

## Exercise 9 - Robust stability and performance

### Problem 9.1: $M\Delta$ -structure

Consider a combined multiplicative and inverse multiplicative uncertainty at the output of the family of multiple-input multiple-output plants

$$\mathbf{G}_p = (\mathbf{I} - \Delta_{iO} \mathbf{W}_{iO})^{-1} (\mathbf{I} + \Delta_O \mathbf{W}_O) \mathbf{G}.$$

The combined uncertainty may be bounded such that  $\|[\Delta_{iO} \ \Delta_O]\|_\infty \leq 1$ . Draw a block diagram of the uncertain plant, and derive a necessary and sufficient condition for robust stability of the closed-loop system.

#### Solution:

Since a multiplication of systems corresponds to an interconnection of blocks, the block diagram can be represented as depicted in Figure 1.

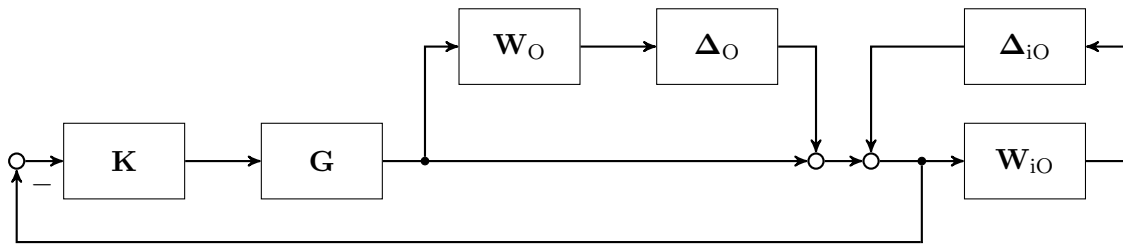


Figure 1: Block diagram of combined output uncertainties.

The nominally stable system  $\mathbf{M}$  is stable for all perturbations satisfying  $\|\Delta\|_\infty < 1$  (i.e. robustly stable) if and only if

$$\|\mathbf{M}\|_\infty < 1$$

where  $\mathbf{M}$  is the transfer function matrix from the output  $\mathbf{u}_\Delta$  to the input  $\mathbf{y}_\Delta$  of the perturbations.

We obtain with the MIMO rule (SKOGESTAD, p. 68) from the diagram that

$$\mathbf{M} = \begin{bmatrix} \mathbf{W}_{iO} \mathbf{S} \\ \mathbf{W}_O \mathbf{T} \end{bmatrix}.$$

Hence, the condition for robust stability is

$$\left\| \begin{bmatrix} \mathbf{W}_{iO} \mathbf{S} \\ \mathbf{W}_O \mathbf{T} \end{bmatrix} \right\|_\infty < 1.$$

### Problem 9.2: Inverse multiplicative uncertainty

Consider a scalar parametric gain uncertainty

$$G_p(s) = k_p G_0(s) \quad \text{with} \quad k_{\min} \leq k_p \leq k_{\max}.$$

Represent this uncertainty as inverse multiplicative uncertainty

$$G_p(s) = G(s)(1 + w_{iI}\Delta_{iI})^{-1}$$

where  $|\Delta_{iI}| \leq 1$  and  $w_{iI} = r_k$  with nominal model  $G(s) = \bar{k}_i G_0(s)$ .

(a) Derive that  $\bar{k}_i = \frac{2k_{\min}k_{\max}}{k_{\min} + k_{\max}}$  if the substitution  $k_p = \bar{k}_i(1 + r_k\Delta_{iI})^{-1}$  is used.

(b) Show that the form  $G_p(s) = G(s)(1 + w_{iI}\Delta_{iI})^{-1}$  with  $|\Delta_{iI}| \leq 1$  does not allow for  $k_p = 0$ .

**Solution:**

(a) Let the substitution be

$$k_p = \bar{k}_i(1 + r_k\Delta_{iI})^{-1} \quad \text{with} \quad r_k = \frac{k_{\max} - k_{\min}}{k_{\max} + k_{\min}}.$$

To derive  $\bar{k}_i$  consider for example the case  $\Delta_{iI} = 1$ :

$$\begin{aligned} k_{\min} &= \bar{k}_i(1 + r_k\Delta_{iI})^{-1} \\ \Leftrightarrow k_{\min} &= \frac{\bar{k}_i}{1 + \frac{k_{\max} - k_{\min}}{k_{\max} + k_{\min}}} \\ \Leftrightarrow k_{\min} &= \frac{\bar{k}_i(k_{\max} + k_{\min})}{2k_{\max}} \\ \Leftrightarrow \bar{k}_i &= \frac{2k_{\min}k_{\max}}{k_{\min} + k_{\max}}. \end{aligned}$$

(b) The expression

$$k_p = \bar{k}_i(1 + r_k\Delta_{iI})^{-1} \stackrel{!}{=} 0$$

with  $\bar{k}_i \neq 0$  implies  $1 + r_k\Delta_{iI} = \infty$ . This is impossible, because  $r_k$  is a real constant and  $|\Delta_{iI}| \leq 1$ .

**Problem 9.3: Linear fractional transformation**

A multiple input multiple output feedback system with combined multiplicative output and inverse additive uncertainty as depicted in Figure 2 is given. Assume, that  $\Delta \in \mathcal{RH}_{\infty}$  with  $\|\Delta\|_{\infty} \leq 1$ .

(a) Let  $\Delta_f = \mathbf{0}$  and bring the system to a general control configuration as depicted in Figure 3. State a condition for robust stability.

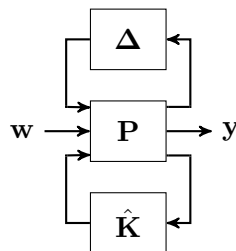


Figure 3: General control configuration.

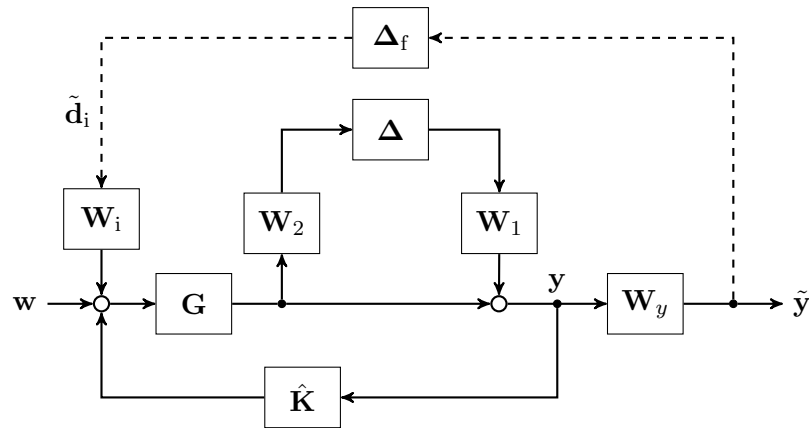


Figure 2: Block diagram of combined uncertainties.

- (b) Now consider the case  $\Delta_f \neq 0$ , bring the system to the configuration shown in Figure 4 and state a condition for robust performance.

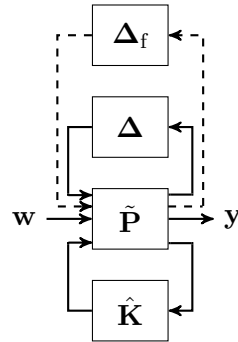


Figure 4: Augmented control configuration.

**Solution:**

- (a) The general control configuration is defined as

$$\begin{bmatrix} y_\Delta \\ y \\ v \end{bmatrix} = \mathbf{P} \begin{bmatrix} u_\Delta \\ w \\ u \end{bmatrix}$$

where the generalised plant model  $\mathbf{P}$  can be obtained from Figure 2 by breaking the loop before and after  $\hat{\mathbf{K}}$  and  $\Delta$  (and keeping in mind that  $\Delta_f = 0$ ). Start from a particular output and write down all blocks as you meet them when moving backwards towards the input. This yields

$$\mathbf{P} = \begin{bmatrix} 0 & \mathbf{W}_2 \mathbf{G} & \mathbf{W}_2 \mathbf{G} \\ \mathbf{W}_1 & \mathbf{G} & \mathbf{G} \\ \mathbf{W}_1 & \mathbf{G} & \mathbf{G} \end{bmatrix}.$$

In order to check robust stability, we use a lower linear fractional transformation to derive

$$\mathbf{N} = \mathbf{F}_l(\mathbf{P}, \hat{\mathbf{K}}) = \mathbf{P}_{11} + \mathbf{P}_{12} \hat{\mathbf{K}} (\mathbf{I} - \mathbf{P}_{22} \hat{\mathbf{K}})^{-1} \mathbf{P}_{21}$$

where  $\mathbf{P}$  is partitioned such that matrix dimensions are consistent:

$$\mathbf{P}_{11} = \begin{bmatrix} \mathbf{0} & \mathbf{W}_2 \mathbf{G} \\ \mathbf{W}_1 & \mathbf{G} \end{bmatrix}, \quad \mathbf{P}_{12} = \begin{bmatrix} \mathbf{W}_2 \mathbf{G} \\ \mathbf{G} \end{bmatrix}, \quad \mathbf{P}_{21} = [\mathbf{W}_1 \quad \mathbf{G}] \quad \text{and} \quad \mathbf{P}_{22} = \mathbf{G}.$$

Hence

$$\begin{aligned} \mathbf{N} &= \begin{bmatrix} \mathbf{0} & \mathbf{W}_2 \mathbf{G} \\ \mathbf{W}_1 & \mathbf{G} \end{bmatrix} + \begin{bmatrix} \mathbf{W}_2 \mathbf{G} \\ \mathbf{G} \end{bmatrix} \hat{\mathbf{K}} (\mathbf{I} - \mathbf{G} \hat{\mathbf{K}})^{-1} [\mathbf{W}_1 \quad \mathbf{G}] \\ &= \begin{bmatrix} \mathbf{W}_2 \mathbf{G} \hat{\mathbf{K}} (\mathbf{I} - \mathbf{G} \hat{\mathbf{K}})^{-1} \mathbf{W}_1 & \mathbf{W}_2 \mathbf{G} + \mathbf{W}_2 \mathbf{G} \hat{\mathbf{K}} (\mathbf{I} - \mathbf{G} \hat{\mathbf{K}})^{-1} \mathbf{G} \\ \mathbf{W}_1 + \mathbf{G} \hat{\mathbf{K}} (\mathbf{I} - \mathbf{G} \hat{\mathbf{K}})^{-1} \mathbf{W}_1 & \mathbf{G} + \mathbf{G} \hat{\mathbf{K}} (\mathbf{I} - \mathbf{G} \hat{\mathbf{K}})^{-1} \mathbf{G} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{W}_2 \mathbf{T} \mathbf{W}_1 & \mathbf{W}_2 \mathbf{G} + \mathbf{W}_2 \mathbf{T} \mathbf{G} \\ \mathbf{W}_1 + \mathbf{T} \mathbf{W}_1 & \mathbf{G} + \mathbf{T} \mathbf{G} \end{bmatrix}. \end{aligned}$$

We check robust stability with the  $\mathbf{M}\Delta$ -structure, where  $\mathbf{M} = \mathbf{N}_{11}$ . Thus, the conditions for robust stability are  $\mathbf{N}$  is internally stable and

$$\|\mathbf{W}_2 \mathbf{T} \mathbf{W}_1\|_{\infty} < 1.$$

(b) With  $\Delta_f \neq \mathbf{0}$  we want to obtain the generalised plant model  $\tilde{\mathbf{P}}$  such that

$$\begin{bmatrix} \mathbf{y}_{\Delta_f} \\ \mathbf{y}_{\Delta} \\ \mathbf{y} \\ \mathbf{v} \end{bmatrix} = \tilde{\mathbf{P}} \begin{bmatrix} \mathbf{u}_{\Delta_f} \\ \mathbf{u}_{\Delta} \\ \mathbf{w} \\ \mathbf{u} \end{bmatrix}$$

with  $\tilde{\mathbf{d}}_i = \mathbf{u}_{\Delta_f}$  and  $\tilde{\mathbf{y}} = \mathbf{y}_{\Delta_f}$  as well as the inputs and outputs as defined in Figure 5:

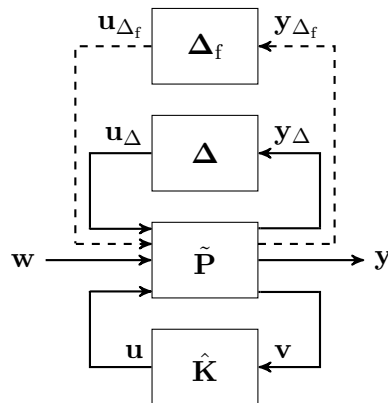


Figure 5: Generalised plant model with multiple uncertainties.

We obtain the generalised plant model again from Figure 2 by breaking the loop before and after  $\hat{\mathbf{K}}$  which yields

$$\tilde{\mathbf{P}} = \begin{bmatrix} \mathbf{W}_y \mathbf{G} \mathbf{W}_i & \mathbf{W}_y \mathbf{W}_1 & \mathbf{W}_y \mathbf{G} & \mathbf{W}_y \mathbf{G} \\ \mathbf{W}_2 \mathbf{G} \mathbf{W}_i & \mathbf{0} & \mathbf{W}_2 \mathbf{G} & \mathbf{W}_2 \mathbf{G} \\ \mathbf{G} \mathbf{W}_i & \mathbf{W}_1 & \mathbf{G} & \mathbf{G} \\ \mathbf{G} \mathbf{W}_i & \mathbf{W}_1 & \mathbf{G} & \mathbf{G} \end{bmatrix}.$$

Partitioning leads to

$$\begin{aligned}\tilde{\mathbf{P}}_{11} &= \begin{bmatrix} \mathbf{W}_y \mathbf{G} \mathbf{W}_i & \mathbf{W}_y \mathbf{W}_1 & \mathbf{W}_y \mathbf{G} \\ \mathbf{W}_2 \mathbf{G} \mathbf{W}_i & \mathbf{0} & \mathbf{W}_2 \mathbf{G} \\ \mathbf{G} \mathbf{W}_i & \mathbf{W}_1 & \mathbf{G} \end{bmatrix}, & \tilde{\mathbf{P}}_{12} &= \begin{bmatrix} \mathbf{W}_y \mathbf{G} \\ \mathbf{W}_2 \mathbf{G} \\ \mathbf{G} \end{bmatrix}, \\ \tilde{\mathbf{P}}_{21} &= [\mathbf{G} \mathbf{W}_i \quad \mathbf{W}_1 \quad \mathbf{G}] & \text{and} & \tilde{\mathbf{P}}_{22} = \mathbf{G}.\end{aligned}$$

Again, we use a lower linear fractional transformation to determine

$$\begin{aligned}\mathbf{N} &= \begin{bmatrix} \mathbf{W}_y \mathbf{G} \mathbf{W}_i & \mathbf{W}_y \mathbf{W}_1 & \mathbf{W}_y \mathbf{G} \\ \mathbf{W}_2 \mathbf{G} \mathbf{W}_i & \mathbf{0} & \mathbf{W}_2 \mathbf{G} \\ \mathbf{G} \mathbf{W}_i & \mathbf{W}_1 & \mathbf{G} \end{bmatrix} + \begin{bmatrix} \mathbf{W}_y \mathbf{G} \\ \mathbf{W}_2 \mathbf{G} \\ \mathbf{G} \end{bmatrix} \hat{\mathbf{K}} (\mathbf{I} - \mathbf{G} \hat{\mathbf{K}})^{-1} [\mathbf{G} \mathbf{W}_i \quad \mathbf{W}_1 \quad \mathbf{G}] \\ &= \begin{bmatrix} \mathbf{W}_y \mathbf{G} \mathbf{W}_i + \mathbf{W}_y \mathbf{T} \mathbf{G} \mathbf{W}_i & \mathbf{W}_y \mathbf{W}_1 + \mathbf{W}_y \mathbf{T} \mathbf{W}_1 & \mathbf{W}_y \mathbf{G} + \mathbf{W}_y \mathbf{T} \mathbf{G} \\ \mathbf{W}_2 \mathbf{G} \mathbf{W}_i + \mathbf{W}_2 \mathbf{T} \mathbf{G} \mathbf{W}_i & \mathbf{W}_2 \mathbf{T} \mathbf{W}_1 & \mathbf{W}_2 \mathbf{G} + \mathbf{W}_2 \mathbf{T} \mathbf{G} \\ \mathbf{G} \mathbf{W}_i + \mathbf{T} \mathbf{G} \mathbf{W}_i & \mathbf{W}_1 + \mathbf{T} \mathbf{W}_1 & \mathbf{G} + \mathbf{T} \mathbf{G} \end{bmatrix}.\end{aligned}$$

In order to state a condition for robust performance, we have to find

$$\mathbf{F} = \mathbf{F}_u(\mathbf{N}, \Delta_c) = \mathbf{N}_{22} + \mathbf{N}_{21} \Delta_c (\mathbf{I} - \mathbf{N}_{11} \Delta_c)^{-1} \mathbf{N}_{12}$$

from a upper linear fractional transformation with the combined uncertainty

$$\Delta_c = \begin{bmatrix} \Delta_f & \mathbf{0} \\ \mathbf{0} & \Delta \end{bmatrix}$$

and the partitioned nominal system

$$\begin{aligned}\mathbf{N}_{11} &= \begin{bmatrix} \mathbf{W}_y \mathbf{G} \mathbf{W}_i + \mathbf{W}_y \mathbf{T} \mathbf{G} \mathbf{W}_i & \mathbf{W}_y \mathbf{W}_1 + \mathbf{W}_y \mathbf{T} \mathbf{W}_1 \\ \mathbf{W}_2 \mathbf{G} \mathbf{W}_i + \mathbf{W}_2 \mathbf{T} \mathbf{G} \mathbf{W}_i & \mathbf{W}_2 \mathbf{T} \mathbf{W}_1 \end{bmatrix}, & \mathbf{N}_{12} &= \begin{bmatrix} \mathbf{W}_y \mathbf{G} + \mathbf{W}_y \mathbf{T} \mathbf{G} \\ \mathbf{W}_2 \mathbf{G} + \mathbf{W}_2 \mathbf{T} \mathbf{G} \end{bmatrix}, \\ \mathbf{N}_{21} &= [\mathbf{G} \mathbf{W}_i + \mathbf{T} \mathbf{G} \mathbf{W}_i \quad \mathbf{W}_1 + \mathbf{T} \mathbf{W}_1] & \text{and} & \mathbf{N}_{22} = \mathbf{G} + \mathbf{T} \mathbf{G}.\end{aligned}$$

Assuming that  $\mathbf{N}$  is internally stable, robust performance is given if and only if

$$\|\mathbf{F}\|_\infty < 1$$

for all  $\Delta_c$  that satisfy  $\|\Delta_c\|_\infty \leq 1$ .

#### Problem 9.4: Non-symmetric nominal value

May the family of single-input single-output plants

$$G_p(s) = \frac{1}{\tau s + 1}$$

have an uncertain time constant  $6 \leq \tau \leq 42$  with  $\bar{\tau} = 10$  and be controlled by the PI-controller

$$K(s) = \frac{10s + 1}{s}.$$

(a) Reformulate the uncertain system to

$$G_p(s) = \frac{1}{\frac{a+\Delta b}{c+\Delta d}s + 1}$$

and determine the parameters  $a$ ,  $b$ ,  $c$  and  $d$  assuming  $|\Delta| \leq 1$ . If you have to choose a parameter, let  $c = 1$ .

- (b) Use MATLAB to transform the closed-loop system in (uncertain) state space form and use the  $\mathbf{M}\Delta$ -structure to determine if the system is robustly stable.

### Solution:

- (a) We want

$$\frac{1}{\tau s + 1} \stackrel{!}{=} \frac{1}{\frac{a+\Delta b}{c+\Delta d}s + 1}.$$

To determine the parameters of the substitution, we check three special cases for the perturbation:

**Case 1:**  $\Delta = 0 \Rightarrow \tau = \bar{\tau}$

$$\frac{1}{\bar{\tau}s + 1} = \frac{1}{\frac{a}{c}s + 1} \Leftrightarrow a = \bar{\tau}c$$

We choose  $c = 1$ , this yields  $a = \bar{\tau}$ .

**Case 2:**  $\Delta = 1 \Rightarrow \tau = \tau_{\max}$

$$\frac{1}{\tau_{\max}s + 1} = \frac{1}{\frac{a+b}{c+d}s + 1} \Leftrightarrow \frac{a+b}{c+d} = \tau_{\max}$$

**Case 3:**  $\Delta = -1 \Rightarrow \tau = \tau_{\min}$

$$\frac{1}{\tau_{\min}s + 1} = \frac{1}{\frac{a-b}{c-d}s + 1} \Leftrightarrow \frac{a-b}{c-d} = \tau_{\min}$$

The solution of this system of equations is either an algebra exercise or can conveniently be derived with MATLAB:

```
% Solve linear system of equations
syms av bv cv dv
[ a, b, c, d ] = solve( av/cv == 10, (av + bv)/(cv + dv) == 42, ...
    (av - bv)/(cv - dv) == 6, cv == 1 )
```

With the assumption  $c = 1$  we find  $a = 10$ ,  $b = -\frac{2}{3}$  and  $d = -\frac{7}{9}$ .

- (b) The system can be brought to an  $\mathbf{M}\Delta$ -structure and checked for robust stability with MATLAB:

```
% Transform system to M delta-structure

% Convert substitution parameters from part a)
a = eval( a ); b = eval( b ); c = eval( c ); d = eval( d );

% Define uncertain state space system
s = tf( 's' );
del = ureal( 'delta', 0 );
Gp = 1 / ( ( a + del * b ) / ( c + del * d ) * s + 1 );
K = ( 10 * s + 1 ) / s;
Gss = uss( feedback( Gp * K, 1 ) );

% Calculate M delta-structure
[ N, Delta ] = lfdata( Gss );
```

```
n = length( Delta.NominalValue );
Ntf = tf( minreal( N ) );
M = Ntf(1:n,1:n);

if hinfnorm( M ) < 1
    disp( 'The system is robustly stable' );
else
    disp( 'The system may NOT be robustly stable' );
end
```

Even though the PI-controlled closed loop system with the family plants  $G_p$  is robustly stable, the conservative condition for complex unstructured uncertainty ( $\|\Delta\|_\infty \leq 1$ ) based on

$$\|\mathbf{M}\|_\infty < 1$$

is not satisfied. This is the price we pay for ignoring the structure of  $\Delta$ .