13

What Are the Chances?

In the long run, we are all dead.
—JOHN MAYNARD KEYNES (ENGLAND, 1883–1946)

1. INTRODUCTION

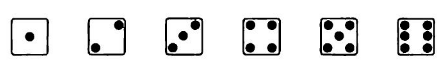
People talk loosely about chance all the time, without doing any harm. What are the chances of getting a job? of meeting someone? of rain tomorrow? But for scientific purposes, it is necessary to give the word *chance* a definite, clear interpretation. This turns out to be hard, and mathematicians have struggled with the job for centuries. They have developed some careful and rigorous theories, but these theories cover just a small range of the cases where people ordinarily speak of chance. This book will present the *frequency theory*, which works best for processes which can be repeated over and over again, independently and under the same conditions.[1] Many games fall into this category, and the frequency theory was originally developed to solve gambling problems. One of the great early masters was Abraham de Moivre, a French Protestant who fled to England to avoid religious persecution. Part of the dedication to his book, *The Doctrine of Chances*, is reproduced in figure 1 on the next page.[2]

Figure 1. De Moivre's dedication to *The Doctrine of Chances*.

> To the Right Honorable the
> Lord CARPENTER.
>
> My Lord,
>
> There are many people in the World who are prepossessed with an Opinion, that the Doctrine of Chances has a Tendency to promote Play; but they soon will be undeceived, if they think fit to look into the general Design of this Book; in the mean time it will not be improper to inform them, that your Lordship is pleased to espouse the Patronage of this second Edition; which your strict Probity, and the distinguished Character you bear in the World, would not have permitted, were not their Apprehensions altogether groundless.
>
> Your Lordship does easily perceive, that this Doctrine is so far from encouraging Play, that it is rather a Guard against it, by setting in a clear light, the Advantages and Disadvantages of those Games wherein Chance is concerned....
>
> Another use to be made of this Doctrine of Chances is that it may serve in conjunction with the other parts of the Mathematicks, as a fit Introduction to the Art of Reasoning: it being known by experience that nothing can contribute more to the attaining of that Art, than the consideration of a long Train of Consequences, rightly deduced from undoubted Principles, of which this Book affords many Examples.

One simple game of chance involves betting on the toss of a coin. The process of tossing the coin can be repeated over and over again, independently and under the same conditions. The chance of getting heads is 50%: in the long run, heads will turn up about 50% of the time.

Take another example. A die (plural, "dice") is a cube with six faces, labelled



1 2 3 4 5 6

When the die is rolled, the faces are equally likely to turn up. The chance of getting an ace—⚀—is 1 in 6, or $16\frac{2}{3}\%$. The interpretation: if the die is rolled over and over again, repeating the basic chance process under the same conditions, in the long run an ace will show about $16\frac{2}{3}\%$ of the time.

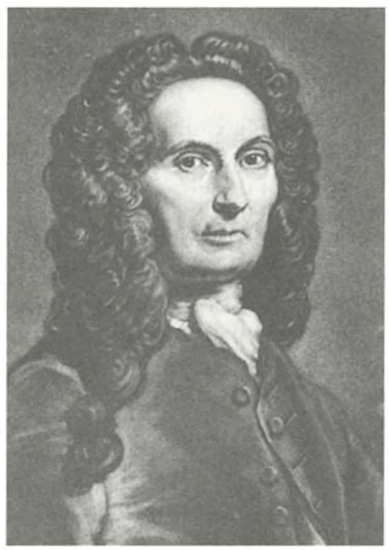
The chance of something gives the percentage of time it is expected to happen, when the basic process is done over and over again, independently and under the same conditions.

If something is impossible, it happens 0% of the time. At the other extreme, if something is sure to happen, then it happens 100% of the time. All chances are between these two extremes.

Chances are between 0% and 100%.

Here is another basic fact. Suppose you are playing a game, and have a 45% chance to win. In other words, you expect to win about 45% of the time. So you must expect to lose the other 55% of the time.

The chance of something equals 100% minus the chance of the opposite thing.



Abraham de Moivre (England, 1667–1754)
Etching by Faber. Copyright © British Museum.

Example 1. A box contains red marbles and blue marbles. One marble is drawn at random from the box (each marble has an equal chance to be drawn). If it is red, you win \$1. If it is blue, you win nothing. You can choose between two boxes:

- box A contains 3 red marbles and 2 blue ones;
- box B contains 30 red marbles and 20 blue ones.

Which box offers a better chance of winning, or are they the same?

Solution. Some people prefer box A, because it has fewer blue marbles. Others prefer B, because it has more red marbles. Both views are wrong. The two boxes offer the same chance of winning, 3 in 5. To see why, imagine drawing many times at random from box A (replacing the marble after each draw, so as not to change the conditions of the experiment). In the long run each of the

5 marbles will appear about 1 time in 5. So the red marbles will turn up about 3/5 of the time. With box A, your chance of drawing a red marble is 3/5, that is, 60%.

Now imagine drawing many times at random with replacement from box B. Each of the 50 marbles will turn up about 1 time in 50. But now there are 30 red marbles. With box B, your chance of winning is 30/50 = 3/5 = 60%, just as for box A. What counts is the ratio

$$\frac{\text{number of red marbles}}{\text{total number of marbles}}.$$

The ratio is the same in both boxes. De Moivre's solution for this example is given in figure 2.

Figure 2. De Moivre's solution.

The Probability of an Event is greater or less, according to the number of Chances by which it may happen, compared with the whole number of Chances by which it may either happen or fail.

Wherefore, if we constitute a Fraction whereof the Numerator be the number of Chances whereby an Event may happen, and the Denominator the number of all the Chances whereby it may either happen or fail, that Fraction will be a proper designation of the Probability of it happening. Thus if an Event has 3 Chances to happen, and 2 to fail, the Fraction 3/5 will fitly represent the Probability of its happening, and may be taken as the measure of it.

The same things may be said of the Probability of failing, which will likewise be measured by a Fraction, whose Numerator is the number of Chances whereby it may fail, and the Denominator the whole number of Chances, both for its happening and failing; thus the Probability of the failing of that Event which has 2 Chances to fail and 3 to happen will be measured by the Fraction 2/5.

The Fractions which represent the Probabilities of happening and failing, being added together, their Sum will always be equal to Unity, since the Sum of their Numerators will be equal to their common Denominator: now it being a certainty that an Event will either happen or fail, it follows that Certainty, which may be conceived under the notion of an infinitely great degree of Probability, is fitly represented by Unity. [By "Unity," de Moivre means the number 1.]

These things will easily be apprehended, if it be considered that the word Probability includes a double Idea: first, of the number of Chances whereby an Event may happen; secondly, of the number of Chances whereby it may either happen or fail.

Many problems, like example 1, take the form of drawing at random from a box. A typical instruction is,

Draw two tickets at random WITH replacement from the box

$$\boxed{\boxed{1}\ \boxed{2}\ \boxed{3}}$$

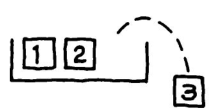
This asks you to imagine the following process: shake the box, draw out one ticket at random (equal chance for all three tickets), make a note of the number on it, put it back in the box, shake the box again, draw a second ticket at random (equal chance for all three tickets), make a note of the number on it, and put the second ticket back in the box. The contrast is with the instruction,

Draw two tickets at random WITHOUT replacement from the box

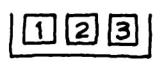
$$\boxed{\boxed{1}\ \boxed{2}\ \boxed{3}}$$

The second instruction asks you to imagine the following process: shake the box, draw out one ticket at random (equal chance for all three tickets), set it aside, draw out a second ticket at random (equal chance for the two tickets left in the box). See figure 3.

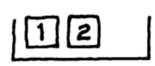
Figure 3. The difference between drawing with or without replacement. Two draws are made at random from the box $\boxed{\boxed{1}\ \boxed{2}\ \boxed{3}}$. Suppose the first draw is $\boxed{3}$.



WITH replacement . . . the second draw is from



WITHOUT replacement . . . the second draw is from



When you draw at random, all the tickets in the box have the same chance to be picked.

Exercise Set A

1. A computer is programmed to compute various chances. Match the numerical answers with the verbal descriptions (which may be used more than once).

Numerical answer	*Verbal description*
(a) −50%	(i) This is as likely to happen as not.
(b) 0%	(ii) This is very likely to happen, but it's not certain.
(c) 10%	(iii) This won't happen.
(d) 50%	(iv) This may happen, but it's not likely.
(e) 90%	(v) This will happen, for sure.
(f) 100%	(vi) There's a bug in the program.
(g) 200%	

2. A coin will be tossed 1,000 times. About how many heads are expected?
3. A die will be rolled 6,000 times. About how many aces are expected?
4. In five-card draw poker, the chance of being dealt a full house (one pair and three of a kind) is 0.14 of 1%. If 10,000 hands are dealt, about how many will be a full house?
5. One hundred tickets will be drawn at random with replacement from one of the two boxes shown below. On each draw, you will be paid the amount shown on the ticket, in dollars. Which box is better and why?

(i) $\boxed{\boxed{1}\ \boxed{2}}$ (ii) $\boxed{\boxed{1}\ \boxed{3}}$

The answers to these exercises are on p. A66.

2. CONDITIONAL PROBABILITIES

This section introduces conditional probabilities. The examples involve cards. A deck of cards has 4 suits: clubs, diamonds, hearts, spades. There are 13 cards in each suit: 2 through 10, jack, queen, king, ace. So there are $4 \times 13 = 52$ cards in the deck.

Example 2. A deck of cards is shuffled and the top two cards are put on a table, face down. You win \$1 if the second card is the queen of hearts.

(a) What is your chance of winning the dollar?
(b) You turn over the first card. It is the seven of clubs. Now what is your chance of winning?

Solution. Part (a). The bet is about the second card, not the first. Initially, this will seem a little strange. Some illustrations may help.

- If the first card is the two of spades and the second is the queen of hearts, you win.
- If the first card is the jack of clubs and the second is the queen of hearts, you win.
- If the first card is the seven of clubs and the second is the king of hearts, you lose.

The bet can be settled without even looking at the first card. The second card is all you need to know.

The chance of winning is 1/52. To see why, think about shuffling the deck. That brings the cards into random order. The queen of hearts has to wind up somewhere. There are 52 possible positions, and they are all equally likely. So there is 1 chance in 52 for her to wind up as the second card in the deck—and bring you the dollar.

Part (b). There are 51 cards left. They are in random order, and the queen of hearts is one of them. She has 1 chance in 51 to be on the table. Your chance goes up a little, to 1/51. That is the answer.

The 1/51 in part (b) is a *conditional* chance. The problem puts a condition on the first card: it has to be the seven of clubs. A mathematician might talk about the conditional probability that the second card is the queen of hearts *given* the first card is the seven of clubs. To emphasize the contrast, the 1/52 in part (a) is called an *unconditional* chance: the problem puts no conditions on the first card.

Exercise Set B

1. Two tickets are drawn at random without replacement from the box $\boxed{\boxed{1}\ \boxed{2}\ \boxed{3}\ \boxed{4}}$.
   (a) What is the chance that the second ticket is 4?
   (b) What is the chance that the second ticket is 4, given the first is 2?
2. Repeat exercise 1, if the draws are made with replacement.
3. A penny is tossed 5 times.
   (a) Find the chance that the 5th toss is a head.
   (b) Find the chance that the 5th toss is a head, given the first 4 are tails.
4. Five cards are dealt off the top of a well-shuffled deck.
   (a) Find the chance that the 5th card is the queen of spades.
   (b) Find the chance that the 5th card is the queen of spades, given that the first 4 cards are hearts.

The answers to these exercises are on pp. A66–67.

Technical notes. (i) Mathematicians write the probability for the second card to be the queen of hearts as follows:

$$\text{P(2nd card is queen of hearts).}$$

The "P" is short for "probability."

(ii) The conditional probability for the second card to be the queen of hearts, given the first was the seven of clubs, is written as follows:

$$\text{P(2nd card is queen of hearts} \mid \text{1st card is seven of clubs).}$$

The vertical bar is read "given."

3. THE MULTIPLICATION RULE

This section will show how to figure the chance that two events happen, by multiplying probabilities.

Example 3. A box has three tickets, colored red, white and blue.

$$\boxed{R}\,\boxed{W}\,\boxed{B}$$

Two tickets will be drawn at random without replacement. What is the chance of drawing the red ticket and then the white?

Solution. Imagine a large group of people. Each of these people holds a box $\boxed{R}\,\boxed{W}\,\boxed{B}$ and draws two tickets at random without replacement. About one third of the people get $\boxed{R}$ on the first draw, and are left with

$$\boxed{W}\,\boxed{B}$$

On the second draw, about half of these people will get $\boxed{W}$. The fraction who draw $\boxed{R}$ $\boxed{W}$ is therefore

$$\frac{1}{2} \text{ of } \frac{1}{3} = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}.$$

The chance is 1 in 6, or $16\frac{2}{3}\%$.

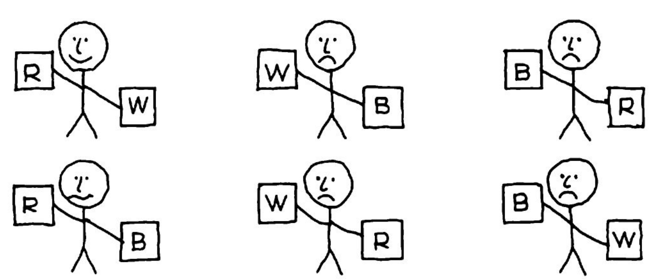
For instance, suppose you start with 600 people. About 200 of them will get $\boxed{R}$ on the first draw. Of these 200 people, about 100 will get $\boxed{W}$ on the second draw. So $100/600 = 1/6$ of the people draw the red ticket first and then the white one. In figure 4, the people who draw $\boxed{R}$ $\boxed{W}$ are at the top left.

Statisticians usually multiply the chances in reverse order:

$$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}.$$

The reason: $1/3$ refers to the first draw, and $1/2$ to the second.

Figure 4. The multiplication rule. Each stick figure corresponds to 100 people.



The method in example 3 is called the multiplication rule.

> *Multiplication Rule.* The chance that two things will both happen equals the chance that the first will happen, multiplied by the chance that the second will happen given the first has happened.

Example 4. Two cards will be dealt off the top of a well-shuffled deck. What is the chance that the first card will be the seven of clubs and the second card will be the queen of hearts?

Solution. This is like example 3, with a much bigger box. The chance that the first card will be the seven of clubs is 1/52. Given that the first card was the seven of clubs, the chance that the second card will be the queen of hearts is 1/51. The chance of getting both cards is

$$\frac{1}{52} \times \frac{1}{51} = \frac{1}{2{,}652}.$$

This is a small chance: about 4 in 10,000, or 0.04 of 1%.

Example 5. A deck of cards is shuffled, and two cards are dealt. What is the chance that both are aces?

Solution. The chance that the first card is an ace equals 4/52. Given that the first card is an ace, there are 3 aces among the 51 remaining cards. So the chance of a second ace equals 3/51. The chance that both cards are aces equals

$$\frac{4}{52} \times \frac{3}{51} = \frac{12}{2{,}652}.$$

This is about 1 in 200, or 1/2 of 1%.

Example 6. A coin is tossed twice. What is the chance of a head followed by a tail?

Solution. The chance of a head on the first toss equals 1/2. No matter how the first toss turns out, the chance of tails on the second toss equals 1/2. So the chance of heads followed by tails equals

$$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

Exercise Set C

1. A deck is shuffled and two cards are dealt.
   (a) Find the chance that the second card is a heart given the first card is a heart.
   (b) Find the chance that the first card is a heart and the second card is a heart.

2. A die is rolled three times.
   (a) Find the chance that the first roll is an ace ⚀.
   (b) Find the chance that the first roll is an ace ⚀, the second roll is a deuce ⚁, and the third roll is a trey ⚂.
3. A deck is shuffled and three cards are dealt.
   (a) Find the chance that the first card is a king.
   (b) Find the chance that the first card is a king, the second is a queen, and the third is a jack.
4. A die will be rolled six times. You have a choice—
   (i) to win \$1 if at least one ace shows
   (ii) to win \$1 if an ace shows on all the rolls

   Which option offers the better chance of winning? Or are they the same? Explain.
5. Someone works example 2(a) on p. 226 this way:

   For me to win, the queen can't be the first card dealt (51 chances in 52) and she must be the second card (1 chance in 51), so the answer is

$$\frac{51}{52} \times \frac{1}{51} = \frac{1}{52}.$$

   Is the multiplication legitimate? Why?
6. "A cat-o'nine-______ can be used to punish ______ of state, but this is seldom done." A coin is tossed twice, to fill in the blanks. What is the chance of the coin getting it right?
7. A coin is tossed 3 times.
   (a) What is the chance of getting 3 heads?
   (b) What is the chance of not getting 3 heads?
   (c) What is the chance of getting at least 1 tail?
   (d) What is the chance of getting at least 1 head?

The answers to these exercises are on p. A67.

4. INDEPENDENCE

This section introduces the idea of independence, which will be used many times in the rest of the book.

> Two things are *independent* if the chances for the second given the first are the same, no matter how the first one turns out. Otherwise, the two things are *dependent*.

Example 7. Someone is going to toss a coin twice. If the coin lands heads on the second toss, you win a dollar.

(a) If the first toss is heads, what is your chance of winning the dollar?
(b) If the first toss is tails, what is your chance of winning the dollar?

(c) Are the tosses independent?

Solution. If the first toss is heads, there is a 50% chance to get heads the second time. If the first toss is tails, the chance is still 50%. The chances for the second toss stay the same, however the first toss turns out. That is independence.

Example 8. Two draws will be made at random with replacement from

$$\left|\ \boxed{1}\ \boxed{1}\ \boxed{2}\ \boxed{2}\ \boxed{3}\ \right|$$

(a) Suppose the first draw is $\boxed{1}$. What is the chance of getting a $\boxed{2}$ on the second draw?
(b) Suppose the first draw is $\boxed{2}$. What is the chance of getting $\boxed{2}$ on the second draw?
(c) Are the draws independent?

Solution. Whether the first draw is $\boxed{1}$ or $\boxed{2}$ or anything else, the chance of getting $\boxed{2}$ on the second draw stays the same—two in five, or 40%. The reason: the first ticket is replaced, so the second draw is always made from the same box $\left|\ \boxed{1}\ \boxed{1}\ \boxed{2}\ \boxed{2}\ \boxed{3}\ \right|$. The draws are independent.

Example 9. As in example 8, but the draws are made without replacement.

Solution. If the first draw turns out to be $\boxed{1}$ then the second draw is from the box $\left|\ \quad \boxed{1}\ \boxed{2}\ \boxed{2}\ \boxed{3}\ \right|$. The chance for the second draw to be $\boxed{2}$ is 50%. On the other hand, if the first draw turns out to be $\boxed{2}$, then the second draw is from the box $\left|\ \boxed{1}\ \boxed{1}\ \quad \boxed{2}\ \boxed{3}\ \right|$. Now there is only a 25% chance for the second to be $\boxed{2}$. The draws are dependent.

> When drawing at random with replacement, the draws are independent. Without replacement, the draws are dependent.

What does independence of the draws mean? To answer this question, think about bets which can be settled on one draw: for instance, that the draw will be 3 or more. Then the conditional chance of winning the bet must stay the same, no matter how the other draws turn out.

Example 10. A box has three tickets, colored red, white, and blue.

$$\left|\ \boxed{R}\ \boxed{W}\ \boxed{B}\ \right|$$

Two tickets will be drawn at random with replacement. What is the chance of drawing the red ticket and then the white?

Solution. The draws are independent, so the chance is

$$\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}.$$

Compare this with example 3. The answers are different. Independence matters. And it's easier this time: you don't need to work out conditional probabilities.

> If two things are independent, the chance that both will happen equals the product of their unconditional probabilities. This is a special case of the multiplication rule (p. 229).

Exercise Set D

1. For each of the following boxes, say whether color and number are dependent or independent.

   (a) | 1 2 2 1 2 2 | (c) | 1 2 3 1 2 2 |

   (b) | 1 2 1 2 1 2 |

2. (a) In the box shown below, each ticket has two numbers.

   | 1 2   1 3   4 2   4 3 |

   (For instance, the first number on 4 2 is 4 and the second is 2.) A ticket is drawn at random. Are the two numbers dependent or independent?

   (b) Repeat, for the box

   | 1 2   1 3   1 3   4 2   4 3   4 3 |

   (c) Repeat, for the box

   | 1 2   1 3   1 3   4 2   4 2   4 3 |

3. Every week you buy a ticket in a lottery that offers one chance in a million of winning. What is the chance that you never win, even if you keep this up for ten years?

4. Two draws are made at random without replacement from the box | 1 2 3 4 |. The first ticket is lost, and nobody knows what was written on it. True or false, and explain: the two draws are independent.

5. Suppose that in a certain class, there are
   - 80% men and 20% women;
   - 15% freshmen and 85% sophomores.

   (a) The percentage of sophomore women in the class can be as small as ______.
   (b) This percentage can be as large as ______.

6. One student is chosen at random from the class described in the previous exercise.

   (a) The chance of getting a sophomore woman can be as small as ______.
   (b) This chance can be as large as ______.

7. In 2002, about 50.9% of the population of the United States was female. Also, 1.6% of the population was age 85 and over.[3] True or false, and explain: the percentage of the population consisting of women age 85 and over is

$$50.9\% \text{ of } 1.6\% = 0.509 \times 1.6\% \approx 0.8 \text{ of } 1\%$$

8. (Hard.) In a certain psychology experiment, each subject is presented with three ordinary playing cards, face down. The subject takes one of these cards. The subject also takes one card at random from a separate, full deck of playing cards. If the two cards are from the same suit, the subject wins a prize. What is the chance of winning? If more information is needed, explain what you need, and why.

The answers to these exercises are on pp. A67–68.

5. THE COLLINS CASE

People v. Collins is a law case in which there was a major statistical issue. A black man and a white woman were charged with robbery. The facts were described by the court as follows.[4]

> On June 18, 1964, about 11:30 A.M. Mrs. Juanita Brooks, who had been shopping, was walking home along an alley in the San Pedro area of the City of Los Angeles. She was pulling behind her a wicker basket carryall containing groceries and had her purse on top of the packages. She was using a cane. As she stooped down to pick up an empty carton, she was suddenly pushed to the ground by a person whom she neither saw nor heard approach. She was stunned by the fall and felt some pain. She managed to look up and saw a young woman running from the scene. According to Mrs. Brooks the latter appeared to weigh about 145 pounds, was wearing "something dark," and had hair "between a dark blond and a light blond," but lighter than the color of defendant Janet Collins' hair as it appeared at trial. Immediately after the incident, Mrs. Brooks discovered that her purse, containing between $35 and $40, was missing.
>
> About the same time as the robbery, John Bass, who lived on the street at the end of the alley, was in front of his house watering his lawn. His attention was attracted by "a lot of crying and screaming" coming from the alley. As he looked in that direction, he saw a woman run out of the alley and enter a yellow automobile parked across the street from him. He was unable to give the make of the car. The car started off immediately and pulled wide around another parked vehicle so that in the narrow street it passed within six feet of Bass. The latter then saw that it was being driven by a male Negro, wearing a mustache and beard. At the trial Bass identified defendant as the driver of the yellow automobile. However, an attempt was made to impeach his identification by his admission that at the preliminary hearing he testified to an uncertain identification at the police lineup shortly after the attack on Mrs. Brooks, when defendant was beardless.
>
> In his testimony Bass described the woman who ran from the alley as a Caucasian, slightly over five feet tall, of ordinary build, with her hair in a dark blond ponytail, and wearing dark clothing. He further testified that her ponytail was "just like" one which Janet had in a police photograph taken on June 22, 1964.

The prosecutor then had a mathematics instructor at a local state college explain the multiplication rule, without paying much attention to independence, or the distinction between conditional and unconditional probabilities. After this testimony, the prosecution assumed the following chances:

Yellow automobile	1/10	Woman with blond hair	1/3
Man with mustache	1/4	Black man with beard	1/10
Woman with ponytail	1/10	Interracial couple in car	1/1,000

When multiplied together, these come to 1 in 12,000,000. According to the prosecution, this procedure gave the chance "that any [other] couple possessed the distinctive characteristics of the defendants." If no other couple possessed these characteristics, the defendants were guilty. The jury convicted. On appeal, the Supreme Court of California reversed the verdict. It found no evidence to support the assumed values for the six chances. Furthermore, these were presented as unconditional probabilities. The basis for multiplying them, as the mathematics instructor should have explained, was independence. And there was no evidence to support that assumption either. On the contrary, some factors were clearly dependent—like "Black man with beard" and "interracial couple in car."

> Blindly multiplying chances can make real trouble. Check for independence, or use conditional probabilities.

There is another objection to the prosecutor's reasoning. Probability calculations like the multiplication rule were developed for dealing with games of chance, where the basic process can be repeated independently and under the same conditions. The prosecutor was trying to apply this theory to a unique event: something that either happened—or didn't happen—on June 18, 1964, at 11:30 A.M. What does chance mean, in this new context? It was up to the prosecutor to answer this question, and to show that the theory applied to his situation.[5]

In the 1990s, DNA evidence began to be used for identification of criminals: the idea is to match a suspect's DNA with DNA left at the scene of the crime—for instance, in bloodstains. Matching is done on a set of characteristics of DNA. The technical issues are similar to those raised by the Collins case: Can you estimate the fraction of the population with a given characteristic? Are those characteristics independent? Is the lab work accurate? Many experts believe that such questions have satisfactory answers, others are quite skeptical.[6]

6. REVIEW EXERCISES

When a die is rolled, each of the six faces is equally likely to come up. A deck of cards has 4 suits (clubs, diamonds, hearts, spades) with 13 cards in each suit—2, 3, . . . , 10, jack, queen, king, ace. See pp. 222 and 226.

1. True or false, and explain:
   (a) If something has probability 1,000%, it is sure to happen.
   (b) If something has probability 90%, it can be expected to happen about nine times as often as its opposite.

2. Two cards will be dealt off the top of a well-shuffled deck. You have a choice:
   (i) To win $1 if the first is a king.

(ii) To win \$1 if the first is a king and the second is a queen.

Which option is better? Or are they equivalent? Explain briefly.

3. Four cards will be dealt off the top of a well-shuffled deck. There are two options:

   (i) To win \$1 if the first card is a club and the second is a diamond and the third is a heart and the fourth is a spade.
   (ii) To win \$1 if the four cards are of four different suits.

   Which option is better? Or are they the same? Explain.

4. A poker hand is dealt. Find the chance that the first four cards are aces and the fifth is a king.

5. One ticket will be drawn at random from the box below. Are color and number independent? Explain.

$$\boxed{1}\ \boxed{1}\ \boxed{8}\ \boxed{1}\ \boxed{1}\ \boxed{8}$$

6. A deck of cards is shuffled and the top two cards are placed face down on a table. True or false, and explain:

   (a) There is 1 chance in 52 for the first card to be the ace of clubs.
   (b) There is 1 chance in 52 for the second card to be the ace of diamonds.
   (c) The chance of getting the ace of clubs and then the ace of diamonds is $1/52 \times 1/52$.

7. A coin is tossed six times. Two possible sequences of results are

   (i) H T T H T H (ii) H H H H H H

   (The coin must land H or T in the order given; H = heads, T = tails.) Which of the following is correct? Explain.[7]

   (a) Sequence (i) is more likely.
   (b) Sequence (ii) is more likely.
   (c) Both sequences are equally likely.

8. A die is rolled four times. What is the chance that—

   (a) all the rolls show 3 or more spots?
   (b) none of the rolls show 3 or more spots?
   (c) not all the rolls show 3 or more spots?

9. A die is rolled 10 times. Find the chance of—

   (a) getting 10 sixes.
   (b) not getting 10 sixes.
   (c) all the rolls showing 5 spots or less.

10. Which of the two options is better, or are they the same? Explain briefly.

    (i) You toss a coin 100 times. On each toss, if the coin lands heads, you win \$1. If it lands tails, you lose \$1.
    (ii) You draw 100 times at random with replacement from $\boxed{1}\ \boxed{0}$. On each draw, you are paid (in dollars) the number on the ticket.

11. In the box shown below, each ticket should have two numbers:

$$\boxed{\boxed{1\,\square}\ \boxed{1\,2}\ \boxed{1\,2}\ \boxed{1\,3}\ \boxed{3\,1}\ \boxed{3\,2}\ \boxed{3\,\square}\ \boxed{3\,\square}}$$

A ticket will be drawn at random. Can you fill in the blanks so the two numbers are independent?

12. You are thinking about playing a lottery. The rules: you buy a ticket, choose 3 different numbers from 1 to 100, and write them on the ticket. The lottery has a box with 100 balls numbered from 1 through 100. Three balls are drawn at random without replacement. If the numbers on these balls are the same as the numbers on your ticket, you win. (Order doesn't matter.) If you decide to play, what is your chance of winning?

7. SUMMARY

1. The *frequency theory* of chance applies most directly to chance processes which can be repeated over and over again, independently and under the same conditions.

2. The chance of something gives the percentage of times the thing is expected to happen, when the basic process is repeated over and over again.

3. Chances are between 0% and 100%. Impossibility is represented by 0%, certainty by 100%.

4. The chance of something equals 100% minus the chance of the opposite thing.

5. The chance that two things will both happen equals the chance that the first will happen, multiplied by the *conditional* chance that the second will happen given that the first has happened. This is the *multiplication rule*.

6. Two things are *independent* if the chances for the second one stay the same no matter how the first one turns out.

7. If two things are independent, the chance that both will happen equals the product of their unconditional chances. This is a special case of the multiplication rule.

8. When you draw at random, all the tickets in the box have the same chance to be picked. Draws made at random with replacement are independent. Without replacement, the draws are dependent.

9. Blindly multiplying chances can make real trouble. Check for independence, or use conditional chances.

10. The mathematical theory of chance only applies in some situations. Using it elsewhere can lead to ridiculous results.