}(1 - lnx) + x^{-2}\left(0 - \frac{1}{x}\right)$$

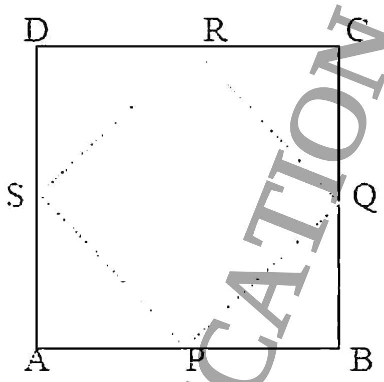
$$= -2x^{-3}(1 - lnx) + x^{-3}$$

$$= -x^{-3}(2 - 2\ln x + 1)$$

$$= \frac{2lnx - 3}{x^3}$$

Q8: If the mid-points of the consecutive sides of a quadrilateral are joined, then show (by using vectors) that they form a parallelogram. **(4 Marks)**

Solution : Let $\vec{a}$. $\vec{b}$. $\vec{c}$ $\vec{d}$ be the positon vectors of the vertices A. B. C. D of the quadrilateral ABCD. Let *P, Q, R, S* be the mid-points of sides AB. BC. CD. DA respectively. Then the position vectors of P. Q. R and S are $\frac{1}{2}(\vec{a}+\vec{b}), \frac{1}{2}(\vec{b}+\vec{c}), \frac{1}{2}(\vec{c}+\vec{d})$ and $\frac{1}{2}(\vec{d}+\vec{a})$ respectively.



$$Now.\ \overline{PQ} = \overline{PQ} - \overline{PQ} = \frac{1}{2}(\vec{b}+\vec{c}) - \frac{1}{2}(\vec{a}+\vec{b}) = \frac{1}{2}(\vec{c}-\vec{a})$$

or $\quad \overline{AB} + \overline{BC} + \overline{CA} \quad (\because \overline{CA} = -\overline{AC})$

$\therefore \quad \overline{PQ} = \overline{SR}$

$\Rightarrow \quad PQ = SR$ and also $PQ \parallel SR$.

Since a pair of opposite sides are equal and parallel. therefore. PQRS is a parallelogram.

Q9: Solve the equation $2x^3 - 15x^2 + 37x - 30 = 0$, given that the roots of the equation are in A.P. **(4 Marks)**

Solution : Recall three numbers in A.P. can be taken as $\alpha - \beta,\ \alpha,\ \alpha + \beta$.

If $\alpha - \beta,\ \alpha,\ \alpha + \beta$ are roots of (1), then $(\alpha - \beta) + \alpha + (\alpha + \beta) = 15/2 \Rightarrow 3\alpha = 15/2$

$\Rightarrow \alpha = 5/2$

Next,

$\alpha(\alpha - \beta) + \alpha(\alpha + \beta)\ (\alpha - \beta)(\alpha + \beta) = 37/2$

$\Rightarrow \alpha^2 - \alpha\beta + \alpha^2 + \alpha\beta + \alpha^2 - \beta^2 = 37/2$

$\Rightarrow 3\alpha^2 - \beta^2 = 37/2$

$$\Rightarrow \beta^2 = 3\alpha^2 - \frac{37}{2} = 3 \times \frac{25}{4} - \frac{37}{2} = \frac{1}{4}$$

$$\Rightarrow \beta = \pm\frac{1}{2}$$

When $\beta = ½$, the roots are

$$\frac{5}{2} - \frac{1}{2}, \frac{5}{2}, \frac{5}{2} + \frac{1}{2}, \text{ or } 2, \frac{5}{2}, 3$$

When $\beta = -\frac{1}{2}$, the roots are 3, 5/2 2.

It is easily to check that these are roots of (1).

Q10: A young child is flying a kite which is at height of 50 m. The wind is carrying the kite horizontally away from the child at a speed of 6.5 m/s. How fast must the kite string be let out when the string is 130m ? **(4 Marks)**

Solution : Let h be the horizontal distance of the kite from the point directly over the child's head 5. Let l be the length of kite string from the child to the kite at time t. [See Fig. 1] Then

$$l^2 = h^2 + 50^2$$

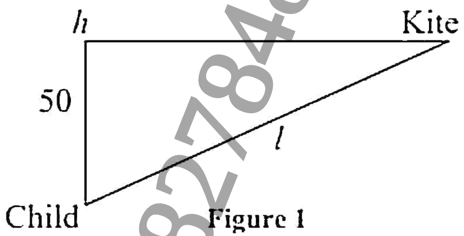


Figure 1

Differentiating both the sides with respect to t, we get

$$2l\,\frac{dl}{dt} = 2h\,\frac{dh}{dt} \text{ or } l\,\frac{dl}{dt} = h\,\frac{dh}{dt}.$$

We are given $\frac{dh}{dt} = 6.5\ m/s$. We are interested to find dl/dt when $l = 130$.
But when $l = 130$, $h^2 = l^2 - 50^2 = 130^2 - 50^2 = 14400$ or $h = 120$.

$$\text{Thus, } \frac{dl}{dt} = \frac{120}{130} \times 6.5 = 5\ m/s.$$

This shows that the string should be let out at a rate of 6 m/s.

Q11: Using first derivative test, find the local maxima and minima of the function $f(x) = x^3 - 12x$. **(4 Marks)**

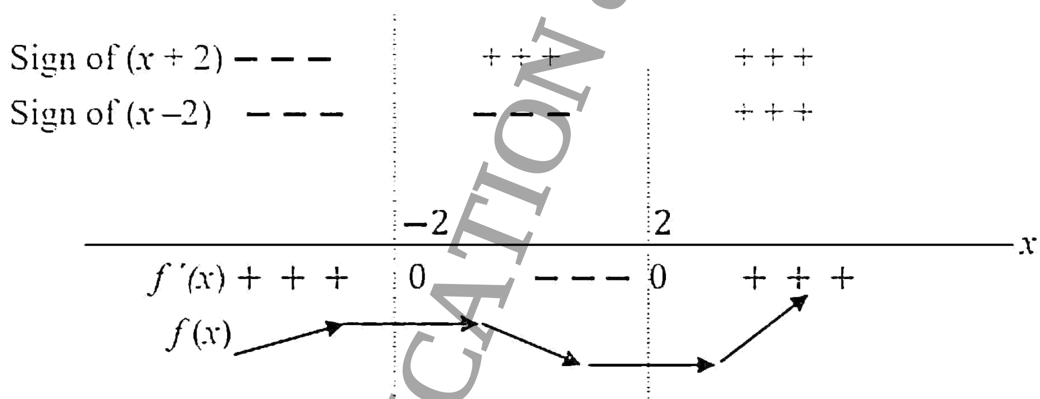
Answer: -

$f(x) = x^3 - 12x$

Differentiating w.r.t. x, we get

$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 2(x-2)(x+2)$

Setting $f'(x) = 0$, we obtain $x = 2, -2$. Thus, $x = -2$, and $x = 2$ are the only critical numbers of f. Fig. shows the sign of derivative f' in three intervals.



Figure

From figure it is clear that if $x < -2$, $f'(x) > 0$; if $-2 < x < 2$, $f'(x) < 0$ and if $x > 2$, $f'(x) > 0$.

Using the first derivative test, we conclude that

$f(x)$ has a local maximum at $x = -2$ and a local minimum *at* $x = 2$.

Now, $f(-2) = (-2)^3 - 12(-2) = -8 + 24 = 16$ is the value of local maximum at $x = -2$ and $f(2) = 2^3 - 12(2) = 8 - 24 = -16$ is the value of the local minimum at $x = 2$.

Q12: Evaluate the integral $I = \int \frac{x^2}{(x+1)^3} dx$ **(4 Marks)**

Solution : To evaluate an integral of the form

$$\int \frac{P(x)}{(a+bx)^r}\, dx, \text{we put } a + bx = t.$$

So, we put $x + 1 = t \Rightarrow dx = dt$

$$\text{and I} = \int \frac{(t+1)^2}{t^3}\, dt = \int \frac{t^2 + 2t + 1}{t^3}\, dt$$

$$= \int \left(\frac{1}{t} - 2t^{-2} + t^{-3}\right) dt$$

$$= \log|t| - \frac{2t^{-1}}{-1} + \frac{t^{-2}}{-2} + c$$

$$= \log|t| + \frac{2}{t} - \frac{1}{2t^2} + c$$

$$= \log|x+1| - \frac{2}{x+1} + \frac{1}{2(x+1)^2} + c$$

Q13: Find the scalar component of projection of the vector $\vec{a} = 2\hat{i} + 3\hat{j} + 5\hat{k}$ on the vector $\vec{b} = 2\hat{i} - 2\hat{j} - \hat{k}$. **(4 Marks)**

Solution : Scalar projection of $\vec{a}$ on $\vec{b}$ = $\dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

Here. $\vec{a}.\vec{b} = 2.2 + 3(-2) + 5(-1) = -7$

and $|\vec{b}| = \sqrt{2^2 + (-2)^2 + (-1)^2} = 3$

$\therefore$ Scalar projection of $\vec{a}$ on $\vec{b}$ = $\dfrac{-7}{3}$

Q14: If 1, ω, $\omega2$ are cube roots unity, show that $(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11}) = 49$. **(4 Marks)**

Answer: -

Since $\omega^{10} = (\omega^3)^3\ \omega = \omega$

and $\omega^{11} = (\omega^3)^3\ \omega^2 = \omega^2$.

Thus $(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11})$

$= (2-\omega)(2-\omega^2)(2-\omega)(2-\omega^2)$

$= [(2-\omega)(2-\omega^2)]^2$

$= [4 - 2\omega - 2\omega^2 + \omega^3]^2$

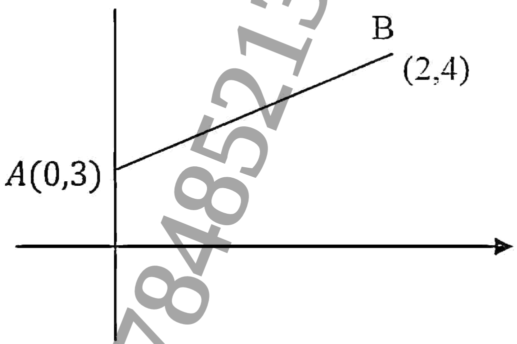
$= [4 - 2(\omega + \omega^2) + 1]^2$

$= [4 - 2(-1) + 1]^2$ $\quad [\because \omega + \omega^2 = -1]$

$= 7^2 = 49$

Q15: Find the length of the curve $y = 3 + \frac{x}{2}$ from (0, 3) to (2, 4). **(4 Marks)**

Answer: -



We have

$$\frac{dy}{dx} = \frac{1}{2}$$

Required length

$$= \int_0^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^2 \sqrt{1 + \frac{1}{4}}\, dx = \frac{\sqrt{5}}{2}\int_0^2 dx = \frac{\sqrt{5}}{2}\, x\Big|_0^2 = \sqrt{5} \text{ units}$$

Q16: Evaluate the determinant given below, where ω is a cube root of unity. **(4 Marks)**

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$$

Solution : $\begin{vmatrix} 1 & \omega & 2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$

$1 + \omega + \omega^2$

$$= \begin{vmatrix} 1+\omega+\omega^2 & \omega & \omega^2 \\ 1+\omega+\omega^2 & \omega^2 & 1 \\ 1+\omega+\omega^2 & 1 & \omega \end{vmatrix} \quad (\text{By } C_1 \to C_1 + C_2 - C_3)$$

$$= \begin{vmatrix} 0 & \omega & \omega^2 \\ 0 & \omega^2 & 1 \\ 0 & 1 & \omega \end{vmatrix} \quad (\because 1 + \omega + \omega^2 = 0)$$

$= 0$ [$\because C_1$ consists of all zero entries].

Q17: Using determinant, find the area of the triangle whose vertices are $(-3,\ 5), (3, -6)$ and $(7,\ 2)$. **(4 Marks)**

Answer: -

$$\Delta = \frac{1}{2}1\begin{vmatrix} -3 & 5 & 1 \\ 3 & -6 & 1 \\ 7 & 2 & 1 \end{vmatrix}$$

$$= \frac{1}{2}1\begin{vmatrix} -3 & 5 & 1 \\ 6 & -11 & 0 \\ 10 & -3 & 0 \end{vmatrix} \text{(By applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1\text{)}$$

$$= \frac{1}{2}1\ |-18 + 110)|$$

$$= \frac{1}{2} \times 92 = 46 \text{ square units}$$

Q18: Solve the following system of linear equations using Cramer's rule: **(4 Marks)**

$x + y = 0,\ \ y + z = 1,\ \ z + x = 3$

Answer: -

Here,

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} \quad [\text{Applying } C_2 \to C_2 - C_1]$$

$$2 \quad (\text{Expanding along } R_1)$$

Since $\Delta \neq 0$, $\therefore$ the given system has unique solution.

Now, $\Delta x = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix} = 2$

$$\Delta y = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} = -2$$

and $\Delta z = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{vmatrix} = 4$

Hence by Cramer's Rule

$$x = \frac{\Delta x}{\Delta} = \frac{2}{2} = 1$$

$$y = \frac{\Delta y}{\Delta} = \frac{-2}{2} = -1 \text{ and}$$

$$z = \frac{\Delta z}{\Delta} = \frac{4}{2} = 2$$

Q19: If $A=\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$, $B=\begin{bmatrix} \mathbf{a} & 1 \\ \mathbf{b} & -1 \end{bmatrix}$ and $(A+B)^2 = A^2 + B^2$, Find a and b. **(4 Marks)**

We have $(A+B)^2 = (A+B)(A+B)$

$= (A+B)A + (A+B)B$ (Distributive Law)

$= AA + BA + AB + BB$

$= A^2 + BA + AB + B^2$

Therefore, $(A+B)^2 = A^2 + B^2$

$\Rightarrow A^2 + BA + AB + B^2 = A^2 + B^2$

$\Rightarrow BA + AB = 0.$

Thus, we must find a and b such that BA + AB = 0.

We have BA $= \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} a+2 & -a-1 \\ b-2 & -b+1 \end{bmatrix}$

and AB $= \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} = \begin{bmatrix} a-b & 2 \\ 2a-b & 3 \end{bmatrix}$

Therefore,

BA + AB $= \begin{bmatrix} a+2 & -a-1 \\ b-2 & -b+1 \end{bmatrix} + \begin{bmatrix} a-b & 2 \\ 2a-b & 3 \end{bmatrix}$

$$= \begin{bmatrix} 2a-b+2 & -a+1 \\ 2a-2 & -b+4 \end{bmatrix}$$

But $BA + AB = 0$

$\Rightarrow 2a - b + 2 = 0, -a + 1 = 0, 2a - 2 = 0, -b + 4 = 0$

$\Rightarrow a = 1, b = 4$

Q20: Use De Moivre's theorem to find $(\sqrt{3}+i)^3$. **(4 Marks)**

Solution : We first put $\sqrt{3}+i$ in the polar form.

Let $\sqrt{3}+i = r(\cos\theta + i\sin\theta)$

$\Rightarrow \sqrt{3} = r\cos\theta$ and $1 = r\sin\theta$

$\Rightarrow (\sqrt{3})^2 + 1^2 = r^2(cos^2\theta + sin^2\theta)$

$\Rightarrow r^2 = 4 \Rightarrow r = 2$

Thus, $\sqrt{3} + i = 2(cos\theta + i\sin\theta)$

$\Rightarrow \sqrt{3} = 2\cos\theta$ and $1 = 2\sin\theta$

$\Rightarrow 2\cos\theta = \frac{\sqrt{3}}{2}$ and $\sin\theta = \frac{1}{2}$

$\Rightarrow \theta = 30°$.

Now, $(\sqrt{3}+i)^3 = [2\cos(30°) + i\sin(30°)]^3$

$= 2^3[\cos(30°) + i\sin(30°)]^3$

$= 8[\cos(3\times30°) + i\sin(3\times30°))]$ [De Moivre's theorem]

$= 8(\cos 90° + i\sin 90°) = 8(0+i)$

$= 8i$