圏論

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12022-76

Definition 0 (圏) 圏 A は次からなる.

- **對象**が集まり $\operatorname{obj} A$
- **射**が集まり $\operatorname{mor} A$
- 各 $a \in \operatorname{obj} A$ につき, **恆等射** $\operatorname{id}_A a \in \operatorname{obj} A$
- 各 $f \in \operatorname{mor} A$ につき, **域** $\operatorname{dom}_A a \in \operatorname{obj} A$ と**餘域** $\underline{\operatorname{dom}}_A b \in \operatorname{obj} A$
- 各 $f, g \operatorname{mor} A$, 但し $\underline{\operatorname{dom}}_A f = \operatorname{dom}_A g$ につき, **合成射** $f \mid_A g$

但し次を滿たす.

$$\operatorname{dom}_A(\operatorname{id}_A a) = \underline{\operatorname{dom}}_A(\operatorname{id}_A a) = a \tag{0}$$

$$\operatorname{dom}_A(f \mid_A g) = \operatorname{dom}_A f \tag{1}$$

$$\underline{\operatorname{dom}}_A(f \mid_A g) = \underline{\operatorname{dom}}_A g \tag{2}$$

$$\operatorname{dom}_A f = a \Rightarrow \operatorname{id}_A a \mid_A f = f \tag{左單位律}$$

$$\underline{\operatorname{dom}}_A f = b \Rightarrow f \mid_A \operatorname{id}_A b = f \tag{右單位律}$$

$$(f \mid_A g) \mid_A h = f \mid_A (g \mid_A h) \tag{結合律}$$

A の明らかなる時に $\operatorname{id}_A, \operatorname{dom}_A, \ldots$ 等の添へ字を省略す.

definition 0 が條件を圖にせば次となる.

$$\begin{tikzcd} a \arrow[loop left, "\operatorname{id} a"] \end{tikzcd}$$

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "f|g"'] & b \arrow[d, "g"] \\ & c \end{tikzcd}$$

$$\begin{tikzcd} a \arrow[r, equal, "\operatorname{id} a"] \arrow[dr, "\operatorname{id} a|f"'] & a \arrow[d, "f"] \arrow[dr, "f|\operatorname{id} b"] & \\ & b \arrow[r, equal, "\operatorname{id} b"'] & b \end{tikzcd}$$

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "f|g"'] \arrow[drr, bend left=60, "f|(g|h)"] \arrow[drr, bend right=60, "(f|g)|h"'] & b \arrow[d, "g"] \arrow[dr, "g|h"] & \\ & c \arrow[r, "h"'] & d \end{tikzcd}$$

次なる略記を使ふ.

$$a \in A \Leftrightarrow a \in \operatorname{obj} A \tag{3}$$
$$f \in A \Leftrightarrow f \in \operatorname{mor} A \tag{4}$$
$$f : a \to b \Leftrightarrow a \xrightarrow{f} b \Leftrightarrow \operatorname{dom} f = a \wedge \underline{\operatorname{dom}} f = b \tag{5}$$
$$f \in A(a, b) \Leftrightarrow f \in A \wedge f : a \to b \tag{6}$$

Example 0
- $\operatorname{obj} \mathbb{1} = \{*\}, \operatorname{mor} \mathbb{1} = \{\operatorname{id} *\}$ なる $\mathbb{1}$

Definition 1 (同型) 對象 a, b が**同型**なる $\Leftrightarrow$

$$\exists (f \in A(a, b), g \in A(b, a)),\ \ f \mid g = \operatorname{id} a \wedge g \mid f = \operatorname{id} b \tag{7}$$

Definition 2 (函手) 圏 A, B につき, 函手 F は次からなる.

- $\operatorname{obj} A$ から $\operatorname{obj} B$ が写像 F_{obj}
- $\operatorname{mor} A$ から $\operatorname{mor} B$ が写像 F_{mor}

但し次を満たす.

$$F_{\mathrm{mor}}(\operatorname{id}_A a) = \operatorname{id}_B(F_{\mathrm{obj}} a) \tag{8}$$
$$F_{\mathrm{mor}}(f \mid_A g) = (F_{\mathrm{mor}} f) \mid_B (F_{\mathrm{mor}} g) \tag{9}$$

次なる略記を使ふ.

$$Fa = F_{\mathrm{obj}} a \tag{10}$$
$$Ff = F_{\mathrm{mor}} f \tag{11}$$
$$\tag{12}$$

Definition 3 (終對象) 次を満たす對象 1 は A が**終對象**なり.

$$\forall a \in A,\ \exists_1 u,\ \ u : a \to 1 \tag{13}$$

$$\forall a \overset{u}{\dashrightarrow} 1$$

Definition 4 (餘終對象) 次を満たす對象 0 は A が**餘終對象** (始對象) なり.

$$\forall a \in A,\ \exists_1 u,\ \ u : 0 \to a \tag{14}$$

$$\forall a \underset{u}{\dashleftarrow} 0$$

Definition 5 (雙對圏) 圏 A が**雙對圏** $\underline{A}$ を次にて定義す.

$$\operatorname{obj} \underline{A} = \operatorname{obj} A \tag{15}$$
$$\operatorname{mor} \underline{A} = \operatorname{mor} A \tag{16}$$
$$\underline{\operatorname{dom}}_{\underline{A}} = \operatorname{dom}_A \tag{17}$$
$$\operatorname{dom}_{\underline{A}} = \underline{\operatorname{dom}}_A \tag{18}$$
$$g \mid_{\underline{A}} f = f \mid_A g \tag{19}$$

Example 1 圏 A は終對象 1 を持てば, $\forall a, \exists_1 u, u : 1 \to a$. 從ひて $\underline{A}$ にて $\forall a, \exists_1 u, u : a \to 1$.
終對象は雙對圏にて餘終對象なり. 以ちて餘終對象は終對象が**雙對**なると言ふ. 逆も真なり.

Definition 6 (積) $a_0, a_1, a_0 \times a_1 \in A, \pi_0 : a_0 \times a_1 \to a_0, \pi_1 : a_0 \times a_1 \to a_1$ とす. 次を滿たす $(a_0 \times a_1, \pi_0, \pi_1)$ は a_0, a_1 が積なり.

$$(\forall i \in \{0, 1\},\ \forall f_i : b \to a_i),\ \exists_1 f_0 \times f_1,\ f_i = g \mid \pi_i \tag{20}$$

$$\begin{tikzcd}
 & b \arrow[dl, "f_0"'] \arrow[d, dotted, "f_0 \times f_1"] \arrow[dr, "f_1"] & \\
a_0 & a_0 \times a_1 \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & a_1
\end{tikzcd}$$

Definition 7 (餘積) 餘積 (和) $(a_0 + a_1, \underline{\pi}_0, \underline{\pi}_1)$ は積が雙對なり.

$$\begin{tikzcd}
 & b & \\
a_0 \arrow[ur, "f_0"] \arrow[r, "\underline{\pi}_0"'] & a_0 + a_1 \arrow[u, dotted, "f_0 + f_1"] & a_1 \arrow[ul, "f_1"'] \arrow[l, "\underline{\pi}_1"]
\end{tikzcd}$$

Definition 8 (自然變換) 圏 A, B, 函手 $F, G : A \to B$ につき, **自然變換** $\alpha : F \to G$ は射の集合 $\{\alpha_a\}_{a \in A}$ なり. 但し次を滿たす.

$$\forall f \in A(a, b),\ \alpha_a \mid_B Gf = Ff \mid_B \alpha_b \tag{21}$$

$$\begin{tikzcd}
Fa \arrow[r, "Ff"] \arrow[d, "\alpha_a"] & Fb \arrow[d, "\alpha_b"] \\
Ga \arrow[r, "Gf"] & Gb
\end{tikzcd}$$

Definition 9 (函手圏) 圏 A, B につき, **函手圏** B^A は次なる圏なり.

$$\operatorname{obj} B^A = \{F; F : A \to B\} \tag{22}$$
$$\operatorname{mor} B^A = \{\alpha; \alpha : F \to G \wedge F, G : A \to B\} \tag{23}$$
$$\alpha \mid \beta = \{\alpha_a \mid \beta_a; a \in A\} \tag{24}$$

Definition 10 (定函手) 圏 A につき $\Delta_a : A \to \{a\}$ は $a \in A$ が**定函手**なり. 但し次を滿たす.

$$\forall b,\ \Delta_a b = a \tag{25}$$
$$\forall f,\ \Delta_a f = \operatorname{id} a \tag{26}$$

Definition 11 (comma 圏) $B \overset{F}{\to} A \overset{G}{\leftarrow} C$ につき, **comma 圏** F/G は次なる圏なり.

$$F/G \ni (b, c, f) \Leftrightarrow b \in \operatorname{obj} B \wedge c \in \operatorname{obj} C \wedge f : Fb \to Gc \tag{27}$$

$$F/G \ni (b, c, f) \overset{(g,h)}{\to} (b', c', f') \Leftrightarrow b \overset{g}{\to} b' \in B \wedge c \overset{h}{\to} c' \in C \wedge f \mid_A Gh = Fg \mid_A f' \tag{28}$$

$$\begin{tikzcd}
b \arrow[r, "g"] & b' & Fb \arrow[r, "Fg"] \arrow[d, "f"] & Fb' \arrow[d, "f'"] & & \\
 & & Gc \arrow[r, "Gh"] & Gc' & c \arrow[r, "h"] & c'
\end{tikzcd}$$

Definition 12 (slice 圏) slice 圏 A/a は $A \overset{\operatorname{id} A}{\to} A \overset{a}{\leftarrow} \mathbb{1}$ が comma 圏なり.

$$(b, f) \in F/G \Leftrightarrow f : b \to a \tag{29}$$

$$(b, f) \overset{g}{\to} (b', f') \in F/G \Leftrightarrow b \overset{g}{\to} b' \in B \wedge f = g \mid f' \tag{30}$$

$$\begin{tikzcd} b \arrow[r, "g"] \arrow[dr, "f"'] & b' \arrow[d, "f'"] \\ & a \end{tikzcd}$$

Definition 13 (錐) 圏 A, 函手 $F : I \to A$ につき, F への**錐**とは $(a \in A, \alpha : \Delta_a \to F)$ なり.

$$\begin{tikzcd} & & a \arrow[d, "\alpha_i"'] \arrow[dr, "\alpha_j"] & \\ i \arrow[r, "f"] & j & Fi \arrow[r, "Ff"'] & Fj \end{tikzcd}$$

Definition 14 (極限) $F : I \xrightarrow{F} A$ につき, F が**極限**とは錐 $(\lim F, \alpha)$ なり. 但し次を満たす.

$$\forall(b, \beta),\ \exists_1 u : b \to \lim F,\ \forall i,\ \beta_i = u \mid \alpha_i \tag{31}$$

$$\begin{tikzcd} \forall b \arrow[r, dotted, "u"] \arrow[dr, "\beta_i"'] & \lim F \arrow[d, "\alpha_i"] \\ & Fi \end{tikzcd}$$

Example 2 (終對象と積は極限)

- 終對象は空圏 $\emptyset$ からの函手が極限なり.
- 積は 2 元離散圏 $\{*, *'\}$ からの函手が極限なり.

Definition 15 (普遍射) 函手 $F : A \to B$ につき, F から b への**普遍射**は comma 圏 F/a が終對象 (a, f) なり. 即ち

$$\forall f' : Fa \to b,\ \exists_1 g : a \to a',\ f' = Fg \mid f \tag{32}$$

$$\begin{tikzcd} a' \arrow[r, dotted, "g"] & a & Fa' \arrow[r, "Fg"] \arrow[dr, "\forall f'"'] & Fa \arrow[d, "f"] \\ & & & b \end{tikzcd}$$

Example 3 (極限は普遍射)

$$\Delta : A \to A^I \tag{33}$$
$$\Delta a : I \to A \tag{34}$$
$$\Delta a = \Delta_a \tag{35}$$
$$\Delta f : A^I \to A^I \tag{36}$$
$$F : I \to A \tag{37}$$

とせば (a, α) は F への錐なり.

Δ から F への普遍射は極限 $(\lim F, \alpha)$ なり.

$$\begin{tikzcd} b \arrow[r, dotted, "f"] & \lim F & \Delta b \arrow[r, "\Delta f"] \arrow[dr, "\forall \beta"'] & \Delta \lim F \arrow[d, "\alpha"] \\ & & & F \end{tikzcd}$$

Definition 16 (随伴) $F : A \to B$ は $G : A \leftarrow B$ が**左随伴**なり $\Leftrightarrow \forall a \in A, \exists a$ から G への普遍射 $\Leftrightarrow$

$$\forall a \in A,\ \forall f : Fx \to a,\ \exists_1 g : a' \to a,\ f = Fg \mid \varepsilon_x \tag{38}$$

$$
\begin{tikzcd}
Fb \arrow[d, "Fg"] \arrow[dr, "f"] & & \\
F(Ga) \arrow[r, "\varepsilon_x"'] & x & Ga
\end{tikzcd}
$$

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