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More about Chance

Some of the Problems about Chance having a great appearance of Simplicity, the Mind is easily drawn into a belief, that their Solution may be attained by the meer Strength of natural good Sense; which generally proving otherwise and the Mistakes occasioned thereby being not unfrequent, 'tis presumed that a Book of this Kind, which teaches to distinguish Truth from what seems so nearly to resemble it, will be looked upon as a help to good Reasoning.

—ABRAHAM DE MOIVRE (ENGLAND, 1667–1754)[1]

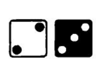
1. LISTING THE WAYS

A *probabilist* is a mathematician who specializes in computing the probabilities of complex events. In the twentieth century, two of the leading probabilists were A. N. Kolmogorov (Russia, 1903–1987) and P. Lévy (France, 1886–1971). The techniques they developed are beyond the scope of this book, but we can look at more basic methods, developed by earlier mathematicians.

When trying to figure chances, it is sometimes very helpful to list all the possible ways that a chance process can turn out. If this is too hard, writing down a few typical ones is a good start.

Example 1. Two dice are thrown. What is the chance of getting a total of 2 spots?

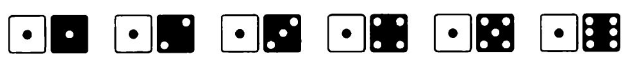
Solution. The chance process here consists of throwing the two dice. What matters is the number of spots shown by each die. To keep the dice separate, imagine that one is white and the other black. One way for the dice to fall is

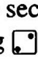


This means the white die showed 2 spots, and the black die showed 3. The total number of spots is 5.

How many ways are there for the two dice to fall? To begin with, the white die can fall in any one of 6 ways:

When the white die shows , say, there are still 6 possible ways for the black die to fall:

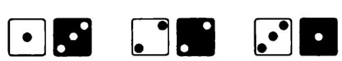


We now have 6 of the possible ways that the two dice can fall. These ways are shown in the first row of figure 1. Similarly, the second row shows another 6 ways for the dice to fall, with the white die showing . And so on. The figure shows there are $6 \times 6 = 36$ possible ways for the dice to fall. They are all equally likely, so each has 1 chance in 36. There is only one way to get a total of 2 spots: . The chance is $1/36$. That is the answer.

There may be several methods for answering questions about chance. In figure 1, for example, the chance for each of the 36 outcomes can also be worked out using the multiplication rule: $1/6 \times 1/6 = 1/36$.

Example 2. A pair of dice are thrown. What is the chance of getting a total of 4 spots?

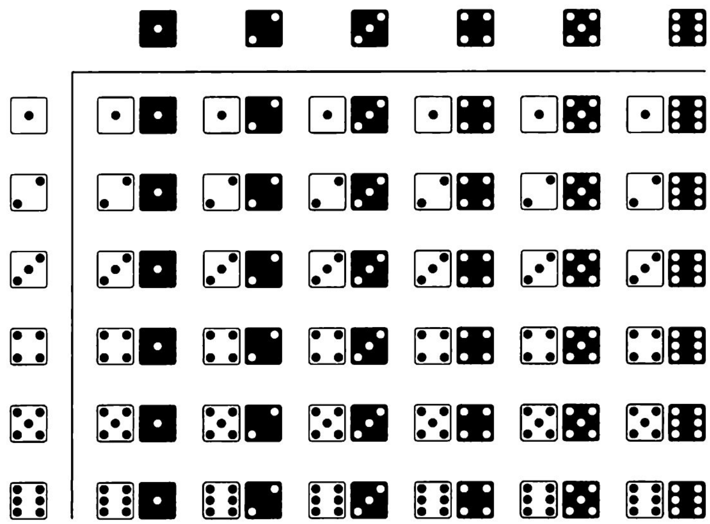
Solution. Look at figure 1. There are 3 ways to get a total of four spots:



The chance is 3 in 36. That is the answer.

What about three dice? A three-dimensional picture like figure 1 would be a bit much to absorb, but similar reasoning can be used. In the seventeenth century, Italian gamblers used to bet on the total number of spots rolled with three dice. They believed that the chance of rolling a total of 9 ought to equal the chance of

Figure 1. Throwing a pair of dice. There are 36 ways for the dice to fall, shown in the body of the diagram; all are equally likely.



rolling a total of 10. For instance, they said, one combination with a total of 9 spots is

1 spot on one die, 2 spots on another die, 6 spots on the third die.

This can be abbreviated as "1 2 6." There are altogether six combinations for 9:

1 2 6 1 3 5 1 4 4 2 3 4 2 2 5 3 3 3

Similarly, they found six combinations for 10:

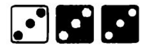
1 4 5 1 3 6 2 2 6 2 3 5 2 4 4 3 3 4

Thus, argued the gamblers, 9 and 10 should by rights have the same chance. However, experience showed that 10 came up a bit more often than 9.

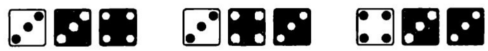
They asked Galileo for help, and he reasoned as follows. Color one of the dice white, another one grey, and another one black—so they can be kept apart. This won't affect the chances. How many ways can the three dice fall? The white die can land in 6 ways. Corresponding to each of them, the grey die can land in 6 ways, making 6×6 possibilities. Corresponding to each of these possibilities, there are still 6 for the black die. Altogether, there are $6 \times 6 \times 6 = 6^3$ ways for three dice to land. (With 4 dice, there would be 6^4; with 5 dice, 6^5 and so on.)

Now $6^3 = 216$ is a lot of ways for three dice to fall. But Galileo sat down and listed them. Then he went through his list and counted the ones with a total of 9 spots. He found 25. And he found 27 ways to get a total of 10 spots. He concluded that the chance of rolling 9 is $25/216 \approx 11.6\%$, while the chance of rolling 10 is $27/216 = 12.5\%$.

The gamblers made a basic error: they didn't get down to the different ways for the dice to land. For instance, the triplet 3 3 3 for 9 only corresponds to one way for the dice to land:



But the triplet 3 3 4 for 10 corresponds to three ways for the dice to land:



The gamblers' argument is corrected in table 1.

Table 1. The chance of getting 9 or 10 spots with three dice.

Triplets for 9	*Number of ways to roll each triplet*	*Triplets for 10*	*Number of ways to roll each triplet*
1 2 6	6	1 4 5	6
1 3 5	6	1 3 6	6
1 4 4	3	2 2 6	3
2 3 4	6	2 3 5	6
2 2 5	3	2 4 4	3
3 3 3	1	3 3 4	3
Total	25	Total	27

Galileo (Italy, 1564–1642)

Wolff-Leavenworth Collection, courtesy of the Syracuse University Art Collection.

Exercise Set A

1. Look at figure 1 and make a list of the ways to roll a total of 5 spots. What is the chance of throwing a total of 5 spots with two dice?
2. Two draws are made at random with replacement from the box | [1] [2] [3] [4] [5] |.

Draw a picture like figure 1 to represent all possible results. How many are there? What is the chance that the sum of the two draws turns out to equal 6?

3. A pair of dice is thrown 1,000 times. What total should appear most often? What totals should appear least often?

4. (a) In the box shown below, each ticket has two numbers.

$$\boxed{1\mid 2}\ \boxed{1\mid 3}\ \boxed{3\mid 1}\ \boxed{3\mid 2}$$

(For instance, the first number on $\boxed{3\mid 1}$ is 3 and the second is 1.) A ticket is drawn at random. Find the chance that the sum of the two numbers is 4.

(b) Repeat, for the box

$$\boxed{1\mid 2}\ \boxed{1\mid 3}\ \boxed{1\mid 3}\ \boxed{3\mid 2}\ \boxed{3\mid 3}\ \boxed{3\mid 3}$$

(c) Repeat, for the box

$$\boxed{1\mid 2}\ \boxed{1\mid 3}\ \boxed{1\mid 3}\ \boxed{3\mid 1}\ \boxed{3\mid 2}\ \boxed{3\mid 3}$$

The answers to these exercises are on p. A68.

2. THE ADDITION RULE

This section is about the chance that at least one of two specified things will happen: either the first happens, or the second, or both. The possibility of both happening turns out to be a complication, which can sometimes be ruled out.

> Two things are *mutually exclusive* when the occurrence of one prevents the occurrence of the other: one excludes the other.

Example 3. A card is dealt off the top of a well-shuffled deck. The card might be a heart. Or, it might be a spade. Are these two possibilities mutually exclusive?

Solution. If the card is a heart, it can't be a spade. These two possibilities are mutually exclusive.

We can now state a general principle for figuring chances. It is called the addition rule.

> *Addition Rule.* To find the chance that at least one of two things will happen, check to see if they are mutually exclusive. If they are, add the chances.

Example 4. A card is dealt off the top of a well-shuffled deck. There is 1 chance in 4 for it to be a heart. There is 1 chance in 4 for it to be a spade. What is the chance for it to be in a major suit (hearts or spades)?

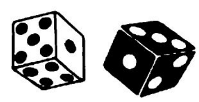
Solution. The question asks for the chance that one of the following two things will happen:

- the card is a heart;
- the card is a spade.

As in example 3, if the card is a heart then it can't be a spade: these are mutually exclusive events. So it is legitimate to add the chances. The chance of getting a card in a major suit is $1/4 + 1/4 = 1/2$. (A check on the reasoning: there are 13 hearts and 13 spades, so $26/52 = 1/2$ of the cards in the deck are in a major suit.)

Example 5. Someone throws a pair of dice. True or false: the chance of getting at least one ace is $1/6 + 1/6 = 1/3$.

Solution. This is false. Imagine one of the dice is white, the other black.



The question asks for the chance that one of the following two things will happen:

- the white die lands ace ⚀;
- the black die lands ace ⚀.

A white ace does not prevent a black ace. These two events are not mutually exclusive, so the addition rule does not apply. Adding the chances gives the wrong answer.

Look at figure 1. There are 6 ways for the white die to show ⚀. There are 6 ways for the black die to show ⚀. But the number of ways to get at least one ace is not $6 + 6$. Addition double counts the outcome ⚀⚀ at the top left corner. The chance of getting at least one ace is

$$(6 + 6 - 1)/36 = 11/36, \quad \text{not} \quad (6 + 6)/36 = 12/36 = 1/3.$$

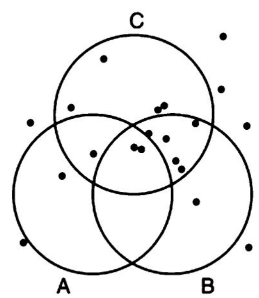
> If you want to find the chance that at least one event occurs, and the events are not mutually exclusive, do not add the chances: the sum will be too big.

Blindly adding chances can give the wrong answer, by double counting the chance that two things happen. With mutually exclusive events, there is no double counting: that is why the addition rule works.

Exercise Set B

1. Fifty children went to a party where cookies and ice cream were served: 12 children took cookies; 17 took ice cream. True or false: 29 children must have had cookies or ice cream. Explain briefly.

2. There are 20 dots in the diagram below, and 3 circles. The circles are labeled A, B, and C. One of the dots will be chosen at random.
   (a) What is the probability that the dot falls inside circle A?
   (b) What is the probability that the dot falls inside circle B?
   (c) What is the probability that the dot falls inside circle C?
   (d) What is the probability that the dot falls inside at least one of the circles?



3. Two cards are dealt off the top of a well-shuffled deck. You have a choice:
   (i) to win \$1 if the first card is an ace or the second card is an ace;
   (ii) to win \$1 if at least one of the two cards is an ace.

   Which option is better? or are they the same? Explain briefly.

4. Two dicc will be rolled. The chance that the first one lands ⚀ is 1/6. The chance that the second one lands ⚁ is 1/6. True or false: the chance that the first one lands ⚀ or the second one lands ⚁ equals $1/6 + 1/6$. Explain briefly.

5. A box contains 10 tickets numbered 1 through 10. Five draws will be made at random with replacement from this box. True or false: there are 5 chances in 10 of getting $\boxed{7}$ at least once. Explain briefly.

6. A number is drawn at random from a box. There is a 20% chance for it to be 10 or less. There is a 10% chance for it to be 50 or more. True or false: the chance of getting a number between 10 and 50 (exclusive) is 70%. Explain briefly.

The answers to these exercises are on pp. A68–69.

3. TWO FAQs (FREQUENTLY ASKED QUESTIONS)

- What's the difference between mutually exclusive and independent?
- When do I add and when do I multiply?

"Mutually exclusive" is one idea; independence is another. Both ideas apply to pairs of events, and say something about how the events are related. However, the relationships are quite different.

- Two events are mutually exclusive if the occurrence of one prevents the other from happening.
- Two events are independent if the occurrence of one does not change the chances for the other.

The addition rule, like the multiplication rule, is a way of combining chances. However, the two rules solve different problems (pp. 229 and 241).

- The addition rule finds the chance that at least one of two things happens.
- The multiplication rule finds the chance that two things both happen.

So, the first step in deciding whether to add or to multiply is to read the question: Do you want to know P(A or B), P(A and B), or something else entirely? But there is also a second step—because the rules apply only if the events are related in the right way.

- Adding the probabilities of two events requires them to be mutually exclusive.[2]
- Multiplying the unconditional probabilities of two events requires them to be independent. (For dependent events, the multiplication rule uses conditional probabilities.)

Example 6. A die is rolled 6 times; a deck of cards is shuffled.

(a) The chance that the first roll is an ace or the last roll is an ace equals ______.

(b) The chance that the first roll is an ace and the last roll is an ace equals ______.

(c) Thc chancc that thc top card is thc acc of spades or the bottom card is the ace of spades equals ______.

(d) The chance that the top card is the ace of spades and the bottom card is the ace of spades equals ______.

Options for parts (a) and (b):

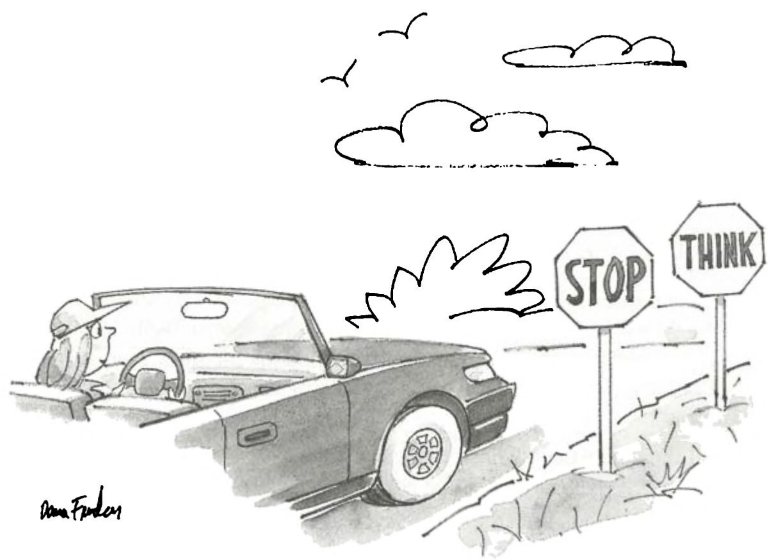
(i) $\frac{1}{6} + \frac{1}{6}$ (ii) $\frac{1}{6} \times \frac{1}{6}$ (iii) neither of these

Options for parts (c) and (d):

(i) $\frac{1}{52} + \frac{1}{52}$ (ii) $\frac{1}{52} \times \frac{1}{52}$ (iii) neither of these

Solution. Part (a). You want the chance that at least one of the two things will happen, so the addition rule looks relevant. However, the two things are not mutually exclusive. Do not use the addition rule, it will give the wrong answer (example 5). If you can't add, maybe you can multiply? The two events are independent, but you do not want the chance that both happen. Do not use the multiplication rule either, it too will give the wrong answer. Choose option (iii).

Part (b). You want the chance that both events happen, and they are independent. Now is the time to multiply. Choose option (ii).



Part (c). The chance the top card is the ace of spades equals 1/52. The chance that the bottom card is the ace of spades—computed before looking at any of the cards (example 2 on p. 226) also equals 1/52. The two events are mutually exclusive; you want the chance that at least one of the two will occur. This is when the addition rule shines. Choose (i).

Part (d). The two events are mutually exclusive, but you do not want the chance that at least one of the two will occur. Therefore, do not use the addition rule, it will give the wrong answer. You want the chance that both things happen, so multiplication may be relevant. However, the events are dependent. Do not multiply the unconditional probabilities, you will get the wrong answer. Choose (iii). (The chance is 0: the ace of spades cannot turn up in both places.)

As example 6 indicates, you may not be able either to add or to multiply. Then more thinking is needed. (The cartoon is trying to tell you something.) The next section gives an example—The Paradox of the Chevalier de Méré.

Technical notes. The chance of two aces is 1/36, so the chance in example 6(a) can be figured as

$$\frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36}$$

However, if the die is rolled 3 times, the chance of getting at least one ace is not

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} - \left(\frac{1}{6}\right)^3$$

Think about 12 rolls! This sort of problem will be solved in the next section.

In example 6(d), the multiplication rule can be used, with conditional probabilities—although this is very fussy. The chance that the top card is the ace of spades equals 1/52. Given that the top card is the ace of spades, the conditional chance that the bottom card is the ace of spades equals 0. The chance that both things happen equals $1/52 \times 0 = 0$.

Exercise Set C

1. A large group of people are competing for all-expense-paid weekends in Philadelphia. The Master of Ceremonies gives each contestant a well-shuffled deck of cards. The contestant deals two cards off the top of the deck, and wins a weekend in Philadelphia if the first card is the ace of hearts or the second card is the king of hearts.
   (a) All the contestants whose first card was the ace of hearts are asked to step forward. What fraction of the contestants do so?
   (b) The contestants return to their original places. Then, the ones who got the king of hearts for their second card are asked to step forward. What fraction of the contestants do so?
   (c) Do any of the contestants step forward twice?
   (d) True or false, and explain: the chance of winning a weekend in Philadelphia is $1/52 + 1/52$.

"PHILADELPHIA, PLEASE."

2. A large group of people are competing for all-expense-paid weekends in Philadelphia. The Master of Ceremonies gives each contestant a well-shuffled deck of cards. The contestant deals two cards off the top of the deck, and wins a weekend in Philadelphia if the first card is the ace of hearts or the second card is the ace of hearts. (This is like exercise 1, but the winning cards are a little different.)
   (a) All the contestants whose first card was the ace of hearts are asked to step forward. What fraction of the contestants do so?
   (b) The contestants return to their original places. Then, the ones who got the ace of hearts for their second card are asked to step forward. What fraction of the contestants do so?
   (c) Do any of the contestants step forward twice?
   (d) True or false, and explain: the chance of winning a weekend in Philadelphia is $1/52 + 1/52$.

3. A deck of cards is shuffled. True or false, and explain briefly:
   (a) The chance that the top card is the jack of clubs equals $1/52$.
   (b) The chance that the bottom card is the jack of diamonds equals $1/52$.
   (c) The chance that the top card is the jack of clubs or the bottom card is the jack of diamonds equals $2/52$.
   (d) The chance that the top card is the jack of clubs or the bottom card is the jack of clubs equals $2/52$.
   (e) The chance that the top card is the jack of clubs and the bottom card is the jack of diamonds equals $1/52 \times 1/52$.
   (f) The chance that the top card is the jack of clubs and the bottom card is the jack of clubs equals $1/52 \times 1/52$.

4. The unconditional probability of event A is $1/2$. The unconditional probability of event B is $1/3$. Say whether each of the following is true or false, and explain briefly.
   (a) The chance that A and B both happen must be $1/2 \times 1/3 = 1/6$.
   (b) If A and B are independent, the chance that they both happen must be $1/2 \times 1/3 = 1/6$.
   (c) If A and B are mutually exclusive, the chance that they both happen must be $1/2 \times 1/3 = 1/6$.
   (d) The chance that at least one of A or B happens must be $1/2 + 1/3 = 5/6$.
   (e) If A and B are independent, the chance that at least one of them happens must be $1/2 + 1/3 = 5/6$.
   (f) If A and B are mutually exclusive, the chance that at least one of them happens must be $1/2 + 1/3 = 5/6$.

5. Two cards are dealt off the top of a well-shuffled deck.
   (a) Find the chance that the second card is an ace.
   (b) Find the chance that the second card is an ace, given the first card is a king.
   (c) Find the chance that the first card is a king and the second card is an ace.

The answers to these exercises are on pp. A69–70.

4. THE PARADOX OF THE CHEVALIER DE MÉRÉ

In the seventeenth century, French gamblers used to bet on the event that with 4 rolls of a die, at least one ace would turn up: an ace is ⚀. In another game, they bet on the event that with 24 rolls of a pair of dice, at least one double-ace would turn up: a double-ace is a pair of dice which show ⚀⚀.

The Chevalier de Méré, a French nobleman of the period, thought the two events were equally likely. He reasoned this way about the first game:

- In one roll of a die, I have $1/6$ of a chance to get an ace.
- So in 4 rolls, I have $4 \times 1/6 = 2/3$ of a chance to get at least one ace.

His reasoning for the second game was similar:

- In one roll of a pair of dice, I have $1/36$ of a chance to get a double-ace.
- So in 24 rolls, I must have $24 \times 1/36 = 2/3$ of a chance to get at least one double-ace.

By this argument, both chances were the same, namely $2/3$. However, the gamblers found that the first event was a bit more likely than the second. This contradiction became known as the *Paradox of the Chevalier de Méré*.

De Méré asked the philosopher Blaise Pascal about the problem, and Pascal solved it with the help of his friend, Pierre de Fermat. Fermat was a judge and a member of parliament, who is remembered today for the mathematical research he did after hours. Fermat saw that de Méré was adding chances for events that were not mutually exclusive. In fact, pushing de Méré's argument a little further, it shows the chance of getting an ace in 6 rolls of a die to be $6/6$, or 100%. Something had to be wrong.

The question is how to calculate the chances correctly. Pascal and Fermat solved this problem, with a typically indirect piece of mathematical reasoning—

Blaise Pascal (France, 1623–1662)
Wolff-Leavenworth Collection, courtesy of the Syracuse University Art Collection.

Pierre de Fermat (France, 1601–1665)
From the *Oeuvres Complètes*

the kind that always leaves non-mathematicians feeling a bit cheated. Of course, a direct attack like Galileo's (section 1) could easily bog down. With 4 rolls of a die, there are $6^4 = 1{,}296$ outcomes to worry about. With 24 rolls of a pair of dice, there are $36^{24} \approx 2.2 \times 10^{37}$ outcomes.

The conversation between Pascal and Fermat is lost to history, but here is a reconstruction.[3]

Pascal. Let's look at the first game first.

Fermat. Bon. The chance of winning is hard to compute, so let's work out the chance of the opposite event—losing. Then

$$\text{chance of winning} = 100\% - \text{chance of losing}.$$

Pascal. D'accord. The gambler loses when none of the four rolls shows an ace. But how do you work out the chances?

Fermat. It does look complicated. Let's start with one roll. What's the chance that the first roll doesn't show an ace?

Pascal. It has to show something from 2 through 6, so the chance is $5/6$.

Fermat. C'est ça. Now, what's the chance that the first two rolls don't show aces?

Pascal. We can use the multiplication rule. The chance that the first roll doesn't give an ace and the second doesn't give an ace equals $5/6 \times 5/6 = (5/6)^2$. After all, the rolls are independent, n'est-ce pas?

Fermat. What about 3 rolls?

Pascal. It looks like $5/6 \times 5/6 \times 5/6 = (5/6)^3$.

Fermat. Oui. Now what about 4 rolls?

Pascal. Must be $(5/6)^4$.

Fermat. Sans doute, and that's about 0.482, or 48.2%.

Pascal. So there is a 48.2% chance of losing. Now

$$\begin{aligned}\text{chance of winning} &= 100\% - \text{chance of losing}\\ &= 100\% - 48.2\% = 51.8\%.\end{aligned}$$

Fermat That settles the first game. The chance of winning is a little over 50%. Now what about the second?

Pascal Eh bien, in one roll of a pair of dice, there is 1 chance in 36 of getting a double-ace, and 35 chances in 36 of not getting a double-ace. By the multiplication rule, in 24 rolls of a pair of dice the chance of getting no double-aces must be

$$(35/36)^{24}.$$

Fermat Entendu. That's about 50.9%. So we have the chance of losing. Now

$$\begin{aligned}\text{chance of winning} &= 100\% - \text{chance of losing}\\ &= 100\% - 50.9\% = 49.1\%.\end{aligned}$$

Pascal Le résultat is a bit less than 50%. Voilà. That's why you win the second game a bit less frequently than the first. But you have to roll a lot of dice to see the difference.

> If the chance of an event is hard to find, try to find the chance of the opposite event. Then subtract from 100%. (See p. 223.) This is useful when the chance of the opposite event is easier to compute.

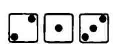
Exercise Set D

1. A die is rolled three times. You bet $1 on some proposition. Below is a list of 6 bets, and then a list of 3 outcomes. For each bet, find all the outcomes where you win. For instance, with (a), you win on (i) only.

   Bets

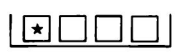
   (a) all aces
   (b) at least one ace
   (c) no aces
   (d) not all aces
   (e) 1st roll is an ace, or 2nd roll is an ace, or 3rd roll is an ace
   (f) 1st roll is an ace, and 2nd roll is an ace, and 3rd roll is an ace

   Outcomes

   (i)  (ii)  (iii) 

2. In exercise 1, which is a better bet—(a) or (f)? Or are they same? What about (b) and (e)? What about (c) and (d)? (You do not need to compute the chances.)

3. A box contains four tickets, one marked with a star, and the other three blank:

   Two draws are made at random with replacement from this box.

   (a) What is the chance of getting a blank ticket on the first draw?
   (b) What is the chance of getting a blank ticket on the second draw?
   (c) What is the chance of getting a blank ticket on the first draw and a blank ticket on the second draw?
   (d) What is the chance of not getting the star in the two draws?
   (e) What is the chance of getting the star at least once in the two draws?

4. (a) A die is rolled 3 times. What is the chance of getting at least one ace?
   (b) Same, with 6 rolls.
   (c) Same, with 12 rolls.

5. A pair of dice is rolled 36 times. What is the chance of getting at least one double-ace?

6. According to de Moivre, in eighteenth-century England people played a game similar to modern roulette. It was called "Royal Oak." There were 32 "points" or num-

bered pockets on a table. A ball was thrown in such a way that it landed in each pocket with an equal chance, 1 in 32.

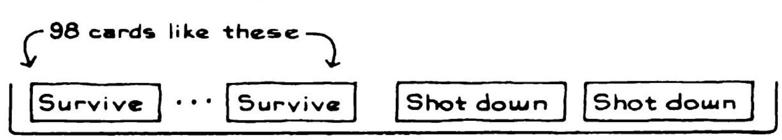
If you bet 1 pound on a point and it came up, you got your stake back, together with winnings of 27 pounds. If your point didn't come up, you lost your pound. The players (or "Adventurers," as de Moivre called them) complained that the game was unfair, and they should have won 31 pounds if their point came up. (They were right; section 1 of chapter 17.) De Moivre continues:

> The Master of the Ball maintained they had no reason to complain; since he would undertake that any particular point of the Ball should come up in Two-and-Twenty Throws: of this he would offer to lay a Wager, and actually laid it when required.The seeming contradiction between the Odds of One-and-Thirty to One, and Twenty-two Throws for any [point] to come up, so perplexed the Adventurers, that they begun to think the Advantage was on their side: for which reason they played on and continued to lose. [Two-and-Twenty is 22, One-and-Thirty is 31.]

What is the chance that the point 17, say, will come up in Two-and-Twenty Throws? (The Master of the Ball laid this wager at even money, so if the chance is over 50%, he shows a profit here too.)

7. In his novel *Bomber*, Len Deighton argues that a World War II pilot had a 2% chance of being shot down on each mission. So in 50 missions he is "mathematically certain" to be shot down: $50 \times 2\% = 100\%$. Is this a good argument?

   Hint: To make chance calculations, you have to see how the situation is like a game of chance. The analogy here is getting the card "survive" every time, if you draw 50 times at random with replacement from the box



The answers to these exercises are on p. A70.

5. ARE REAL DICE FAIR?

According to Galileo (section 1), when a die is rolled it is equally likely to show any of its 6 faces. Galileo was thinking of an ideal die which is perfectly symmetric. This is like ignoring friction in the study of physics: the results are only a first approximation. What does Galileo's calculation say about real dice?

- For real dice, the 216 possible ways three dice can land are close to being equally likely.
- If these ways were equally likely, the chance of rolling a total of 9 spots would be exactly 25 in 216.
- So for real dice, the chance of rolling a total of 9 spots is just about 25 in 216.

For loaded dice, the calculations would be badly off. But ordinary dice, coins, and the like are very close to fair—in the sense that all the outcomes are equally likely. Of course, you have to put some effort into shaking the dice or flipping the coins. And the games of chance based on these fair mechanisms may be quite unfair (chapter 17).

In a similar way, if you are told that a ticket is drawn at random, you should assume that each ticket in the box is equally likely to be drawn. If the tickets are close to the same size, shape, and texture, and the box is well shaken, this is quite a reasonable approximation.

6. REVIEW EXERCISES

Review exercises may cover material from previous chapters.

When a die is rolled, each of the 6 faces is equally likely to come up. A deck of cards has 4 suits (clubs, diamonds, hearts, spades) with 13 cards in each suit—2, 3, . . . , 10, jack, queen, king, ace. See pp. 222 and 226.

1. A pair of dice are thrown.
   (a) Find the chance that both dice show 3 spots.
   (b) Find the chance that both dice show the same number of spots.
2. In the game of Monopoly, a player rolls two dice, counts the total number of spots, and moves that many squares. Find the chance that the player moves 11 squares (no more and no less).
3. True or false, and explain:
   (a) If a die is rolled three times, the chance of getting at least one ace is $1/6 + 1/6 + 1/6 = 1/2$.
   (b) If a coin is tossed twice, the chance of getting at least one head is 100%.
4. Two cards will be dealt off the top of a well-shuffled deck. You have a choice:

(i) to win $1 if at least one of the two cards is a queen.
(ii) to win $1 if the first is a queen.

Which option is better? Or are they equivalent? Explain.

5. The chance of A is 1/3; the chance of B is 1/10. True or false, and explain:
   (a) If A and B are independent, they must also be mutually exclusive.
   (b) If A and B are mutually exclusive, they cannot be independent.

6. One event has chance 1/2, another has chance 1/3. Fill in the blanks, using one phrase from each pair below, to make up two true sentences. Write out both sentences.

   "If you want to find the chance that __(i)__ will happen, check to see if they are __(ii)__. If so, you can __(iii)__ the chances."

   (i) at least one of the two events, both events
   (ii) independent, mutually exclusive
   (iii) add, multiply

7. Four draws are going to be made at random with replacement from the box | $\boxed{1}$ $\boxed{2}$ $\boxed{2}$ $\boxed{3}$ $\boxed{3}$ |. Find the chance that $\boxed{2}$ is drawn at least once.

8. Repeat exercise 7, if the draws are made at random without replacement.

9. One ticket will be drawn at random from each of the two boxes shown below:

   (A) | $\boxed{1}$ $\boxed{2}$ $\boxed{3}$ | (B) | $\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$ |

   Find the chance that:
   (a) The number drawn from A is larger than the one from B.
   (b) The number drawn from A equals the one from B.
   (c) The number drawn from A is smaller than the one from B.

10. There are two options:
    (i) A die will be rolled 60 times. Each time it shows an ace or a six, you win $1; on the other rolls, you win nothing.
    (ii) Sixty draws will be made at random with replacement from the box | $\boxed{1}$ $\boxed{1}$ $\boxed{1}$ $\boxed{0}$ $\boxed{0}$ $\boxed{0}$ |. On each draw, you will be paid the amount shown on the ticket, in dollars.

    Which option is better? or are they the same? Explain briefly.

11. Three cards are dealt from a well-shuffled deck.
    (a) Find the chance that all of the cards are diamonds.
    (b) Find the chance that none of the cards are diamonds.
    (c) Find the chance that the cards are not all diamonds.

12. A coin is tossed 10 times. True or false, and explain:
    (a) The chance of getting 10 heads in a row is 1/1,024.
    (b) Given that the first 9 tosses were heads, the chance of getting 10 heads in a row is 1/2.

Exercises 13 and 14 are more difficult.

13. A box contains 2 red marbles and 98 blue ones. Draws are made at random with replacement. In ______ draws from the box, there is better than a 50% chance for a red marble to appear at least once. Fill in the blank with the smallest number that makes the statement true. (You will need a calculator.)

14. In Lotto 6-53, there is a box with 53 balls, numbered from 1 to 53. Six balls are drawn at random without replacement from the box. You win the grand prize if the numbers on your lottery ticket are the same as the numbers on the six balls; order does not matter.

    Person A bought two tickets, with the following numbers:

Ticket #1	5	12	21	30	42	51
Ticket #2	5	12	23	30	42	49

    Person B bought two tickets, with the following numbers:

Ticket #1	7	11	25	28	34	50
Ticket #2	9	14	20	22	37	45

    Which person has the better chance of winning? Or are their chances the same? Explain briefly.

7. SUMMARY

1. When figuring chances, one helpful strategy is to write down a complete list of all the possible ways that the chance process can turn out. If this is too hard, at least write down a few typical ways, and count how many ways there are in total.

2. The chance that at least one of two things will happen equals the sum of the individual chances, provided the things are mutually exclusive. Otherwise, adding the chances will give the wrong answer—double counting.

3. If you are having trouble working out the chance of an event, try to figure out the chance of its opposite; then subtract from 100%.