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Master Course in Electrical Engineering

Assignment description: Multivariable Control Problems

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1 Introduction

This document contains the problems needed to be solved for the coursework in *Advanced Control Systems*. However, the problems provided in this coursework are not given for a single application but cover different multivariable analysis and control questions. The first part of this coursework covers problems in multivariable control system analysis, where the second part consists of control design related problems. A description of the different problems will be given below.

2 Control system analysis

2.1 System analysis

2.1.1 System zeros: Part I

Consider a state-space system, given in terms of matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$ (not necessarily a minimal realisation). Assume that the system is a Multiple Input-Multiple Output (MIMO) system that has the same amount of inputs as outputs, i.e. $p = m$ and $p \geq 2$. Develop your own procedure written in Matlab (no *Control System Toolbox* functions) that is able to (a) compute the system zeros and (b) suiting initial states $\boldsymbol{x}(t=0) = \boldsymbol{x}_0$ as well as input vector $\boldsymbol{u}(t=0) = \boldsymbol{u}_0$. Provide a detailed description of the involved formulas of your approach in your coursework and discuss your results. How are the zeros called that you compute? Furthermore, test your own Matlab-procedure with the following example:

$$\boldsymbol{G}(s) = \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+1)} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix} = \frac{1}{s(s+1)} \begin{bmatrix} s+1 & 1 \\ s & s \end{bmatrix} = \frac{1}{d(s)} \boldsymbol{N}(s)$$

Give the numerical matrices for the minimal realised state space model of $\boldsymbol{G}(s)$ and state the results of the zero (if any) of your algorithm.

Hint: The Rosenbrock-matrix can be used to compute a special class of zeros

$$\boldsymbol{P}(s) = \begin{bmatrix} s\boldsymbol{I} - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} .$$

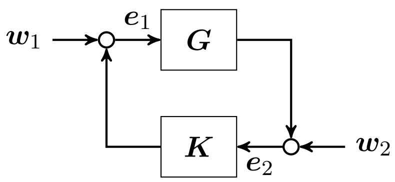

Figure 1: Feedback control loop.

2.1.2 System zeros: Part II

Consider a state-space system, given in terms of matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$ (not necessarily given in a minimal realisation). Assume again a MIMO system that now has a different number of inputs than outputs $p \neq m$. However, it must hold $p \geq 2$ and $m \geq 2$. Develop your own numerically stable procedure written in Matlab (no *Control System Toolbox* functions) that is able to (a) compute the system zeros and (b) suiting initial states $\boldsymbol{x}(t = 0) = \boldsymbol{x}_0$ as well as input vector $\boldsymbol{u}(t = 0) = \boldsymbol{u}_0$. Provide a detailed description of the involved formulas of your approach in your coursework and discuss your results. Furthermore, test your own Matlab-procedure with the following example:

$$\boldsymbol{G}(s) = \frac{1}{(s+1)(s+2)(s-1)} \begin{bmatrix} (s-1)(s+2) & 0 & (s-1)^2 \\ -(s+1)(s+2) & (s-1)(s+1) & (s-1)(s+1) \end{bmatrix}$$

Give the numerical matrices for the minimal realised state space model of $\boldsymbol{G}(s)$ and state the results of the zero (if any) of your algorithm.

Hint: The Rosenbrock-matrix can be used to compute a special class of zeros

$$\boldsymbol{P}(s) = \begin{bmatrix} s\boldsymbol{I} - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}.$$

2.1.3 System perturbation

We assume, that the following system is given

$$\boldsymbol{G}(s) = \begin{bmatrix} 7 & 8 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{2}{s+2} \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 6 & 7 \end{bmatrix}^{-1}, \tag{1}$$

and a controller $\boldsymbol{K} = -\boldsymbol{I}$ in feedback configuration like shown in Fig. 1.

(a) Determine internal stability of the closed loop. Check all corresponding conditions and document your findings in your coursework. Is it necessary to check all conditions?

(b) Assume dynamic additive and multiplicative output uncertainty with corresponding weighting function $\boldsymbol{W}_1 = \boldsymbol{W}_2 = \boldsymbol{I}$, respectively. Calculate the stability margins for additive and multiplicative output perturbations and determine the worst case of this perturbation to γ. At which frequency ω^* does the worst case occur? Provide a singular value plot to underline your findings.

(c) Construct a system $\boldsymbol{\Delta} \in \mathcal{RH}_\infty$ that has $\|\boldsymbol{\Delta}(s)\|_\infty \leq \gamma$ and moreover has $\bar{\sigma}(\boldsymbol{\Delta}(j\omega^*)) = \gamma$. Show that the interconnection of the closed-loop and $\boldsymbol{\Delta}$ is unstable.

2.2 MIMO-systems

2.2.1 Proof of equivalence

Consider the following equivalence

$$
\begin{aligned}
&E\left\{\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \boldsymbol{z}(t)^T \boldsymbol{z}(t) dt\right\} = \text{trace} E\left\{\boldsymbol{z}(t)\boldsymbol{z}(t)^T\right\} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}\left[\boldsymbol{F}(j\omega)\boldsymbol{F}(j\omega)^H\right] d\omega = \|\boldsymbol{F}\|_2^2 = \|\mathcal{F}_l(\boldsymbol{P},\boldsymbol{K})\|_2^2
\end{aligned}
$$

with regard to the generalised plant $\boldsymbol{P}$, the controller $\boldsymbol{K}$ and a lower linear fractional transformation (LFT) $\mathcal{F}_l(\boldsymbol{P},\boldsymbol{K})$. Provide a detailed mathematical proof of each of the statements, along with a discussion for each of the steps. For details on the concepts and nomenclature refer to the lecture.

2.2.2 System zero and down squaring

Consider the following system, which is given by the state-space model in minimal realisation

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{u} \\
y &= \begin{bmatrix} 1 & -1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{u}
\end{aligned} \tag{2}
$$

Calculate the set of transmission zeros of the system. Provide your calculations in your coursework. Is it possible to design a stable, proper pre-compensator $\boldsymbol{k} = \begin{bmatrix} k_1 & k_2 \end{bmatrix}^T$ in such a way, as to place the resulting system zero of the reduced SISO system in the open left-half plane? Provide the detailed steps of your analytical investigation in your coursework.

2.2.3 Stabilisation via Youla-controller

Consider the following transfer function

$$G(s) = \frac{s-1}{s^3+s^2-4s-4} = \frac{m(s)}{m(s)}.$$

Bring the transfer function to a minimum realised transfer function

$$\boldsymbol{G}(s) \stackrel{s}{=} \left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{C} & \boldsymbol{D} \end{array}\right],$$

and employ the procedure based on an observer gain $\boldsymbol{H}$ and a state feedback gain $\boldsymbol{F}$, from ACS, Exercise 07, to design a Youla controller with the following coprime factorisation:

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{U} \\ \boldsymbol{N} & \boldsymbol{V} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|c} \boldsymbol{A}+\boldsymbol{B}\boldsymbol{F} & \begin{bmatrix} \boldsymbol{B} & -\boldsymbol{H} \end{bmatrix} \\ \hline \begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{C}+\boldsymbol{D}\boldsymbol{F} \end{bmatrix} & \begin{bmatrix} \boldsymbol{I} & 0 \\ \boldsymbol{D} & \boldsymbol{I} \end{bmatrix} \end{array}\right],$$

where $\boldsymbol{A}+\boldsymbol{B}\boldsymbol{F}$ and $\boldsymbol{A}+\boldsymbol{H}\boldsymbol{C}$ are both stable and satisfy the following modified Bezout's identity:

$$\boldsymbol{V}\boldsymbol{M} - \boldsymbol{U}\boldsymbol{N} = \boldsymbol{I}.$$

Design a stabilising controller for $\boldsymbol{Q} = \boldsymbol{0}$ is given by

$$\boldsymbol{K} = -\boldsymbol{V}^{-1}\boldsymbol{U},$$

where $U(s), V(s) \in \mathcal{RH}_\infty$. Check internal stability for your controller. Is it fulfilled for the given $G(s)$? Document all your steps by the corresponding formula and numerical matrices and come to a conclusion.

2.3 Stability and LFT

2.3.1 Stability and LFT

Consider the following dynamical system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{m_p}\left(-c_p x_2 - k_p x_1 + F\right) \\ y &= h_p x_1 \end{aligned}$$

where input $u(t) = F(t)$ and the output is $y(t)$. Furthermore m_p, c_p, k_p and h_p are uncertain parameters. Assume that we have the following parametric uncertainties:

m: $[-50 \ldots +80]$ % of nominal value $\bar{m} = 10$ (no units specified)

c: $[-40 \ldots +40]$ % of nominal value $\bar{c} = 10$ (no units specified)

k: $[-20 \ldots +80]$ % of nominal value $\bar{k} = 1$ (no units specified)

h: $[-20 \ldots +80]$ % of nominal value $\bar{h} = 2.5$ (no units specified)

For example, the parameter m can vary in the bounds $m_p \in [5 \ldots 18]$.

(a) Rearrange the uncertain system to the generalised plant configuration $\boldsymbol{P}$ by **manually** introducing the subsequent LFT elements (w.r.t. controller and uncertainty). Give single formulas for your LFT elements and derive the block diagram in a step by step procedure (similar to the example in lecture 09). *Hint*: Note that the parametric uncertainty approach $m_p = \bar{m}(1 + r_m\Delta)$, for $|\Delta| \leq 1$ is not suited. Instead a different parametric uncertainty approach should be chosen. Assume that the controller is in a stabilising configuration for the loop (i.e. $r = 0$) and there is a disturbance d that enters at output y, i.e. $y = h_p x_1 + d$. Note that the output y and the actuator effort u are weighted by $W_1(s)$ and $W_2(s)$, respectively. As values choose:

$$W_1(s) = \frac{2s}{s + 0.1127}\,,$$

and

$$W_2(s) = 0.3\frac{0.7s + 1}{0.07s + 1}\,.$$

How many uncertainty channels do you have? Compare the results to the results given by the robust control toolbox. What do you observe?

(b) Design a PI-controller including a lead-lag filter for the nominal model. Compute robust stability (RS) for this controller by rearranging the plant to the $\boldsymbol{M}\Delta$-structure. Compare your results with the automatically computed $\boldsymbol{M}\Delta$-structure by the robust control toolbox. Is your stability statement conservative? If yes, explain why. Try to reduce conservatism in your model by applying a numerically better suited procedure.

(c) Does you controller fulfil nominal performance (NP)? Show your results with respect to the generalised plant and visualise your results in frequency domain analysis.

(d) Introduce an uncertain time-delay by a first-order Padé-approximation with nominal value $\bar{\tau} = 0.1$ and range $\tau_p = \pm 30$ %, and repeat the previous tasks for the previously assumed uncertainties including the new case of time-delay. Compare your new results with your previous findings from (a)-(c).

2.3.2 Sensitivity integrals

Consider the Bode's sensitivity integral:

$$\int_0^\infty \ln|S(j\omega)|d\omega = \pi\sum_i^{N_p} Re(p_i)\,, \tag{3}$$

for which the open compensated loop is assumed to have a relative degree of at least 2. In equation (3), $S(j\omega)$ denotes the frequency response of the sensitivity transfer function.

(a) Provide a proof of Bode's sensitivity formula and extend the proof from natural logarithm (base e) to the common or decadic logarithm (base 10). Discuss your findings.

(b) Consider the following statement:

For each function G consisting of stable and proper real rational transfer functions, there exist an all-pass function G_{ap} and a minimum-phase function G_{mp} such that $G = G_{ap}G_{mp}$. The factors are unique up to sign.

Provide a proof of the statement, as well as a proof of uniqueness.

2.4 Coprime factorisations and Youla Parametrisation

2.4.1 HIMAT example system analysis

Consider the linearised dynamics of the NASA Highly Maneuverable Aircraft Technology (HIMAT) aircraft, which is shown in Figure 2. The longitudinal dynamics of the linearised aircraft system, trimmed at 25000 ft. can be given by the linear state-space model of 6th-order with matrices

$$\mathbf{A} = \begin{bmatrix} -2.2567\cdot 10^{-2} & -36.617 & -18.897 & -32.090 & 3.2509 & -0.76257 \\ 9.2572\cdot 10^{-5} & -1.8997 & 0.98312 & -7.2562\cdot 10^{-4} & -0.1708 & -4.9652\cdot 10^{-3} \\ 1.2338\cdot 10^{-2} & 11.720 & -2.6316 & 8.7582\cdot 10^{-4} & -31.604 & 22.396 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -30.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -30.000 \end{bmatrix} \tag{4}$$

Figure 2: NASA highly maneuverable aircraft technology (HIMAT) research aircraft.

$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 30 & 0 \\ 0 & 30 \end{bmatrix} \tag{5}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{6}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{7}$$

An analysis of the system shows two faster real poles lying at LHP at 30 rad/s. Using model reduction techniques the system can be reduced to 4th-order

[hsv_stab, hsv_unstab] = hankelsv(sys, 'ncf', 'log');

sys_red = reduce(sys, 4, 'errortype', 'ncf');

where *sys* is the state-space model with matrices (4) to (7). The reduced order model *sys_red* is of fourth order and includes a new RHP-zero. Use the Smith decomposition to bring the fourth order-model to a transfer function common denominator form and decompose the system to

Smith-form (for example use the Smith-decomposition implemented in Matlab). What do you observe? Discuss your results. Determine the singular values of the full order and the reduced order model. Discuss your results.

2.4.2 Coprime factorisation

Implement a program in Matlab that calculates the doubly coprime factorisation for the plant of reduced order, refer to [1]. Describe the basic function of your algorithm by introducing formulas, if needed for understanding, and discuss your results.

2.4.3 Youla parametrisation

Use the doubly coprime factorisation for the 2x2 reduced plant to design a stabilising controller via the Youla-parametrisation. Describe the performance of your controller in the frequency domain via the singular values of the sensitivity and the complementary sensitivity matrix. Discuss your results.

2.4.4 Bonus: Stabilising controller by optimisation

Design a stabilising controller with your own optimisation function by using your optimisation criterion (e.g. damping and natural frequency of closed-loop poles of the system). Limit the range of the parameters of $\mathbf{Q}(s)$ to stable transfer functions. Discuss your results by using singular value plots of sensitivity and complementary sensitivity function and time domain performance.

3 Bonus: Robust controller design

Consider the linear state-space model of the HIMAT example given by equations (4) to (7). The model was presented in [4] and describes the motion in the vertical plane around the equilibrium point of Mach 0.9 and an altitude of 25000 ft. above sea level. The model states are according to the lateral-directional dynamics $\mathbf{x}^T = \begin{bmatrix} \delta V & \alpha & q & \theta \end{bmatrix}$, where δV are the perturbations along the velocity vector, α is the angle-of-attack, q is the rate-of-change of aircraft attitude angle and θ is the aircraft attitude angle. Two more states are attributed to the dynamics of the control actuating system with the elevon (δ_e) and the canard (δ_c). These dynamics are modelled by first-order lag transfer functions with poles located at 30 rad/s.

3.1 Uncertainty modelling

The model of 6^{th}-order is assumed to be valid up to frequencies of 100 rad/s, with less than 30% variation, the model compared to the real-world aircraft at low frequencies. However, the aircraft is assumed to be a rigid body in the linearisation and therefore body bending at frequencies above 100 rad/s is neglected. Additional loop gain caused by body bending resonance can be as much as 20 dB (factor 10 or 1000%), which must be attenuated at frequencies higher than 100 rad/s. Model the uncertainty of the multivariable plant by suggesting a reasonable type of dynamic uncertainty. Describe and discuss your results.

3.2 Coprime factor uncertainty

Assume that the complex conjugated poles at $s_p = 0.6898 \pm j0.2488$ are uncertain for different trimmed angle-of-attack and may wander between RHP and LHP by assuming values of a natural frequency of $\omega_n = [0.7 \cdots 2]$ rad/s and the damping of $\zeta = [0.7 \cdots 0.95]$ (for the complex conjugated poles either in the RHP or in the LHP). Use coprime factor uncertainty, refer to Figure 3, to model the system uncertainty with a dynamic weighting using the uncertainty assumptions of Question 3.1. Describe the approach of coprime factor uncertainty modelling and discuss your results.

3.3 Controller design

Design your controller by using the the $\mathcal{H}_\infty$-controller via robust stabilisation of normalised coprime factor plant description, as proposed by [3]. Note that the Matlab function is given in [5]. A weighting function of

$$\mathbf{W}_p(s) = \frac{8}{s}\mathbf{I}, \tag{8}$$

should be used, where $\mathbf{I}$ is the identity matrix. Show the result of the controller design by discussing the singular value plots of the sensitivity and the complementary sensitivity function. Compare the controller performance to the loop-shaping approach of the Matlab demo. Discuss your results. Does the controller guarantee robust stability for the modelled coprime factor uncertainty? How can you make a statement about the possible perturbations?

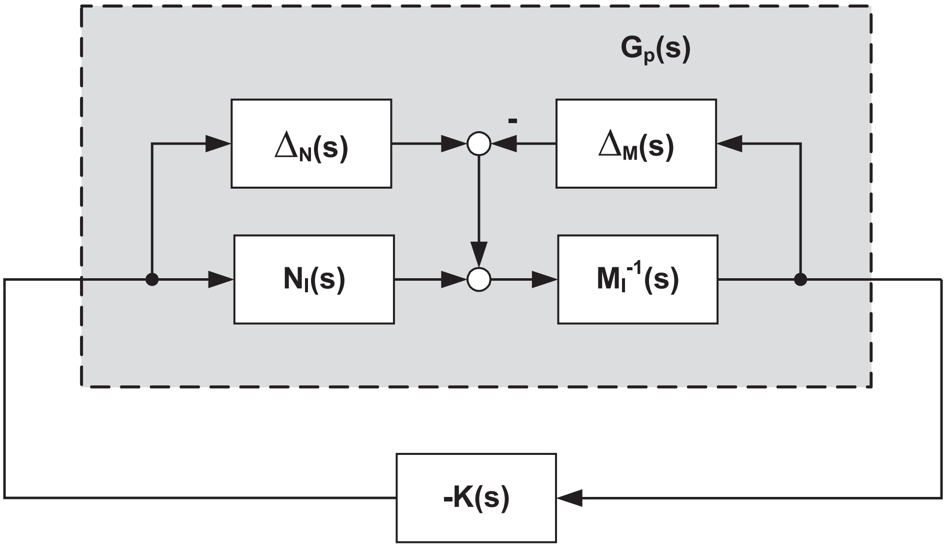

Figure 3: Coprime factor uncertainty.

3.4 Redesign of coprime factor $\mathcal{H}_\infty$-controller

Redesign the normalised coprime factor $\mathcal{H}_\infty$-controller by using the reduced order model of the HIMAT-aircraft (without actuation system dynamics). Determine robust performance with the full-order perturbed (uncertainty) model. Test your controller with the full order system including uncertainty. Present and discuss your results and compare to Matlab loop-shaping controller.

4 List of provided files and information for report

4.1 List of provided files

Included to the zip-file that is available from the online system are the following files:

List of model files with description

- himat_example.m: Script file containing the model and order reduction.
- himat_loopShaping.m: Script file containing the Matlab demo for loop shaping design.
- HIMATdemo.pdf: PDF-document that describes the Robust Control Toolbox HIMAT loop-shaping example.

4.2 Information for writing the report

The deadline for submission of the report is announced via the online system of the course. Concerning guidelines of how to write the report, please regard the following information.

4.2.1 Software used

Please hand-in the Matlab code for your assignment, in addition to your report. Follow the submission guidelines, that are given in the online system. Do not hand in temporary files of, for example, early stages of your work. Software files provided, should be adjusted to the latest version of your work.

If you use software from third sources that solve or help to solve the given problems, provide code or references. You do not need to mention standard Matlab functions or toolbox code (like e.g. functions contained in the Robust Control Toolbox). If you develop your own code in addition to existing Matlab functions, it must be provided along with the other files. **All submitted code, as well as Simulink models, must be compatible with Matlab 2022b or lower versions**.

4.2.2 Report writing

The report for this coursework can be written in English or German. The submitted report should be well organised and contain about 15-25 pages including introduction and bibliography and it must not exceed 30 pages. Identify formulas by reference numbers and refer to them accordingly. Use standard lecture nomenclature for formulas. If necessary provide a discussion and conclusion at the end of each particular question. In case you are not used to mathematical writing, refer to the literature, for example [2].

4.2.3 Plagiarism

If you copy from third sources (figures, code or text), provide references. You are responsible for your report, so do not give it away. Even though you can solve the problems in a team, you must write your own report, providing your individual interpretation of the achieved results. Do not copy discussions from your teammates. Of course, there is a penalty on this.

4.2.4 Figures

Figures in the report should be provided with appropriate axes description and units. If no units apply, denote this in the axis, e.g. y [-]. Provide figures that are easily readable, including legends to identify various lines. Provide only most important results using figures. Limit the number of figures in your report and discuss the results presented in each figure sufficiently.

References

[1] Bruce Allen Francis. *A course in H [infinity] control theory*. Springer-Verlag, Berlin, 1987.

[2] Paul R Halmos. How to write mathematics. *Enseign. Math*, 16(2):123–152, 1970.

[3] Duncan McFarlane and Keith Glover. *Robust Controller Design Using Normalized Coprime Factor Plant Descriptions*, volume 138 of *Lecture Notes in Control and Information Sciences*. Springer, Berlin, 1990.

[4] M. G. SAFONOV, A. J. LAUB, and G. L. HARTMANN. Feedback properties of multivariable systems - the role and use of the return difference matrix. *IEEE TRANSACTIONS ON AUTOMATIC CONTROL*, 26(1):47–65, 1981.

[5] Sigurd Skogestad and Ian Postlethwaite. *Multivariable feedback control: Analysis and design*. J. Wiley, Chichester and England, 2nd edition, 2005.