BCS-12: BASIC MATHEMATICS
Guess Paper-1

Q. Compute the following determinants:

(a) $\begin{vmatrix} 3 & 5 \\ -2 & 6 \end{vmatrix}$ (b) $\begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix}$

(c) $\begin{vmatrix} \alpha + i\beta & \gamma + is \\ -\gamma + is & \alpha - i\beta \end{vmatrix}$ (d) $\begin{vmatrix} \omega & \omega \\ -1 & \omega \end{vmatrix}$

(e) $\begin{vmatrix} x-1 & 1 \\ x^3 & x^2+x+1 \end{vmatrix}$ (f) $\begin{vmatrix} \dfrac{1-t^2}{1+t^2} & \dfrac{2t}{1+t^2} \\ \dfrac{-2t}{1+t^2} & \dfrac{1-t^2}{1+t^2} \end{vmatrix}$

Solutions :

(a) $\begin{vmatrix} 3 & 5 \\ -2 & 6 \end{vmatrix} = 18 - (-10) = 28$

(b) $\begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix} = a^2b^2 - (ab)^2 = 0$

(c) $\begin{vmatrix} \alpha + i\beta & \gamma + is \\ -\gamma + is & \alpha - i\beta \end{vmatrix} = \alpha^2 + \beta^2 + \gamma^2 + s^2$

$(\because (a+ib)(a-ib) = a^2 + b^2)$

(d) $\begin{vmatrix} \omega & \omega \\ -1 & \omega \end{vmatrix} = \omega^2 + \omega = -1$ because $\omega^2 + \omega + 1 = 0$

(e) $\begin{vmatrix} x-1 & 1 \\ x^3 & x^2+x+1 \end{vmatrix} = (x-1)(x^2+x+1) - x^3 = x^3 - 1 - x^3 = -1$

(f) $\begin{vmatrix} \dfrac{1-t^2}{1+t^2} & \dfrac{2t}{1+t^2} \\ \dfrac{-2t}{1+t^2} & \dfrac{1-t^2}{1+t^2} \end{vmatrix} = \left(\dfrac{1-t^2}{1+t^2}\right)^2 + \dfrac{4t^2}{(1+t^2)^2}$

$$= \frac{(1-t^2)^2 + 4t^2}{(1+t^2)^2} = \frac{(1-t^2)^2}{(1+t^2)^2} = 1 \ [\because (a-b)^2 + 4ab = (a+b)^2 \ ]$$

Q. Write down the minor and cofactors of each element of the determinant.

$\begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix}$.

Solution: Hence, $\Delta = \begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix}$

$M_{11} = |5| = 5$ $M_{12} = |2| = 2$

$M_{21} = |-1| = -1$ $M_{22} = |3| = 3$

Q. Show that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)\,(c-a)(c-b)$$

Solution : By applying $R_2 \rightarrow R_2 - R_1$, and $R_3 \rightarrow R_3 - R_1$ we get,

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

Taking $(b-a)$ common from R_2 and $(c-a)$ common from R_3, we get

$$\Delta = (b-a)\,(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+c \\ 0 & 1 & c+1 \end{vmatrix}$$

Expanding along C_1, we get

$$\Delta = (b-a)\,(c-a) \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$

$$= (b-a)\,(c-a)[(c+a)-(b+a)]$$

$$= (b-a)(c-a)(c-b)$$

Q. Show that

$$\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Solution : Denote the determinant on the L.H.S. by Δ. Then applying $C_1 \rightarrow C_1 + C_2 + C_3$ we get

$$\Delta = \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \\ 2(a+b+c) & b+c & c+a \end{vmatrix}$$

Taking 2 common from C_1 and applying $C_2 \rightarrow C_2 - C_1$, and $C_3 \rightarrow C_3 - C_1$, we get

Taking 2 common from C_1 and applying $C_2 \rightarrow C_2 - C_1$, and $C_3 \rightarrow C_3 - C_1$, we get

$$\Delta = 2\begin{vmatrix} (a+b+c) & -b & -c \\ (a+b+c) & -c & -a \\ (a+b+c) & -a & -b \end{vmatrix}$$

Applying $C_1 \rightarrow C_2$, $C_2 + C_3$ and taking (-1) common from both C_2 and C_2, we get

$$\Delta = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Q. Using determinants, find the area of the triangle whose vertices are
(a) A(1, 4) , B(2,3) and C(–5,–3)
(b) A(–2,4), B(2,–6) and C(5,4)

Solution :

$$\text{Area of } \Delta ABC = \frac{1}{2} \left| \begin{vmatrix} 1 & 4 & 1 \\ 2 & 3 & 1 \\ -5 & -3 & 1 \end{vmatrix} \right|$$

$$= \frac{1}{2} \left| \begin{vmatrix} 1 & 4 & 1 \\ 1 & -1 & 0 \\ -6 & -7 & 0 \end{vmatrix} \right| \text{ (using } R1 \rightarrow R2 - R1, \text{ and } R3 \rightarrow R3 - R1)$$

$$= \frac{1}{2}|-7-6|$$

$$= \frac{13}{2} \text{ square units}$$

$$\text{Area of } \Delta ABC = \frac{1}{2} \left| \begin{vmatrix} -2 & 4 & 1 \\ 2 & -6 & 1 \\ 5 & 4 & 1 \end{vmatrix} \right|$$

$$= \frac{1}{2}|70|$$

$$= 35 \text{ square units}$$

Q. Solve the system of linear homogeneous equation:
$2x - y + 3z = 0$,
$x + 5y - 7z = 0$,
$x - 6y + 10z = 0$

Solution : We first evaluate Δ. We have

$$\Delta = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 5 & -7 \\ 1 & -6 & 10 \end{vmatrix} \text{ Applying } R_1 \rightarrow R_1 - 2R_2 \text{ and } R_2 \rightarrow R_2 - R_3 \text{, we get}$$

$= -20$ (expanding along C_1)

$$\Delta = \begin{vmatrix} 0 & -11 & 17 \\ 0 & 11 & -17 \\ 1 & -6 & 10 \end{vmatrix} = 0$$

(because R_1 and R_2 are proportional)

Therefore, the given system of linear homogeneous equations has an infinite number of solutions. Let us find these solutions. We can rewrite the first two equations as :

$$\begin{aligned} 2x - y &= -3z \\ x + 5y &= 7z \qquad \qquad \ldots\ldots (1) \end{aligned}$$

Now, we have $\Delta' = \begin{vmatrix} 2 & -1 \\ 2 & 5 \end{vmatrix} = 10 - (-1) = 11.$

As $\Delta' \neq 0$, the system of equation in (1) has a unique solution. We have

$$\Delta x = \begin{vmatrix} -3z & -1 \\ 7z & 5 \end{vmatrix} = -15z - (-7z) = -8z \text{ and}$$

$$\Delta x = \begin{vmatrix} 2 & -3z \\ 1 & 7z \end{vmatrix} = 14z - (-3z) = 17z$$

By Cramer's Rule, $x = \frac{\Delta x}{\Delta'} = \frac{-8z}{11} = \frac{-8}{11}z$ and $y = \frac{\Delta y}{\Delta'} = \frac{17z}{11} = \frac{17}{11}z.$

We now check that this solution satisfies the last equation. We have

$$\begin{aligned} x - 6y + 10z &= \frac{-8}{11}z = -6\left(\frac{17}{11}z\right) + 10z \\ &= \frac{1}{11}(-8z - 102z + 110z) = 0. \end{aligned}$$

Therefore, the infinite number of the given system of equations are given by

$x = \frac{-8}{11}k,\ y = \frac{17}{11}k$ and $z = k$, where k is any real number.

Q. First, we note that by f(A) we mean $A^2 - 4A + 7I_2$. That is, we replace x by A and multiply the constant term by I, the unit matrix. Therefore, f(A) = $A^2 - 4A + 7I_2$

$$= \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & 6+6 \\ -2-2 & -3+4 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ -4 & 8 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ -4 & 8 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1-8+7 & 12-12+0 \\ -4+4+0 & 1-8+7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O_{2\times 2}.$$

Hence, $A^2 - 4A - 7I_2$, from which we get

$$\begin{aligned} A^3 &= A^2 A = (4A - 7I_2)A \\ &= 4A^2 - 7I_2A = 4(4A - 7I_2) - 7A \quad [\because I_2 A = A] \\ &= 9A \quad 28 I_2 \\ \Rightarrow A^5 &= A^2 A^3 = (4A - 7I_2)(9A - 28I_2) \end{aligned}$$

$= 36A^2 - 63I_2A - 112AI_2 + 196\, I_2\, I_2$ (Distributive Law)

$= \ 36\,(4A - 7I_2) - 63A - 112A + 196\, I_2$

$= 144A - 252\, I_2 - 175\, A + 196\, I_2$

$= -31A \ -56\ I_2$

$= -31 \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} - 56 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -62 & -93 \\ 31 & -62 \end{bmatrix} - \begin{bmatrix} 56 & 0 \\ 0 & 56 \end{bmatrix}$

$= \begin{bmatrix} -118 & -93 \\ 31 & -118 \end{bmatrix}$

Q. Find the inverse of A =
$\begin{bmatrix} -3 & 5 \\ 2 & 4 \end{bmatrix}$

We have $A_{11} = (-1)^{1+1}\ |4| = 4$ and $A_{12} = (-1)^{1+2}\ |2| = 2$.

We know that $|A| = a_{11}A_{11} + a_{12}A_{12} = (-3)(4) + 5(-2) = -22$.

Since $|A| \neq 0$ the matrix A is invertible. Also,
$A_{21} = (-1)^{2+1}\ |5| = -5$ and $A_{22} = (-1)^{2+2}\ |-3| = -3$. Therefore,

$\text{adj } A = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} = \begin{pmatrix} 4 & -5 \\ -2 & -3 \end{pmatrix}$

Hence $A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{-22}\begin{pmatrix} 4 & -5 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} -2/11 & 5/22 \\ 1/11 & 3/22 \end{pmatrix}$

Q.

Find the inverse of $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

and verify that $A^{-1}A = I_3$.

Solution : Evaluating the cofactors of the elements in the first row of A, we get

$A_{11} = (-1)^{1+1} \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = 2, \quad A_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = -3,$

and $A_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = 5.$

$\therefore \quad |A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$= (1)(2) + (2)(-3) + (5)(5) = 21$

Since $|A| \neq 0$, A is invertible. Also,

$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} = 3,\qquad A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 5 \\ -1 & 1 \end{vmatrix} = 6,$

$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = -3,\qquad A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} = -13,$

$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix} = 9,\qquad A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1,$

$$\therefore \text{adj A} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{21}\begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2/21 & 3/21 & -13/21 \\ -3/21 & 6/21 & 9/21 \\ 5/21 & -3/21 & -1/21 \end{bmatrix}$$

To verify that this is the inverse of A, we have

$$A^{-1}A = \begin{bmatrix} 2/21 & 3/21 & -13/21 \\ -3/21 & 6/21 & 9/21 \\ 5/21 & -3/21 & -1/21 \end{bmatrix}\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{21}+\frac{6}{21}+\frac{13}{21} & \frac{4}{21}+\frac{9}{21}+\frac{-13}{21} & \frac{10}{21}+\frac{3}{21}+\frac{-13}{21} \\ \frac{-3}{21}+\frac{12}{21}+\frac{9}{21} & \frac{-6}{21}+\frac{18}{21}+\frac{9}{21} & \frac{-15}{21}+\frac{6}{21}+\frac{19}{21} \\ \frac{5}{21}-\frac{6}{21}+\frac{1}{21} & \frac{10}{21}-\frac{9}{21}-\frac{1}{21} & \frac{25}{21}-\frac{3}{21}-\frac{1}{21} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

BCS-12: BASIC MATHEMATICS
Guess Paper-2

Q. Solve the following system of equations by using matrix inverse:
3x + 4 y +7z = 14, 2x – y + 3z = 4, 2x + 2y – 3z = 0

Solution : We can put the given system of equations into the single matrix equation $AX = B$, where

$$A = \begin{pmatrix} 3 & 4 & 7 \\ 2 & -1 & 3 \\ 1 & 2 & -3 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } B = \begin{pmatrix} 14 \\ 4 \\ 0 \end{pmatrix}$$

$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} = 3,\quad A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 5 \\ -1 & 1 \end{vmatrix} = 6,$

$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = -3,\quad A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} = -13,$

$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix} = 9,\quad A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1,$

$$\therefore \text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{21}\begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2/21 & 3/21 & -13/21 \\ -3/21 & 6/21 & 9/21 \\ 5/21 & -3/21 & -1/21 \end{bmatrix}$$

To verify that this is the inverse of A, we have

Since $|A| \neq 0$, A is non–singular (invertible). Its remaining cofactors are

$A_{21} = (-1)^{2+1}\begin{vmatrix} 4 & 7 \\ 2 & -3 \end{vmatrix} = 26,\quad A_{22} = (-1)^{2+2}\begin{vmatrix} 3 & 7 \\ 1 & -3 \end{vmatrix} = -16,$

$A_{23} = (-1)^{2+3}\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = -2,\quad A_{31} = (-1)^{3+1}\begin{vmatrix} 4 & 7 \\ -1 & -3 \end{vmatrix} = 19,$

$A_{32} = (-1)^{3+2}\begin{vmatrix} 3 & 7 \\ 2 & 3 \end{vmatrix} = 5,\quad A_{33} = (-1)^{3+3}\begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} = -11.$

The adjoint of matrix A is given by

$$\therefore\ A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{62}\begin{pmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{pmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{62}\begin{pmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{62}\begin{pmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{pmatrix}\begin{pmatrix} 14 \\ 4 \\ 0 \end{pmatrix}$$

BCS-12: BASIC MATHEMATICS
Guess Paper-3

Q. Solve the following system of equations by using matrix inverse:
3x + 4 y +7z = 14, 2x – y + 3z = 4, 2x + 2y – 3z = 0

Solution : We can put the given system of equations into the single matrix equation $AX = B$, where

Since $|A| \neq 0$, A is non-singular (invertible). Its remaining cofactors are

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 4 & 7 \\ 2 & -3 \end{vmatrix} = 26, \quad A_{22} = (-1)^{2+2}\begin{vmatrix} 3 & 7 \\ 1 & -3 \end{vmatrix} = -16,$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = -2, \quad A_{31} = (-1)^{3+1}\begin{vmatrix} 4 & 7 \\ -1 & -3 \end{vmatrix} = 19,$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 3 & 7 \\ 2 & 3 \end{vmatrix} = 5, \quad A_{33} = (-1)^{3+3}\begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} = -11.$$

The adjoint of matrix A is given by

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{62}\begin{pmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{pmatrix}$$

$$\text{Also}, X = A^{-1}B = \frac{1}{62}\begin{pmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{pmatrix}\begin{pmatrix} 14 \\ 4 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{62}\begin{pmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{pmatrix}\begin{pmatrix} 14 \\ 4 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{62}\begin{pmatrix} -42 + 104 \\ 126 - 64 \\ 70 - 8 \end{pmatrix} = \frac{1}{62}\begin{pmatrix} 62 \\ 62 \\ 62 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Hence $x = 1, y = 1, z = 1$ is the required solution.

Q. Solve the following system of homogeneous linear equation by the Matrix method:
2x – y + 2z = 0, 5x + 3y – z = 0, x +5y –5 z = 0

Solution :

We can rewrite the above system of equations as the single matrix equation AX =0, where

$$A = \begin{pmatrix} 2 & -1 & -2 \\ 5 & 3 & -1 \\ 1 & 5 & -5 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The cofactors of |A| are

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 3 & -1 \\ 5 & -5 \end{vmatrix} = -10$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 5 & -1 \\ 1 & -5 \end{vmatrix} = 24$$

$$\text{and} \quad A_{13} = (-1)^{1+3} \begin{vmatrix} 5 & 3 \\ 1 & 5 \end{vmatrix} = 22.$$

$$\therefore |A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = (2)(-10) + (-1)(24) + (2)(22) = 0.$$

Therefore, A is singular matrix. We can rewrite the first two equation as follows: $2x - y = -2z$, $5x + 3y = z$ or in the matrix form as

$$A = \begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} -2z \\ z \end{bmatrix}.$$

Now, we have $A_{11} = (-1)^{1+1}|3| = 3$ and $A_{12} = (-1)^{1+2}|5| = -5$.

$$\therefore |A| = a_{11}A_{11} + a_{12}A_{12} = (2)(3) + (-1)(-5) = 11 \neq 0.$$

Thus, A is non singular (invertible). Also, $A_{21} = (-1)^{2+1}|-1| = 1$ and $A_{22} = (-1)^{2+2}|2| = 2$. Therefore, the adjoint of A is given by

$$\text{adj } A = \begin{pmatrix} 3 & 1 \\ -5 & 2 \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{11} \begin{pmatrix} 3 & 1 \\ -5 & 2 \end{pmatrix}.$$

Therefore, from $X = A^{-1}B$, we get

Thus, all the equation are satisfied by the values

$$\Rightarrow \quad x = -\frac{5}{11}z, \quad y = \frac{12}{11}z, \qquad z = z.$$

Where z is any complex number. Hence, the given system of equation has an infinite number of solutions.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{11} \begin{pmatrix} 3 & 1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} -2z \\ z \end{pmatrix} = \frac{1}{11} \begin{pmatrix} -6z + z \\ 10z + 2z \end{pmatrix} = \begin{bmatrix} -\frac{5}{11}z \\ \frac{12}{11}z \end{bmatrix}$$

Q.

Show that matrix A = $\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}$ is row equivalent to the matrix.

$B = \begin{bmatrix}1 & 2 & 3\\0 & -3 & -6\\0 & 0 & 0\end{bmatrix}$

Solution : We have $A = \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}$

Applying $R_2 \to R_2 - 4R_1$, we have

$$A \sim \begin{bmatrix}1 & 2 & 3\\0 & -3 & -6\\7 & 8 & 9\end{bmatrix}$$

Applying $R_3 \to R_3 - 7R_1$ to the matrix on R. H. S. we get.

$$A \sim \begin{bmatrix}1 & 2 & 3\\0 & -3 & -6\\0 & -6 & -12\end{bmatrix}$$

Now Applying $R_3 \to R_3 - 2R_2$, we have

$$A \sim \begin{bmatrix}1 & 2 & 3\\0 & -3 & -6\\0 & 0 & 0\end{bmatrix} = B$$

The matrix B in above example is a triangular matrix.

Q. Determine the rank of the matrix.

$$A = \begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\2 & 5 & 8\end{bmatrix}$$

Solution : Here, $|A| = \begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\2 & 5 & 8\end{bmatrix}$

$= 0$

So, rank of A cannot be 3.

Now $\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$ is a square submatrix of A such that $\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix} = 1 \neq 0$

$\therefore$ rank of A = 2.

Q. Reduce the matrix

$$A = \begin{bmatrix}5 & 3 & 8\\0 & 1 & 1\\1 & -1 & 0\end{bmatrix}$$

to normal form by elementary operations.

Solution : $A = \begin{bmatrix} 5 & 3 & 8 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

Applying $R_1 \leftrightarrow R_3$, we have

$$A \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 5 & 3 & 8 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 - 5R_1$, we have

$$A \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 8 & 8 \end{bmatrix}$$

Applying elementary row operations $R_1 \rightarrow R_1 + R_2$ and $R_3 \rightarrow R_3 - 8R_2$, we have

$$A \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, we apply elementary column operation $C_3 \rightarrow C_3 - C_2$, to get

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Again, applying $C_3 \rightarrow C_3 - C_1$, we have

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We have thus reduced A to normal form.

Also, note that the rank of a matrix remains unaltered under elementary operations.

Thus, rank of A in above example is 2 because rank of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is 2.

Q. Find the first term and the common difference of each of the following arithmetic progressions.

(i) 7, 11, 15, 19, 23..................

(ii) $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

(iii) $a + 2b, a + b, a, a - b, a - 2b, \ldots\ldots\ldots\ldots\ldots\ldots$

Solution :

	First term	Common difference
(i)	7	4
(ii)	$\frac{1}{6}$	$\frac{1}{3}$
(iii)	$a+2b$	$-b$

Q If pth term of an A.P. is q and its qth term is p, show that its rth term is p+q − r. What is its (p+q)th term ?

Solution : If d is the common difference of the A.P., then

$$a_p - a_q = (p - q)\, d$$

$$\Rightarrow q - p = (p - q)d$$

$$\Rightarrow d = \frac{q - p}{p - q} = -1$$

Now,

$$a_r - a_p = (r - p)\, d = (r - p)(-1)$$

$$\Rightarrow a_r = a_p - r + p$$

$$= q - r + p = p + q - r$$

$$\therefore a_{p+q} = p + q - (p + q) = 0 \qquad [\text{ put } r = p - q]$$

Q. If sum of the p[th], q[th] and r[th] terms of an AP are a, b, c respectively, show that

$$(q-r)\,\frac{a}{p} + (r-p)\,\frac{b}{q} + (p-q)\,\frac{c}{r} = 0 \qquad (1)$$

Solution : Let the first term of the AP be A and the common difference be D.

We are given :

$$a = S_p = \frac{p}{2}\,[2A + (p-1)D] \qquad (2)$$

$$b = S_q = \frac{q}{2}\,[2A + (q-1)D] \qquad (3)$$

$$c = S_r = \frac{r}{2}\,[2A + (r-1)D] \qquad (4)$$

Form (2), (3) and (4), we get

$$\frac{2a}{p} = (2A - D) + pD \qquad (4)$$

$$\frac{2b}{q} = (2A - D) + qD \qquad (5)$$

$$\frac{2c}{r} = (2A - D) + rD \qquad (6)$$

Multiplying (4) by $q-r$, (5) $r-p$ and (6) by $p-q$, we get

$$2(q-r)\,\frac{a}{p} + 2(r-p)\,\frac{b}{q} + 2(p-q)\,\frac{c}{r}$$

$$= (2A - D)(q - r) + p(q - r)D$$

$$+ (2A - D)(r - p) + q(r - p)D$$

$$+ (2A - D)(p - q) + r(p - q)D$$

$$= (2A - D)\{q - r + r - p + p - q\}$$

$+ (pq - pr + qr - qp + rp - rq)D$

$= (2A - D)(0) + (0)D = 0$

Dividing both the sides by 2 we get (1).

Q. Three numbers are in A.P. and their sum is 15. If 1, 3, 9 be added to them respectively, they form a G.P. find the numbers.

Solution : Let the three numbers in AP be $a - d, a, a + d$. we are given

$$(a - d) + a + (a + d) = 15 \Rightarrow 3a = 15 \text{ or } a = 5$$

According to the given condition $a - d + 1$, $a + 3$ and $a + d + 9$ are in GP

$$\Rightarrow \frac{a+3}{a-d+1} = \frac{a+d+9}{a+3}$$

$$\Rightarrow (a+3)^2 = (a-d+1)(a+d+9) \Rightarrow (5+3)^2 = (5-d+1)(5+d+9)$$

$$\Rightarrow 64 = (6-d)(14+d) \Rightarrow 64 = 84 - 8d - d^2$$

Q. Evaluate the following limits:

(i) $\lim_{x\to 2} [(x-1)2 + 6]$ (ii) $\lim_{x\to 0} \frac{ax+b}{cx+d}$ $(d \neq 0)$

(iii) $\lim_{x\to 2} \frac{x^2+5x+7}{x^2+8}$ (iv) $\lim_{x\to -1} \sqrt{x+17}$

Solution: (i) $\lim_{x\to 2} [(x-1)^2 + 6] = (2-1)^2 - 6 = 1 + 6 = 7$

(ii) Since $\lim_{x\to 0} cx + d = d \neq 0$,

$$\lim_{x\to 0} \frac{ax+b}{cx+d} = \frac{a(0)+b}{c(0)+d} = \frac{b}{d}$$

(iii) Since $\lim_{x\to 3} (x^2 - 8) = 3^2 - 8 = 17 \neq 0$,

$$\therefore \lim_{x\to 3} \frac{x^2+5x+7}{x^2+8} = \frac{3^2+5(3)+7}{3^2+8} = \frac{31}{17}$$

(iv) Since $\lim_{x\to -1} x + 17 = -1 + 17 = 16$, we have

$$\lim_{x\to -1} \sqrt{x+17} = \sqrt{16} = 4$$