

Exercise 9 - Robust stability and performance

Problem 9.1: MΔ-structure

Consider a combined multiplicative and inverse multiplicative uncertainty at the output of the family of multiple-input multiple-output plants

$$\mathbf{G}_P = (\mathbf{I} - \Delta_{iO} \mathbf{W}_{iO})^{-1} (\mathbf{I} + \Delta_O \mathbf{W}_O) \mathbf{G}.$$

The combined uncertainty may be bounded such that $\|[\Delta_{iO} \quad \Delta_O]\|_\infty \leq 1$. Draw a block diagram of the uncertain plant, and derive a necessary and sufficient condition for robust stability of the closed-loop system.

Solution:

Since a multiplication of systems corresponds to an interconnection of blocks, the block diagram can be represented as depicted in Figure 1.

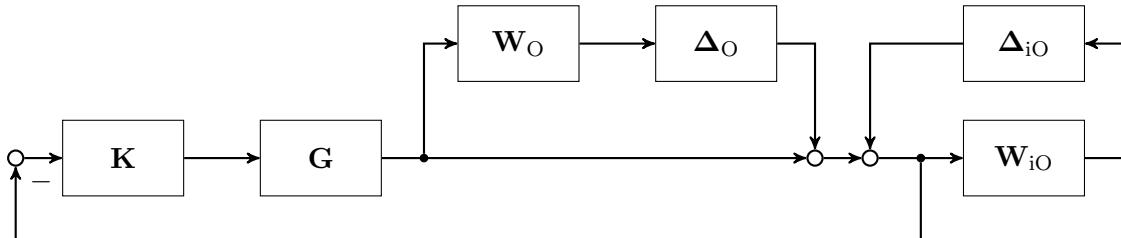


Figure 1: Block diagram of combined output uncertainties.

The nominally stable system \mathbf{M} is stable for all perturbations satisfying $\|\Delta\|_\infty < 1$ (i.e. robustly stable) if and only if

$$\|\mathbf{M}\|_\infty < 1$$

where \mathbf{M} is the transfer function matrix from the output \mathbf{u}_Δ to the input \mathbf{y}_Δ of the perturbations.

We obtain with the MIMO rule (SKOGESTAD, p. 68) from the diagram that

$$\mathbf{M} = \begin{bmatrix} \mathbf{W}_{iO} \mathbf{S} \\ \mathbf{W}_O \mathbf{T} \end{bmatrix}.$$

Hence, the condition for robust stability is

$$\left\| \begin{bmatrix} \mathbf{W}_{iO} \mathbf{S} \\ \mathbf{W}_O \mathbf{T} \end{bmatrix} \right\|_\infty < 1.$$

Problem 9.2: Inverse multiplicative uncertainty

Consider a scalar parametric gain uncertainty

$$G_p(s) = k_p G_0(s) \quad \text{with} \quad k_{\min} \leq k_p \leq k_{\max}.$$

Represent this uncertainty as inverse multiplicative uncertainty

$$G_p(s) = G(s)(1 + w_{ii}\Delta_{ii})^{-1}$$

where $|\Delta_{ii}| \leq 1$ and $w_{ii} = r_k$ with nominal model $G(s) = \bar{k}_i G_0(s)$.

- (a) Derive that $\bar{k}_i = \frac{2k_{\min}k_{\max}}{k_{\min}+k_{\max}}$ if the substitution $k_p = \bar{k}_i(1 + r_k\Delta_{ii})^{-1}$ is used.
- (b) Show that the form $G_p(s) = G(s)(1 + w_{ii}\Delta_{ii})^{-1}$ with $|\Delta_{ii}| \leq 1$ does not allow for $k_p = 0$.

Solution:

- (a) Let the substitution be

$$k_p = \bar{k}_i(1 + r_k\Delta_{ii})^{-1} \quad \text{with} \quad r_k = \frac{k_{\max} - k_{\min}}{k_{\max} + k_{\min}}.$$

To derive \bar{k}_i consider for example the case $\Delta_{ii} = 1$:

$$\begin{aligned} k_{\min} &= \bar{k}_i(1 + r_k\Delta_{ii})^{-1} \\ \Leftrightarrow k_{\min} &= \frac{\bar{k}_i}{1 + \frac{k_{\max} - k_{\min}}{k_{\max} + k_{\min}}} \\ \Leftrightarrow k_{\min} &= \frac{\bar{k}_i(k_{\max} + k_{\min})}{2k_{\max}} \\ \Leftrightarrow \bar{k}_i &= \frac{2k_{\min}k_{\max}}{k_{\min} + k_{\max}}. \end{aligned}$$

- (b) The expression

$$k_p = \bar{k}_i(1 + r_k\Delta_{ii})^{-1} \stackrel{!}{=} 0$$

with $\bar{k}_i \neq 0$ implies $1 + r_k\Delta_{ii} = \infty$. This is impossible, because r_k is a real constant and $|\Delta_{ii}| \leq 1$.

Problem 9.3: Linear fractional transformation

A multiple input multiple output feedback system with combined multiplicative output and inverse additive uncertainty as depicted in Figure 2 is given. Assume, that $\Delta \in \mathcal{RH}_{\infty}$ with $\|\Delta\|_{\infty} \leq 1$.

- (a) Let $\Delta_f = \mathbf{0}$ and bring the system to a general control configuration as depicted in Figure 3. State a condition for robust stability.

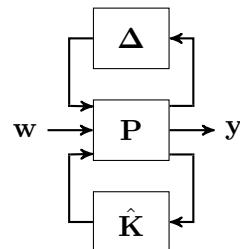


Figure 3: General control configuration.

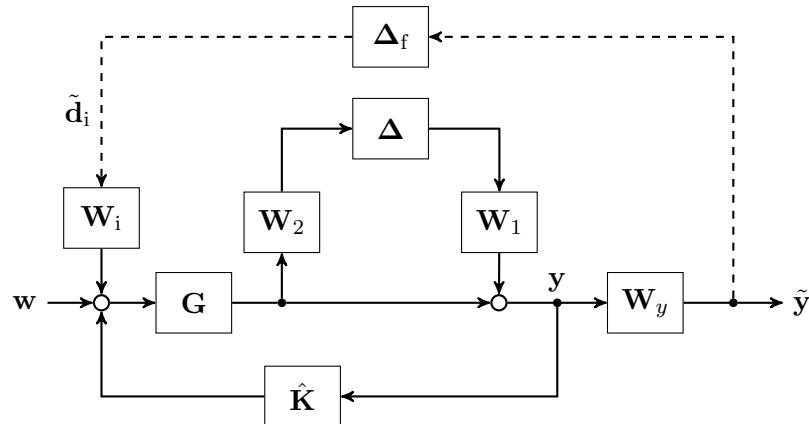


Figure 2: Block diagram of combined uncertainties.

- (b) Now consider the case $\Delta_f \neq 0$, bring the system to the configuration shown in Figure 4 and state a condition for robust performance.

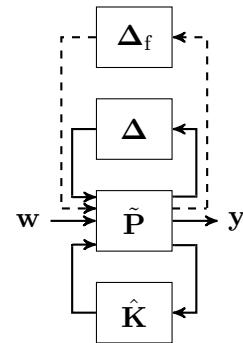


Figure 4: Augmented control configuration.

Solution:

- (a) The general control configuration is defined as

$$\begin{bmatrix} y_\Delta \\ y \\ v \end{bmatrix} = P \begin{bmatrix} u_\Delta \\ w \\ u \end{bmatrix}$$

where the generalised plant model P can be obtained from Figure 2 by breaking the loop before and after \hat{K} and Δ (and keeping in mind that $\Delta_f = 0$). Start from a particular output and write down all blocks as you meet them when moving backwards towards the input. This yields

$$P = \begin{bmatrix} 0 & W_2G & W_2G \\ W_1 & G & G \\ W_1 & G & G \end{bmatrix}.$$

In order to check robust stability, we use a lower linear fractional transformation to derive

$$N = F_l(P, \hat{K}) = P_{11} + P_{12}\hat{K} \left(I - P_{22}\hat{K} \right)^{-1} P_{21}$$

where \mathbf{P} is partitioned such that matrix dimensions are consistent:

$$\mathbf{P}_{11} = \begin{bmatrix} \mathbf{0} & \mathbf{W}_2\mathbf{G} \\ \mathbf{W}_1 & \mathbf{G} \end{bmatrix}, \quad \mathbf{P}_{12} = \begin{bmatrix} \mathbf{W}_2\mathbf{G} \\ \mathbf{G} \end{bmatrix}, \quad \mathbf{P}_{21} = [\mathbf{W}_1 \quad \mathbf{G}] \quad \text{and} \quad \mathbf{P}_{22} = \mathbf{G}.$$

Hence

$$\begin{aligned} \mathbf{N} &= \begin{bmatrix} \mathbf{0} & \mathbf{W}_2\mathbf{G} \\ \mathbf{W}_1 & \mathbf{G} \end{bmatrix} + \begin{bmatrix} \mathbf{W}_2\mathbf{G} \\ \mathbf{G} \end{bmatrix} \hat{\mathbf{K}} (\mathbf{I} - \mathbf{G}\hat{\mathbf{K}})^{-1} [\mathbf{W}_1 \quad \mathbf{G}] \\ &= \begin{bmatrix} \mathbf{W}_2\mathbf{G}\hat{\mathbf{K}}(\mathbf{I} - \mathbf{G}\hat{\mathbf{K}})^{-1}\mathbf{W}_1 & \mathbf{W}_2\mathbf{G} + \mathbf{W}_2\mathbf{G}\hat{\mathbf{K}}(\mathbf{I} - \mathbf{G}\hat{\mathbf{K}})^{-1}\mathbf{G} \\ \mathbf{W}_1 + \mathbf{G}\hat{\mathbf{K}}(\mathbf{I} - \mathbf{G}\hat{\mathbf{K}})^{-1}\mathbf{W}_1 & \mathbf{G} + \mathbf{G}\hat{\mathbf{K}}(\mathbf{I} - \mathbf{G}\hat{\mathbf{K}})^{-1}\mathbf{G} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{W}_2\mathbf{T}\mathbf{W}_1 & \mathbf{W}_2\mathbf{G} + \mathbf{W}_2\mathbf{T}\mathbf{G} \\ \mathbf{W}_1 + \mathbf{T}\mathbf{W}_1 & \mathbf{G} + \mathbf{T}\mathbf{G} \end{bmatrix}. \end{aligned}$$

We check robust stability with the $\mathbf{M}\Delta$ -structure, where $\mathbf{M} = \mathbf{N}_{11}$. Thus, the conditions for robust stability are \mathbf{N} is internally stable and

$$\|\mathbf{W}_2\mathbf{T}\mathbf{W}_1\|_\infty < 1.$$

(b) With $\Delta_f \neq \mathbf{0}$ we want to obtain the generalised plant model $\tilde{\mathbf{P}}$ such that

$$\begin{bmatrix} \mathbf{y}_{\Delta_f} \\ \mathbf{y}_\Delta \\ \mathbf{y} \\ \mathbf{v} \end{bmatrix} = \tilde{\mathbf{P}} \begin{bmatrix} \mathbf{u}_{\Delta_f} \\ \mathbf{u}_\Delta \\ \mathbf{w} \\ \mathbf{u} \end{bmatrix}$$

with $\tilde{\mathbf{d}}_i = \mathbf{u}_{\Delta_f}$ and $\tilde{\mathbf{y}} = \mathbf{y}_{\Delta_f}$ as well as the inputs and outputs as defined in Figure 5:

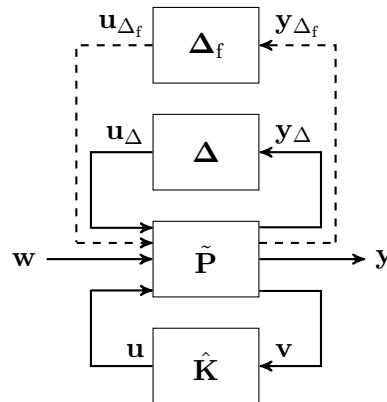


Figure 5: Generalised plant model with multiple uncertainties.

We obtain the generalised plant model again from Figure 2 by breaking the loop before and after $\hat{\mathbf{K}}$ which yields

$$\tilde{\mathbf{P}} = \begin{bmatrix} \mathbf{W}_y\mathbf{G}\mathbf{W}_i & \mathbf{W}_y\mathbf{W}_1 & \mathbf{W}_y\mathbf{G} & \mathbf{W}_y\mathbf{G} \\ \mathbf{W}_2\mathbf{G}\mathbf{W}_i & \mathbf{0} & \mathbf{W}_2\mathbf{G} & \mathbf{W}_2\mathbf{G} \\ \mathbf{G}\mathbf{W}_i & \mathbf{W}_1 & \mathbf{G} & \mathbf{G} \\ \mathbf{G}\mathbf{W}_i & \mathbf{W}_1 & \mathbf{G} & \mathbf{G} \end{bmatrix}.$$

Partitioning leads to

$$\tilde{\mathbf{P}}_{11} = \begin{bmatrix} \mathbf{W}_y \mathbf{G} \mathbf{W}_i & \mathbf{W}_y \mathbf{W}_1 & \mathbf{W}_y \mathbf{G} \\ \mathbf{W}_2 \mathbf{G} \mathbf{W}_i & \mathbf{0} & \mathbf{W}_2 \mathbf{G} \\ \mathbf{G} \mathbf{W}_i & \mathbf{W}_1 & \mathbf{G} \end{bmatrix}, \quad \tilde{\mathbf{P}}_{12} = \begin{bmatrix} \mathbf{W}_y \mathbf{G} \\ \mathbf{W}_2 \mathbf{G} \\ \mathbf{G} \end{bmatrix},$$

$$\tilde{\mathbf{P}}_{21} = [\mathbf{G} \mathbf{W}_i \quad \mathbf{W}_1 \quad \mathbf{G}] \quad \text{and} \quad \tilde{\mathbf{P}}_{22} = \mathbf{G}.$$

Again, we use a lower linear fractional transformation to determine

$$\mathbf{N} = \begin{bmatrix} \mathbf{W}_y \mathbf{G} \mathbf{W}_i & \mathbf{W}_y \mathbf{W}_1 & \mathbf{W}_y \mathbf{G} \\ \mathbf{W}_2 \mathbf{G} \mathbf{W}_i & \mathbf{0} & \mathbf{W}_2 \mathbf{G} \\ \mathbf{G} \mathbf{W}_i & \mathbf{W}_1 & \mathbf{G} \end{bmatrix} + \begin{bmatrix} \mathbf{W}_y \mathbf{G} \\ \mathbf{W}_2 \mathbf{G} \\ \mathbf{G} \end{bmatrix} \hat{\mathbf{K}} (\mathbf{I} - \mathbf{G} \hat{\mathbf{K}})^{-1} [\mathbf{G} \mathbf{W}_i \quad \mathbf{W}_1 \quad \mathbf{G}]$$

$$= \begin{bmatrix} \mathbf{W}_y \mathbf{G} \mathbf{W}_i + \mathbf{W}_y \mathbf{T} \mathbf{G} \mathbf{W}_i & \mathbf{W}_y \mathbf{W}_1 + \mathbf{W}_y \mathbf{T} \mathbf{W}_1 & \mathbf{W}_y \mathbf{G} + \mathbf{W}_y \mathbf{T} \mathbf{G} \\ \mathbf{W}_2 \mathbf{G} \mathbf{W}_i + \mathbf{W}_2 \mathbf{T} \mathbf{G} \mathbf{W}_i & \mathbf{W}_2 \mathbf{T} \mathbf{W}_1 & \mathbf{W}_2 \mathbf{G} + \mathbf{W}_2 \mathbf{T} \mathbf{G} \\ \mathbf{G} \mathbf{W}_i + \mathbf{T} \mathbf{G} \mathbf{W}_i & \mathbf{W}_1 + \mathbf{T} \mathbf{W}_1 & \mathbf{G} + \mathbf{T} \mathbf{G} \end{bmatrix}.$$

In order to state a condition for robust performance, we have to find

$$\mathbf{F} = \mathbf{F}_u(\mathbf{N}, \Delta_c) = \mathbf{N}_{22} + \mathbf{N}_{21} \Delta_c (\mathbf{I} - \mathbf{N}_{11} \Delta_c)^{-1} \mathbf{N}_{12}$$

from a upper linear fractional transformation with the combined uncertainty

$$\Delta_c = \begin{bmatrix} \Delta_f & \mathbf{0} \\ \mathbf{0} & \Delta \end{bmatrix}$$

and the partitioned nominal system

$$\mathbf{N}_{11} = \begin{bmatrix} \mathbf{W}_y \mathbf{G} \mathbf{W}_i + \mathbf{W}_y \mathbf{T} \mathbf{G} \mathbf{W}_i & \mathbf{W}_y \mathbf{W}_1 + \mathbf{W}_y \mathbf{T} \mathbf{W}_1 \\ \mathbf{W}_2 \mathbf{G} \mathbf{W}_i + \mathbf{W}_2 \mathbf{T} \mathbf{G} \mathbf{W}_i & \mathbf{W}_2 \mathbf{T} \mathbf{W}_1 \end{bmatrix}, \quad \mathbf{N}_{12} = \begin{bmatrix} \mathbf{W}_y \mathbf{G} + \mathbf{W}_y \mathbf{T} \mathbf{G} \\ \mathbf{W}_2 \mathbf{G} + \mathbf{W}_2 \mathbf{T} \mathbf{G} \end{bmatrix},$$

$$\mathbf{N}_{21} = [\mathbf{G} \mathbf{W}_i + \mathbf{T} \mathbf{G} \mathbf{W}_i \quad \mathbf{W}_1 + \mathbf{T} \mathbf{W}_1] \quad \text{and} \quad \mathbf{N}_{22} = \mathbf{G} + \mathbf{T} \mathbf{G}.$$

Assuming that \mathbf{N} is internally stable, robust performance is given if and only if

$$\|\mathbf{F}\|_\infty < 1$$

for all Δ_c that satisfy $\|\Delta_c\|_\infty \leq 1$.

Problem 9.4: Non-symmetric nominal value

May the family of single-input single-output plants

$$G_p(s) = \frac{1}{\tau s + 1}$$

have an uncertain time constant $6 \leq \tau \leq 42$ with $\bar{\tau} = 10$ and be controlled by the PI-controller

$$K(s) = \frac{10s + 1}{s}.$$

(a) Reformulate the uncertain system to

$$G_p(s) = \frac{1}{\frac{a+\Delta b}{c+\Delta d}s + 1}$$

and determine the parameters a, b, c and d assuming $|\Delta| \leq 1$. If you have to choose a parameter, let $c = 1$.

- (b) Use MATLAB to transform the closed-loop system in (uncertain) state space form and use the **MΔ**-structure to determine if the system is robustly stable.

Solution:

- (a) We want

$$\frac{1}{\tau s + 1} \stackrel{!}{=} \frac{1}{\frac{a+\Delta b}{c+\Delta d}s + 1}.$$

To determine the parameters of the substitution, we check three special cases for the perturbation:

Case 1: $\Delta = 0 \Rightarrow \tau = \bar{\tau}$

$$\frac{1}{\bar{\tau}s + 1} = \frac{1}{\frac{a}{c}s + 1} \Leftrightarrow a = \bar{\tau}c$$

We choose $c = 1$, this yields $a = \bar{\tau}$.

Case 2: $\Delta = 1 \Rightarrow \tau = \tau_{\max}$

$$\frac{1}{\tau_{\max}s + 1} = \frac{1}{\frac{a+b}{c+d}s + 1} \Leftrightarrow \frac{a+b}{c+d} = \tau_{\max}$$

Case 3: $\Delta = -1 \Rightarrow \tau = \tau_{\min}$

$$\frac{1}{\tau_{\min}s + 1} = \frac{1}{\frac{a-b}{c-d}s + 1} \Leftrightarrow \frac{a-b}{c-d} = \tau_{\min}$$

The solution of this system of equations is either an algebra exercise or can conveniently be derived with MATLAB:

```
%> Solve linear system of equations
syms av bv cv dv
[a, b, c, d] = solve( av/cv == 10, (av + bv)/(cv + dv) == 42, ...
(av - bv)/(cv - dv) == 6, cv == 1 )
```

With the assumption $c = 1$ we find $a = 10$, $b = -\frac{2}{3}$ and $d = -\frac{7}{9}$.

- (b) The system can be brought to an **MΔ**-structure and checked for robust stability with MATLAB:

```
%> Transform system to M delta-structure
% Convert substitution parameters from part a)
a = eval(a); b = eval(b); c = eval(c); d = eval(d);

% Define uncertain state space system
s = tf('s');
del = ureal('delta', 0);
Gp = 1 / ( (a + del * b) / (c + del * d) * s + 1 );
K = (10 * s + 1) / s;
Gss = uss(feedback(Gp * K, 1));

% Calculate M delta-structure
[N, Delta] = lftdata(Gss);
```

```
n      = length( Delta.NominalValue );
Ntf   = tf( minreal( N ) );
M     = Ntf(1:n,1:n);

if hinfnorm( M ) < 1
    disp( 'The system is robustly stable' );
else
    disp( 'The system may NOT be robustly stable' );
end
```

Even though the PI-controlled closed loop system with the family plants G_p is robustly stable, the conservative condition for complex unstructured uncertainty ($\|\Delta\|_\infty \leq 1$) based on

$$\|\mathbf{M}\|_\infty < 1$$

is not satisfied. This is the price we pay for ignoring the structure of Δ .